THE FUNDAMENTALS OF
DIFFERENTIAL
CALCULIS

WITH THE USE OF WHICH

IN FINITE ANALYSIS

AND

IN THE PRINCIPLES OF SERIES

BY THE AUTHOR

LEONHARD EULER

DIRECTOR, PRUSSIAN ROYAL ACAD. SCIENCE AND FINE LITT.
PROF. HONOR. IMP. ACAD. SCIENCE ST.PETERSBURGH, AND FELLOW OF THE ROYAL SOCIETIES OF PARIS AND LONDON.

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PREFACE

What Differential Calculus and in general the Infinitesimal Analysis may be, for those who hitherto have been imbued with none of this knowledge, can scarcely be set out here nor is it permitted, as usually happens in other disciplines, that the introduction of the treatise be taken from a convenient definition. Not because a clear definition of this calculus may not be given, but because in order to understand that, there is a need for notions of this kind, since not only ideas in everyday [mathematical] usage , but truly also ideas less used from that finite Analysis, are accustomed at last to perform the calculations and to set out the explanations pertaining to the differential calculus; with which done, so that the definition of this unable before is now able to be perceived, as the principles of which should be evident enough.

Therefore here in the first place the calculus is about variable quantities ; for although all the quantities by their nature can be increased or diminished indefinitely, yet, while a calculation is directed according to a certain precept, other quantities are considered to retain the same magnitude, truly the others are to be varied by all orders of increase or decrease ; indeed a distinction is required to be noted between these constant quantities and those quantities usually called variables, so that thus the nature of this distinction may be suitably put in place, not only into the nature of the matter [in general] but also according to the nature of the investigation, to which the calculation is referring.

Because this difference between constant and variable quantities will be greatly illustrated by an example, we may consider the throwing of a ball from a cannon by the force of a gunpowder explosion, if indeed this example may be considered suitable for the elucidation in the first place. Here therefore many quantities occur, of which an account is required to be considered in this investigation: evidently first the quantity of gunpowder ; then the elevation of the gun above the horizon; in the third place the distance thrown on a horizontal place; in the fourth place the time, in which the exploded ball is turning in the air ; and unless the experiments are put in place with the same gun, in addition the length of this with the weight of the ball must be drawn into the calculation. Indeed here we may remove the mind from the variation of cannon and ball, lest we fall upon exceedingly complicated questions. But if therefore with the same quantity of gunpowder preserved, the elevation of the gun may be continually unchanged, and the length thrown with the time the transit time of the globe through the air required, in this question the quantity of powder or the impulse force will be a constant quantity; but the elevation of the gun with the length thrown and with the duration will be referred to as variable quantities, if indeed for all the elevation degrees we may wish to define this quantity, so that thence it may become known, how great the changes in the length and the duration of the throw may arise from all the variations of the elevation. But there will be another question, if, with the gun keeping the same elevation, the quantity of gunpowder may be changed continually and the changes must be defined which thence may be many in the throw; for here the elevation of the gun will be a constant quantity, truly the quantity of gunpowder against both the length and duration of the throw are variable quantities. Hence thus it may be apparent, how with a change of the question in place the same quantity at one time may be counted among the constants and at another time among the variables ; but likewise hence it is understood, to which it is to be attended to especially in this matter, how the other variable quantities thus may depend on these others, so that with one changed by necessity the others receive changes. Evidently in the first case, in which the quantity of gunpowder will remain the same, with a change in the elevation of the gun also the length and duration of the throw are changed and therefore the length and the duration of the throw are variable quantities depending on the elevation of the gun and with this change likewise certain sure changes are allowed; indeed in the latter case they depend on the quantity of gunpowder, with the change in which certain changes must be produced in these.

But which quantities depend on the others in this manner, so that from these changes these themselves also undergo changes, the latter are accustomed to be called functions of the former; which denomination extends most widely and all the ways, by which one quantity can be determined by others, is included in itself. Therefore if x may denote a variable quantity, all the quantities, which depend on x in some manner or may be determined through that, are called functions of this ; the quadratic of xx or of some other power of this are of this kind as well as any quantities composed from these, so that also transcending and in general whatever thus depend on x, so that with an increase or decrease of x itself they receive changes. Hence now the question arises, by which it is sought, if the quantity x may be increased or diminished by a given quantity, by how much in some manner these functions of this may be changed, or what amount of increase or decrease they may take. Indeed in the more simple cases this question can be resolved easily; for if the quantity x may be augmented by the quantity ω, the square of this xx hence may take the increment $2x\omega+\omega\omega$ and thus the increment of x itself to the increment of xx may be considered as ω to $2x\omega+\omega\omega$, that is as 1 to $2x+\omega$; and in a similar manner in other cases the ratio of the increment of x to the increment or decrement, which whatever function of this thence is come upon, is accustomed to be considered. Truly the investigation of a ratio of increments of this kind itself is worthy of the greatest interest, but also the whole of the infinitesimal Analysis leans on that. So that by which it may appear clearer, we may take the example of the above square xx, the increment of which $2x\omega+\omega\omega$, which is taken, while the quantity x itself may be augmented by the increment ω, we will see according to this ration to hold as $2x+\omega$ to 1; from which it is evident, with which increment ω being taken less, there the closer that ratio approaches to the ratio $2x$ to 1; yet clearly nor will it change into this ratio before that increment ω plainly may vanish. Hence we understand, if the increment ω of the variable quantity x becomes nothing, then also the increment of this square xx thence arising indeed vanishes, yet truly so that the ratio is maintained as $2x$ to 1; and what has been said here regarding the quadratic, is required to be understood concerning all functions of x ; evidently the vanishing increments of which, which they take, while the increment of the quantity x taken itself vanishing, will hold a sure and assignable ratio to that. And in this manner we have deduced the definition *The Differential Calculus*, which is the *method of determining the ratio of vanishing increments, which any functions accept, while to the variable quantity, of which they are functions, is attributed a vanishing increment;* and in this definition the true nature of the differential calculus is to be contained and to be drawn from those especially, which will be readily apparent to those who are not unfamiliar with the subject.

Therefore the calculus of the differential to be sought is occupied not only with these vanishing increments, which evidently are zero, but also in the ratio and mutual proportion of these requiring to be scrutinised ; and since these ratios may be expressed by finite quantities, also here the calculus is considered to be concerned with finite quantities. Indeed whatever the precepts, as they are usually applied generally to these vanishing increments to be defined in an appropriate manner, yet at no time with these to be considered absolutely, but rather conclusions are deduced from the ratio of these always. Now in a similar manner in the integral calculus an account has been prepared, which may be defined thus must conveniently, as it may be said to be *the method from a know ratio of the vanishing increments, these functions themselves are to be found, of which they are the increments.*

But where these ratios can be deduced easily and may be able to be represented in the calculation, these vanishing increments themselves, even if they shall be zero, are customarily designated by certain signs ; from which with nothing given standing in the way, there is no reason why certain names should not be imposed on these. And thus these may be called the differentials, which, since from the quantity they have abandoned, also may be called infinitely small, which therefore by their nature are to be interpreted thus, so that they may be regarded as nothing at all in general or equal to zero [Recall that Latin has no word for the number zero, as it was not recognised as a number in Roman times, and this translation uses zero for this occasionally]. Thus if ω is the attributed increment of the quantity x , so that it may change into $x+\omega$, the square of this xx will change into $xx+2x\omega+\omega\omega$ and thus it takes the increment $2x\omega+\omega\omega$; whereby the increment of x, which is ω, itself may be considered to the increment of the square, which is itself $2x\omega+\omega\omega$, as 1 to $2x+\omega$; which ratio then at last becomes 1 to $2x$, since ω vanishes. Therefore there is made $\omega=0$ and the ratio of these vanishing increments, which is seen only in the differential calculus, everywhere is as 1 to 2 x; nor in turn may this ratio be considered to be true, unless that increment ω actually may vanish and becomes completely equal to nothing. But if hence this nothing indicated by ω may refer to the increment of the quantity x, because this may be itself considered to the increment of the square xx as 1 to $2x$, the increment of the square $xx = 2x\omega$ and thus also equal to nothing; from which likewise it may be agreed that the annihilation of these increments does not stand in the way, by which the ratio of these, which is as 1 to $2x$ any the less shall be determined. Because now here the letter ω may be presenting nothing, that in differential calculus, because it may be considered as the increment of the quantity x, to be represented by the sign dx and is usually called the differential of this ; and on putting dx in place of ω the differential of xx will be $2xdx$. In a similar manner it is shown that the differential of the cube x^3 becomes $=3xxdx$ and in the case of each power x^n the differential becomes $=nx^{n-1}dx$. Moreover, whatever other functions of x may be proposed, the rules of finding these differentials are to be treated in the differential calculus; truly it is to be understood always, since these absolute differentials shall be zero, from these nothing other can be concluded apart from the mutual ratios of these, which certainly are reduced to finite quantities. But since in this manner, which alone is agreeable to the account, the principles of the differential Calculus are stabilized, all the detractions, which are usually advanced against the calculus, at once fall down ; which yet retain great strength, if the differentials or infinitely small quantities may not clearly be annihilated.

But by several, who propound the precepts of the differential Calculus, it is considered that the differentials are distinct from absolutely zero and to be in a singular order of quantities of the infinitely small, which do not vanish completely, but which retain a certain quantity to be put in place, which indeed is to be assigned very small for all ; for these therefore it is justly objected that the rigor of geometry has been ignored and conclusions thence deduced, because hence infinitely small quantities of this kind may be disregarded, are deservedly to be suspect [by us]; for however small these infinitely small quantities may be considered to be, yet not only with a single, but also with several and also with innumerable small parts likewise rejected [by us] an immense error may result still. Which perverse objection they try to overcome with examples of this kind, in which the same conclusions are elicited by differential calculus and elementary geometry; for if these infinitely small parts which are ignored in the calculation are not zero, thence by necessity an error must result and this greater from that, when these are more added together ; and this is to be attributed if it should happen less, that occurs rather by the fault of the calculation, where sometimes errors are compensated by other errors, so that the calculus itself may be freed from the suspicion of errors. But if moreover a balancing of items is made with no new error of this kind, from such examples that which I want may itself prevail splendidly : those quantities which were to be disregarded, entirely and absolutely are to be considered as zero and not as infinitely small, which are propounded in the [version of the] differential calculus that disagrees from zero absolutely. Also the operation is put together most easily, when they are described thus by several infinitely small parts, so that the image of dust particles may be considered in regard of a great mountain or also of the whole globe of the earth ; for although which magnitude of the whole earth should enter in determining the calculation, the error for that is not of one grain, but may be accustomed to be given by several thousands of grains of dust, yet geometrical rigor also abhors such a small error and there shall be this exceedingly weighty objection, if it were to retain any strength. Then also with the difficulty stated, what gain thence do they hope for, for those who wish to distinguish the infinitely small from zero; but they fear, lest, if plainly the increments vanish, also the preparations of these are removed, by which they consider that the whole matter to be established; just as indeed in no manner is it conceived possible to profess [by them] that absolute zeros are able to be compared between each other. Therefore by necessity they think from these that some magnitude remains, from which in some degree they consider that they can put the preparation in place; yet they compel this magnitude to be admitted so small, so that as if it should be zero, it is able to be considered and to be ignored in the calculation without error. Yet nor do they dare to assign a certain definite magnitude to this, on account of being incomprehensibly small ; indeed always, that may be assumed two or three times smaller, and the preparations are able to be considered in the same manner. From which it is evident clearly that nothing is brought to the preparation to be put in place and this clearly is not removed, even if that magnitude vanishes completely.

But from what has been said above it is evident that the preparation, which is considered in the differential calculus, indeed may not be considered, unless these increments therefore vanish ; indeed the increment of the quantity *x*, which in general we have indicated by ω, to the increment of the square *xx*, which is $2x\omega+\omega\omega$, has the ratio as 1 to $2x+\omega$; which always differs from the ratio to *2x*, unless there shall be $\omega=0$; but if we put in place to be $\omega=0$, then at last we are able to affirm truly this ratio to become exactly as 1 to 2*x*. Meanwhile still it is evident, so that that increment ω may be taken smaller, with that approaching nearer to this ratio ; from which not only is it permitted, but also it agrees with the nature of the thing these increments are to be considered as finite and also in figures, if there is a need for illustrating this to be represented finitely; then truly these increments by thinking are considered to become continually smaller and thus the ratio of these continually may be found to approach a certain limit, which then at last they may reach, since plainly they have departed into nothing. But here the limit, which as if it establishes an ultimate ratio of the increments, truly is the object of the differential calculus; therefore the first fundamentals of which that have been laid down is required to be evaluated, for which these ultimate ratios that first come to mind are to be considered, according to which the increments of the variable quantity, while continually made smaller, approach and since they vanish, then at last they reach [the final ratio].

But we fall upon the vestiges of this speculation in the writings of the ancient authors, from which a certain polished idea concerned with the infinitesimal analysis cannot be denied. Then little by little this science has taken greater advancement nor suddenly has it been carried to that peak, at which now it is discerned, even if indeed in that much more at this stage shall be hidden than dragged forwards to the light. For since the differential calculus may be extended to functions of all kinds, in whatever manner they may be composed, the method has become known gradually how the vanishing increments of all functions clearly are to be prepared among themselves; but slowly this discovery has advanced continually to more complicated functions. Because evidently the ultimate ratio concerning rational functions, which the vanishing increments hold between themselves, was able to be assigned much before the times of Newton and Leibnitz, thus so that the differential calculus, as far as it may be applied to rational functions only, shall be agreed to have been found long before this time. Then indeed there is no doubt, that we must accept to refer to Newton concerning that part of the differential calculus about irrational functions; according to which his distinguished theorem concerning the evaluation of the general powers of the binomial has been happily deduced, from which excellent discovery the limits of the differential calculus now have been wonderfully extended. But we are no less obliged to Leibnitz, because this calculus previously was viewed just as individual artifices, only he reduced that to the form of a discipline and the precepts of which he gathered as it were into a system and which he explained clearly. Hence indeed great aids were being suggested towards further improving the calculus and these, which at this point were desired, were to be elicited from certain principles. Therefore soon from the enthusiasm of Leibniz and then of the Bernoulli's the fast moving boundaries of the differential calculus had moved forwards from that also to transcending functions, which part was treated at this time, then indeed also the most solid foundations of the integral calculus were established; with which pursuits, which they elaborated on in this manner, continually more advancements were added on. Newton now also had given the fullest models of the integral calculus, the first discovery of this thus cannot be put in place absolutely, since it could scarcely be separated from the first beginnings of the differential calculus ; and because a great part of this still remains to be improved, here the calculus now indeed cannot be considered to have been completely discovered, but rather, how much resources may be granted to anyone to bring this project towards perfection, that we must acknowledge with a grateful mind. And I judge concerning this glory of the invention of the calculus, it is to be born in mind that before there was indeed so much held in dispute.

But because according to the various names, which the mathematicians of different countries are accustomed to impose on the calculus, it may be held, that all here correspond, so that here they agree uncommonly well with the given definition; indeed these vanishing increments, the ratio of which may be considered, may be called either differentials or fluxions, these are required to be understood always to be equal to zero; truly in which the idea of the infinitely small must be established. Hence indeed also everything, which concerns the differentials of the second or of higher orders oddly have been disputed more than they should have been, are to be delivered most clearly, since everything by themselves vanish equally, nor are these at any time to be considered by themselves, but rather the mutual relation of these are accustomed to be considered. For since the ratio, which the vanishing increments of two functions hold, again is expressed by a certain function, and if both the increments of this function vanish when brought together with the other, the matter must be referred to the second differential; and thus again the progression to differentials of higher orders ought to be understood, thus so that always finite quantities actually are kept in mind and only the differential signs may be used that represent these conveniently. Indeed in the first place by considering that description of the infinitesimal analysis, it may appear both exceedingly sterile and trivial, although those arcane infinitely small kinds of quantities are no longer promised in the matter ; truly if we may understand correctly the ratios, which exist between the vanishing increments of some functions, this understanding on many occasions by itself is of great interest, then thus indeed it is necessary in most investigations, and with these of the greatest difficulty, as clearly nothing can be understood without the aid of these. Just as if the question shall be concerned with the motion of the ball from the cannon explosion and likewise the resistance of the air must be considered, how the motion through a finite distance shall become known, cannot be defined at once in any way, as well as the direction of the path in which the ball advances, while the speed of this on which the resistance depends, is changed at some instant. But we may consider a much smaller distance in which the motion is made, there the variation will be less and it will be possible from that to reach a true understanding ; but if moreover we may return that distance as clearly vanishing, because now all the inequalities both in the direction as well as in the speed are removed, the effect of the resistance by the laws of motion accurately define the motion and it will be permitted to assign change of the motion at an instant of time. But with these instantaneous changes or rather, since these shall be zero, now we are to gain the most from the mutual relation of these; and there is a need for the integral calculus to infer the change in the motion through a finite distance. But I consider it hardly to be necessary to show by more examples the use of differential calculus and of infinite analysis, since now indeed it shall be examined well enough, even if the most trivial investigation, in which the motion of bodies either of solids or fluids are undertaken, we may wish to set up more carefully, that can only be prepared with the use of infinitesimal analysis, but this skill itself often may not be developed well enough, so that we may wish to explain the matter thoroughly. Evidently through all the parts of mathematics the uses of this higher analysis may be extended thus, so that everything, which without the intervention of this still may be allowed to be established, are to considered as amounting to nothing.

Therefore I have put in place in this book the whole differential calculus to be derived from the true principles and thus to be treated copiously, so that I may omit none of these things which hitherto have indeed been found pertaining to that. The work is divided into two parts, in the first of which with the foundations of differential calculus laid down I have explained the method, functions of all kinds requiring to be differentiated not only the differentials of the first order, but also of higher orders to be found, either involving functions of a single variable or of two or more variables. But in the other part I have explained at great length the use of this calculus in finite analysis and in the principles of series; where also I have explained clearly especially the theorem of maxima and minima. But concerning the use of this calculus in the curvature of lines in geometry at this point I offer nothing , because that will be less desirable, since this part thus shall be treated copiously in other works, so that certainly in this manner the first principles of differential calculus shall be sought as it were from geometry and according to this skill applied with the greatest care, since scarcely have the principles been set out well enough. But thus here everything must be contained within the limits of pure analysis, so that indeed there would be no need for any figure requiring to be explained according to all the precepts of this calculus.

THE FUNDAMENTALS OF DIFFERENTIAL CALCULUS

FIRST PART

CONTAINING

A COMPLETE EXPLANATION OF THIS CALCULUS.

CHAPTER I

CONCERNING FINITE DIFFERENCES

1. From these, which have been explained in the above book concerning variable quantities and functions, [i.e. Euler's *Introductio in analysin infinitorum*, referred to several time here, and to be had elsewhere in translation], it is evident, as the quantity [or magnitude] of the variable actually is varied, thus all the functions of this undergo a variation. Thus, if the magnitude of the variable x takes the increment ω, thus so that for x there may be written $x+\omega$, all the functions of x, of which kind are xx, x^3, $\frac{a+x}{xx+aa}$, may adopt other values: evidently xx will change into $xx+2x\omega+\omega\omega$, x^3 will change into $x^3+3xx\omega+3x\omega\omega+\omega^3$ and $\frac{a+x}{xx+aa}$ will be changed into $\frac{a+x+\omega}{aa+xx+2x\omega+\omega\omega}$. Therefore an alteration of this kind always will arise, unless the function only pretends to be a variable quantity, but actually it shall be constant, just as x^0; in which case such a function remains invariant, in whatever manner the quantity x may be changed.

