A CONSIDERATION OF THE EXPERIMENTS RECENTLY CONDUCTED INTO THE FIRING OF CANNONS

Concerning the motion of cannon balls, two items enter into the computation : the motion of the ball in the cannon and the motion outside the cannon, of which motions any is required to be investigated separately. Moreover the first to be investigated is the motion outside the cannon, which can be determined from the time in which the ball has remained in the air, the diameter of the ball, and the ratio of the specific gravity of the ball to the air. And from these given, the height can be determined, to which the ball reaches, and the initial speed, with which it bursts forth from the cannon, and also the ascent and descent time separately. With which defined we are able to progress to the consideration of the ball within the cannon and from the velocity with which it emerges from the cannon, the strength of the gunpowder will be known and many other things of great use in pyrotechnics. But here I assume the direction of the cannon to be vertical, so that the body may describe a right line in the ascent and descent; for motion at an angle is determined by a curved line of higher order.

Let c designate the diameter of the ball in scruples of Rhenish feet [*i.e.* thousandth parts of a Rh.ft.], m: n the ratio of the specific gravity of the ball to the specific gravity of the air or of the medium, in which the ball is moving. Let t be the time of the duration of the ball in air, in seconds, again let x be the height sought, to which the body rises ; e which is 2,7182818... may be written for the number, the [natural] logarithm of which is one, of which the logarithm [to base 10] is 0,4342944, following Vlaq [Recall that the first complete set of base 10 log tables were published by Vlaq, following on Briggs's work]. Again N shall indicate the number of degrees of the arc, of which the tangent is

$$\sqrt{e^{\frac{3nx}{4mc}}-1},$$

with the total sine being $=1$. The height sought x must be found from this equation :

$$t=\frac{m\sqrt{c}}{447650\sqrt{3n(m-n)}}\left(125N-7162l\left(\sqrt{e^{\frac{3nx}{4mc}}}-\sqrt{e^{\frac{3nx}{4mc}}-1}\right)\right).$$

[See Vol. I of Euler's *Mechanica*, § 450, to be found in translation on this website.]
We shall say, so that the calculation becomes easier :

$$\sqrt{e^{\frac{3nx}{4mc}}-1}=y\,;$$

and N will be the number of degrees of the arc, whose tangent is y; there will be

$$t = \frac{m\sqrt{c}}{447650\sqrt{3n(m-n)}}\left(125N - 7162l\left(\sqrt{yy+1} - y\right)\right).$$

In order that Vlaq's table of logarithms may be used, the logarithm must be multiplied by 2,7182818.
[The editor F.R. Sherrer in the *O.O.* edition (1922) has made the correction, that the logarithms are converted to base 10 on multiplying by $l10 = 2{,}3025851$; these corrections are made subsequently in a table at the end of this work ; thus the errors introduced by Euler are present until that point, as it would not be possible to retrace the iterations as Euler has made them. We should note that this was first introduction of *e* into the language of mathematics by Euler, at the age of 21. Thus, as Clifford Truesdale points out in *An Idiot's Fugitive Essays on Science*…, a great deal of Euler's ideas had already been formed at an early age, and his life's work consisted in elaborating on them.]
A may be written in place of

$$\frac{447650\sqrt{3n(m-n)}}{m\sqrt{c}};$$

there will be

$$At = 125N - 19468\log.(\sqrt{yy+1} - y);$$

therefore there will be

$$N = \frac{At + 19468\log.(\sqrt{yy+1} - y)}{125} = \frac{8At + 155746\log.(\sqrt{yy+1} - y)}{1000}.$$

From which a trial value for *y* must be put in place, while other and still other values must be substituted for *y*, until equality may be found.

EXPERIMENT I

Made on the 21st day of Aug., 1727

An iron ball of 225 scrup. diameter was fired vertically, the time of the duration in the air was 45 seconds.

Therefore there is $c = 225,\ t = 45,\ m = 7000$ and $n = 1$.

Therefore there will be

$$A = 618, \text{ hence } At = 27816 \text{ and } 8At = 222530 .$$

Therefore

$$N = \frac{222530 + 155746\log.(\sqrt{yy+1} - y)}{1000}.$$

Putting y = 2,70; there will be,

$$\sqrt{yy+1} = 2,879, \quad \text{therefore} \quad y\sqrt{yy+1} - y = 0,179,$$

consequently

$$\log.(\sqrt{yy+1} - y) = -0,7471 \text{ et } N = 69\tfrac{41}{60} = \tfrac{69683}{1000},$$

but from the equation it is found that $N = \frac{106173}{1000}$. Therefore y must be assumed larger.

Let $y = 3,00$; there will be

$$\sqrt{yy+1} = 3,162, \quad \text{therefore} \quad \sqrt{yy+1} - y = 0,162,$$

from which the log. of this is $-0,790$, from which there is produced $N = 99°$.

Let $y = 4,00$; there will be

$$\sqrt{yy+1} = 4,123, \quad \text{and} \quad \sqrt{yy+1} - y = 0,123,$$

the log. of which is 0,9100. Therefore $N = 80,802$, but there must be $N = 75°58'$.

Let y = 4,10; there will be

$$\sqrt{yy+1} - y = 0,12,$$

whose $\log. = -0,9208$. Therefore there is $N = 79°12'$, but there must be $N = 76°18'$.

By continuing this there is found: $y = 4,31$; in this case the equation is solved almost exactly, as it is correct to the hundredth part. And there will be $N = 76°\ 56'$.