2. Since which shall have been explained well enough, we may approach these well disposed functions, on which the whole of the infinitesimal calculus is supported. Therefore let y be some function of the variable quantity x, for which successively the values may be substituted in arithmetical progression, evidently $x, x+\omega, x+2\omega, x+3\omega, x+4\omega,$ etc., and let y^{I} denote the value, which the function y adopts, if in that in place of x there is substituted $x+\omega$; in a similar manner let y^{II} be this value of y, if in place of x there may be written $x+2\omega$, and by like reasoning $y^{\mathrm{III}}, y^{\mathrm{IV}}, y^{\mathrm{V}}$, etc. may denote the values of y, which arise, while in place of x there are placed $x+3\omega, x+4\omega, x+5\omega$ etc., thus so that these different values of x and y may correspond in the following way to themselves:

$$x,\ x+\omega,\ x+2\omega, x+3\omega,\ x+4\omega,\ x+5\omega,\ \text{etc.}$$
$$y,\quad y^{\mathrm{I}},\quad y^{\mathrm{II}},\quad y^{\mathrm{III}},\quad y^{\mathrm{IV}},\quad y^{\mathrm{V}},\quad \text{etc.}$$

3. Just as the arithmetical series $x, x+\omega, x+2\omega$ etc. is able to be continued indefinitely, thus the series arising from the function $y: y, y^{\mathrm{I}}, y^{\mathrm{II}}$ etc. also may progress indefinitely and the nature of this will depend on the character of function y. Thus, if there should be y

= *x* or *y* = *ax* +*b*, then the series $y, y^{\mathrm{I}}, y^{\mathrm{II}}$ etc. also will be arithmetical; if it were $y = \frac{a}{bx+c}$ then the harmonic series will be produced; but if it should be $y = a^x$, a geometric series will be had. Nor can any series be devised, which does not arise in this way from a certain function of *y* ; moreover it is customary to call a function of this kind of *x* on account of the series, from which thus it arises, the *general term* of this*;* whereby, since all the series should have a general term, that in turn arises from a certain function of x, as it is usual in the principles of series to be explained further.

4. But here we attend chiefly to the differences, by which the terms of the series $y, y^{\mathrm{I}}, y^{\mathrm{II}}, y^{\mathrm{III}}$ etc. disagree between themselves; so that we may adapt ourselves to the nature of the differences, we may indicate which by the following signs so that there shall be

$$y^{\mathrm{I}} - y = \Delta y, \quad y^{\mathrm{II}} - y^{\mathrm{I}} = \Delta y^{\mathrm{I}}, \quad y^{\mathrm{III}} - y^{\mathrm{II}} = \Delta y^{\mathrm{II}} \quad \text{etc.}$$

Therefore Δy expresses the increment, which the function *y* takes, if in place of *x* there is placed $x + \omega$ with ω denoting some number assumed as it pleases. Indeed in the teaching of series it is accustomed to take $\omega = 1$; truly here it is arranged according to our custom to be used with a general value, which by choice is able to be increased or diminished. Also it is accustomed to call this increment Δy the *difference* of this function *y,* by which the following value y^{I} exceeds the value *y*, and continually as the increments may be considered, even if more often perhaps a decrement may be shown, that is recognised from the negative value of this.

5. Because y^{II} arises from *y,* if in place of *x* there may be written $x + 2\omega$, it is evident the same quantity will arise, if first for *x* there is put in place $x + \omega$ and then anew $x + \omega$ may be put in place of x. Hence y^{II} may arise from y^{I}, if in place of *x* there may be written $x + \omega$*;* and evidently the increment of y^{I} will be Δy^{I}, which is taken on putting $x + \omega$ in place of *x;* and thus Δy^{I} in a similar manner is called the *difference* of y^{I}. Again by the same reason there will be the *difference* Δy^{II} of y^{II} or the increment of this, which it accepts, if in place of *x* there is put $x + \omega$*;* and Δy^{III} will be the *difference* or increment of y^{III} and thus again. With this agreed upon from the series of values of y, which are $y, y^{\mathrm{I}}, y^{\mathrm{II}}, y^{\mathrm{III}}$ etc., there will be obtained the series of the differences $\Delta y, \Delta y^{\mathrm{I}}, \Delta y^{\mathrm{II}}$ etc., which are found, if any term of that series is subtracted from the following.

6. With the series of the differences found if from that anew as many as you please differences may be taken by subtracting from the sequence, the differences of the differences will arise, which are called the *second differences* and may be represented conveniently by characters in this manner, so that there is indicated

$$\begin{aligned}
\Delta\Delta y &= \Delta y^{\mathrm{I}} - \Delta y\\
\Delta\Delta y^{\mathrm{I}} &= \Delta y^{\mathrm{II}} - \Delta y^{\mathrm{I}}\\
\Delta\Delta y^{\mathrm{II}} &= \Delta y^{\mathrm{III}} - \Delta y^{\mathrm{II}}\\
\Delta\Delta y^{\mathrm{III}} &= \Delta y^{\mathrm{IV}} - \Delta y^{\mathrm{III}}\\
&\text{etc.}
\end{aligned}$$

And thus $\Delta\Delta y$ is called the second difference of y, $\Delta\Delta y^{\mathrm{I}}$ the second difference of y^{I} and thus so on. Moreover in a similar manner from the second differences, if anew the differences of these may be taken, there will be produced the third differences to be written in this way $\Delta^3 y$, $\Delta^3 y^{\mathrm{I}}$ etc., and hence again the fourth order $\Delta^4 y$, $\Delta^4 y^{\mathrm{I}}$ etc. and thus further, as far as it should please.

7. We may represent these series of differences thus outlined, from which the connection of these is obvious to see:

The arithmetic progression

$$x,\quad x+\omega,\quad x+2\omega,\quad x+3\omega,\quad x+4\omega,\quad x+5\omega,\ \text{etc.}$$

Values of the function

$$y,\quad y^{\mathrm{I}},\quad y^{\mathrm{II}},\quad y^{\mathrm{III}},\quad y^{\mathrm{IV}},\quad y^{\mathrm{V}}\quad \text{etc.}$$

First differences

$$\Delta y,\quad \Delta y^{\mathrm{I}},\quad \Delta y^{\mathrm{II}},\quad \Delta y^{\mathrm{III}},\quad \Delta y^{\mathrm{IV}}\quad \text{etc.}$$

Second differences

$$\Delta\Delta y,\quad \Delta\Delta y^{\mathrm{I}},\quad \Delta\Delta y^{\mathrm{II}},\quad \Delta\Delta y^{\mathrm{III}}\quad \text{etc.}$$

Third differences

$$\Delta^3 y,\quad \Delta^3 y^{\mathrm{I}},\quad \Delta^3 y^{\mathrm{II}},\quad \text{etc.}$$

Fourth differences

$$\Delta^4 y,\quad \Delta^4 y^{\mathrm{I}},\quad \text{etc.}$$

Fifth differences

$$\Delta^5 y,\quad \text{etc.}$$

etc.

any of which arises from the preceding, and which terms are required to be subtracted for the following.

Therefore with whatever function of x substituted in place of y, because the values $y^{\mathrm{I}}, y^{\mathrm{II}}, y^{\mathrm{III}}$ etc. may be formed easily from recognised arrangements, the individual series of the differences may be found without effort.

8. We may put to be $y = x$ and there will be $y^{\mathrm{I}} = x^{\mathrm{I}} = x + \omega, y^{\mathrm{II}} = x^{\mathrm{II}} = x + 2\omega$ etc. and so on thus. From the differences assumed there will be $\Delta x = \omega, \Delta x^{\mathrm{I}} = \omega, \Delta x^{\mathrm{II}} = \omega$ etc. and thus all the first differences of x are constant and therefore the second differences all vanish and equally the third differences and all of the following orders. Therefore since there shall be $\Delta x = \omega$, on account of the analogy in place of the letter ω this character Δx will be used conveniently. Hence the variable of magnitude x, the of which the successive values $x, x^{\mathrm{I}}, x^{\mathrm{II}}, x^{\mathrm{III}}$ etc. assume an arithmetic progression $\Delta x, \Delta x^{\mathrm{I}}, \Delta x^{\mathrm{II}}$ etc. are constants and equal to each other; and therefore there will be $\Delta\Delta x = 0,\ \Delta^3 x = 0,\ \Delta^4 x = 0$ and so on thus.

9. For the values of x, which these are given successively, here we have an arithmetical progression, thus so that the first differences of the values of these shall be constant, the second and the rest all vanish. Because although it may depend on our choice, since equally we may be able to use some other progression, yet an arithmetical progression is accustomed most conveniently to be taken before all the others, because which shall be the most simple and by being understood most easily, then indeed especially, because certainly all the values which indeed x is able to adopt, shall be apparent. Indeed on being attributed both negative as well as positive values of ω in this series of values of x generally all the quantities to be included are real quantities, which can be substituted in place of x ; but on the other hand if we should have selected a geometric series, it would be apparent that negative values could not be used. Because of this reason, the variability of the functions y is decided most suitably from the values of an arithmetic progression of x put in place.

10. As there is $\Delta y = y^{\mathrm{I}} - y$, thus the further differences also can be defined from the terms of the first series $y, y^{\mathrm{I}}, y^{\mathrm{II}}, y^{\mathrm{III}}$ etc.

For since there shall be

$$\Delta y^{\mathrm{I}} = y^{\mathrm{II}} - y^{\mathrm{I}},$$

there will be

$$\Delta\Delta y = y^{\mathrm{II}} - 2y^{\mathrm{I}} + y \quad \text{and} \quad \Delta\Delta y^{\mathrm{I}} = y^{\mathrm{III}} - 2y^{\mathrm{II}} + y^{\mathrm{I}}$$

and thus

$$\Delta^3 y = \Delta\Delta y^{\mathrm{I}} - \Delta\Delta y = y^{\mathrm{III}} - 3y^{\mathrm{II}} + 3y^{\mathrm{I}} - y\,;$$

and in a similar manner there will be

$$\Delta^4 y = y^{\mathrm{IV}} - 4y^{\mathrm{III}} + 6y^{\mathrm{II}} - 4y^{\mathrm{I}} + y \quad \text{and} \quad \Delta^5 y = y^{\mathrm{V}} - 5y^{\mathrm{IV}} + 10y^{\mathrm{III}} - 10y^{\mathrm{II}} + 5y^{\mathrm{I}} - y,$$

the numerical coefficients of which formulas keep the same rule, which is observed in powers of the binomial. Therefore just as the first difference may be determined from the first two terms of the series $y, y^{\mathrm{I}}, y^{\mathrm{II}}, y^{\mathrm{III}}$ etc., thus the second difference is determined from three terms, the third difference from four, and thus with the others. But with the difference of any order of *y* known, in a similar manner the differences of all the orders of $y^{\mathrm{I}}, y^{\mathrm{II}}$ etc. may be defined.

11. Therefore with any proposed function of *y*, the individual differences of that both with the first as well as the following are able to be found, indeed for which the corresponding values of *x* are progressing by the difference ω. Nor indeed is there a need for that, so that the series of values of *y* may be continued further; for just as the first difference Δy is found, if in y in place of x there may be written $x+\omega$ and from the value arising Δy^{I} that function *y* may be subtracted, thus the second difference $\Delta\Delta y$ will be obtained, if in the first difference Δy there is put $x+\omega$ in place of *x*, so that there may arise Δy^{I}, and Δy may be subtracted from Δy^{I}. In a similar manner if the second difference $\Delta\Delta y$ is taken, that on subtracting from the value which it adopts, if in place of *x* there is put $x+\omega$, the third difference is come upon $\Delta^3 y$ and hence again in the same way the fourth difference $\Delta^4 y$ etc. Therefore it may become known how to investigate the first difference of any function, from which likewise the second difference can be found, the third and all the following, because the second difference of y therefore is nothing other than the first difference of Δy and the third difference of *y* nothing other than the first difference of $\Delta\Delta y$ and so henceforth for the remainder.

12. if the function *y* were prepared from two or mere parts, so that there shall be $y = p + q + r + \text{etc.}$, then, because there is $y^{\mathrm{I}} = p^{\mathrm{I}} + q^{\mathrm{I}} + r^{\mathrm{I}} + \text{etc.}$, the difference will be

$$\Delta y = \Delta p + \Delta q + \Delta r + \text{etc.}$$

and in a similar manner again

$$\Delta\Delta y = \Delta\Delta p + \Delta\Delta q + \Delta\Delta r + \text{etc.},$$

from which the finding of the differences, if the proposed function were prepared from parts, is returned somewhat easier. But if indeed the function *y* were produced from the two functions *p* and *q*, certainly $y = pq$, because there shall be $y^{\mathrm{I}} = p^{\mathrm{I}} q^{\mathrm{I}}$ and $p^{\mathrm{I}} = p + \Delta p$ as well as $q^{\mathrm{I}} = q + \Delta q$,
there arises

$$p^{\mathrm{I}} q^{\mathrm{I}} = pq + p\Delta q + q\Delta p + \Delta p\Delta q$$

and hence

$$\varDelta y = p\varDelta q + q\varDelta p + \varDelta p\varDelta q .$$

From which , if p shall be a constant quantity = a, on account of $\varDelta a = 0$ the first difference of the function $y = aq$ will be the $\varDelta y = a\varDelta q$ and in a similar manner the second difference $\varDelta\varDelta y = a\varDelta\varDelta q$, the third $\varDelta^3 y = a\varDelta^3 q$ and thus so on.

13. Because an integral rational function is a sum from some powers of x, we are able to find all the differences of the rational whole numbers, only if we may know how to show the differences of the powers of the whole numbers.
On account of this matter, we may investigate the differences of the individual powers of the variable quantity x in the following examples.

Moreover since there shall be $x^0 = 1$, then there will be $\varDelta x^0 = 0$, therefore because x^0 does not vary; even if x may change into $x + \omega$.

Then truly we may consider to be $\varDelta x = \omega$ and $\varDelta\varDelta x = 0$ and likewise the differences of the following orders vanish. Since which shall be evident, we may begin from the second power.

EXAMPLE 1

To find the differences of all orders of the power x^2.

Since here there shall be $y = x^2$, there will be $y^{\mathrm{I}} = (x + \omega)^2$ and thus

$$\varDelta y = 2\omega x + \omega\omega ,$$

which is the first difference. Now on account of the constant quantity ω there will be $\varDelta\varDelta y = 2\omega\omega$ and $\varDelta^3 y = 0,\ \varDelta^4 y = 0$ etc.

EXAMPLE 2

To find the differences of all the orders of the power x^3.

There may be put in place $y = x^3$, and since there shall be $y^{\mathrm{I}} = (x + \omega)^3$, there will be

$$\varDelta y = 3\omega xx + 3\omega^2 x + \omega^3 ,$$

which is the first difference. Successively on account of $\varDelta xx = 2\omega x + \omega\omega$ there will be

$$\varDelta .3\omega xx = 6\omega\omega x + 3\omega^3 ,\ \ \varDelta .3\omega^2 x = 3\omega^3, \text{ and } \ \varDelta\omega^3 = 0;$$

with which gathered together there will be

$$\varDelta\varDelta y = 6\omega^2 x + 6\omega^3 \text{ and } \ \varDelta^3 y = 6\omega^3.$$

Truly the following differences vanish.

EXAMPLE 3

To find the differences of all the orders of the power x^4.

On putting $y = x^4$ on account of $y^{\mathrm{I}} = (x+\omega)^4$ there will be

$$\Delta y = 4\omega x^3 + 6\omega^2 x^2 + 4\omega^3 x + \omega^4,$$

which is the first difference. Then from the preceding there is

$$\begin{aligned} \Delta.4\omega x^3 &= 12\omega^2 x^2 + 12\omega^3 x + 4\omega^4 \\ \Delta.6\omega^2 x^2 &= \ldots\ldots\ldots\ldots + 12\omega^3 x + 6\omega^4 \\ \Delta.4\omega^3 x &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 4\omega^4 \\ \Delta.\ \omega^4 &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 0. \end{aligned}$$

With these gathered together the second difference will be

$$\Delta\Delta y = 12\omega^2 x^2 + 24\omega^3 x + 14\omega^4.$$

Because then again there is

$$\begin{aligned} \Delta.12\omega^2 x^2 &= 24\omega^3 x + 12\omega^4 \\ \Delta.24\omega^3 x &= \ldots\ldots\ldots\ldots 24\omega^4\ . \\ \Delta.14\omega^4 &= \ldots\ldots\ldots\ldots 0, \end{aligned}$$

the third difference will be produced

$$\Delta^3 y = 24\omega^3 x + 36\omega^4$$

and finally the fourth difference

$$\Delta^4 y = 24\omega^4;$$

which since it shall be constant, the differences of the following orders vanish.

EXAMPLE 4

To find the differences of any order of the power x^n

There may be put in place $y = x^n$, and since there shall be
$y^{\mathrm{I}} = (x+\omega)^n$, $y^{\mathrm{II}} = (x+2\omega)^n$, $y^{\mathrm{III}} = (x+E\omega)^n$ etc., the powers expanded out shall be

$$y = x^n$$
$$y^{\mathrm{I}} = x^n + \tfrac{n}{1}\omega x^{n-1} + \tfrac{n(n-1)}{1\cdot 2}\omega^2 x^{n-2} + \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\omega^3 x^{n-3} + \text{etc.}$$
$$y^{\mathrm{II}} = x^n + \tfrac{n}{1}2\omega x^{n-1} + \tfrac{n(n-1)}{1\cdot 2}4\omega^2 x^{n-2} + \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}8\omega^3 x^{n-3} + \text{etc.}$$
$$y^{\mathrm{III}} = x^n + \tfrac{n}{1}3\omega x^{n-1} + \tfrac{n(n-1)}{1\cdot 2}9\omega^2 x^{n-2} + \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}27\omega^3 x^{n-3} + \text{etc,}$$
$$y^{\mathrm{IV}} = x^n + \tfrac{n}{1}4\omega x^{n-1} + \tfrac{n(n-1)}{1\cdot 2}16\omega^2 x^{n-2} + \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}64\omega^3 x^{n-3} + \text{etc}$$

Hence with the differences assumed there will be produce

$$\Delta y = \tfrac{n}{1}\omega x^{n-1} + \tfrac{n(n-1)}{1\cdot 2}\omega^2 x^{n-2} + \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\omega^3 x^{n-3} + \text{etc.}$$
$$\Delta y^{\mathrm{I}} = \tfrac{n}{1}\omega x^{n-1} + \tfrac{n(n-1)}{1\cdot 2}3\omega^2 x^{n-2} + \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}7\omega^3 x^{n-3} + \text{etc.}$$
$$\Delta y^{\mathrm{II}} = \tfrac{n}{1}\omega x^{n-1} + \tfrac{n(n-1)}{1\cdot 2}5\omega^2 x^{n-2} + \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}19\omega^3 x^{n-3} + \text{etc.}$$
$$\Delta y^{\mathrm{III}} = \tfrac{n}{1}\omega x^{n-1} + \tfrac{n(n-1)}{1\cdot 2}7\omega^2 x^{n-2} + \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}37\omega^3 x^{n-3} + \text{etc.}$$

The differences may be taken anew and there will be obtained

$$\Delta\Delta y = n(n-1)\omega^2 x^{n-2} + \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}6\omega^3 x^{n-3} + \tfrac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}14\omega^4 x^{n-4} + \text{etc.}$$
$$\Delta\Delta y^{\mathrm{I}} = n(n-1)\omega^2 x^{n-2} + \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}12\omega^3 x^{n-3} + \tfrac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}50\omega^4 x^{n-4} + \text{etc.}$$
$$\Delta\Delta y^{\mathrm{II}} = n(n-1)\omega^2 x^{n-2} + \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}18\omega^3 x^{n-3} + \tfrac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}110\omega^4 x^{n-4} + \text{etc.}$$

From these by subtraction there is elicited further

$$\Delta^3 y = n(n-1)(n-2)\omega^3 x^{n-3} + \tfrac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}36\omega^4 x^{n-4} + \text{etc.}$$
$$\Delta^4 y^{\mathrm{I}} = n(n-1)(n-2)\omega^3 x^{n-3} + \tfrac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}60\omega^4 x^{n-4} + \text{etc.}$$

and again

$$\varDelta^4 y = n(n-1)(n-2)(n-3)\omega^4 x^{n-4} + \text{etc.}$$

14. So that the law, following which these differences of the powers x^n proceed, may be examined more easily, we may put for brevity first therefore

$$A = \tfrac{n}{1}$$
$$B = \tfrac{n(n-1)}{1\cdot 2}$$
$$C = \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}$$
$$D = \tfrac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}$$
$$E = \tfrac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}$$
etc.

Then the following table may be formed, which will serve to look after the individual differences :

y	1	0	0	0	0	0	0	0 etc.
$\varDelta y$	0	1	1	1	1	1	1	1 etc.
$\varDelta^2 y$	0	0	2	6	14	30	62	126	254 etc.
$\varDelta^3 y$	0	0	0	6	36	150	540	1806	5796 etc.
$\varDelta^4 y$	0	0	0	0	24	240	1560	8400	40824 etc.
$\varDelta^5 y$	0	0	0	0	0	120	1800	16800	126000 etc.
$\varDelta^6 y$	0	0	0	0	0	0	720	15120	191520 etc.
$\varDelta^7 y$	0	0	0	0	0	0	0	5040	141120 etc.

In which table any number of the series you wish may be found, if the preceding term of the same series may be added to the number placed above and the sum multiplied by the fixed index of the character $\varDelta$. Thus in the series of the differences $\varDelta^5 y$ with the corresponding term 16800 found, if the preceding1800 be added to the 1560 written above and the sum 3360 may be multiplied by 5.

15. Therefore with this table set up the individual differences of the powers $x^n = y$ may be considered in the following manner:

$$\Delta y = A\omega x^{n-1} + B\omega^2 x^{n-2} + C\omega^3 x^{n-3} + D\omega^4 x^{n-4} + \text{etc.}$$
$$\Delta^2 y = 2B\omega^2 x^{n-2} + 6C\omega^3 x^{n-3} + 14D\omega^4 x^{n-4} + \text{etc.}$$
$$\Delta^3 y = 6C\omega^3 x^{n-3} + 36D\omega^4 x^{n-4} + 150E\omega^5 x^{n-5} + \text{etc.}$$
$$\Delta^4 y = 24D\omega^4 x^{n-4} + 240E\omega^5 x^{n-5} + 1560F\omega^6 x^{n-6} + \text{etc.}$$

But generally the difference of the powers x^n of order *m*, or $\Delta^m y$, may be expressed in the following manner.

Let there be

$$I = \frac{n(n-1)(n-2)\cdots(n-m+1)}{1\cdot 2\cdot 3\cdots m},$$
$$K = \frac{n-m}{m+1}I, \quad L = \frac{n-m-1}{m+2}K, \quad M = \frac{n-m-2}{m+3}L, \quad \text{etc.}$$

Truly successively there shall be

$$\alpha = (m+1)^m - \frac{m}{1}m^m + \frac{m(m-1)}{1\cdot 2}(m-1)^m - \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}(m-2)^m + \text{etc.}$$
$$\beta = (m+1)^{m+1} - \frac{m}{1}m^{m+1} + \frac{m(m-1)}{1\cdot 2}(m-1)^{m+1} - \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}(m-2)^{m+1} + \text{etc.}$$
$$\gamma = (m+1)^{m+2} - \frac{m}{1}m^{m+1} + \frac{m(m-1)}{1\cdot 2}(m-1)^{m+2} - \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}(m-2)^{m+2} + \text{etc.}$$

etc;

with which values found there will be

$$\Delta^m y = \alpha I\omega^m x^{n-m} + \beta K\omega^{m+1} x^{n-m-1} + \gamma L\omega^{m+2} x^{n-m-2} + \text{etc}$$

an account of which expression from the manner, in which the individual differences are elucidated from the values $y, y^{\text{I}}, y^{\text{II}}, y^{\text{III}}$ etc., follows automatically.

16. From these it is evident, if the exponent n were a positive whole number, finally constants are come upon for the differences and with these beyond all are $=0$. Thus there will be

$$\varDelta.x=\omega$$
$$\varDelta^2.x^2=2\omega^2$$
$$\varDelta^3.x^3=6\omega^3$$
$$\varDelta^4.x^4=24\omega^4 \text{ and finally}$$
$$\varDelta^n.x^n=1\cdot2\cdot3\cdot4......n\cdot\omega^n.$$

Therefore a whole function of all rational numbers finally may be deduced to constant differences. Evidently a function of x of the first order $ax+b$ now may have a constant first difference $=a\omega$. A function of the second order $axx+bx+c$ will have a constant second difference $=2a\omega\omega$. Moreover with functions of the third order the third difference will be constant, of the fourth order the fourth, and thus so on.

17. But the manner, by which we have found the differences of the powers of x^n, also may be more accessible and be extended to these powers, of which the exponent n is a negative number or a fraction or even irrational. So that which may appear clearer, we will show only the first differences of this particular kind of powers, because the law of the second differences and of the following may not be easily discerned ; therefore there will be

$$\varDelta.x=\omega$$
$$\varDelta.x^2=2\omega x+\omega^2$$
$$\varDelta.x^3=3\omega x^2+3\omega^2x+\omega^3$$
$$\varDelta.x^3=4\omega x^2+6\omega^2x^2+4\omega^3x+\omega^4$$
$$\text{etc.}$$

In a similar manner there will be

$$\varDelta.x^{-1}=-\tfrac{\omega}{x^2}+\tfrac{\omega^2}{x^3}-\tfrac{\omega^3}{x^4}+\text{etc.}$$
$$\varDelta.x^{-2}=-\tfrac{2\omega}{x^3}+\tfrac{3\omega^2}{x^4}-\tfrac{4\omega^3}{x^5}+\text{etc.}$$
$$\varDelta.x^{-3}=-\tfrac{3\omega}{x^4}+\tfrac{6\omega^2}{x^5}-\tfrac{10\omega^3}{x^6}+\text{etc.}$$
$$\varDelta.x^{-4}=-\tfrac{4\omega}{x^5}+\tfrac{10\omega^2}{x^6}-\tfrac{20\omega^3}{x^7}+\text{etc.}$$

and thence for the rest. Equally there will be

$$\Delta.x^{\frac{1}{2}} = \frac{\omega}{2x^{\frac{1}{2}}} - \frac{\omega^2}{8x^{\frac{3}{2}}} + \frac{\omega^3}{16x^{\frac{5}{2}}} - \text{etc.}$$

$$\Delta.x^{\frac{1}{3}} = \frac{\omega}{3x^{\frac{2}{3}}} - \frac{\omega^2}{9x^{\frac{5}{3}}} + \frac{5\omega^3}{81x^{\frac{8}{3}}} - \text{etc.}$$

$$\Delta.x^{-\frac{1}{2}} = -\frac{\omega}{2x^{\frac{3}{2}}} + \frac{3\omega^2}{8x^{\frac{5}{2}}} - \frac{5\omega^3}{16x^{\frac{7}{2}}} + \text{etc.}$$

$$\Delta.x^{-\frac{1}{3}} = -\frac{\omega}{3x^{\frac{4}{3}}} + \frac{2\omega^2}{9x^{\frac{7}{3}}} - \frac{14\omega^3}{81x^{\frac{10}{3}}} + \text{etc.}$$

18. And thus it is apparent these differences, if the exponent of x were not a positive integer, to progress to infinity or to depend on an infinite number of terms. Yet meanwhile the same differences are able to be shown also by a finite expression. For since putting $y = x^{-1} = \frac{1}{x}$ there shall be $y^{\mathrm{I}} = \frac{1}{x+\omega}$, there will be

$$\Delta.\frac{1}{x} = \frac{1}{x+\omega} - \frac{1}{x};$$

from which, if the fraction may be converted into a series $\frac{1}{x+\omega}$, the above expression will be produced. In a similar manner there will be

$$\Delta.x^{-2} = \Delta.\frac{1}{xx} = \frac{1}{(x+\omega)^2} - \frac{1}{xx}$$

and for the irrationals there will be

$$\Delta.\sqrt{x} = \sqrt{(x+\omega)} - \sqrt{x} \quad \text{and} \quad \Delta.\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{(x+\omega)}} - \frac{1}{\sqrt{x}};$$

which formulas, if in the usual custom may be set out in series, present the above expressions.

19. Truly in this manner also the differences of functions either of fractions or of irrationals are able to be found; thus, if the first difference of the fraction $\frac{1}{aa+xx}$ is sought, there may be put $y = \frac{1}{aa+xx}$, and because there shall be $y^{\mathrm{I}} = \frac{1}{aa+xx+2\omega x+\omega^2}$, there becomes

$$\Delta y = \Delta.\frac{1}{aa+xx} = \frac{1}{aa+xx+2\omega x+\omega\omega} - \frac{1}{aa+xx},$$

which expression can be converted into an infinite series also.

There may be put $aa + xx = P$ and $2\omega x + \omega\omega = Q$*;* then there will be

$$\frac{1}{P+Q} = \frac{1}{P} - \frac{Q}{P^2} + \frac{Q^2}{P^3} - \frac{Q^3}{P^4} + \text{etc.}$$

and

$$\varDelta y = -\frac{Q}{P^2} + \frac{Q^2}{P^3} - \frac{Q^3}{P^4} + \text{etc.}$$

Therefore with the values restored in place of P and Q there will be

$$\varDelta y = \varDelta.\frac{1}{aa+xx} = -\frac{2\omega x+\omega\omega}{(aa+xx)^2} + \frac{4\omega\omega xx+4\omega^3 x+\omega^4}{(aa+xx)^3} - \frac{8\omega^3 x^3+12\omega^4 x^2+6\omega^5 x+\omega^6}{(aa+xx)^4} + \text{etc.};$$

which terms if the following powers of ω may be put in order, will be

$$\varDelta.\frac{1}{aa+xx} = -\frac{2\omega x}{(aa+xx)^2} + \frac{\omega^2(3xx-aa)}{(aa+xx)^3} - \frac{4\omega^3(x^3-aax)}{(aa+xx)^4} + \text{etc.}$$

20. The differences of irrational functions also can be expressed by similar infinite series. Let this proposed function be $y = \sqrt{(aa+xx)}$, and since there shall be

$$y^{\mathrm{I}} = \sqrt{\left(aa+xx+2\omega x+\omega^2\right)},$$

there may be put

$$aa+xx = P \text{ and } 2\omega x+\omega\omega = Q\,;$$

then there will be

$$\varDelta y = \sqrt{(P+Q)} - \sqrt{P} = \frac{Q}{2\sqrt{P}} - \frac{QQ}{8P\sqrt{P}} + \frac{Q^3}{16P^2\sqrt{P}} - \text{etc.},$$

from which there becomes

$$\varDelta y = \varDelta.\sqrt{(aa+xx)} = \frac{2\omega x+\omega\omega}{2\sqrt{(aa+xx)}} - \frac{4\omega^2 x^2+4\omega^3 x+\omega^4}{8(aa+xx)\sqrt{(aa+xx)}} + \text{etc.}$$

or

$$= \frac{\omega x}{2\sqrt{(aa+xx)}} + \frac{aa\omega^2}{2(aa+xx)\sqrt{(aa+xx)}} - \frac{aa\omega^3 x}{2(aa+xx)^2\sqrt{(aa+xx)}} + \text{etc.}$$

And hence thus we may gather together the difference of any function of *x*, which shall be *y*, with this form possible to be expressed, so that

$$\varDelta y = P\omega + Q\omega^2 + R\omega^3 + S\omega^4 + \text{etc.}$$

with certain functions of *x* *P*, *Q*, *R*, *S* etc. present, which in whatever case are able to be defined from the function *y*.

21. Nor also from this form of functions are differences of transcending functions to be excluded, that which from the following examples will be appear clearer.

EXAMPLE 1

To find the first difference of the hyperbolic logarithm of x.

There may be put $y = lx$, and since there shall be $y^{\mathrm{I}} = l(x+\omega)$, there will be

$$\Delta y = y^{\mathrm{I}} - y = l(x+\omega) - lx = l\left(1+\tfrac{\omega}{x}\right).$$

But above we have instructed that the logarithm is expressed by an infinite series ; with which given there will be

$$\Delta y = \Delta.lx = \tfrac{\omega}{x} - \tfrac{\omega^2}{2xx} + \tfrac{\omega^3}{3x^3} - \tfrac{\omega^4}{4x^4} + \text{etc.}$$

EXAMPE 2

To find the first difference of the magnitude of the exponential a^x.

On putting $y = a^x$ there will be $y^{\mathrm{I}} = a^{x+\omega} = a^x \cdot a^{\omega}$; but above we have shown that

$$a^{\omega} = 1 + \tfrac{\omega la}{1} + \tfrac{\omega^2 (la)^2}{1\cdot 2} + \tfrac{\omega^3 (la)^3}{1\cdot 2\cdot 3} + \text{etc.};$$

from which value introduced there will be

$$\Delta.a^{\omega} = y^{\mathrm{I}} - y = \Delta y = \tfrac{a^x \omega la}{1} + \tfrac{a^x \omega^2 (la)^2}{1\cdot 2} + \tfrac{a^x \omega^3 (la)^3}{1\cdot 2\cdot 3} + \text{etc.}$$

EXAMPLE 3

In the circle, the radius of which = 1, to find the first difference of the sine of the arc x.

Let the $\sin x = y$; there will be $y^{\mathrm{I}} = \sin(x+\omega)$, from which there arises

$$\Delta y = y^{\mathrm{I}} - y = \sin(x+\omega) - \sin x.$$

But there is $\sin(x+\omega) = \cos\omega \cdot \sin x + \sin\omega \cdot \cos x$ and we have shown by infinite series to be

$$\cos\omega = 1 - \tfrac{\omega^2}{1\cdot 2} + \tfrac{\omega^4}{1\cdot 2\cdot 3\cdot 4} - \tfrac{\omega^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \text{etc.}$$

and

$$\sin\omega = \omega - \tfrac{\omega^3}{1\cdot 2\cdot 3} + \tfrac{\omega^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \tfrac{\omega^7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} + \text{etc.};$$

with which series substituted there will be

$$\Delta.\sin x = \omega\cos x - \tfrac{\omega^2}{2}\sin x - \tfrac{\omega^3}{6}\cos x + \tfrac{\omega^4}{24}\sin x + \tfrac{\omega^5}{120}\cos x - \text{ etc.}$$

EXAMPLE 4

In the circle, the radius of which = 1, *to find the first difference of the cosine of the arc x.*

On putting
$y = \cos x$ on account of $y^{\mathrm{I}} = \cos(x + \omega)$ there will be $y^{\mathrm{I}} = \cos\omega \cdot \cos x - \sin\omega \cdot \sin x$ and

$$\Delta y = \cos\omega \cdot \cos x - \sin\omega \cdot \sin x - \cos x\,.$$

Therefore from the series used set out before there will be produced

$$\Delta.\cos x = -\omega\sin x - \tfrac{\omega^2}{2}\cos x + \tfrac{\omega^3}{6}\sin x + \tfrac{\omega^4}{24}\cos x - \tfrac{\omega^5}{120}\sin x - \text{etc.}$$

22. Therefore since with any proposed function of *x* either algebraic or transcendent, which shall be y, the first difference of this kind may always have a form of this kind, so that there shall be

$$\Delta y = P\omega + Q\omega^2 + R\omega^3 + S\omega^4 + \text{etc.},$$

if the difference of this is taken anew, it will appear that the second difference of y shall have a form of this kind

$$\Delta\Delta y = P\omega^2 + Q\omega^3 + R\omega^4 + \text{etc.}$$

and in a similar manner the third difference of *y* will be of this kind

$$\Delta^3 y = P\omega^3 + Q\omega^4 + R\omega^5 + \text{etc.}$$

and thus again.

Where it is to be observed that the letters *P*, *Q*, *R* etc. here are not to be given for determined values nor are the same letters to denote the same function of *x* in diverse differences; thus indeed I only use the same letters, lest the number sufficing of different letters be deficient.

Moreover these forms of the differences are to be noted properly, since the offer the maximum use in infinite analysis.

23. Therefore since I may have established the manner, by which the first difference of any functions of this kind and from that again the differences of the following orders are able to be found, which obviously are found from the successive values of the function y $y^{\mathrm{I}}, y^{\mathrm{II}}, y^{\mathrm{III}}, y^{\mathrm{IV}}$ etc., in turn from the given differences of each order of y those varied values of y can themselves be elicited. Indeed there shall be

$$\begin{aligned} y^{\mathrm{I}} &= y + \Delta y \\ y^{\mathrm{II}} &= y + 2\Delta y + \Delta\Delta y \\ y^{\mathrm{III}} &= y + 3\Delta y + 3\Delta\Delta y + \Delta^3 y \\ y^{\mathrm{IV}} &= y + 4\Delta y + 6\Delta\Delta y + 4\Delta^3 y + \Delta^4 y \\ &\text{etc.,} \end{aligned}$$

where the numerical coefficients in turn arise from the binomial expansion. Therefore just as $y^{\mathrm{I}}, y^{\mathrm{II}}, y^{\mathrm{III}}$ etc. are the values of y, which arise, if in place of x successively there may be placed these values $x+\omega,\ x+2\omega,\ x+3\omega$ etc., we can immediately assign the value of $y^{(n)}$, which will be produced, if in place of x there may be written $x+n\omega$; evidently this will be the value

$$y + \tfrac{n}{1}\Delta y + \tfrac{n(n-1)}{1\cdot 2}\Delta^2 y + \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\Delta^3 y + \text{etc.}$$

[Thus, symbolically, $y^{(n)} = (1+\Delta)^n y$; while if $n = -1, -2, \text{etc.}$, in what follows, $y^{(-1)} = (1-\Delta)^{-1} y$, $y^{(-2)} = (1-\Delta)^{-2} y$, etc. We may note that all equations involving the Δ sign will relate eventually to differential quantities, while those below with the Σ sign will relate to integral quantities.]
And hence indeed also the values of y can be given, if n were a negative number. Thus, if in place of x there may be put $x-\omega$, the function y will change into this form

$$y - \Delta y + \Delta^2 y - \Delta^3 y + \Delta^4 y - \text{etc.}$$

but if in place of x there may be put $x-2\omega$, the function y will change into

$$y - 2\Delta y + 3\Delta^2 y - 4\Delta^3 y + 5\Delta^4 y - \text{etc.}$$

24. We may add only a certain few by the inverse method; from which if the difference may be given, from that function itself, of which it is the difference, has to be found. But since this shall be most difficult and on many occasions it may require infinitesimal analysis, we will only set out certain easier cases. Therefore at first on regressing, if we have come upon a certain difference of the function, in turn from this proposed difference that function itself, from which it has arisen, can be shown. Thus, since the difference of the function $ax+b$ shall be $a\omega$, if the function may be sought the difference of which shall be $a\omega$, the response will be to bring out that function to be $ax+b$. Therefore in this there is found the constant quantity b, which was not present in the difference and which therefore depends on our choice. But always, if the difference of some function P were Q, also the difference of the function $P+A$ (with A denoting some constant quantity) will be Q. Hence, if that difference Q is proposed, then the function, from which that has arisen, will be $P+A$ and on that account may not have a determined value, since the constant A may depend on choice.

25. We may call that function sought, the difference of which is proposed, the *sum* ; because the name may be used conveniently, when, because the sum of the differences is usually place opposite, then also, because the function sought actually shall be the sum of the values of all the preceding differences. Just as indeed there is

$$y^{\mathrm{I}} = y + \Delta y \quad \text{and} \quad y^{\mathrm{II}} = y + \Delta y + \Delta y^{\mathrm{I}},$$

if the values of y may be continued backwards, thus so that that, which corresponds to the value $x-\omega$, may be written y_{I} and this preceding y_{II} and which precede further $y_{\mathrm{III}}, y_{\mathrm{IV}}, y_{\mathrm{V}}$ etc. and hence the series may be formed backwards with their differences

$$y_{\mathrm{V}}, y_{\mathrm{IV}}, y_{\mathrm{III}},\ y_{\mathrm{II}},\ y_{\mathrm{I}},\ y \quad \text{and} \quad \Delta y_{\mathrm{V}},\ \Delta y_{\mathrm{IV}},\ \Delta y_{\mathrm{III}},\ \Delta y_{\mathrm{II}},\ \Delta y_{\mathrm{I}}$$

there will be $y = \Delta y_{\mathrm{I}} + y_{\mathrm{I}}$ and on account of $y_{\mathrm{I}} = \Delta y_{\mathrm{II}} + y_{\mathrm{II}}$ and again $y_{\mathrm{II}} = \Delta y_{\mathrm{III}} + y_{\mathrm{III}}$ and certainly there will be

$$y = \Delta y_{\mathrm{I}} + \Delta y_{\mathrm{II}} + \Delta y_{\mathrm{III}} + \Delta y_{\mathrm{IV}} + \Delta y_{\mathrm{V}} + \text{etc.}$$

and thus the function y will be, the difference of which is Δy, the sum of all the preceding differences Δy, which arise, if in place of x there may be written the preceding values $x-\omega,\ x-2\omega,\ x-3\omega$ etc.

26. Just as we have been accustomed to specify the difference by the sign Δ, thus we will indicate the sum by the sign Σ; evidently if the difference of the function y were z, there will be $z = \Delta y$; from which, if y may be given, the difference z is found we have shown before. But if moreover the difference z shall be given and the sum of this y must be found, $y = \Sigma z$ is made and evidently from the equation $z = \Delta y$ on regressing this equation will have the form $y = \Sigma z$, where some constant quantity can be added on account of the reasons given above; from which the equation $z = \Delta y$, if it may be inverted, also will give

$y = \Sigma z + C$. Then, since the difference of the quantity ay shall be $a\Delta y = az$, there will be $\Sigma az = ay$, if indeed a shall be a constant quantity. Therefore because $\Delta x = \omega$, there will be $\Sigma\omega = x + C$ and $\Sigma a\omega = ax + C$ and on account of the constant quantity ω there will be $\Sigma\omega^2 = \omega x + C$, $\Sigma\omega^3 = \omega^2 x + C$ and so on thus.