So that the height may be found, pertaining to the body, there will be

$$\sqrt{e^{\frac{3nx}{4mc}} - 1} = y$$

and thus

$$e^{\frac{3nx}{4mc}} = 19,5761,$$

therefore

$$\frac{3nx}{4mc} \cdot 0,4342944 = 1,2915908$$

or

$$x = \frac{2100000 \cdot 1,2915908}{0,4342944} = 6245 \text{ Rhen.ft.}$$

Hence the initial velocity or height becomes known, to which it may reach by the same force in a vacuum ; for there is

$$e^{\frac{3nx}{4mc}} = \frac{4c(m-n)+3nK}{4c(m-n)}$$

with K denoting the height described in a vacuum; therefore there will be

$$K = 20997\cdot 1857,61 \text{ scrup.} = 39004 \text{ Rhen.ft.}$$

The time which the ball takes in rising, is equal to

$$\frac{mN\sqrt{c}}{3581\sqrt{3n(m-n)}} \text{ sec.,}$$

that is (on account of $N = 76,93$ and $\sqrt{c} = 15$)

$$15\tfrac{1}{2} \text{ sec.}$$

Therefore the time of descent is $29\frac{1}{2}$ sec., so that thus the difference between the time of ascent and descent shall be 14 sec.

EXPERIMENT II

conducted on the same day

A ball was fired from the same cannon with half the quantity of powder, that remained in the air for 34 sec.

Therefore there will be

$$c = 225,\ t = 34,\ m = 7000,\ n = 1 \text{ and } A = 618;$$

there will be

$$At = 21012 \text{ and } 8At = 168096.$$

Therefore

$$N = \frac{168096 + 155746\log.(\sqrt{yy+1} - y)}{1000}.$$

There is put $y = 2,00$; there will be

$$\sqrt{yy+1} - y = 0,236,$$

of which the log. is $=-0,6270$; hence there is found $N=70,91$ and there must be 63° 26'. In this manner assumed there is found to be in place finally $y=2,185$; there will be $N=65°\ 25'$; therefore there will be

$$\sqrt{e^{\frac{3nx}{4mc}}-1}=\ 2,185 \text{ and } e^{\frac{3nx}{4mc}}=5,7742.$$

Therefore

$$\frac{3nx}{4mc}=\frac{\log 5,7742}{0,43429}=\frac{0,76149}{0,43429},$$

from which,

$$x=\frac{2100000\cdot 0,76149}{0,43429}\text{ scrup.}=3682\text{ Rhen.ft.}$$

Hence the height, to which it reaches in a vacuum, is 10025,862 Rhen.ft. The time of ascent is 13,19 sec. Therefore the time of descent is 20,81 sec.

EXPERIMENT III

made on the 23rd day of Aug., 1727

The same ball of diameter 225 scrup. was fired vertically and the time was 2 sec., the quantity of powder was 1 Loth [*i.e.* $\frac{1}{2}$ lb.], or the $\frac{1}{8}$ th part of the preceding.

Therefore as above there is:

$$c=225,\ m=7000,\ n=1,\text{ but } t=2.$$

Therefore on account of $A=618$ there is:

$$At=1236,\text{ there } 8At=9888.$$

Consequently there will be :

$$N=\frac{9888+155746\log(\sqrt{yy+1}-y)}{1000}.$$

By testing, what shall be substituted in place of *y*, there is found to be $y=0,075$, hence there is $N=4°19'$. Therefore there is :

$$\sqrt{e^{\frac{3nx}{4mc}}-1}=0,075 \quad \text{hence} \quad e^{\frac{3nx}{4mc}}=1,005625.$$

Hence

$$\frac{3nx}{4mc}=\frac{0,002300}{0,4343},$$

hence

$$x=\frac{2100000\cdot 0,0023}{0,4343}=11121 \text{ scrup.};$$

and therefore the ball reaches a height of 11 ft.

Thereafter

$$0,005625=\frac{3nK}{4c(m-n)}.$$

Hence $K=2099700\cdot 0,005625=11800$ scrup. Therefore the difference in heights in a vacuum and in air is 678 scrup. But the time of ascent is $\frac{7000\cdot 4,32}{3581\cdot 144}=0,88$ sec., therefore the time of descent is 1,12 sec.

In these experiments the length of the cannon was 7260 scruples. Moreover in the following, the same cannon was used, but shortened so that its length was only 5808 scruples. In the first experiment the quantity of powder was 16 Loth, in the second 8 Loth, and in the third 1 Loth.
[1 Loth is equal to half a pound, or 8 ounces ; 1000 scruples is equal to 1 Rh.ft.]

EXPERIMENT IV

performed on the 2. Sept. 1727

The same ball of diameter 225 scrup. is fired vertically with 1 Loth of powder [*i.e.* $\frac{1}{2}$ lb.] and falls finally after 8 sec. Again there is:

$$c=225,\ m=7000,\ n=1,\ \text{but}\ \ t=8.$$

From which there shall be

$$N=\frac{39552+155746\log(\sqrt{yy+1}-y)}{1000}.$$

Finally there is found $y=0,33$. Now there will be $N=18°25'$. Therefore there is

$$e^{\frac{3nx}{mc}}=1{,}1089,\text{ hence } x=\frac{2100000\cdot 0{,}04458}{0{,}4343}.$$

Therefore 215 ft., $1\frac{7}{12}$ inch is the height to which the ball ascends, but the height to which the ball arrives *in vacuo*, is

$$K=2099700\cdot 0{,}1089=228\text{ ft. }5\tfrac{8}{12}\text{ inch.}$$

Moreover, the ascent time is

$$=\frac{7000\cdot 18{,}41\cdot 15}{3581\cdot 144}=3{,}7\text{ sec.}$$

Therefore the descent time $=4{,}3$ sec.