27. Therefore if we may invert the differences of the sums of powers of x found above, there will be [note the idea of the index suffix is lacking at this stage]

$$\Sigma\omega = x \quad \text{and hence} \quad \Sigma 1 = \tfrac{x}{\omega}.$$

Then we may consider

$$\Sigma\left(2\omega x + \omega^2\right) = x^2$$

[since $\Sigma\left(\omega + x\right)^2 - \Sigma x^2 = x^2$, where the understood final value x_f^2 appears on the right hand side,]
from which there is made

$$\Sigma x = \tfrac{x^2}{2\omega} - \Sigma\tfrac{\omega}{2} = \tfrac{x^2}{2\omega} - \tfrac{x}{2}.$$

Again there is

$$\Sigma\left(3\omega xx + 3\omega^2 x + \omega^3\right) = x^3$$

or

$$3\omega\Sigma x^2 + 3\omega^2\Sigma x + \omega^3\Sigma 1 = x^3;$$

hence

$$\Sigma x^2 = \tfrac{x^3}{3\omega} - \omega\Sigma x - \tfrac{\omega^2}{3}\Sigma 1$$

or

$$\Sigma x^2 = \tfrac{x^3}{3\omega} - \tfrac{x^2}{2} + \tfrac{\omega x}{6}.$$

In a similar manner there will be

$$\Sigma x^3 = \tfrac{x^4}{4\omega} - \tfrac{3\omega}{2}\Sigma x^2 - \omega^2\Sigma x - \tfrac{\omega^3}{4}\Sigma 1,$$

where if in place of $\Sigma x^2, \Sigma x$ and $\Sigma 1$ the values found before may be substituted, there may be found

$$\Sigma x^3 = \tfrac{x^4}{4\omega} - \tfrac{x^3}{2} + \tfrac{\omega xx}{4}.$$

Then since there shall be

$$\Sigma x^4 = \tfrac{x^5}{5\omega} - 2\omega \Sigma x^3 - 2\omega^2 \Sigma x^2 - \tfrac{\omega^5}{5}\Sigma 1,$$

there will be from the substitutions brought to bear

$$\Sigma x^4 = \tfrac{x^5}{5\omega} - \tfrac{1}{2}x^4 + \tfrac{1}{3}\omega x^3 - \tfrac{1}{30}\omega^3 x.$$

In a similar manner on progressing further there may be found

$$\Sigma x^5 = \tfrac{x^6}{6\omega} - \tfrac{1}{2}x^5 + \tfrac{5}{12}\omega x^4 - \tfrac{1}{12}\omega^3 x^2,$$

and

$$\Sigma x^6 = \tfrac{x^7}{7\omega} - \tfrac{1}{2}x^6 + \tfrac{1}{2}\omega x^5 - \tfrac{1}{6}\omega^3 x^3 + \tfrac{1}{42}\omega^5 x,$$

which expressions we will show how to find more easily below.

28. Therefore if the proposed difference were a rational function of *x*, the sum of this (or that function, of which that is the difference) is found from these formulas easily. For because the difference will depend upon some powers of *x*, the sum is sought of one of the terms and all these sums are gathered together.

EXAMPLE 1

The function is sought, the difference of which shall be = axx +bx +c.

The sums of the individual terms are sought with the help of the formulas found before ; there will be

$$\Sigma axx = \tfrac{ax^3}{3\omega} - \tfrac{axx}{2} + \tfrac{a\omega x}{6}$$

and

$$\Sigma ax = ...+ \tfrac{bxx}{2\omega} - \tfrac{bx}{2}$$

and

$$\Sigma c = ...+ \tfrac{cx}{\omega}.$$

Hence on gathering together these sums there will be

$$\Sigma\left(axx + bx + c\right) = \tfrac{a}{3\omega}x^3 - \tfrac{a\omega - b}{2\omega}x^2 + \tfrac{a\omega^2 - 3b\omega + 6c}{6\omega}x + C,$$

which is the function sought, of which the difference is $axx + bx + c$.

EXAMPLE 2

The function is sought , the difference of which is $x^4 - 2\omega^2 xx + \omega^4$.

The operation may be considered to be put in place in the same manner

$$\Sigma x^4 = \tfrac{1}{5\omega}x^5 - \tfrac{1}{2}x^4 + \tfrac{\omega}{3}x^3 \qquad - \tfrac{\omega^3}{30}x \quad \text{and}$$
$$-\Sigma 2\omega^2 x^2 = \ldots\ldots\ldots\ldots - \tfrac{2\omega}{3}x^3 + \omega^2 x^2 - \tfrac{\omega^3}{3}x \quad \text{and}$$
$$+\Sigma\omega^4 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \omega^4 x,$$

from which the function sought will be

$$\tfrac{1}{5\omega}x^5 - \tfrac{1}{2}x^4 - \tfrac{1}{3}\omega x^3 + \omega^2 x^2 + \tfrac{19}{30}\omega^3 x + C.$$

Indeed if here in place of x there may be put $x+\omega$ and from the resulting quantity that found may be subtracted, the proposed difference will remain $x^4 - 2\omega^2 x^2 + \omega^4$.

29. If the sums, which we have found for the powers of x, we should examine more carefully, indeed we will observe a certain law in the first, second and soon in the third terms, by which these following individual powers are progressing ; but the law of the remainder of the terms thus is not evident, so that the sum of the powers x^n thence in general is possible to be deduced. Yet meanwhile in the following [§ 132, of the second part] it will be shown to be

$$\Sigma x^n = \frac{x^{n+1}}{(n+1)\omega} - \frac{1}{2}x^n + \frac{1}{2}\cdot\frac{n\omega}{2\cdot3}x^{n-1} - \frac{1}{2}\cdot\frac{n(n-1)(n-2)\omega^3}{2\cdot3\cdot4\cdot5}x^{n-3}$$
$$+\frac{1}{6}\cdot\frac{n(n-1)(n-2)(n-3)(n-4)\omega^5}{2\cdot3\cdot4\cdot5\cdot6\cdot7}x^{n-5}$$
$$-\frac{3}{10}\cdot\frac{n(n-1)\ldots\ldots(n-6)\omega^7}{2\cdot3\ldots\ldots8\cdot9}x^{n-7}$$
$$+\frac{5}{6}\cdot\frac{n(n-1)\ldots\ldots(n-8)\omega^9}{2\cdot3\ldots\ldots10\cdot11}x^{n-9}$$
$$-\frac{691}{210}\cdot\frac{n(n-1)\ldots\ldots(n-10)\omega^{11}}{2\cdot3\ldots\ldots12\cdot13}x^{n-11}$$
$$+\frac{35}{2}\cdot\frac{n(n-1)\ldots\ldots(n-12)\omega^{13}}{2\cdot3\ldots\ldots14\cdot15}x^{n-13}$$
$$-\frac{3617}{30}\cdot\frac{n(n-1)\ldots\ldots(n-14)\omega^{15}}{2\cdot3\ldots\ldots16\cdot17}x^{n-15}$$
$$+\frac{43867}{42}\cdot\frac{n(n-1)\ldots\ldots(n-16)\omega^{17}}{2\cdot3\ldots\ldots18\cdot19}x^{n-17}$$
$$-\frac{1222277}{110}\cdot\frac{n(n-1)\ldots\ldots(n-18)\omega^{19}}{2\cdot3\ldots\ldots20\cdot21}x^{n-19}$$
$$+\frac{854513}{6}\cdot\frac{n(n-1)\ldots\ldots(n-20)\omega^{21}}{2\cdot3\ldots\ldots22\cdot23}x^{n-21}$$
$$-\frac{1181820455}{546}\cdot\frac{n(n-1)\ldots\ldots(n-22)\omega^{23}}{2\cdot3\ldots\ldots24\cdot25}x^{n-23}$$
$$+\frac{76977927}{2}\cdot\frac{n(n-1)\ldots\ldots(n-24)\omega^{25}}{2\cdot3\ldots\ldots26\cdot27}x^{n-25}$$
$$-\frac{23749461029}{30}\cdot\frac{n(n-1)\ldots\ldots(n-26)\omega^{27}}{2\cdot3\ldots\ldots28\cdot29}x^{n-27}$$
$$+\frac{81615841276005}{462}\cdot\frac{n(n-1)\ldots\ldots(n-28)\omega^{29}}{2\cdot3\ldots\ldots30\cdot31}x^{n-29}$$
$$\text{etc.} + C,$$

there is placed special interest in the purely numerical coefficients of this progression; which in whatever manner it may be formed, here is not yet the place where it can be explained.

30. But it is apparent, unless *n* shall be a positive whole number, this expression of the sum progresses to infinity nor in this manner is it possible for the summation to be shown in a finite form. Moreover here it is to be noted that not all the proposed powers of *x* occur less than x^n; indeed they lack the terms $x^{n-2}, x^{n-4}, x^{n-6}, x^{n-8}$, etc., obviously the coefficients of these are $= 0$, even if the coefficient of the second term x^n may not follow this rule, but shall be equal to $= -\frac{1}{2}$. Therefore with the aid of this expression the sums of powers, of which the exponents are negative or fractions, are to be shown in an infinite form, with the only case excepted, in which $n=-1$, because then the term $\frac{x^{n+1}}{(n+1)\omega}$ shall be made infinite , on account of $n+1=0$. Thus on putting $n=-2$ there will be

$$\Sigma\frac{1}{xx}=C-\frac{1}{\omega x}-\frac{1}{2xx}-\frac{1}{2}\cdot\frac{\omega}{3x^3}+\frac{1}{6}\cdot\frac{\omega^3}{5x^5}-\frac{1}{6}\cdot\frac{\omega^5}{7x^7}+\frac{3}{10}\cdot\frac{\omega^7}{9x^9}-\frac{5}{6}\cdot\frac{\omega^9}{11x^{11}}+\frac{691}{210}\cdot\frac{\omega^{11}}{13x^{13}}$$
$$-\frac{35}{2}\cdot\frac{\omega^{13}}{15x^{15}}+\frac{3617}{30}\cdot\frac{\omega^{15}}{17x^{17}}-\text{etc.}$$

31. Therefore if the proposed difference were any power of *x*, the sum of which hence can be assigned always or the function, of which that shall be the difference , can be shown. But if the proposed difference may have another form, so that it may be unable to be set out as parts in powers of *x*, then the sum with the greatest difficulty and on may occasions in short cannot be found, unless perhaps it may be apparent that it arises from some function. On account of this case it may be agreed to examine several differences of functions and to note these properly, so that, if whenever a difference of this kind may be proposed, the sum of this or the function, from which it has arisen, may be able to be shown at once.
Yet meanwhile the method of the infinitesimals supplies several rules, with the aid of which the discovery of the sums will be supported wonderfully.

32. But often the sum sought may be found more easily, if the proposed difference may depend on simple factors, which establish an arithmetical progression, the difference of which shall be the quantity ω itself. Thus, if the proposed function were $(x+\omega)(x+2\omega)$, the difference of which may be sought, because on putting $x+\omega$ in place of x this function will change into $(x+2\omega)(x+3\omega)$, the difference of this will be $2\omega(x+2\omega)$. Whereby in turn, if the difference $2\omega(x+2\omega)$ were proposed, the sum of this will be $(x+2\omega)(x+3\omega)$; hence therefore there will be

$$\Sigma(x+2\omega)=\frac{1}{2\omega}(x+\omega)(x+2\omega).$$

In a similar manner, if the function $(x+n\omega)(x+(n+1)\omega)$ were proposed, since the difference of this will be $2\omega(x+(n+1)\omega)$, there will be

$$\Sigma(x+(n+1)\omega)=\frac{1}{2\omega}(x+n\omega)(x+(n+1)\omega)$$

and

$$\Sigma(x+n\omega)=\frac{1}{2\omega}(x+(n-1)\omega)(x+n\omega).$$

33. If the function should depend on several factors, so that there may be

$$y=(x+(n-1)\omega)(x+n\omega)(x+(n+1)\omega),$$

since there shall be

$$y^{\mathrm{I}} = (x+n\omega)\big(x+(n+1)\omega\big)\big(x+(n+2)\omega\big),$$

there will be

$$\Delta y = 3\omega(x+n\omega)\big(x+(n+1)\omega\big)$$

and therefore

$$\Sigma\,(x+n\omega)\big(x+(n+1)\omega\big) = \tfrac{1}{3\omega}\big(x+(n-1)\omega\big)(x+n\omega)\big(x+(n+1)\omega\big).$$

In a like manner there may be found to be

$$\Sigma\,(x+n\omega)\big(x+(n+1)\omega\big)\big(x+(n+2)\omega\big)$$
$$= \tfrac{1}{4\omega}\big(x+(n-1)\omega\big)(x+n\omega)\big(x+(n+1)\omega\big)\big(x+(n+2)\omega\big),$$

from which the rule for finding the sums may at once apparent, if the difference may depend on several factors of this kind. But although these differences shall be integral rational functions, yet the some of these are found more easily in this manner than by the preceding method.

34. Hence also the way may be apparent for finding the sums of differences of fractions. Indeed let the proposed fraction be

$$y = \tfrac{1}{x+n\omega};$$

because there shall be

$$y^{\mathrm{I}} = \tfrac{1}{x+(n+1)\omega},$$

then there will be

$$\Delta y = \tfrac{1}{x+(n+1)\omega} - \tfrac{1}{x+n\omega} = \tfrac{-\omega}{(x+n\omega)\big(x+(n+1)\omega\big)}$$

and therefore

$$\Sigma\,\tfrac{1}{(x+n\omega)\big(x+(n+1)\omega\big)} = -\tfrac{1}{\omega}\cdot\tfrac{1}{x+n\omega}.$$

Again let there be

$$y = \tfrac{1}{(x+n\omega)\big(x+(n+1)\omega\big)};$$

on account of

$$y^{\mathrm{I}} = \tfrac{1}{\big(x+(n+1)\omega\big)\big(x+(n+2)\omega\big)}$$

there will be

$$\Delta y = \frac{-2\omega}{(x+n\omega)(x+(n+1)\omega)(x+(n+2)\omega)}$$

Hence therefore there comes about

$$\Sigma \frac{1}{(x+n\omega)(x+(n+1)\omega)(x+(n+2)\omega)} = \frac{-1}{2\omega} \cdot \frac{1}{(x+n\omega)(x+(n+1)\omega)}.$$

Again in a similar manner there will be

$$\Sigma \frac{1}{(x+n\omega)(x+(n+1)\omega)(x+(n+2)\omega)(x+(n+3)\omega)} = \frac{-1}{3\omega} \cdot \frac{1}{(x+n\omega)(x+(n+1)\omega)(x+(n+2)\omega)}.$$

35. This method of summation is to be understood correctly, because the sum of differences of this kind cannot be found by the preceding method. But if moreover the above difference may have the numerator, or the factors of the denominator, may not proceed in an arithmetical progression, then the safest manner of finding the sum is, that the proposed differences may be resolved into its simple factors ; of which the individual terms even if they may be unable to be summed, yet with two taken together the whole sum can be found, as often as that may be allowed to happen; for only will it be required to be considered, that each sum may be found with the aid of this formula

$$\Sigma \frac{1}{x+(n+1)\omega} - \Sigma \frac{1}{x+n\omega} = \frac{1}{x+n\omega};$$

although indeed neither of these sums are themselves able to be shown, yet the difference of these is known.

36. Therefore from these cases the matter returns to the resolution of each fraction into their simple fractions, which has been shown in the above book in profusion [Euler's *Introductio* referred to here on several occasions, of which a translation exists.] Therefore just as the sums may be found by the benefit of this, we will show by several examples.

EXAMPLE 1

The sum is sought, the difference of which shall be $\frac{3x+2\omega}{x(x+\omega)(x+2\omega)}$.

This difference may be resolved into its simple fractions, which will be

$$\frac{1}{\omega} \cdot \frac{1}{x} + \frac{1}{\omega} \cdot \frac{1}{x+\omega} - \frac{2}{\omega} \cdot \frac{1}{x+2\omega}.$$

Now since from the above formula there shall be

$$\Sigma\frac{1}{x+n\omega}=\Sigma\frac{1}{x+(n+1)\omega}-\frac{1}{x+n\omega},$$

there will be

$$\Sigma\frac{1}{x}=\Sigma\frac{1}{x+\omega}-\frac{1}{x}.$$

Hence the sum sought will be

$$\frac{1}{\omega}\Sigma\frac{1}{x}+\frac{1}{\omega}\Sigma\frac{1}{x+\omega}-\frac{2}{\omega}\Sigma\frac{1}{x+2\omega}=\frac{2}{\omega}\Sigma\frac{1}{x+\omega}-\frac{2}{\omega}\Sigma\frac{1}{x+2\omega}-\frac{1}{\omega x};$$

but there is

$$\Sigma\frac{1}{x+\omega}=\Sigma\frac{1}{x+2\omega}-\frac{1}{x+\omega};$$

from which the sum sought will be

$$-\frac{1}{\omega x}-\frac{2}{\omega(x+\omega)}=\frac{-3x-\omega}{\omega x(x+\omega)}.$$

EXAMPLE 2

The sum is sought the difference of which is $\frac{3\omega}{x(x+3\omega)}$.

On putting this difference $=z$ there will be $z=\frac{1}{x}-\frac{1}{x+3\omega}$, and thus

$$\Sigma z=\Sigma\frac{1}{x}-\Sigma\frac{1}{x+3\omega}=\Sigma\frac{1}{x+\omega}-\Sigma\frac{1}{x+3\omega}-\frac{1}{x}$$
$$=\Sigma\frac{1}{x+2\omega}-\Sigma\frac{1}{x+3\omega}-\frac{1}{x}-\frac{1}{x+\omega}=-\frac{1}{x}-\frac{1}{x+\omega}-\frac{1}{x+2\omega},$$

which is the sum sought. Therefore as often as the summation signs Σ are themselves finally removed, so the proposed difference will be able to show the sum ; but if this cancellation may not succeed, from this the indication is that the sum cannot be found.

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PRAEFATIO

Quid sit Calculus Differentialis atque in genere Analysis Infinitorum, iis, qui nulla adhuc eius cognitione sunt imbuti, vix explicari potest neque hic, uti in aliis disciplinis fieri solet, exordium tractationis a definitione commode sumere licet. Non quod huius calculi nulla plane detur definitio, sed quoniam ad eam intelligendam eiusmodi opus est notionibus non solum in vita communi, verum etiam in ipsa Analysi finitorum minus usitatis, quae demum in Calculi differentialis pertractatione evolvi atque explicari solent; quo fit, ut eius definitio non ante percipi queat, quam eius principia iam satis dilucide fuerint perspecta.

Primum igitur hic calculus circa quantitates variabiles versatur; etsi enim omnis quantitas sua natura in infinitum augeri et diminui potest, tamen, dum calculus ad certum quoddam institutum dirigitur, aliae quantitates constanter eandem magnitudinem retinere concipiuntur, aliae vero per omnes gradus auctionis ac diminutionis variari; ad quam distinctionem notandam illae quantitates constantes, hae vero variabiles vocari solent, ita ut hoc discrimen non tam in rei natura quam in quaestionis, ad quam calculus refertur, indole sit positum.

Quoniam haec differentia inter quantitates constantes et variabiles exemplo maxime illustrabitur, consideremus iactum globi ex tormento bellico vi pulveris pyrii explosi, siquidem hoc exemplum ad rem dilucidandam imprimis idoneum videtur. Plures igitur hic occurrunt quantitates, quarum ratio in ista investigatione est habenda: primo scilicet quantitas pulveris pyrii; tum elevatio tormenti supra horizontem; tertio longitudo iactus super plano horizontali; quarto tempus, quo globus explosus in aere versatur; ac nisi experimenta eodem tormento instituantur, insuper eius longitudo cum pondere globi in computum trahi deberet. Verum hic a varietate tormenti et globi animum removeamus, ne in quaestiones nimium implicatas incidamus. Quodsi ergo servata perpetuo eadem pulveris pyrii quantitate elevatio tormenti continuo immutetur iactusque longitudo cum tempore transitus globi per aerem requiratur, in hac quaestione copia pulveris seu vis impulsus erit quantitas constans; elevatio autem tormenti cum longitudine iactus eiusque duratione ad quantitates variabiles referri debebunt, siquidem pro omnibus elevationis gradibus has res definire velimus, ut inde innotescat, quantae mutationes in longitudine ac duratione iactus ab omnibus elevationis variationibus oriantur. Alia autem erit quaestio, si servata eadem tormenti elevatione quantitas pulveris pyrii continuo mutetur et mutationes, quae inde in iactum redundant, definiri debeant; hic enim elevatio tormenti erit quantitas constans, contra vero quantitas pulveris pyrii et longitudo ac duratio iactus quantitates variabiles. Sic igitur patet, quomodo mutato quaestionis statu eadem quantitas modo inter constantes modo inter variabiles numerari queat; simul autem hinc intelligitur, ad quod in hoc negotio maxime est attendendum, quomodo quantitates variabiles aliae ab aliis ita pendeant, ut mutata una reliquae necessario immutationes recipiant. Priori scilicet casu, quo quantitas pulveris pyrii eadem manebat, mutata tormenti elevatione etiam longitudo et duratio iactus mutantur suntque ergo longitudo et duratio iactus quantitates variabiles pendentes ab elevatione tormenti hacque mutata simul certas quasdam mutationes patientes; posteriori vero casu pendent a quantitate pulveris pyrii, cuius mutatio in illis certas mutationes producere debet.