EXPERIMENT V
made on the same day

The same ball from the same cannon with a charge of 4 Loth [*i.e.* 2 lb.] was discharged and the time in which it remained in the air was 20 sec.

Therefore there is :

$$c=225,\ m=7000,\ n=1,\ t=20.$$

Hence there is

$$N=\frac{988880+155746\log(\sqrt{yy+1}-y)}{1000}.$$

Therefore $y=0{,}93$, and hence

$$N=42°\ 56',\quad e^{\frac{3nx}{mc}}=1{,}8649.$$

Therefore

$$x=\frac{2100000\cdot 0{,}27044}{0{,}43429}=1307{,}707\text{ ft.}$$

From which

$$K=2099700\cdot 0{,}8694=1816{,}025\text{ ft.}$$

Moreover the time of ascent is

$$=\frac{7000\cdot 42{,}93\cdot 15}{3581\cdot 144}=\frac{210\cdot 4293}{103849}=8{,}6\text{ sec.}$$

Hence the descent time was 11,4 sec.

EXPERIMENT VI
made on the same day

The same ball was fired from the same cannon with a charge of 8 Loth [*i.e.* 4 lb.] and the time, in which it remained in the air, was 28 sec.

Therefore there is :

$$c = 225,\ m = 7000,\ n = 1,\ t = 28.$$

Hence

$$N = \frac{138432 + 155746\log(\sqrt{yy+1} - y)}{1000}.$$

Hence there is found $y = 1,52$ and

$$N = 56°\ 39',\quad e^{\frac{3nx}{mc}} = 3,3104,$$

from which

$$x = \frac{2100000 \cdot 0,51982}{0,43429} = 2513,621 \text{ Rhen.ft.}$$

But

$$K = 20997 \cdot 31,04 = 4851,150 \text{ Rhen.ft.}$$

Moreover the time of the ascent is :

$$\frac{7000 \cdot 56,66 \cdot 15}{3581 \cdot 144} = 11,45 \text{ sec.}$$

Therefore the descent time $= 16,55$ sec..

EXPERIMENT VII
performed on the said day

From the same cannon, but with a charge of 12 Loth [or 6 lb.], the same ball is ejected and the time until it falls was 32 sec. On account of

$$c = 225,\ m = 7000,\ n = 1,\ t = 32$$

there will be

$$N=\frac{158\ 202+155746\log(\sqrt{yy+1}-y)}{1000}.$$

From which consequently there will be $y=1,93$, therefore $N=62°27'$.
There will be

$$e^{\frac{3nx}{4mc}}=4,7249,$$

therefore

$$x=\frac{2100000\cdot 0,6733099}{0,43429}=3255,776 \text{ Rhen.ft. or } 3255776 \text{ scrup.}$$

But there will be

$$K=20997\cdot 372,49=7821,172 \text{ Rhen.ft.}$$

The ascent time is

$$=\frac{210\cdot 6261}{103849}=12,67 \text{ sec.}$$

and the descent time will be 19,33 sec.

Table containing the corrections of this dissertation.

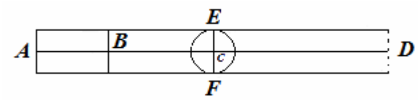
Experi-ment	Powder charge in $\frac{1}{2}$ lbs.	t	N	x Rhen.ft.	Ascent time expressed in sec.	Descent time expressed in sec.	K Rhen.ft.
I	16	45	80° 25'	7530	16,27	28,73	73558
II	8	34	69° 39'	4436	14,09	19,91	15263
III	1	2	4° 56'	15,59	0,998	1,002	15,644
IV	1	8	19° 35'	250,3	3,96	4,04	265,74
V	4	20	46° 22'	1559	9,38	10,62	2311,6
VI	8	28	60° 58'	3036	12,34	15,66	6813
VII	12	32	66° 58$\frac{1}{2}$'	3943	13,55	18,45	11625

[See Daniel Bernoulli's *Hydodynamicae* p.234 onwards for his description of the original experiments.]

FRAGMENT EXTRACTED FROM EULER'S MATHEMATICAL MEMORANDA
[Taken from Euler's *Opera Postuma*.]

CONCERNING THE MOTION OF A BALL FIRED THROUGH THE BORE OF A CANNON

Initially the length AB was filled with gunpowder, so that there shall be $AB = a$; which in turn is equivalent to the air compressed into a space n less, and which is ignited. Now the ball arrives at ECF and there shall be $AC = x$. The weight of the ball shall be $= A$, the radius $CE = c$, the speed of the centre $\sqrt{v}$, the rotational speed of the point



$E = m\sqrt{v}$.
[m is not defined apart from by this equation, and may be considered to be some fraction lying between 0 and 1, depending on how rapidly the ball spins.]
The *vis viva* of the ball will be $= Av + \frac{2}{5}Ammv.$

Let the height of the atmosphere be $= f$ and the pressure of an atmospheric column acting on the base on an equal small great circle $= p$; [the acting on the cannon ball due to the atmospheric air pressure will be] equivalent to the weight of an air column $= \pi ccf$ .The force [*i.e.* pressure] acting will be

$$= \frac{nap}{x} - p\,,$$

truly the resistance will be [*i.e.* proportional to the speed squared]

$$= \tfrac{1}{2}\pi ccv.$$

Therefore there will be [equation derived from the work done by the ball falling an increment *dv in vacuo* equated to the work done in the Boyle's law type expansion minus the work done against the atmospheric pressure and a term proportional to the speed squared; where we recall that velocity squared is equated to height fallen in the *vis viva* model ; we note that F. R. Sherrer made a number of corrections to Euler's original paper that have been included here]:

$$A\left(1+\tfrac{2}{5}mm\right)dv = \pi cc\left(\frac{nafdx}{x} - fdx - \tfrac{1}{2}vdx\right).$$

Let

$$\frac{A\left(1+\frac{2}{5}mm\right)}{\pi cc} = b;$$

there will be

$$dv + \frac{vdx}{2b} = \frac{nafdx}{bx} - \frac{fdx}{b}\,,$$

therefore

$$e^{\frac{x}{2b}}v=+\frac{naf}{b}\int\frac{e^{\frac{x}{2b}}dx}{x}-\frac{f}{b}\int e^{\frac{x}{2b}}dx.$$

Let $x=a+t$; there will be

$$e^{\frac{t}{2b}}v=\frac{naf}{b}\int\frac{e^{\frac{t}{2b}}dt}{a+t}-\frac{f}{b}\int e^{\frac{t}{2b}}dt$$

$$=\frac{f}{b}\int e^{\frac{t}{2b}}dt\left(\frac{(n-1)a-t}{a+t}\right),$$

hence

$$v=\frac{fe^{-\frac{t}{2b}}}{b}\int e^{\frac{t}{2b}}dt\left(\frac{(n-1)a-t}{a+t}\right);$$

or

$$2abdv+2btdv+avdt+vtdt=2nafdt-2fadt-2ftdt.$$

and

$$\begin{aligned}v=\frac{(n-1)ft}{b}-\frac{(n-1)ftt}{4bb}+\frac{(n-1)ft^3}{24b^3}-\frac{(n-1)ft^4}{4\cdot6\cdot8b^4}&+\text{etc.}\\ -\frac{nftt}{2ab}+\frac{nft^3}{12abb}-\frac{nft^4}{8\cdot12ab^3}&+\text{etc.}\\ +\frac{nft^3}{3aab}-\frac{nft^4}{8\cdot3aabb}&+\text{etc.}\\ -\frac{nft^4}{4a^3b}&+\text{etc.}\\ &+\text{etc.,}\end{aligned}$$

therefore

$$\begin{aligned}v&=-\frac{ft}{b}+\frac{naf}{b}l\left(1+\frac{t}{a}\right)\\ &+\frac{ftt}{4bb}-\frac{naaf}{2bb}\left(\left(1+\frac{t}{a}\right)l\left(1+\frac{t}{a}\right)-\frac{t}{a}\right)\\ &-\frac{ftt}{24b^3}+\frac{na^3f}{8b^3}\left(\left(1+\frac{t}{a}\right)^2l\left(1+\frac{t}{a}\right)-\frac{t}{a}-\frac{3tt}{2aa}\right)\\ &\qquad\text{etc.}\\ &=-2f\left(1-e^{\frac{-t}{2b}}\right)+\frac{naf}{b}e^{\frac{-a-t}{2b}}l\left(1+\frac{t}{a}\right)+\frac{nft}{b}\left(1-e^{\frac{-a}{2b}}\right)-\text{etc.}\end{aligned}$$

Putting $2b=g$; there will be

$$\int \frac{e^{\frac{t}{2b}}dt}{a+t} = e^{-\frac{a}{g}} \left\{ \begin{array}{l} l\left(1+\frac{t}{a}\right)+\frac{a}{g}\left(\frac{t}{a}\right)+\frac{aa}{2gg}\left(\frac{t}{a}+\frac{tt}{2aa}\right) \\ +\frac{a^3}{6g^3}\left(\frac{t}{a}+\frac{tt}{aa}+\frac{t^3}{3a^3}\right)+\frac{a^4}{24g^4}\left(\frac{t}{a}+\text{etc.}\right) \end{array} \right\}.$$

Hence there will be

$$v=\frac{2naf}{g}e^{-\frac{t}{g}}\int \frac{e^{\frac{t}{2b}}dt}{a+t} -2f\left(1-e^{-\frac{t}{g}}\right)$$

$$=\frac{2naf}{g}e^{\frac{-a-t}{g}}l\left(1+\frac{t}{a}\right)+\frac{2naf}{g}e^{\frac{-t}{g}} \left\{ \begin{array}{l} \frac{a}{g}\left(\frac{t}{a}\right)-\frac{aa}{2gg}\left(\frac{t}{a}-\frac{tt}{2aa}\right)+\frac{a^3}{6g^3}\left(\frac{t}{a}-\frac{tt}{aa}+\frac{t^3}{3a^3}\right) \\ -\frac{a^4}{24g^4}\left(\frac{t}{a}-\frac{tt}{aa}+\frac{t^3}{3a^3}-\frac{t^4}{4a^4}+\text{etc.}\right) \end{array} \right\}$$

$$-2f+2fe^{\frac{-t}{g}};$$

but there is

$$g=\frac{2A\left(1+\frac{2}{5}mm\right)}{\pi cc}.$$

First therefore the length of the bore must be defined, so that the ball may exit with the maximum speed ; moreover hence there will be, [from $dv+\frac{vdx}{2b}=\frac{nafdx}{bx}-\frac{fdx}{b}$,]