Quae autem quantitates hoc modo ab aliis pendent, ut his mutatis etiam ipsae mutationes subeant, eae harum function.es appellari solent; quae denominatio latissime patet atque omnes modos, quibus una quantitas per alias determinari potest, in se complectitur. Si igitur x denotet quantitatem variabilem, omnes quantitates, quae utcunque ab x pendent seu per eam determinantur, eius functiones vocantur; cuiusmodi sunt quadratum eius xx aliaeve potentiae quaecunque nec non quantitates ex his utcunque compositae, quin etiam transcendentes et in genere, quaecunque ita ab x pendent, ut aucta vel diminuta x ipsae mutationes recipiant. Hinc iam nascitur quaestio, qua quaeritur, si quantitas x data quantitate sive augeatur sive diminuatur, quantum inde quaevis eius functiones immutentur seu quantum incrementum decrementumve accipiant.
Casibus quidem simplicioribus haec quaestio facile resolvitur; si enim quantitas x augeatur quantitate ω, eius quadratum xx hinc incrementum capiet $2x\omega+\omega\omega$ sicque incrementum ipsius x se habebit ad incrementum ipsius xx ut ω ad $2x\omega+\omega\omega$, hoc est ut 1 ad $2x+\omega$; similique modo in allis casibus ratio incrementi ipsius x ad incrementum vel decrementum, quod quaevis eius functio inde adipiscitur, considerari solet. Est vero investigatio rationis huiusmodi incrementorum ipsa non solum maximi momenti, sed ei etiam universa Analysis infinitorum innititur. Quod quo clarius appareat, sumamus exemplum superius quadrati xx, cuius incrementum $2x\omega+\omega\omega$, quod capit, dum ipsa quantitas x incremento ω augetur, vidimus ad hoc rationem tenere ut $2x+\omega$ ad 1; unde perspicuum est, quo minus sumatur incrementum ω, eo propius istam rationem accedere ad rationem $2x$ ad 1; neque tamen ante prorsus in hanc rationem abit, quam incrementum illud ω plane evanescat. Hinc intelligimus, si quantitatis variabilis x incrementum ω in nihilum abeat, tum etiam quadrati eius xx incrementum inde oriundum quidem evanescere, verumtamen ad id rationem tenere ut $2x$ ad 1; et quod hic de quadrato est dictum, de omnibus allis functionibus ipsius x est intelligendum; quippe quarum incrementa evanescentia, quae capiunt, dum ipsa quantitas x incrementum evanescens sumit, ad hoc ipsum certam et assignabilem rationem tenebunt. Atque hoc modo sumus deducti ad definitionem *Calculi Differentialis*, qui est *methodus determinandi rationem incrementorum evanescentium, quae functiones quaecunque accipiunt, dum quantitati variabili, cuius sunt functiones, incrementum evanescens tribuitur;* hacque definitione veram indolem Calculi differentialis contineri atque adeo exhauriri iis, qui in hoc genere non sunt hospites, facile erit perspicuum.

Calculus igitur differentialis non tam in his ipsis incrementis evanescentibus, quippe quae sunt nulla, exquirendis quam in eorum ratione ac proportione mutua scrutanda occupatur; et cum hae rationes finitis quantitatibus exprimantur, etiam hic calculus circa quantitates finitas versari est censendus. Quamvis enim praecepta, uti vulgo tradi solent, ad ista incrementa evanescentia definienda videantur accommodata, nunquam tamen ex iis absolute spectatis, sed potius semper ex eorum ratione conclusiones deducuntur. Simili vero modo Calculi integralis ratio est comparata, qui convenientissime ita definitur, ut dicatur esse *methodus ex cognita ratione incrementorum evanescentium ipsas illas functiones, quarum sunt incrementa, inveniendi.*

Quo autem facilius hae rationes colligi atque in calculo repraesentari possint, haec ipsa incrementa evanescentia, etiamsi sint nulla, tamen certis signis denotari solent; quibus adhibitis nihil obstat, quominus iis certa nomina imponantur. Vocantur itaque differentialia, quae, cum quantitate destituantur, infinite parva quoque dicuntur, quae igitur sua natura ita sunt interpretanda, ut omnino nulla seu nihilo aequalia reputentur. Ita si quantitati x incrementum tribuatur ω, ut abeat in $x+\omega$, eius quadratum xx abibit in $xx+2x\omega+\omega\omega$ ideoque incrementum capit $2x\omega+\omega\omega$*;* quare incrementum ipsius x,quod est ω, se habebit ad incrementum quadrati, quod est $2x\omega+\omega\omega$, uti 1 ad $2x+\omega$; quae ratio abit in 1 ad $2x$ tum demum, cum ω evanescit. Fiat igitur $\omega=0$ et ratio istorum incrementorum evanescentium, quae sola in Calculo differentiali spectatur, utique est ut 1 ad 2 x*;* neque vicissim haec ratio veritati esset consentanea, nisi revera illud incrementum ω evanesceret penitusque nihilo fieret aequale. Quodsi ergo hoc nihilum per ω indicatum referat incrementum quantitatis x, quia hoc se habet ad incrementum quadrati xx ut 1 ad $2x$, erit quadrati xx incrementum $=2x\omega$ ideoque etiam nihilo aequale; unde simul constat annihilationem horum incrementorum non obstare, quominus eorum ratio, quae est ut 1 ad $2x$, sit determinata. Quod nihilum iam hic littera ω exhibetur, id in Calculo differentiali, quia ut incrementum quantitatis x spectatur, signo dx repraesentari eiusque differentiale vocari solet; positoque dx loco ω ipsius xx differentiale erit $2xdx$. Simili modo ostenditur fore cubi x^3 differentiale $=3xxdx$ et in genere cuiusque dignitatis x^n differentiale fore $=nx^{n-1}dx$. Quaecunque autem aliae functiones ipsius x proponantur, in Calculo differentiali regulae traduntur earum differentialia inveniendi; verum perpetuo tenendum est, cum haec differentialia absolute sint nihila, ex iis nihil aliud concludi nisi eorum rationes mutuas, quae utique ad quantitates finitas reducuntur. Cum autem hoc modo, qui solus est rationi consentaneus, principia Calculi differentialis stabiliuntur, omnes obtrectationes, quae contra hunc calculum proferri sunt solitae, sponte corruunt; quae tamen summam vim retinerent, si differentialia seu infinite parva non plane annihilarentur.

Pluribus autem, qui Calculi differentialis praecepta tradidere, visum est differentialia a nihilo absoluto secernere peculiaremque ordinem quantitatum infinite parvarum, quae non penitus evanescant, sed quantitatem quandam, quae quidem esset omni assignabili minor, retineant, constituere; his igitur iure est obiectum rigorem geometricum negligi et conclusiones inde deductas, propterea quod huiusmodi infinite parva negligerentur, merito esse suspectas; quantumvis enim exigua haec infinite parva concipiantur, tamen non solum singulis, sed etiam pluribus atque adeo innumerabilibus simul reiiciendis errorem tandem inde enormem resultare posse. Quam obiectionem perperam eiusmodi exemplis, quibus per Calculum differentialem eaedem conclusiones ac per Geometriam elementarem eliciuntur, infringere conantur; nam si ea infinite parva, quae in calculo negliguntur, non sunt nihil, inde necessario error isque eo maior, quo magis ea coacervantur, resultare debet; hocque si minus eveniat, id potius vitio calculi, quo nonnunquam errores per alios errores compensantur, esset tribuendum, quam ipse calculus ab erroris suspicione liberaretur. Quodsi autem nullo novo errore huiusmodi compensatio fiat, talibus exemplis luculenter id ipsum, quod volo, evincitur: ea, quae fuerint neglecta, omnino et absolute pro nihilo esse habenda neque infinite parva, quae in Calculo differentiali tractantur, a nihilo absoluto discrepare. Minime etiam negotium conficitur, quando a nonnullis infinite parva ita describuntur, ut instar pulvisculorum respectu vasti montis vel etiam totius globi terrestris spectari debeant; etsi enim qui magnitudinem totius globi terrestris calculo determinare susceperit, ei error non unius, sed plurium milium pulvisculorum facile condonari soleat, tamen rigor geometricus etiam a tantillo errore abhorret nimisque gravis esset haec obiectio, si ullam vim retineret. Deinde etiam difficile dictu est, quid lucri inde sperent, qui infinite parva a nihilo distingui volunt; metuunt autem, ne, si plane evanescant, etiam comparatio eorum, ad quam totum negotium perduci sentiunt, tollatur; quomodo enim absolute nihila inter se comparari queant, nullo modo concipi posse profitentur. Necesse ergo putant iis aliquam magnitudinem relinquere, quo habeant aliquid, in quo comparationem instituant; hanc tamen magnitudinem tam parvam admittere coguntur, ut, quasi esset nulla, spectari ac sine errore in calculo negligi possit. Neque tamen certam ac definitam ipsi magnitudinem, licet incomprehensibiliter parvam, assignare audent; semper enim, si eam bis terve minorem assumerent, eodem modo comparationes se essent habiturae. Ex quo perspicuum est nihil plane ipsam magnitudinem ad comparationem instituendam conferre hancque adeo non tolli, etiamsi illa magnitudo penitus evanescat.

Ex dictis autem supra manifestum est eam comparationem, quae in Calculo differentiali spectatur, ne locum quidem habere, nisi illa incrementa prorsus evanescant; incrementum enim quantitatis x, quod in genere indicavimus per ω, ad incrementum quadrati xx, quod est $2x\omega+\omega\omega$, rationem habet ut 1 ad $2x+\omega$; quae semper differt a ratione 1 ad $2x$, nisi sit $\omega=0$; at si statuamus esse $\omega=0$, tum demum vere affirmare possumus hanc rationem fieri exacte ut 1 ad $2x$. Interim tamen perspicitur, quo minus illud incrementum ω accipiatur, eo propius ad hanc rationem accedi; unde non solum licet, sed etiam naturae rei convenit haec incrementa primum ut finita considerare atque etiam in figuris, si quibus opus est ad rem illustrandam, finite repraesentare; deinde vero haec incrementa cogitatione continuo minora fieri concipiantur sicque eorum ratio continuo magis ad certum quendam limitem appropinquare reperietur, quem autem tum demum attingant, cum plane in nihilum abierint. Hic autem limes, qui quasi rationem ultimam incrementorum illorum constituit, verum est obiectum Calculi differentialis; cuius igitur prima fundamenta is iecisse existimandus est, cui primum in mentem venit has rationes ultimas, ad quas quantitatum variabilium incrementa, dum continuo magis diminuuntur, appropinquant et, cum evanescunt, tum demum attingunt, contemplari.

Huius autem speculationis vestigia deprehendimus apud antiquissimos Auctores, quibus idcirco idea quaedam levisque cognitio Analysis infinitorum abiudicari nequit. Paullatim deinde haec scientia maiora accepit incrementa neque subito ad id fastigium, in quo nunc cernitur, est evecta, etiamsi quidem in ea multo plura adhuc sint occulta quam in lucem protracta. Cum enim Calculus differentialis ad omnis generis functiones, utcunque sint compositae, extendatur, non repente methodus innotuit omnium plane functionum incrementa evanescentia inter se comparandi; sed sensim haec inventio ad functiones continuo magis complicatas processit. Quod scilicet ad functiones rationales attinet, ratio ultima, quam earum incrementa evanescentia inter se tenent, multo ante NEUTONI ac LEIBNIZII tempora assignari potuit, ita ut Calculus differentialis, quatenus ad solas functiones rationales applicatur, diu ante haec tempora inventus sit censendus. Tum vero nullum est dubium, quin NEUTONO eam Calculi differentialis partem, quae circa functiones irrationales versatur, acceptam referre debeamus; ad quam insigni suo Theoremate de evolutione generali potestatum binomii feliciter est deductus, quo eximio invento limites Calculi differentialis iam mirifice erant amplificati. LEIBNIZIO autem non minus sumus obstricti, quod hunc calculum antehac tantum velut singulare artificium spectatum in formam disciplinae redegerit eiusque praecepta tanquam in systema collegerit ac dilucide explicaverit. Hinc enim maxima subsidia suggerebantur ad hunc calculum ulterius excolendum et ea, quae adhuc desiderabantur, ex cenis principiis elicienda. Mox igitur studio cum ipsius LEIBNIZII tum BERNOULLIORUM ad hoc ab eo incitatorum fines Calculi differentialis etiam ad functiones transcendentes, quae pars adhuc fuerat inculta, sunt promoti, tum vero etiam solidissima fundamenta Calculi integralis constituta; quibus insistentes, qui deinceps in hoc genere elaborarunt, continuo maiora incrementa addiderunt. NEUTONUS vero etiam amplissima dederat specimina Calculi integralis, cuius prima inventio, cum a prima origine Calculi differentialis vix separari queat, non ita absolute constitui potest; et quoniam maxima eius pars adhuc excolenda restat, hic calculus ne nunc quidem pro absolute invento haberi potest, sed potius, quantum cuique pro viribus ad eius perfectionem conferre contigerit, id grata mente agnoscere debemus. Atque haec de gloria inventionis huius calculi tenenda esse iudico, de qua quidem antehac tantopere est disceptatum.

Quod autem ad varia nomina, quae isti calculo a diversarum nationum Mathematicis imponi solent, attinet, ea omnia huc redeunt, ut cum data hic definitione egregie consentiant; sive enim incrementa illa evanescentia, quorum ratio consideratur, differentialia vocentur sive fluxiones, ea semper nihilo aequalia sunt intelligenda; in quo vera notio infinite parvorum constitui debet. Hinc vero etiam omnia, quae de differentialibus secundi et altiorum ordinum curiose magis quam utiliter sunt disputata, reddentur planissima, cum omnia per se aeque evanescant neque ea unquam per se, sed potius eorum relatio mutua spectari soleat. Cum enim ratio, quam duarum functionum incrementa evanescentia tenent, iterum per functionem quandam exprimatur, si et huius functionis incrementum evanescens cum aliis conferatur, res ad differentialia secunda referri est censenda; sicque porro progressio ad differentialia altiorum graduum intelligi debet, ita ut semper quantitates finitae revera animo obversentur signaque differentialium tantum ad eas commode repraesentandas adhibeantur. Primo quidem intuitu ista Analysis infinitorum descriptio plerisque levis ac nimis sterilis videatur, etsi species illa arcana infinite parvorum re haud plus polliceatur; verum si rationes, quae inter incrementa evanescentia functionum quarumvis intercedunt, probe cognoscamus, haec cognitio saepenumero per se maximi est momenti, tum vero in plerisque iisque maxime arduis investigationibus ita est necessaria, ut sine eius adminiculo nihil plane intelligi possit. Veluti si quaestio sit de motu globi ex tormento explosi simulque ratio resistentiae aeris haberi debeat, quomodo motus per spatium finitum sit futurus, nullo modo statim definire licet, dum tam directio semitae, in qua globus incedit, quam ipsius celeritas, a qua resistentia pendet, quovis momento immutatur. Quo minus autem spatium, per quod motus fiat, consideremus, eo minor erit illa variabilitas eoque facilius ad cognitionem veri pertingere licebit; quodsi autem illud spatium plane evanescens reddamus, quia iam omnis inaequalitas tam in directione viae quam in celeritate tollitur, effectum resistentiae per regulas motus accurate definire motusque mutationem puncto temporis productam assignare licebit. Cognitis autem his mutationibus momentaneis seu potius, cum ipsae sint nullae, earum relatione mutua iam plurimum sumus lucrati; atque Calculi integralis opus est exinde motum per spatium finitum variatum concludere. Minime autem necesse esse arbitror usum Calculi differentialis atque Analyseos infinitorum in genere pluribus ostendere, cum nunc quidem satis sit exploratum, si vel levissimam investigationem, in quam motus corporum tam solidorum quam fluidorum ingrediatur, accuratius instituere velimus, id non solum non sine Analysi infinitorum praestari posse, sed hanc ipsam scientiam saepe nondum satis excultam esse, ut rem penitus explicare valeamus. Per omnes scilicet Matheseos partes usus huius Analyseos sublimioris usque adeo diffunditur, ut omnia, quae sine eius interventu adhuc expedire licuit, pro nihilo propemodum sint habenda.

Constitui igitur in hoc libro universum Calculum differentialem ex veris princips derivare atque ita copiose pertractare, ut nihil praetermitterem eorum, quae quidem adhuc eo pertinentia sunt inventa. In duas opus divisi partes, in quarum priori iactis Calculi differentialis fundamentis methodum exposui omnis generis functiones differentiandi neque tantum differentialia primi ordinis, sed etiam superiorum ordinum inveniendi, sive functiones unicam variabilem sive duas pluresve involvant. In altera autem parte amplissimum huius calculi usum in ipsa Analysi finitorum ac doctrina serierum exposui; ubi etiam imprimis Theoriam maximorum ac minimorum dilucide explicavi. De usu autem huius calculi in Geometria linearum curvarum nihil adhuc affero, quod eo minus desiderabitur, cum in aliis operibus haec pars ita copiose sit pertractata, ut adeo prima Calculi differentialis principia quasi ex Geometria sint petita ad hancque scientiam, cum vix satis essent evoluta, summa cura applicata. Hic autem omnia ita intra Analyseos purae limites continentur, ut ne ulla quidem figura opus fuerit ad omnia huius calculi praecepta explicanda.

INSTITUTIONUM
CALCULI DIFFERENTIALIS

PARS PRIOR

CONTINENS

COMPLETAM HUlUS CALCULI

EXPLICATIONEM.

CAPUT I

DE DIFFERENTIIS FINITIS

1. Ex iis, quae in libro superiori de quantitatibus variabilibus atque functionibus sunt exposita, perspicuum est, prout quantitas variabilis actu variatur, ita omnes eius functiones variationem pati. Sic, si quantitas variabilis x capiat incrementum ω, ita ut pro x scribatur $x+\omega$, omnes functiones ipsius x, cuiusmodi sunt xx, x^3, $\frac{a+x}{xx+aa}$, alios induent valores: scilicet abibit in $xx+2x\omega+\omega\omega$, x^3 abibit in $x^3+3xx\omega+3x\omega\omega+\omega^3$ et $\frac{a+x}{xx+aa}$ transmutabitur in $\frac{a+x+\omega}{aa+xx+2x\omega+\omega\omega}$. Huiusmodi ergo alteratio semper orietur, nisi functio speciem tantum quantitatis variabilis mentiatur, revera autem sit quantitas constans, veluti x^0; quo casu talis functio invariata manet, utcunque quantitas x immutetur.