$$v=\frac{2naf}{a+f}-2f$$

Therefore we have

$$v=\frac{2(n-1)ft}{g}-\frac{(n-1)ftt}{1\cdot 2gg}+\frac{2(n-1)ft^3}{1\cdot 2\cdot 3g^3}-\frac{2(n-1)ft^4}{1\cdot 2\cdot 3\cdot 4g^4}+\frac{(n-1)ft^5}{1\cdot 2\cdot 3\cdot 4\cdot 5g^4}-\text{etc.}$$

$$-\frac{1\cdot 2nftt}{1\cdot 2ag}+\frac{1\cdot 2nft^3}{1\cdot 2\cdot 3agg}-\frac{1\cdot 2nft^4}{1\cdot 2\cdot 3\cdot 4ag^3}+\frac{1\cdot 2nft^5}{1\cdot 2\cdot 3\cdot 4\cdot 5ag^4}-\text{etc.}$$

$$+\frac{1\cdot 2\cdot 2nft^3}{1\cdot 2\cdot 3agg}-\frac{1\cdot 2\cdot 2nft^4}{1\cdot 2\cdot 3\cdot 4aagg}+\frac{1\cdot 2\cdot 2nft^5}{1\cdot 2\cdot 3\cdot 4\cdot 5aag^3}-\text{etc.}$$

$$-\frac{1\cdot 2\cdot 3\cdot 2nft^4}{1\cdot 2\cdot 3\cdot 4a^3g}+\frac{1\cdot 2\cdot 3\cdot 2nft^5}{1\cdot 2\cdot 3\cdot 4\cdot 5a^3gg}-\text{etc.}$$

$$+\frac{1\cdot 2\cdot 3\cdot 4\cdot 2nft^5}{1\cdot 2\cdot 3\cdot 4\cdot 5a^4gg}-\text{etc.}$$

$$-\text{etc.}$$

The specific gravity of the ball to the specific gravity of air shall be as *r* to 1; there will be

$$A = \tfrac{4}{3}\pi c^3 r$$

hence

$$b = \tfrac{4}{3} rc(1 + \tfrac{2}{5} mm) = \tfrac{1}{2} g,$$

$$g = \tfrac{8}{3} rc(1 + \tfrac{2}{5} mm).$$

MEDITATIO IN EXPERIMENTA EXPLOSIONE TORMENTORUM NUPER INSTITUTA

Commentatio 853 indicis ENESTROEMIANI Opera postuma 2, 1862, p. 800-804

Circa motum globorum duo in computum veniunt, motus globi in tormento et motus extra tormentum, de quorum motuum quolibet seorsim agendum est. Primum autem excutiendus est motus extra tormentum, qui determinari poterit ex tempore, quo globus in aëre commoratus est, diametro globi et ratione gravitatum specificarum globi et aeris. Ex hisce datis innotescit altitudo, ad quam globus pervenit, et velocitas initialis, quae tormento erumpit, tempus quoque ascensus et descensus seorsim. Quibus definitis progredi poterimus ad contemplandum motum globi intra tormentum et ex velocitate, qua globus egreditur, cognita innotescet vis pulveris pyrii multaque alia maximi usus in Pyrotechnia. Suppono autem hic directionem tormenti esse verticalem, ut corpus lineam rectam ascensu et descensu describat; motus enim obliquus in linea curva altioris est indaginis.

Designet c diametrum globi in scrupulis pedis Rhenan, m: n rationem gravitatis specificae globi ad gravitatem specificam aeris seu medii, in quo globus movetur. Sit t tempus durationis globi in aëre, in minutis secundis, sit porro altitudo quaesita, ad quam corpus ascendit, x. Scribatur pro numero, cuius logarithmus est unitas, e, qui est 2,7182818 ... , cuius logarithmus secundum Vlacquium est 0,4342944. Indicat porro N numerum graduum arcus, cuius tangens est

$$\sqrt{e^{\frac{3nx}{4mc}}-1}$$

existente sinu toto $=1$. Altitudo quaesita x ex hac aequatione erui debet

$$t=\frac{m\sqrt{c}}{447650\sqrt{3n(m-n)}}\left(125N-7162l\left(\sqrt{e^{\frac{3nx}{4mc}}}-\sqrt{e^{\frac{3nx}{4mc}}-1}\right)\right).$$

Vocemus, ut calculus facilior evadat,

$$\sqrt{e^{\frac{3nx}{4mc}}-1}=y\;;$$

erit N numerus graduum arcus, cuius tangens est y; erit

$$t=\frac{m\sqrt{c}}{447650\sqrt{3n(m-n)}}\left(125N-7162l\left(\sqrt{yy+1}-y\right)\right).$$

Ut logarithmis VLACQUII uti liceat, multiplicari debet logarithmus per 2,7182818. Scribatur A loco

$$\frac{447650\sqrt{3n(m-n)}}{m\sqrt{c}};$$

erit

$$At = 125N - 19468\log.(\sqrt{yy+1} - y);$$

erit ergo

$$N = \frac{At + 19468\log.(\sqrt{yy+1} - y)}{125} = \frac{8At + 155746\log.(\sqrt{yy+1} - y)}{1000}.$$

Ex qua aequatione tentando *y* erui debebit, tamdiu alios atque alios substituendo valores loco *y*, donec resultet aequalitas.

EXPERIMENTUM I

factum d. 21 Aug. a. 1727

Globus ferreus diametri 225 scrup. explodebatur verticaliter, tempus durationis in aëre erat 45 minut. secund.

Est ergo $c = 225,\ t = 45,\ m = 7000$ et $n = 1$.