2. Quae cum sint satis exposita, propius accedamus ad eas functionum affectiones, quibus universa analysis infinitorum innititur. Sit igitur y functio quaecunque quantitatis variabilis x, pro qua successive valores in arithmetica progressione procedentes substituantur, scilicet $x,\ x+\omega,\ x+2\omega,\ x+3\omega,\ x+4\omega,$ etc., ac denotet y^{I} valorem, quem functio y induit, si in ea loco x substituatur $x+\omega$; simili modo sit y^{II} is ipsius y valor, si loco x scribatur $x+2\omega$, parique ratione denotent $y^{\mathrm{III}}, y^{\mathrm{IV}}, y^{\mathrm{V}}$, etc. valores ipsius y, qui emergunt, dum loco x ponuntur $x+3\omega,\ x+4\omega,\ x+5\omega$ etc., ita ut isti diversi valores ipsarum x et y sequenti modo sibi respondeant:

$$\begin{array}{lllllll} x, & x+\omega, & x+2\omega, & x+3\omega, & x+4\omega, & x+5\omega, & \text{etc.} \\ y, & y^{\mathrm{I}}, & y^{\mathrm{II}}, & y^{\mathrm{III}}, & y^{\mathrm{IV}}, & y^{\mathrm{V}}, & \text{etc.} \end{array}$$

3. Quemadmodum series arithmetica $x,\ x+\omega,\ x+2\omega$ etc. in infinitum continuari potest, ita series ex functione y orta $y, y^{\mathrm{I}}, y^{\mathrm{II}}$ etc. quoque in infinitum progredietur eiusque natura pendebit ab indole functionis y. Sic, si fuerit $y = x$ vel $y = ax + b$, series $y, y^{\mathrm{I}}, y^{\mathrm{II}}$ etc. quoque erit arithmetica; si fuerit $y=\frac{a}{bx+c}$ series prodibit harmonica; sin autem sit $y=a^x$, habebitur series geometrica. Neque ulla excogitari potest series, quae non hoc modo ex certa functione ipsius y oriri queat; vocari autem solet huiusmodi functio ipsius x ratione seriei, quae ex ita oritur, eius *terminus generalis;* quare, cum omnis series certa lege formata habeat terminum generalem, ea vicissim ex certa ipsius *x* functione oritur, uti in doctrina de seriebus fusius explicari solet.

4. Hic autem potissimum ad differentias, quibus termini seriei $y, y^{\text{I}}, y^{\text{II}}, y^{\text{III}}$ etc. inter se discrepant, attendimus; quas ut ad differentialium naturam accommodemus, sequentibus signis indicemus, ut sit

$$y^{\text{I}} - y = \Delta y, \quad y^{\text{II}} - y^{\text{I}} = \Delta y^{\text{I}}, \quad y^{\text{III}} - y^{\text{II}} = \Delta y^{\text{II}} \quad \text{etc.}$$

Exprimet ergo Δy incrementum, quod functio *y* capit, si in ea loco *x* ponatur $x+\omega$ denotante ω numerum quemcunque pro lubitu assumtum. In doctrina quidem serierum sumi solet $\omega = 1$; verum hic ad nostrum institutum expedit valore generali uti, qui pro arbitrio augeri diminuive queat. Vocari quoque solet hoc incrementum Δy functionis *y* eius *differentia,* qua sequens valor y^{I} primum *y* superat, atque perpetuo tanquam incrementum consideratur, etiamsi saepius revera decrementum exhibeat, id quod ex eius valore negativo agnoscitur.

5. Quoniam y^{II} oritur ex *y,* si loco *x* scribatur $x+2\omega$, manifestum est eandem quantitatem esse orituram, si primum pro *x* ponatur $x+\omega$ tumque denuo $x+\omega$ loco *x* statuatur. Hinc y^{II} orietur ex y^{I}, si in hoc loco *x* scribatur $x+\omega$*;* eritque ideo Δy^{I} incrementum ipsius y^{I}, quod capit posito $x+\omega$ loco *x;* sicque Δy^{I} vocatur simili modo *differentia* ipsius y^{I}. Pari ratione porro erit Δy^{II} *differentia* ipsius y^{II} seu eius incrementum, quod accipit, si loco *x* ponatur $x+\omega$*;* atque Δy^{III} erit *differentia* seu incrementum ipsius y^{III} et ita porro. Hoc pacto ex serie valorum ipsius *y*, qui sunt $y, y^{\text{I}}, y^{\text{II}}, y^{\text{III}}$ etc., obtinebitur series differentiarum $\Delta y, \Delta y^{\text{I}}, \Delta y^{\text{II}}$ etc., quae inveniuntur, si quilibet terminus illius seriei a sequente subtrahatur.

6. Inventa serie differentiarum si ex ea denuo differentiae capiantur quamlibet a sequente subtrahendo, orientur differentiae differentiarum, quae vocantur *differentiae secundae* hocque modo per characteres convenientissime repraesentantur, ut significet

$$\begin{aligned}
\Delta\Delta y &= \Delta y^{\text{I}} - \Delta y \\
\Delta\Delta y^{\text{I}} &= \Delta y^{\text{II}} - \Delta y^{\text{I}} \\
\Delta\Delta y^{\text{II}} &= \Delta y^{\text{III}} - \Delta y^{\text{II}} \\
\Delta\Delta y^{\text{III}} &= \Delta y^{\text{IV}} - \Delta y^{\text{III}} \\
&\text{etc.}
\end{aligned}$$

Vocatur itaque $\Delta\Delta y$ differentia secunda ipsius y, $\Delta\Delta y^{\mathrm{I}}$ differentia secunda ipsius y^{I} et ita porro. Simili autem modo ex differentiis secundis, si denuo earum differentiae capiantur, prodibunt differentiae tertiae hoc modo scribendae $\Delta^3 y$, $\Delta^3 y^{\mathrm{I}}$ etc. hincque porro differentiae quartae $\Delta^4 y$, $\Delta^4 y^{\mathrm{I}}$ etc. sicque ultra, quousque libuerit.

7. Repraesentemus singulas has differentiarum series ita in schemate, quo earum nexus facilius in oculos incidat:

$$\textit{Progressio arithmetica}$$
$$x,\quad x+\omega,\quad x+2\omega,\quad x+3\omega,\quad x+4\omega,\quad x+5\omega,\ \text{etc.}$$
$$\textit{Valores functionis}$$
$$y,\quad y^{\mathrm{I}},\quad y^{\mathrm{II}},\quad y^{\mathrm{III}},\quad y^{\mathrm{IV}},\quad y^{\mathrm{V}}\quad \text{etc.}$$
$$\textit{Differentiae primae}$$
$$\Delta y,\quad \Delta y^{\mathrm{I}},\quad \Delta y^{\mathrm{II}},\quad \Delta y^{\mathrm{III}},\quad \Delta y^{\mathrm{IV}}\quad \text{etc.}$$
$$\textit{Differentiae secundae}$$
$$\Delta\Delta y,\quad \Delta\Delta y^{\mathrm{I}},\quad \Delta\Delta y^{\mathrm{II}},\quad \Delta\Delta y^{\mathrm{III}}\quad \text{etc.}$$
$$\textit{Differentiae tertiae}$$
$$\Delta^3 y,\quad \Delta^3 y^{\mathrm{I}},\quad \Delta^3 y^{\mathrm{II}},\quad \text{etc.}$$
$$\textit{Differentiae quartae}$$
$$\Delta^4 y,\quad \Delta^4 y^{\mathrm{I}},\quad \text{etc.}$$
$$\textit{Differentiae quintae}$$
$$\Delta^5 y,\quad \text{etc.}$$
$$\text{etc.}$$

quarum quaelibet ex praecedente oritur quosque terminos a sequentibus subtrahendo. Quacunque ergo functione ipsius x loco y substituta, quoniam valores $y^{\mathrm{I}}, y^{\mathrm{II}}, y^{\mathrm{III}}$ etc. per notas compositiones facile formantur, ex iis sine labore singulae differentiarum series invenientur.

8. Ponamus esse $y = x$ eritque $y^{\mathrm{I}} = x^{\mathrm{I}} = x+\omega$, $y^{\mathrm{II}} = x^{\mathrm{II}} = x+2\omega$ etc. ita porro. Unde differentiis sumendis erit $\Delta x = \omega$, $\Delta x^{\mathrm{I}} = \omega$, $\Delta x^{\mathrm{II}} = \omega$ etc. ideoque omnes differentiae primae ipsius x erunt constantes ac proinde differentiae secundae omnes evanescent pariterque differentiae tertiae et sequentium ordinum omnes. Cum igitur sit $\Delta x = \omega$, ob analogiam loco litterae ω iste character Δx commode adhibebitur. Quantitatis ergo variabilis *x*, cuius valores successivi $x, x^{\mathrm{I}}, x^{\mathrm{II}}, x^{\mathrm{III}}$ etc. arithmeticam progressionem constituere assumuntur, differentiae $\Delta x, \Delta x^{\mathrm{I}}, \Delta x^{\mathrm{II}}$ etc. erunt constantes atque inter se aequales; ac propterea erit $\Delta\Delta x = 0,\ \Delta^3 x = 0,\ \Delta^4 x = 0$ sicque porro.

9. Pro valoribus ipsius *x*, qui ipsi successive tribuuntur, progressionem arithmeticam hic assumsimus, ita ut horum valorum differentiae primae sint constantes, secundae ac reliquae omnes evanescant. Quod etsi ab arbitrio nostro pendet, cum aliam quamcunque progressionem aeque adhibere potuissemus, tamen progressio arithmetica prae reliquis omnibus commodissime usurpari solet, cum quod sit simplicissima atque intellectu facillima, tum vero maxime, quod ad omnes omnino valores, quos quidem *x* induere potest, pateat. Tribuendo enim ipsi ω valores tam negativos quam affirmativos in hac serie valorum ipsius *x* omnes omnino continentur quantitates reales, quae in locum ipsius *x* substitui possunt; contra autem si seriem geometricam elegissemus, ad valores negativos nullus aditus patuisset. Hanc ob causam variabilitas functionum *y* ex valoribus ipsius *x* progressionem arithmeticam constituentibus aptissime diiudicatur.

10. Uti est $\Delta y = y^{\mathrm{I}} - y$, ita differentiae ulteriores quoque ex terminis primae seriei $y, y^{\mathrm{I}}, y^{\mathrm{II}}, y^{\mathrm{III}}$ etc. definiri possunt.

Cum enim sit

$$\Delta y^{\mathrm{I}} = y^{\mathrm{II}} - y^{\mathrm{I}},$$

erit

$$\Delta\Delta y = y^{\mathrm{II}} - 2y^{\mathrm{I}} + y \quad \text{et} \quad \Delta\Delta y^{\mathrm{I}} = y^{\mathrm{III}} - 2y^{\mathrm{II}} + y^{\mathrm{I}}$$

ideoque

$$\Delta^3 y = \Delta\Delta y^{\mathrm{I}} - \Delta\Delta y = y^{\mathrm{III}} - 3y^{\mathrm{II}} + 3y^{\mathrm{I}} - y;$$

simili modo erit

$$\Delta^4 y = y^{\mathrm{IV}} - 4y^{\mathrm{III}} + 6y^{\mathrm{II}} - 4y^{\mathrm{I}} + y \quad \text{et} \quad \Delta^5 y = y^{\mathrm{V}} - 5y^{\mathrm{IV}} + 10y^{\mathrm{III}} - 10y^{\mathrm{II}} + 5y^{\mathrm{I}} - y,$$

quarum formularum coefficientes numerici eandem legem tenent, quae in potestatibus binomii observatur. Quemadmodum ergo differentia prima ex duobus terminis seriei $y, y^{\mathrm{I}}, y^{\mathrm{II}}, y^{\mathrm{III}}$ etc. determinatur, ita differentia secunda determinatur ex tribus, tertia ex quatuor et ita de ceteris. Cognitis autem differentiis cuiusque ordinis ipsius *y* simili modo differentiae omnium ordinum ipsius $y^{\mathrm{I}}, y^{\mathrm{II}}$ etc. definientur.

11. Proposita ergo quacunque functione *y* singulae eius differentiae tam prima quam sequentes, quae quidem differentiae ω, qua valores ipsius *x* progrediuntur, respondent, poterunt inveniri. Neque vero ad hoc opus est, ut series valorum ipsius *y* ulterius continuetur; quemadmodum enim differentia prima Δy reperitur, si in *y* loco *x* scribatur

$x+\omega$ atque a valore orto Δy^{I} ipsa functio *y* subtrahatur, ita differentia secunda $\Delta\Delta y$ obtinebitur, si in differentia prima Δy loco *x* ponatur $x+\omega$, ut oriatur Δy^{I}, atque Δy a Δy^{I} subtrahatur. Simili modo si differentiae secundae $\Delta\Delta y$ capiatur differentia, eam subtrahendo a valore, quem induit, si loco *x* ponatur $x+\omega$, proveniet differentia tertia $\Delta^3 y$ hincque porro eodem modo differentia quarta $\Delta^4 y$ etc. Dummodo ergo quis noverit differentiam primam cuiusque functionis investigare, simul poterit differentiam secundam, tertiam omnesque sequentes invenire, propterea quod differentia secunda ipsius *y* nil aliud est nisi differentia prima ipsius Δy et differentia tertia ipsius *y* nil aliud nisi differentia prima ipsius $\Delta\Delta y$ sicque porro de reliquis.

12. Si functio *y* fuerit ex duabus pluribusve partibus composita, ut sit $y=p+q+r+\text{etc.}$, tum, quia est $y^{\mathrm{I}}=p^{\mathrm{I}}+q^{\mathrm{I}}+r^{\mathrm{I}}+\text{etc.}$, erit differentia

$$\Delta y=\Delta p+\Delta q+\Delta r+\text{etc.}$$

similique modo porro

$$\Delta\Delta y=\Delta\Delta p+\Delta\Delta q+\Delta\Delta r+\text{etc.},$$

unde inventio differentiarum, si functio proposita ex partibus fuerit composita, non parum facilior redditur. Quodsi vero functio *y* fuerit productum ex duabus functionibus *p* et *q*, nempe $y=pq$, quia erit $y^{\mathrm{I}}=p^{\mathrm{I}}q^{\mathrm{I}}$ et $p^{\mathrm{I}}=p+\Delta p$ atque $q^{\mathrm{I}}=q+\Delta q$, fiet

$$p^{\mathrm{I}}q^{\mathrm{I}}=pq+p\Delta q+q\Delta p+\Delta p\Delta q$$

hincque

$$\Delta y=p\Delta q+q\Delta p+\Delta p\Delta q.$$

Unde, si sit *p* quantitas constans = *a*, ob $\Delta a=0$ erit functionis $y=aq$ differentia prima $\Delta y=a\Delta q$ similique modo differentia secunda $\Delta\Delta y=a\Delta\Delta q$, tertia $\Delta^3 y=a\Delta^3 q$ et ita porro.

13. Quoniam omnis functio rationalis integra est aggregatum ex aliquot potestatibus ipsius *x*, omnes differentias functionum rationalium integrarum invenire poterimus, si differentias potestatum tantum exhibere noverimus.
Hanc ob rem singularum potestatum quantitatis variabilis *x* differentias investigemus in sequentibus exemplis.

Cum autem sit $x^0=1$, erit $\Delta x^0=0$, propterea quod x^0 non variatur; etiamsi *x* abeat in $x+\omega$.

Tum vero vidimus esse $\Delta x=\omega$ et $\Delta\Delta x=0$ simulque differentiae sequentium ordinum evanescunt. Quae cum sint manifesta, a potestate secunda incipiamus.

EXEMPLUM 1

Invenire differentias omnium ordinum potestatis x^2.

Cum hic sit $y = x^2$, erit $y^{\mathrm{I}} = (x+\omega)^2$ ideoque

$$\Delta y = 2\omega x + \omega\omega,$$

quae est differentia prima. Iam ob ω quantitatem constantem erit $\Delta\Delta y = 2\omega\omega$ et $\Delta^3 y = 0$, $\Delta^4 y = 0$ etc.

EXEMPLUM 2

Invenire differentias omnium ordinum potestatis x^3.

Ponatur $y = x^3$, et cum sit $y^{\mathrm{I}} = (x+\omega)^3$, erit

$$\Delta y = 3\omega xx + 3\omega^2 x + \omega^3,$$

quae est differentia prima. Deinde ob $\Delta xx = 2\omega x + \omega\omega$ erit

$$\Delta 3\omega xx = 6\omega\omega x + 3\omega^3 \text{ et } \Delta .3\omega^2 x = 3\omega^3 \text{ et } \Delta\omega^3 = 0;$$

quibus collectis erit

$$\Delta\Delta y = 6\omega^2 x + 6\omega^3 \text{ atque } \Delta^3 y = 6\omega^3.$$

Differentiae vero sequentes evanescent.

EXEMPLUM 3

Invenire differentias omnium ordinum potestatis x^4.

Posito $y = x^4$ ob $y^{\mathrm{I}} = (x+\omega)^4$ erit

$$\Delta y = 4\omega x^3 + 6\omega^2 x^2 + 4\omega^3 x + \omega^4,$$

quae est differentia prima. Tum ex praecedentibus est

$$\Delta.4\omega x^3 \quad = 12\omega^2 x^2 + 12\omega^3 x + 4\omega^4$$
$$\Delta.6\omega^2 x^2 = ... + 12\omega^3 x + 6\omega^4$$
$$\Delta.4\omega^3 x \quad = ...4\omega^4$$
$$\Delta.\ \omega^4 \qquad = ...0.$$

His colligendis erit differentia secunda

$$\Delta\Delta y = 12\omega^2 x^2 + 24\omega^3 x + 14\omega^4.$$

Quia deinde porro est

$$\Delta.12\omega^2 x^2 = 24\omega^3 x + 12\omega^4$$
$$\Delta.24\omega^3 x \quad = ...24\omega^4\ .$$
$$\Delta.14\omega^4 \quad = ...0,$$

prodibit differentia tertia

$$\Delta^3 y = 24\omega^3 x + 36\omega^4$$

atque tandem differentia quarta

$$\Delta^4 y = 24\omega^4\,;$$

quae cum sit constans, differentiae sequentium ordinum evanescent.

EXEMPLUM 4

Invenire differentias cuiusvis ordinis potestatis x^n

Ponatur $y = x^n$, et cum sit $y^{\mathrm{I}} = (x+\omega)^n$, $y^{\mathrm{II}} = (x+2\omega)^n$, $y^{\mathrm{III}} = (x+E\omega)^n$ etc., potestates evolutae dabunt

$$y = x^n$$
$$y^{\mathrm{I}} = x^n + \tfrac{n}{1}\omega x^{n-1} + \tfrac{n(n-1)}{1\cdot 2}\omega^2 x^{n-2} + \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\omega^3 x^{n-3} + \text{etc.}$$
$$y^{\mathrm{II}} = x^n + \tfrac{n}{1}2\omega x^{n-1} + \tfrac{n(n-1)}{1\cdot 2}4\omega^2 x^{n-2} + \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}8\omega^3 x^{n-3} + \text{etc.}$$
$$y^{\mathrm{III}} = x^n + \tfrac{n}{1}3\omega x^{n-1} + \tfrac{n(n-1)}{1\cdot 2}9\omega^2 x^{n-2} + \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}27\omega^3 x^{n-3} + \text{etc,}$$
$$y^{\mathrm{IV}} = x^n + \tfrac{n}{1}4\omega x^{n-1} + \tfrac{n(n-1)}{1\cdot 2}16\omega^2 x^{n-2} + \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}64\omega^3 x^{n-3} + \text{etc}$$

Hinc differentiis sumendis prodibit

$$\Delta y = \frac{n}{1}\omega x^{n-1} + \frac{n(n-1)}{1\cdot 2}\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\omega^3 x^{n-3} + \text{etc.}$$
$$\Delta y^{\text{I}} = \frac{n}{1}\omega x^{n-1} + \frac{n(n-1)}{1\cdot 2}3\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}7\omega^3 x^{n-3} + \text{etc.}$$
$$\Delta y^{\text{II}} = \frac{n}{1}\omega x^{n-1} + \frac{n(n-1)}{1\cdot 2}5\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}19\omega^3 x^{n-3} + \text{etc.}$$
$$\Delta y^{\text{III}} = \frac{n}{1}\omega x^{n-1} + \frac{n(n-1)}{1\cdot 2}7\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}37\omega^3 x^{n-3} + \text{etc.}$$

Sumantur denuo differentiae atque obtinebitur

$$\Delta\Delta y = n(n-1)\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}6\omega^3 x^{n-3} + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}14\omega^4 x^{n-4} + \text{etc.}$$
$$\Delta\Delta y^{\text{I}} = n(n-1)\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}12\omega^3 x^{n-3} + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}50\omega^4 x^{n-4} + \text{etc.}$$
$$\Delta\Delta y^{\text{II}} = n(n-1)\omega^2 x^{n-2} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}18\omega^3 x^{n-3} + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}110\omega^4 x^{n-4} + \text{etc.}$$

Ex his per subtractionem ulterius eruitur

$$\Delta^3 y = n(n-1)(n-2)\omega^3 x^{n-3} + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}36\omega^4 x^{n-4} + \text{etc.}$$
$$\Delta^4 y^{\text{I}} = n(n-1)(n-2)\omega^3 x^{n-3} + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}60\omega^4 x^{n-4} + \text{etc.}$$

atque porro

$$\Delta^4 y = n(n-1)(n-2)(n-3)\omega^4 x^{n-4} + \text{etc.}$$

14. Quo lex, secundum quam istae differentiae potestatis x^n progrediuntur, facilius perspiciatur, ponamus primo brevitatis ergo

$$A = \frac{n}{1}$$
$$B = \frac{n(n-1)}{1\cdot 2}$$
$$C = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}$$
$$D = \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}$$
$$E = \frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}$$
etc.