Erit ergo

$$A = 618,\ \text{ergo } At = 27816 \text{ et } 8At = 222530.$$

Erit ergo

$$N = \frac{222530 + 155746\log.(\sqrt{yy+1} - y)}{1000}.$$

Ponatur y = 2,70; erit,

$$\sqrt{yy+1} = 2{,}879, \quad \text{ergo } y\sqrt{yy+1} - y = 0{,}179,$$

consequenter

$$\log.(\sqrt{yy+1} - y) = -0{,}7471 \text{ et } N = 69\tfrac{41}{60} = \tfrac{69683}{1000},$$

sed ex aequatione invenitur $N = \frac{106173}{1000}$. Ergo y maior assumi debet.

Sit $y = 3{,}00$; erit

$$\sqrt{yy+1} = 3{,}162, \quad \text{ergo } \sqrt{yy+1} - y = 0{,}162,$$

unde log. eius est $-\ 0{,}790$, unde prodit $N = 99°$.

Sit $y = 4{,}00$; erit

$$\sqrt{yy+1} = 4{,}123, \text{ et } \sqrt{yy+1} - y = 0{,}123,$$

cuius log. est 0,9100. Est ergo $N = 80{,}802$, sed debebat esse $N = 75°58'$.

Sit y = 4,10; erit

$$\sqrt{yy+1} - y = 0{,}12,$$

cuius $\log. = -\ 0{,}9208$. Est ergo $N =\ 79°12'$, sed debebat esse $N = 76°18'$.

Hoc continuando reperitur $y = 4{,}31$; hoc in casu exacte admodum obtinetur aequatio, ut ne in centesimis erretur. Et erit $N = 76°\ 56'$.

Ut inveniatur altitudo, ad quam corpus pertingit, erit

$$\sqrt{e^{\frac{3nx}{4mc}} - 1} = y$$

adeoque

$$e^{\frac{3nx}{4mc}} = 19{,}5761,$$

ergo

$$\frac{3nx}{4mc} \cdot 0{,}4342944 = 1{,}2915908$$

seu

$$x = \frac{2100000 \cdot 1{,}2915908}{0{,}4342944} = 6245 \text{ ped. Rhen.}$$

Hinc innotescit velocitas initialis seu altitudo, ad quam eodem impetu in vacuo pervenisset; est enim

$$e^{\frac{3nx}{4mc}} = \frac{4c(m-n)+3nK}{4c(m-n)}$$

denotante K altitudinem in vacuo describendam; erit ergo

$$K = 20997 \cdot 1857{,}61 \text{ scrup.} = 39004 \text{ ped. Rhen.}$$

Tempus, quod globus in ascensu consumit, est aequale

$$\frac{mN\sqrt{c}}{3581\sqrt{3n(m-n)}} \text{ minut. secund.,}$$

id est (ob $N = 76{,}93$ et $\sqrt{c} = 15$)

$15\frac{1}{2}$ minut. secund.

Tempus ergo descensus est $29\frac{1}{2}$ minut. secund.,
ut adeo differentia inter tempus ascensus et descensus sit 14 minut. secund.

EXPERIMENTUM II

eodem die institutum

Ex eodem tormento idem globus explodebatur dimidia pulveris quantitate, mansit ille in aëre 34 minut. secund.

Est ergo

$$c = 225,\ t = 34,\ m = 7000,\ n = 1 \ \text{ et } \ A = 618;$$

erit

$$At = 21012 \ \text{ et } \ 8At = 168096.$$

Est ergo

$$N = \frac{168096 + 155746\log.(\sqrt{yy+1} - y)}{1000}.$$

Ponatur $y = 2,00$; erit

$$\sqrt{yy+1} - y = 0,236,$$

cuius log. est $= -0,6270$; hinc invenitur $N = 70,91$ et deberet esse 63° 26'. Hoc modo tentando invenitur tandem sumi debere loco y 2,185; erit $N = 65° \ 25'$; erit ergo

$$\sqrt{e^{\frac{3nx}{4mc}} - 1} = \ 2,185 \quad \text{et} \quad e^{\frac{3nx}{4mc}} = 5,7742.$$

Ergo

$$\frac{3nx}{4mc} = \frac{\log 5,7742}{0,43429} = \frac{0,76149}{0,43429},$$

unde ,

$$x = \frac{2100000 \cdot 0,76149}{0,43429} \ \text{scrup.} = 3682 \ \text{ped.Rhen.}$$

Dein altitudo, ad quam in vacuo pervenisset, est 10025,862 ped. Rhen. Tempus ascensus est 13,19 minut. secund. Ergo tempus descensus est 20,81 minut. secund.

EXPERIMENTUM III

factum d. 23. Aug. a. 1727

Idem globus diametri 225 scrup. explodebatur verticaliter et tempus erat 2 minut. secund., quantitas pulveris 1 Loth seu $\frac{1}{8}$ pars praecedentis.

Est ergo ut supra

$$c = 225,\ m = 7000,\ n = 1,\ \text{sed}\ \ t = 2.$$

Ergo ob $A = 618$ est

$$At = 1236,\ \text{ergo}\ \ 8At = 9888.$$

Consequenter erit

$$N = \frac{9888 + 155746\log(\sqrt{yy+1} - y)}{1000}.$$

Tentando, quid loco y substituendum sit, reperietur esse $y = 0,075$, unde est $N = 4°19'$. Est ergo

$$\sqrt{e^{\frac{3nx}{4mc}} - 1} = 0,075 \quad \text{ergo} \quad e^{\frac{3nx}{4mc}} = 1,005625.$$

Ergo

$$\frac{3nx}{4mc} = \frac{0,002300}{0,4343},$$

ergo

$$x = \frac{2100000 \cdot 0,0023}{0,4343} = 11121\ \text{scrup.};$$

pervenit ergo globus ad altitudinem 11 pedum.

Dein est

$$0,005625 = \frac{3nK}{4c(m-n)}.$$

Ergo $K = 2099700 \cdot 0,005625 = 11800$ scrup. Differentia ergo altitudinum in vacuo et aëre est 678 scrup. Tempus autem ascensus est $\frac{7000 \cdot 4,32}{3581 \cdot 144} = 0,88$ minut. secund., ergo tempus descensus est 1,12 minut. secund.

In his experimentis erat longitudo tormenti 7260 scrupula. In sequentibus autem idem tormentum adhibitum est, sed abbreviatum ut eius longitudo erat saltem 5808 scrupula. In primo experimento erat quantitas pulveris 16 Loth, in secundo 8 Loth, in tertio 1 Loth.

EXPERIMENTUM IV
factum d. 2. Sept. a. 1727

Idem globus diam. 225 scrup. explodebatur verticaliter pulvere 1 Loth et cecidit demum post 8 minut. secund. Est iterum

$$c = 225,\ m = 7000,\ n = 1,\ \text{sed}\ \ t = 8\,.$$

Unde erit

$$N = \frac{39552 + 155746\log(\sqrt{yy+1} - y)}{1000}.$$

Unde reperitur $y = 0,33$. Erit ergo $N = 18°25'$. Est ergo

$$e^{\frac{3nx}{mc}} = 1,1089,\ \text{ergo}\ \ x = \frac{2100000 \cdot 0,04458}{0,4343}.$$

Est ergo altitudo, ad quam globus ascendit, 215 ped. 1 dig. 7 lin., altitudo autem, ad quam in vacuo pervenisset, est

$$K = 2099700 \cdot 0,1089 = 228 \text{ ped. } 5 \text{ dig. } 8 \text{ lin.}$$

Tempus autem ascensus est

$$= \frac{7000 \cdot 18,41 \cdot 15}{3581 \cdot 144} = 3,7 \text{ minut. secund.}$$

Ergo tempus descensus erat = 4,3 minut. secund.

EXPERIMENTUM V
eodem die factum

Idem globus ex eodem tormento pulvere 4 Loth onerato explodebatur et tempus, quo in aëre mansit, fuit 20 minut. secund.

Est ergo

$$c = 225,\ m = 7000,\ n = 1,\ t = 20.$$

Est ergo

$$N = \frac{988880 + 155746\log(\sqrt{yy+1} - y)}{1000}.$$

Est ergo $y = 0,93$, ergo

$$N = 42^\circ\ 56',\quad e^{\frac{3nx}{mc}} = 1,8649.$$

Ergo

$$x = \frac{2100000 \cdot 0,27044}{0,43429} = 1307,707 \text{ ped.}$$

Dein

$$K = 2099700 \cdot 0,8694 = 1816,025 \text{ ped.}$$

Tempus autem ascensus est

$$= \frac{7000 \cdot 42,93 \cdot 15}{3581 \cdot 144} = \frac{210 \cdot 4293}{103849} = 8,6 \text{ minut. secund.}$$

Ergo tempus descensus erat 11,4 minut. secund.

EXPERIMENTUM VI
eodem die factum

Idem globus ex eodem tormento pulvere 8 Loth onerato explodebatur et tempus, quo in aëre mansit, fuit 28 minut. secund.

Est ergo

$$c = 225,\ m = 7000,\ n = 1,\ \text{t} = 28.$$

Ergo

$$N = \frac{138432 + 155746\log(\sqrt{yy+1} - y)}{1000}.$$

Hinc reperitur $y = 1,52$ et

$$N = 56^\circ\ 39',\quad e^{\frac{3nx}{mc}} = 3,3104,$$

unde

$$x = \frac{2100000 \cdot 0,51982}{0,43429} = 2513,621 \text{ ped. Rhen.}$$

At

$$K = 20997 \cdot 31,04 = 4851,150 \text{ ped. Rhen.}$$

Tempus autem ascensus est

$$\frac{7000 \cdot 56,66 \cdot 15}{3581 \cdot 144} = 11,45 \text{ minut. secund.}$$

Tempus ergo descensus est $= 16,55$ minut. secund.

EXPERIMENTUM VII
dicto die institutum

Ex eodem tormento, sed 12 Loth onerato, eiaculabatur globus idem et tempus, donec cecidit, erat 32 minut. secund. Ob

$$c = 225,\ m = 7000,\ n = 1,\ t = 32$$

erit

$$N = \frac{158\,202 + 155746 \log(\sqrt{yy+1} - y)}{1000}.$$

Unde consequitur esse $y = 1,93$, ergo $N = 62°27'$.
Erit

$$e^{\frac{3nx}{4mc}} = 4,7249,$$

ergo

$$x = \frac{2100000 \cdot 0,6733099}{0,43429} = 3255,776 \text{ ped. Rhen. seu } 3255776 \text{ scrup.}$$

Sed erit

$$K = 20997 \cdot 372,49 = 7821,172 \text{ ped. Rhen.}$$

Tempus autem ascensus est

$$= \frac{210 \cdot 6261}{103849} = 12,67 \text{ minut. secund.}$$

et tempus descensus erit 19,33.