Deinde sequens formetur tabula, quae pro singulis differentiis inserviet:

y	1	0	0	0	0	0	0	0	0 etc.
Δy	0	1	1	1	1	1	1	1	1 etc.
$\Delta^2 y$	0	0	2	6	14	30	62	126	254 etc.
$\Delta^3 y$	0	0	0	6	36	150	540	1806	5796 etc.
$\Delta^4 y$	0	0	0	0	24	240	1560	8400	40824 etc.
$\Delta^5 y$	0	0	0	0	0	120	1800	16800	126000 etc.
$\Delta^6 y$	0	0	0	0	0	0	720	15120	191520 etc.
$\Delta^7 y$	0	0	0	0	0	0	0	5040	141120 etc.

In qua tabula numerus cuiusvis seriei invenitur, si eiusdem seriei praecedens ad numerum supra positum addatur atque summa per indicem characteri Δ infixum multiplicetur. Sic in serie differentiae $\Delta^5 y$ respondente terminus 16800 invenitur, si praecedens 1800 ad supra scriptum 1560 addatur atque summa 3360 per 5 multiplicetur.

15. Tabula ergo hac constituta singulae differentiae potestatis $x^n = y$ sequenti modo se habebunt:

$$\Delta y = A\omega x^{n-1} + B\omega^2 x^{n-2} + C\omega^3 x^{n-3} + D\omega^4 x^{n-4} + \text{etc.}$$

$$\Delta^2 y = 2B\omega^2 x^{n-2} + 6C\omega^3 x^{n-3} + 14D\omega^4 x^{n-4} + \text{etc.}$$

$$\Delta^3 y = 6C\omega^3 x^{n-3} + 36D\omega^4 x^{n-4} + 150E\omega^5 x^{n-5} + \text{etc.}$$

$$\Delta^4 y = 24D\omega^4 x^{n-4} + 240E\omega^5 x^{n-5} + 1560F\omega^6 x^{n-6} + \text{etc.}$$

Generatim autem potestatis x^n differentia ordinis *m*, seu $\Delta^m y$, sequenti modo exprimetur.

Sit

$$I = \frac{n(n-1)(n-2)\cdots\cdots(n-m+1)}{1\cdot 2\cdot 3\cdots\cdots m},$$

$$K = \frac{n-m}{m+1} I, \quad L = \frac{n-m-1}{m+2} K, \quad M = \frac{n-m-2}{m+3} L, \quad \text{etc.}$$

Deinde vero sit

$$\alpha = (m+1)^m - \frac{m}{1} m^m + \frac{m(m-1)}{1\cdot 2}(m-1)^m - \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}(m-2)^m + \text{etc.}$$

$$\beta = (m+1)^{m+1} - \frac{m}{1} m^{m+1} + \frac{m(m-1)}{1\cdot 2}(m-1)^{m+1} - \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}(m-2)^{m+1} + \text{etc.}$$

$$\gamma = (m+1)^{m+2} - \frac{m}{1} m^{m+1} + \frac{m(m-1)}{1\cdot 2}(m-1)^{m+2} - \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}(m-2)^{m+2} + \text{etc.}$$

etc;

quibus valoribus inventis erit

$$\Delta^m y = \alpha I\omega^m x^{n-m} + \beta K\omega^{m+1}x^{n-m-1} + \gamma L\omega^{m+2}x^{n-m-2} + \text{etc}$$

cuius expressionis ratio ex modo, quo singulae differentiae ex valoribus $y, y^{\mathrm{I}}, y^{\mathrm{II}}, y^{\mathrm{III}}$ etc. eliciuntur, sponte sequitur.

16. Ex his perspicuum est, si exponens *n* fuerit numerus integer affirmativus, tandem ad differentias perveniri constantes hisque ulteriores omnes esse $= 0$. Sic erit

$$\Delta.x = \omega$$

$$\Delta^2.x^2 = 2\omega^2$$

$$\Delta^3.x^3 = 6\omega^3$$

$$\Delta^4.x^4 = 24\omega^4 \text{ et tandem}$$

$$\Delta^n.x^n = 1\cdot 2\cdot 3\cdot 4......n\cdot\omega^n.$$

Omnis ergo functio rationalis integra tandem ad differentias constantes deducetur. Scilicet functio ipsius *x* primi gradus $ax + b$ differentiam primam iam habet constantem $= a\omega$. Functio secundi gradus $axx + bx + c$ differentiam secundam habebit constantem $= 2a\omega\omega$. Functionis autem tertii gradus differentia tertia erit constans, quarti quarta et ita porro.

17. Modus autem, quo invenimus differentias potestatis x^n, quoque latius patet atque ad eas potestates, quarum exponens *n* est numerus negativus vel fractus vel adeo irrationalis, extenditur. Quod quo clarius appareat, differentias tantum primas praecipuarum huiusmodi potestatum exhibebimus, quoniam lex differentiarum secundarum ac sequentium non tam facile cernitur; erit ergo

$$\Delta.x = \omega$$

$$\Delta.x^2 = 2\omega x + \omega^2$$

$$\Delta.x^3 = 3\omega x^2 + 3\omega^2 x + \omega^3$$

$$\Delta.x^3 = 4\omega x^2 + 6\omega^2 x^2 + 4\omega^3 x + \omega^4$$

etc.

Simili modo vero erit

$$\varDelta.x^{-1} = -\frac{\omega}{x^2} + \frac{\omega^2}{x^3} - \frac{\omega^3}{x^4} + \text{etc.}$$
$$\varDelta.x^{-2} = -\frac{2\omega}{x^3} + \frac{3\omega^2}{x^4} - \frac{4\omega^3}{x^5} + \text{etc.}$$
$$\varDelta.x^{-3} = -\frac{3\omega}{x^4} + \frac{6\omega^2}{x^5} - \frac{10\omega^3}{x^6} + \text{etc.}$$
$$\varDelta.x^{-4} = -\frac{4\omega}{x^5} + \frac{10\omega^2}{x^6} - \frac{20\omega^3}{x^7} + \text{etc.}$$

et inde pro reliquis. Pariter erit

$$\varDelta.x^{\frac{1}{2}} = \frac{\omega}{2x^{\frac{1}{2}}} - \frac{\omega^2}{8x^{\frac{3}{2}}} + \frac{\omega^3}{16x^{\frac{5}{2}}} - \text{etc.}$$
$$\varDelta.x^{\frac{1}{3}} = \frac{\omega}{3x^{\frac{2}{3}}} - \frac{\omega^2}{9x^{\frac{5}{3}}} + \frac{5\omega^3}{81x^{\frac{8}{3}}} - \text{etc.}$$
$$\varDelta.x^{-\frac{1}{2}} = -\frac{\omega}{2x^{\frac{3}{2}}} + \frac{3\omega^2}{8x^{\frac{5}{2}}} - \frac{5\omega^3}{16x^{\frac{7}{2}}} + \text{etc.}$$
$$\varDelta.x^{-\frac{1}{3}} = -\frac{\omega}{3x^{\frac{4}{3}}} + \frac{2\omega^2}{9x^{\frac{7}{3}}} - \frac{14\omega^3}{81x^{\frac{10}{3}}} + \text{etc.}$$

18. Apparet itaque has differentias, si exponens ipsius x non fuerit numerus integer affirmativus, in infinitum progredi seu ex terminorum numero infinito constare. Interim tamen eaedem differentiae quoque per expressionem finitam exhiberi possunt. Cum enim posito $y = x^{-1} = \frac{1}{x}$ sit $y^{\mathrm{I}} = \frac{1}{x+\omega}$, erit

$$\varDelta.\frac{1}{x} = \frac{1}{x+\omega} - \frac{1}{x};$$

unde, si fractio $\frac{1}{x+\omega}$ in seriem convertatur, prodit expressio superior. Simili modo erit

$$\varDelta.x^{-2} = \varDelta.\frac{1}{xx} = \frac{1}{(x+\omega)^2} - \frac{1}{xx}$$

atque pro irrationalibus erit

$$\varDelta.\sqrt{x} = \sqrt{(x+\omega)} - \sqrt{x} \quad \text{et} \quad \varDelta.\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{(x+\omega)}} - \frac{1}{\sqrt{x}};$$

quae formulae, si more solito in series explicentur, superiores expressiones praebent.

19. Hoc vero modo quoque differentiae functionum sive fractarum sive irrationalium inveniri possunt; sic, si quaeratur differentia prima fractionis $\frac{1}{aa+xx}$, ponatur $y = \frac{1}{aa+xx}$, et quia est $y^{\mathrm{I}} = \frac{1}{aa+xx+2\omega x+\omega^2}$, erit

$$\Delta y = \Delta.\frac{1}{aa+xx} = \frac{1}{aa+xx+2\omega x+\omega\omega} - \frac{1}{aa+xx},$$

quae expressio quoque in seriem infinitam converti potest.

Ponatur $aa+xx=P$ et $2\omega x+\omega\omega=Q$; erit

$$\frac{1}{P+Q} = \frac{1}{P} - \frac{Q}{P^2} + \frac{Q^2}{P^3} - \frac{Q^3}{P^4} + \text{etc.}$$

et

$$\Delta y = -\frac{Q}{P^2} + \frac{Q^2}{P^3} - \frac{Q^3}{P^4} + \text{etc.}$$

Restitutis ergo loco P et Q valoribus erit

$$\Delta y = \Delta.\frac{1}{aa+xx} = -\frac{2\omega x+\omega\omega}{(aa+xx)^2} + \frac{4\omega\omega xx+4\omega^3x+\omega^4}{(aa+xx)^3} - \frac{8\omega^3x^3+12\omega^4x^2+6\omega^5x+\omega^6}{(aa+xx)^4} + \text{etc.};$$

qui termini si secundum potestates ipsius ω ordinentur, erit

$$\Delta.\frac{1}{aa+xx} = -\frac{2\omega x}{(aa+xx)^2} + \frac{\omega^2(3xx-aa)}{(aa+xx)^3} - \frac{4\omega^3(x^3-aax)}{(aa+xx)^4} + \text{etc.}$$

20. Similibus seriebus infinitis differentiae functionum irrationalium quoque exprimi possunt.
Sit proposita ista functio $y=\sqrt{(aa+xx)}$, et cum sit

$$y^{\mathrm{I}} = \sqrt{(aa+xx+2\omega x+\omega^2)},$$

ponatur

$$aa+xx=P \text{ et } 2\omega x+\omega\omega=Q;$$

erit

$$\Delta y = \sqrt{(P+Q)} - \sqrt{P} = \frac{Q}{2\sqrt{P}} - \frac{QQ}{8P\sqrt{P}} + \frac{Q^3}{16P^2\sqrt{P}} - \text{etc.},$$

unde fiet

$$\Delta y = \Delta.\sqrt{(aa+xx)} = \frac{2\omega x+\omega\omega}{2\sqrt{(aa+xx)}} - \frac{4\omega^2x^2+4\omega^3x+\omega^4}{8(aa+xx)\sqrt{(aa+xx)}} + \text{etc.}$$

vel

$$= \frac{\omega x}{2\sqrt{(aa+xx)}} + \frac{aa\omega^2}{2(aa+xx)\sqrt{(aa+xx)}} - \frac{aa\omega^3x}{2(aa+xx)^2\sqrt{(aa+xx)}} + \text{etc.}$$

Hincque adeo colligimus functionis cuiuscunque ipsius x, quae sit *y*, differentiam hac forma exprimi posse, ut sit

$$\Delta y = P\omega + Q\omega^2 + R\omega^3 + S\omega^4 + \text{etc.}$$

existentibus *P*, *Q*, *R*, *S* etc. certis ipsius *x* functionibus, quae quovis casu ex functione *y* definiri possunt.

21. Neque etiam ex hac forma differentiae functionum transcendentium excluduntur, id quod ex sequentibus exemplis clarius apparebit.

EXEMPLUM 1

Invenire differentiam primam logarithmi hyperbolici ipsius x.

Ponatur $y = lx$, et cum sit $y^{\text{I}} = l(x+\omega)$, erit

$$\Delta y = y^{\text{I}} - y = l(x+\omega) - lx = l\left(1+\tfrac{\omega}{x}\right).$$

Huiusmodi autem logarithmum supra docuimus per seriem infinitam exprimere; qua adhibita erit

$$\Delta y = \Delta .lx = \tfrac{\omega}{x} - \tfrac{\omega^2}{2xx} + \tfrac{\omega^3}{3x^3} - \tfrac{\omega^4}{4x^4} + \text{etc.}$$

EXEMPLUM 2

Invenire differentiam primam quantitatis exponentialis a^x.

Posito $y = a^x$ erit $y^{\text{I}} = a^{x+\omega} = a^x \cdot a^{\omega}$; at supra ostendimus esse

$$a^{\omega} = 1 + \tfrac{\omega la}{1} + \tfrac{\omega^2 (la)^2}{1\cdot 2} + \tfrac{\omega^3 (la)^3}{1\cdot 2\cdot 3} + \text{etc.};$$

qua valore introducto erit

$$\Delta .a^{\omega} = y^{\text{I}} - y = \Delta y = \tfrac{a^x \omega la}{1} + \tfrac{a^x \omega^2 (la)^2}{1\cdot 2} + \tfrac{a^x \omega^3 (la)^3}{1\cdot 2\cdot 3} + \text{etc.}$$

EXEMPLUM 3

In circulo, cuius radius = 1, *invenire differentiam sinus arcus x.*

Sit $\sin x = y$; erit $y^{\text{I}} = \sin(x+\omega)$, unde

$$\Delta y = y^{\text{I}} - y = \sin(x+\omega) - \sin x.$$

At est $\sin(x+\omega) = \cos\omega \cdot \sin x + \sin\omega \cdot \cos x$ atque per series infinitas ostendimus esse

$$\cos\omega = 1 - \tfrac{\omega^2}{1\cdot 2} + \tfrac{\omega^4}{1\cdot 2\cdot 3\cdot 4} - \tfrac{\omega^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \text{etc.}$$

et

$$\sin\omega = \omega - \frac{\omega^3}{1\cdot2\cdot3} + \frac{\omega^5}{1\cdot2\cdot3\cdot4\cdot5} - \frac{\omega^7}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7} + \text{etc.};$$

quibus seriebus substitutis erit

$$\Delta.\sin x = \omega\cos x - \frac{\omega^2}{2}\sin x - \frac{\omega^3}{6}\cos x + \frac{\omega^4}{24}\sin x + \frac{\omega^5}{120}\cos x - \text{ etc.}$$

EXEMPLUM 4

In circulo, cuius radius = 1, *invenire differentiam cosinus arcus x.*

Posito $y = \cos x$ ob $y^{\text{I}} = \cos(x+\omega)$ erit $y^{\text{I}} = \cos\omega \cdot \cos x - \sin\omega \cdot \sin x$ et

$$\Delta y = \cos\omega \cdot \cos x - \sin\omega \cdot \sin x - \cos x.$$

Seriebus ergo ante expositis adhibendis prodibit

$$\Delta.\cos x = -\omega\sin x - \frac{\omega^2}{2}\cos x + \frac{\omega^3}{6}\sin x + \frac{\omega^4}{24}\cos x - \frac{\omega^5}{120}\sin x - \text{etc.}$$

22. Cum igitur proposita quacunque functione ipsius *x* sive algebraica sive transcendente, quae sit *y*, eius differentia prima eiusmodi habeat formam, ut sit

$$\Delta y = P\omega + Q\omega^2 + R\omega^3 + S\omega^4 + \text{etc.},$$

si huius differentia denuo capiatur, patebit differentiam secundam ipsius *y* huiusmodi formam esse habituram

$$\Delta\Delta y = P\omega^2 + Q\omega^3 + R\omega^4 + \text{etc.}$$

similique modo differentia tertia ipsius *y* erit huiusmodi

$$\Delta^3 y = P\omega^3 + Q\omega^4 + R\omega^5 + \text{etc.}$$

sicque porro.

Ubi notandum est litteras *P*, *Q*, *R* etc. hic non pro valoribus determinatis adhiberi neque eadem littera in diversis differentiis eandem functionem ipsius *x* denotari; ideo enim tantum iisdem litteris utor, ne sufficiens diversarum litterarum numerus deficiat.

Ceterum istae differentiarum formae probe sunt notandae, cum in analysi infinitorum maximum usum offerant.

23. Cum igitur modum exposuerim, quo cuiusvis functionis differentia prima ex eaque porro differentiae sequentium ordinum inveniri queant, quippe quae ex valoribus functionis y successivis $y^{\mathrm{I}}, y^{\mathrm{II}}, y^{\mathrm{III}}, y^{\mathrm{IV}}$ etc. reperiuntur, vicissim ex differentiis ipsius y cuiusque ordinis datis isti ipsi variati valores ipsius y elici poterunt. Erit enim

$$\begin{aligned} y^{\mathrm{I}} &= y + \Delta y \\ y^{\mathrm{II}} &= y + 2\Delta y + \Delta\Delta y \\ y^{\mathrm{III}} &= y + 3\Delta y + 3\Delta\Delta y + \Delta^3 y \\ y^{\mathrm{IV}} &= y + 4\Delta y + 6\Delta\Delta y + 4\Delta^3 y + \Delta^4 y \\ &\text{etc.,} \end{aligned}$$

ubi coefficientes numerici iterum ex evolutione binomii nascuntur. Quemadmodum ergo $y^{\mathrm{I}}, y^{\mathrm{II}}, y^{\mathrm{III}}$ etc. sunt valores ipsius y, qui oriuntur, si loco x successive ponantur hi valores $x+\omega$, $x+2\omega$, $x+3\omega$ etc., statim valorem ipsius $y^{(n)}$ assignare poterimus, qui prodit, si loco x scribatur $x+n\omega$; erit scilicet iste valor

$$y + \tfrac{n}{1}\Delta y + \tfrac{n(n-1)}{1\cdot 2}\Delta^2 y + \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\Delta^3 y + \text{etc.}$$

Hincque adeo etiam valores ipsius y praeberi possunt, si n fuerit numerus negativus. Sic, si loco x ponatur $x-\omega$, functio y abibit in hanc formam

$$y - \Delta y + \Delta^2 y - \Delta^3 y + \Delta^4 y - \text{etc.}$$

sin autem loco x ponatur $x-2\omega$, functio y transibit in

$$y - 2\Delta y + 3\Delta^2 y - 4\Delta^3 y + 5\Delta^4 y - \text{etc.}$$

24. Pauca quaedam addamus de methodo inversa; qua, si detur differentia, ex ea ipsa illa functio, cuius est differentia, investigari debeat. Cum autem hoc sit, difficillimum atque saepenumero ipsam analysin infinitorum requirat, casus tantum quosdam faciliores evolvamus. Primum igitur regrediendo, si functionis cuiuspiam differentiam invenerimus, vicissim hac differentia proposita ipsa illa functio, unde est nata, exhiberi poterit. Sic, cum functionis $ax+b$ differentia sit $a\omega$, si quaeratur, cuiusnam functionis differentia sit $a\omega$, responsio erit in promtu eam functionem esse $ax+b$. In hac igitur reperitur quantitas constans b, quae in differentia non inerat et quae propterea ab arbitrio nostro pendet. Perpetuo autem, si functionis cuiusvis P differentia fuerit Q, quoque functionis $P+A$ (denotante A quantitatem quamcunque constantem) differentia erit Q. Hinc, si ista differentia Q proponatur, functio, ex qua ea est orta, erit $P + A$ atque idcirco determinatum valorem non habet, cum constans A ab arbitrio pendeat.

25. Vocemus eam functionem quaesitam, cuius differentia proponitur, *summam;* quod nomen commode adhibetur, cum, quod summa differentiae opponi solet, tum etiam, quod functio quaesita revera sit summa omnium valorum praecedentium differentiae. Quemadmodum enim est

$$y^{\mathrm{I}} = y + \Delta y \quad \text{et} \quad y^{\mathrm{II}} = y + \Delta y + \Delta y^{\mathrm{I}},$$

si valores ipsius y retro continuentur, ita ut is, qui valori $x - \omega$ respondet, scribatur y_{I} huncque praecedens y_{II} et qui ultra praecedunt $y_{\mathrm{III}}, y_{\mathrm{IV}}, y_{\mathrm{V}}$ etc. hincque series formetur retrograda cum suis differentiis

$$y_{\mathrm{V}}, y_{\mathrm{IV}}, y_{\mathrm{III}}, y_{\mathrm{II}}, y_{\mathrm{I}}, y \quad \text{et} \quad \Delta y_{\mathrm{V}}, \Delta y_{\mathrm{IV}}, \Delta y_{\mathrm{III}}, \Delta y_{\mathrm{II}}, \Delta y_{\mathrm{I}}$$

erit $y = \Delta y_{\mathrm{I}} + y_{\mathrm{I}}$ et ob $y_{\mathrm{I}} = \Delta y_{\mathrm{II}} + y_{\mathrm{II}}$ porroque $y_{\mathrm{II}} = \Delta y_{\mathrm{III}} + y_{\mathrm{III}}$ erit utique

$$y = \Delta y_{\mathrm{I}} + \Delta y_{\mathrm{II}} + \Delta y_{\mathrm{III}} + \Delta y_{\mathrm{IV}} + \Delta y_{\mathrm{V}} + \text{etc.}$$

sicque erit functio y, cuius differentia est Δy, summa omnium valorum antecedentium differentiae Δy, qui oriuntur, si loco x scribantur valores antecedentes $x - \omega,\ x - 2\omega,\ x - 3\omega$ etc.

26. Quemadmodum ad differentiam denotandam usi sumus signo Δ, ita summam indicabimus signo Σ; scilicet si functionis y differentia fuerit z, erit $z = \Delta y$; unde, si y detur, differentiam z invenire ante docuimus. Quodsi autem data sit differentia z eiusque summa y reperiri debeat, fiet $y = \Sigma z$ atque adeo ex aequatione $z = \Delta y$ regrediendo formabitur haec aequatio $y = \Sigma z$, ubi constans quantitas quaecunque adiici poterit ob rationes supra datas; ex quo aequatio $z = \Delta y$, si invertatur, dabit quoque $y = \Sigma z + C$. Deinde, cum quantitatis ay differentia sit $a\Delta y = az$, erit $\Sigma az = ay$, si quidem a sit quantitas constans. Quia ergo est $\Delta x = \omega$, erit $\Sigma\omega = x + C$ et $\Sigma a\omega = ax + C$ atque ob ω quantitatem constantem erit $\Sigma\omega^2 = \omega x + C$, $\Sigma\omega^3 = \omega^2 x + C$ et ita porro.

27. Si igitur differentias potestatum ipsius x supra inventas invertamus, erit

$$\Sigma\omega = x \quad \text{hincque} \quad \Sigma 1 = \frac{x}{\omega}.$$

Deinde habemus

$$\Sigma\left(2\omega x + \omega^2\right) = x^2$$

unde fit

$$\Sigma x = \tfrac{x^2}{2\omega} - \Sigma \tfrac{\omega}{2} = \tfrac{x^2}{2\omega} - \tfrac{x}{2}.$$

Porro est

$$\Sigma\left(3\omega xx + 3\omega^2 x + \omega^3\right) = x^3$$

seu

$$3\omega\Sigma x^2 + 3\omega^2\Sigma x + \omega^3\Sigma 1 = x^3;$$

ergo

$$\Sigma x^2 = \tfrac{x^3}{3\omega} - \omega\Sigma x - \tfrac{\omega^2}{3}\Sigma 1$$

seu

$$\Sigma x^2 = \tfrac{x^3}{3\omega} - \tfrac{x^2}{2} + \tfrac{\omega x}{6}.$$

Simili modo erit

$$\Sigma x^3 = \tfrac{x^4}{4\omega} - \tfrac{3\omega}{2}\Sigma x^2 - \omega^2\Sigma x - \tfrac{\omega^3}{4}\Sigma 1,$$

ubi si loco $\Sigma x^2, \Sigma x$ et $\Sigma 1$ valores ante inventi substituantur, reperietur

$$\Sigma x^3 = \tfrac{x^4}{4\omega} - \tfrac{x^3}{2} + \tfrac{\omega xx}{4}.$$

Deinde cum sit

$$\Sigma x^4 = \tfrac{x^5}{5\omega} - 2\omega\Sigma x^3 - 2\omega^2\Sigma x^2 - \tfrac{\omega^5}{5}\Sigma 1,$$

erit adhibendis substitutionibus

$$\Sigma x^4 = \tfrac{x^5}{5\omega} - \tfrac{1}{2}x^4 + \tfrac{1}{3}\omega x^3 - \tfrac{1}{30}\omega^3 x.$$

Simili modo ulterius progrediendo reperietur

$$\Sigma x^5 = \tfrac{x^6}{6\omega} - \tfrac{1}{2}x^5 + \tfrac{5}{12}\omega x^4 - \tfrac{1}{12}\omega^3 x^2,$$

et

$$\Sigma x^6 = \tfrac{x^7}{7\omega} - \tfrac{1}{2}x^6 + \tfrac{1}{2}\omega x^5 - \tfrac{1}{6}\omega^3 x^3 + \tfrac{1}{42}\omega^5 x,$$

quas expressiones infra facilius invenire docebimus.

28. Si ergo differentia proposita fuerit functio rationalis integra ipsius *x*, eius summa (seu ea functio, cuius ea est differentia) ex his formulis facile invenitur. Quia enim differentia ex aliquot potestatibus ipsius *x* constabit, quaeratur uniuscuiusque termini summa omnesque istae summae colligantur.

EXEMPLUM 1
Quaeratur functio, cuius differentia sit = axx +bx +c.

Quaerantur singulorum terminorum summae ope formularum ante inventarum; erit

$$\Sigma axx = \frac{ax^3}{3\omega} - \frac{axx}{2} + \frac{a\omega x}{6}$$

et

$$\Sigma ax = ...\frac{bxx}{2\omega} - \frac{bx}{2}$$

atque

$$\Sigma c = ...\frac{cx}{\omega}.$$

Hinc colligendo has summas erit

$$\Sigma\left(axx + bx + c\right) = \frac{a}{3\omega}x^3 - \frac{a\omega - b}{2\omega}x^2 + \frac{a\omega^2 - 3b\omega + 6c}{6\omega}x + C,$$

quae est functio quaesita, cuius differentia est $axx + bx + c$.

EXEMPLUM 2
Quaeratur functio, cuiius differentia est $x^4 - 2\omega^2 xx + \omega^4$.

Operationem simili modo instituendo habebitur

$$\Sigma x^4 = \frac{1}{5\omega}x^5 - \frac{1}{2}x^4 + \frac{\omega}{3}x^3 \qquad - \frac{\omega^3}{30}x \quad \text{et}$$
$$-\Sigma 2\omega^2 x^2 = ...- \frac{2\omega}{3}x^3 + \omega^2 x^2 - \frac{\omega^3}{3}x \quad \text{atque}$$
$$+\Sigma\omega^4 = ...+ \omega^4 x,$$

unde functio quaesita erit

$$\frac{x^5}{5\omega} - \frac{1}{2}x^4 - \frac{1}{3}\omega x^3 + \omega^2 x^2 + \frac{19}{30}\omega^3 x + C.$$

Si enim hic loco *x* ponatur $x + \omega$ atque a quantitate resultante subtrahatur ista inventa, remanebit proposita differentia $x^4 - 2\omega^2 x^2 + \omega^4$.

29. Si summas, quas pro potestatibus ipsius *x* invenimus, attentius inspiciamus, in terminis primis, secundis ac tertiis mox quidem legem observabimus, qua illi secundum singulas potestates progrediuntur; reliquorum autem terminorum lex non ita est perspicua, ut summam potestatis x^n in genere inde colligere liceat. Interim tamen in sequentibus [§ 132, partis posterioris] docebitur esse

$$\begin{aligned}\Sigma x^n = {} & \frac{x^{n+1}}{(n+1)\omega} - \frac{1}{2}x^n + \frac{1}{2}\cdot\frac{n\omega}{2\cdot 3}x^{n-1} - \frac{1}{2}\cdot\frac{n(n-1)(n-2)\omega^3}{2\cdot 3\cdot 4\cdot 5}x^{n-3}\\
& + \frac{1}{6}\cdot\frac{n(n-1)(n-2)(n-3)(n-4)\omega^5}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^{n-5}\\
& - \frac{3}{10}\cdot\frac{n(n-1)\ldots\ldots(n-6)\omega^7}{2\cdot 3\ldots\ldots 8\cdot 9}x^{n-7}\\
& + \frac{5}{6}\cdot\frac{n(n-1)\ldots\ldots(n-8)\omega^9}{2\cdot 3\ldots\ldots 10\cdot 11}x^{n-9}\\
& - \frac{691}{210}\cdot\frac{n(n-1)\ldots\ldots(n-10)\omega^{11}}{2\cdot 3\ldots\ldots 12\cdot 13}x^{n-11}\\
& + \frac{35}{2}\cdot\frac{n(n-1)\ldots\ldots(n-12)\omega^{13}}{2\cdot 3\ldots\ldots 14\cdot 15}x^{n-13}\\
& - \frac{3617}{30}\cdot\frac{n(n-1)\ldots\ldots(n-14)\omega^{15}}{2\cdot 3\ldots\ldots 16\cdot 17}x^{n-15}\\
& + \frac{43867}{42}\cdot\frac{n(n-1)\ldots\ldots(n-16)\omega^{17}}{2\cdot 3\ldots\ldots 18\cdot 19}x^{n-17}\\
& - \frac{1222277}{110}\cdot\frac{n(n-1)\ldots\ldots(n-18)\omega^{19}}{2\cdot 3\ldots\ldots 20\cdot 21}x^{n-19}\\
& + \frac{854513}{6}\cdot\frac{n(n-1)\ldots\ldots(n-20)\omega^{21}}{2\cdot 3\ldots\ldots 22\cdot 23}x^{n-21}\\
& - \frac{1181820455}{546}\cdot\frac{n(n-1)\ldots\ldots(n-22)\omega^{23}}{2\cdot 3\ldots\ldots 24\cdot 25}x^{n-23}\\
& + \frac{76977927}{2}\cdot\frac{n(n-1)\ldots\ldots(n-24)\omega^{25}}{2\cdot 3\ldots\ldots 26\cdot 27}x^{n-25}\\
& - \frac{23749461029}{30}\cdot\frac{n(n-1)\ldots\ldots(n-26)\omega^{27}}{2\cdot 3\ldots\ldots 28\cdot 29}x^{n-27}\\
& + \frac{8161584127600 5}{462}\cdot\frac{n(n-1)\ldots\ldots(n-28)\omega^{29}}{2\cdot 3\ldots\ldots 30\cdot 31}x^{n-29}\\
& \text{etc.} + C,\end{aligned}$$

cuius progressionis praecipuum momentum in coefficientibus mere numericis est situm; qui quemadmodum formentur, hic locus nondum est, ubi exponi queat.

30. Apparet autem, nisi *n* sit numerus integer affirmativus, hanc summae expressionem in infinitum progredi neque hoc modo summani in forma finita exhiberi posse. Ceterum hic notandum est non omnes potestates ipsius *x* proposita x^n inferiores occurrere; desunt enim termini $x^{n-2}, x^{n-4}, x^{n-6}, x^{n-8}$, etc., quippe quorum coefficientes sunt = 0, etiamsi termini secundi x^n coefficiens hanc legem non sequatur, sed sit $=-\frac{1}{2}$. Poterunt ergo huius expressionis ope summae potestatum, quarum exponentes sunt vel negativi vel fracti, in forma infinita exhiberi, solo el;cepto casu, quo $n=-1$, quia tum fit terminus $\frac{x^{n+1}}{(n+1)\omega}$ ob $n+1=0$ infinitus. Sic posito $n=-2$ erit

$$\Sigma\frac{1}{xx}=C-\frac{1}{\omega x}-\frac{1}{2xx}-\frac{1}{2}\cdot\frac{\omega}{3x^3}+\frac{1}{6}\cdot\frac{\omega^3}{5x^5}-\frac{1}{6}\cdot\frac{\omega^5}{7x^7}+\frac{3}{10}\cdot\frac{\omega^7}{9x^9}-\frac{5}{6}\cdot\frac{\omega^9}{11x^{11}}+\frac{691}{210}\cdot\frac{\omega^{11}}{13x^{13}}$$
$$-\frac{35}{2}\cdot\frac{\omega^{13}}{15x^{15}}+\frac{3617}{30}\cdot\frac{\omega^{15}}{17x^{17}}-\text{etc.}$$

31. Si ergo differentia proposita fuerit potestas ipsius *x* quaecunque, eius summa hinc perpetuo assignari seu functio, cuius ea sit differentia, exhiberi poterit. Sin autem differentia proposita aliam habeat formam, ut in potestates ipsius *x* tanquam partes distribui nequeat, tum summa difficillime ac saepenumero prorsus non inveniri potest, nisi forte pateat eam ex quapiam functione esse ortam. Hanc ob causam conveniet plurium functionum differentias investigare easque probe notare, ut, si quando huiusmodi differentia proponatur, eius summa seu functio, unde est orta, statim exhiberi queat. Interim tamen methodus infinitorum plures regulas suppeditabit, quarum ope inventio summarum mirifice sublevabitur.

32. Facilius autem saepe summa quaesita reperitur, si differentia proposita ex factoribus simplicibus constet, qui progressionem arithmeticam constituant, cuius differentia sit ipsa quantitas ω. Sic, si proposita fuerit functio $(x+\omega)(x+2\omega)$, cuius differentia quaeratur, quia posito $x+\omega$ loco *x* haec functio abit in $(x+2\omega)(x+3\omega)$, eius differentia erit $2\omega(x+2\omega)$. Quare vicissim, si proponatur differentia $2\omega(x+2\omega)$, eius summa erit $(x+2\omega)(x+3\omega)$; hinc ergo erit

$$\Sigma(x+2\omega)=\tfrac{1}{2\omega}(x+\omega)(x+2\omega).$$

Simili modo, si proponatur functio $(x+n\omega)(x+(n+1)\omega)$, cum sit eius differentia $2\omega(x+(n+1)\omega)$, erit

$$\Sigma(x+(n+1)\omega)=\tfrac{1}{2\omega}(x+n\omega)(x+(n+1)\omega)$$

et

$$\Sigma(x+n\omega)=\tfrac{1}{2\omega}(x+(n-1)\omega)(x+n\omega).$$

33. Si functio ex pluribus factoribus constet, ut sit

$$y=(x+(n-1)\omega)(x+n\omega)(x+(n+1)\omega),$$

cum sit

$$y^{\mathrm{I}} = (x+n\omega)(x+(n+1)\omega)(x+(n+2)\omega),$$

erit

$$\varDelta y = 3\omega(x+n\omega)(x+(n+1)\omega)$$

ac propterea

$$\varSigma(x+n\omega)(x+(n+1)\omega) = \tfrac{1}{3\omega}(x+(n-1)\omega)(x+n\omega)(x+(n+1)\omega).$$

Pari modo reperietur esse

$$\varSigma(x+n\omega)(x+(n+1)\omega)(x+(n+2)\omega)$$
$$= \tfrac{1}{4\omega}(x+(n-1)\omega)(x+n\omega)(x+(n+1)\omega)(x+(n+2)\omega),$$

unde lex inveniendi summas, si differentia ex pluribus huiusmodi factoribus constet, sponte patet. Quamvis autem hae differentiae sint functiones rationales integrae, tamen earum summae hoc modo facilius reperiuntur quam per methodum praecedentem.

34. Hinc quoque via patet ad differentiarum fractarum summas inveniendas.
Sit enim proposita fractio

$$y = \tfrac{1}{x+n\omega};$$

quia erit

$$y^{\mathrm{I}} = \tfrac{1}{x+(n+1)\omega},$$

erit

$$\varDelta y = \tfrac{1}{x+(n+1)\omega} - \tfrac{1}{x+n\omega} = \tfrac{-\omega}{(x+n\omega)(x+(n+1)\omega)}$$

ac propterea

$$\varSigma\tfrac{1}{(x+n\omega)(x+(n+1)\omega)} = -\tfrac{1}{\omega}\cdot\tfrac{1}{x+n\omega}.$$

Sit porro

$$y = \tfrac{1}{(x+n\omega)(x+(n+1)\omega)};$$

ob

$$y^{\mathrm{I}} = \tfrac{1}{(x+(n+1)\omega)(x+(n+2)\omega)}$$

erit

$$\Delta y = \frac{-2\omega}{(x+n\omega)(x+(n+1)\omega)(x+(n+2)\omega)}$$

Hinc ideo fiet

$$\Sigma \frac{1}{(x+n\omega)(x+(n+1)\omega)(x+(n+2)\omega)} = \frac{-1}{2\omega} \cdot \frac{1}{(x+n\omega)(x+(n+1)\omega)}.$$

Simili modo erit porro

$$\Sigma \frac{1}{(x+n\omega)(x+(n+1)\omega)(x+(n+2)\omega)(x+(n+3)\omega)} = \frac{-1}{3\omega} \cdot \frac{1}{(x+n\omega)(x+(n+1)\omega)(x+(n+2)\omega)}.$$

35. Modus iste summandi probe est tenendus, quia huiusmodi differentiarum summae per praecedentem methodum inveniri non possunt. Quodsi autem differentia insuper habeat numeratorem vel factores denominatoris non in arithmetica progressione procedant, tum tutissimus modus investigandi summas est, ut differentia proposita in suas fractiones simplices resolvatur; quarum singulae etsi summari nequeunt, tamen binis coniungendis toties summa inveniri potest, quoties id quidem fieri licet; tantum enim erit dispiciendum, utrum summa ope huius formulae inveniri queat

$$\Sigma \frac{1}{x+(n+1)\omega} - \Sigma \frac{1}{x+n\omega} = \frac{1}{x+n\omega};$$

etsi enim neutra harum summarum per se exhiberi potest, tamen earum differentia cognoscitur.

36. His igitur casibus negotium redit ad resolutionem cuiusque fractionis in fractiones suas simplices, quae in superiori libro fusius est ostensa. Quemadmodum ergo eius beneficio summae inveniri queant, aliquot exemplis docebimus.

EXEMPLUM 1

Queratur summa, cuius differentia sit $\frac{3x+2\omega}{x(x+\omega)(x+2\omega)}$.

Resolvatur haec differentia proposita in suas fractiones simplices, quae erunt

$$\frac{1}{\omega} \cdot \frac{1}{x} + \frac{1}{\omega} \cdot \frac{1}{x+\omega} - \frac{2}{\omega} \cdot \frac{1}{x+2\omega}.$$

Cum iam sit ex superiori formula

$$\Sigma \frac{1}{x+n\omega} = \Sigma \frac{1}{x+(n+1)\omega} - \frac{1}{x+n\omega},$$

erit

$$\Sigma \frac{1}{x} = \Sigma \frac{1}{x+\omega} - \frac{1}{x}.$$

Hinc erit summa quaesita

$$\frac{1}{\omega}\Sigma\frac{1}{x}+\frac{1}{\omega}\Sigma\frac{1}{x+\omega}-\frac{2}{\omega}\Sigma\frac{1}{x+2\omega}=\frac{2}{\omega}\Sigma\frac{1}{x+\omega}-\frac{2}{\omega}\Sigma\frac{1}{x+2\omega}-\frac{1}{\omega x};$$

at est

$$\Sigma\frac{1}{x+\omega}=\Sigma\frac{1}{x+2\omega}-\frac{1}{x+\omega};$$

unde summa quaesita erit

$$-\frac{1}{\omega x}-\frac{2}{\omega(x+\omega)}=\frac{-3x-\omega}{\omega x(x+\omega)}.$$

EXEMPLUM 2

Quaeratur summa cuius differentia est $\frac{3\omega}{x(x+3\omega)}$.

Posita hac differentia = z erit $z=\frac{1}{x}-\frac{1}{x+3\omega}$ ideoque

$$\Sigma z=\Sigma\frac{1}{x}-\Sigma\frac{1}{x+3\omega}=\Sigma\frac{1}{x+\omega}-\Sigma\frac{1}{x+3\omega}-\frac{1}{x}$$
$$=\Sigma\frac{1}{x+2\omega}-\Sigma\frac{1}{x+3\omega}-\frac{1}{x}-\frac{1}{x+\omega}=-\frac{1}{x}-\frac{1}{x+\omega}-\frac{1}{x+2\omega},$$

quae est summa quaesita. Quoties ergo hoc modo signa summatoria Σ sese tandem tollunt, toties differentiae propositae summa exhiberi poterit; sin autem haec destructio non succedat, signum hoc est summam inveniri non posse.