Tabula correctiones huius dissertationis continens

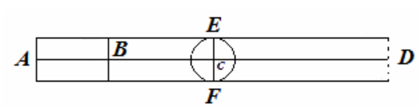
Experimentum	Onus Pulveris Lot	*t*	*N*	*x* ped. Rhen.	Tempus ascencus Secundis expressum	Tempus descensus Secundis expressum	K Ped. Rhen.
I	16	45	80° 25'	7530	16,27	28,73	73558
II	8	34	69° 39'	4436	14,09	19,91	15263
III	1	2	4° 56'	15,59	0,998	1,002	15,644
IV	1	8	19° 35'	250,3	3,96	4,04	265,74
V	4	20	46° 22'	1559	9,38	10,62	2311,6
VI	8	28	60° 58'	3036	12,34	15,66	6813
VII	12	32	66° 58$\frac{1}{2}$'	3943	13,55	18,45	11625

FRAGMENTUM EX ADVERSARIIS MATHEMATICIS DEPROMPTUM

Ex manuscriptis academiae scientiarum Petropolitanae nunc primum editum

DE MOTU GLOBI PER TUBUM TORNATUM EXPLOSI

Fuerit initio spatium *AB* pulvere pyrio repletum, ut sit $AB = a$; qui, simul ac accenditur, aequivalet aeri in spatium *n* vicibus minus compresso. Pervenerit globus iam *ECF* et sit $AC = x$. Sit pondus globi $= A$, radius $CE = c$, celeritas centri $\sqrt{v}$, celeritas



rotatoria puncti $E = m\sqrt{v}$. Erit vis viva globi $= Av + \frac{2}{5}Ammv$.

Sit altitudo atmosphaerae $= f$ et pressio columnae atmosphaerae in basin circolo maximo aequalem $= p$; aequivalebit ponderi columnae aëriae $= nccf$. Erit vis sollicitans

$$= \frac{nap}{x} - p\,,$$

resistentia vero est

$$= \tfrac{1}{2}\pi ccv.$$

Erit ergo,

$$A\left(1 + \tfrac{2}{5}mm\right)dv = \pi cc\left(\tfrac{nafdx}{x} - fdx - \tfrac{1}{2}vdx\right).$$

Sit

$$\frac{A\left(1+\frac{2}{5}mm\right)}{\pi cc}=b;$$

erit

$$dv+\frac{vdx}{2b}=\frac{nafdx}{bx}-\frac{fdx}{b},$$

ergo

$$e^{\frac{x}{2b}}v=+\frac{naf}{b}\int\frac{e^{\frac{x}{2b}}dx}{x}-\frac{f}{b}\int e^{\frac{x}{2b}}dx.$$

Sit $x=a+t$; erit

$$e^{\frac{t}{2b}}v=\frac{naf}{b}\int\frac{e^{\frac{t}{2b}}dt}{a+t}-\frac{f}{b}\int e^{\frac{t}{2b}}dt$$
$$=\frac{f}{b}\int e^{\frac{t}{2b}}dt\left(\frac{(n-1)a-t}{a+t}\right),$$

ergo

$$v=\frac{fe^{-\frac{t}{2b}}}{b}\int e^{\frac{t}{2b}}dt\left(\frac{(n-1)a-t}{a+t}\right);$$

vel

$$2abdv+2btdv+avdt+vtdt=2nafdt-2fadt-2ftdt.$$

et

$$\begin{aligned}v=\frac{(n-1)ft}{b}-\frac{(n-1)ftt}{4bb}+\frac{(n-1)ft^3}{24b^3}-\frac{(n-1)ft^4}{4\cdot6\cdot8b^4}+\text{etc.}\\ -\frac{nftt}{2ab}+\frac{nft^3}{12abb}-\frac{nft^4}{8\cdot12ab^3}+\text{etc.}\\ +\frac{nft^3}{3aab}-\frac{nft^4}{8\cdot3aabb}+\text{etc.}\\ -\frac{nft^4}{4a^3b}+\text{etc.}\\ +\text{etc.},\end{aligned}$$

ergo

$$v=-\frac{ft}{b}+\frac{naf}{b}l\left(1+\frac{t}{a}\right)$$
$$+\frac{ftt}{4bb}-\frac{naaf}{2bb}\left(\left(1+\frac{t}{a}\right)l\left(1+\frac{t}{a}\right)-\frac{t}{a}\right)$$
$$-\frac{ftt}{24b^3}+\frac{na^3f}{8b^3}\left(\left(1+\frac{t}{a}\right)^2 l\left(1+\frac{t}{a}\right)-\frac{t}{a}-\frac{3tt}{2aa}\right)$$

etc.

$$=-2f\left(1-e^{\frac{-t}{2b}}\right)+\frac{naf}{b}e^{\frac{-a-t}{2b}}l\left(1+\frac{t}{a}\right)+\frac{nft}{b}\left(1-e^{\frac{-a}{2b}}\right)-\text{etc.}$$

Ponatur $2b=g$; erit

$$\int\frac{e^{\frac{t}{2b}}dt}{a+t}=e^{-\frac{a}{g}}\left\{\begin{array}{l} l\left(1+\frac{t}{a}\right)+\frac{a}{g}\left(\frac{t}{a}\right)+\frac{aa}{2gg}\left(\frac{t}{a}+\frac{tt}{2aa}\right)\\ +\frac{a^3}{6g^3}\left(\frac{t}{a}+\frac{tt}{aa}+\frac{t^3}{3a^3}\right)+\frac{a^4}{24g^4}\left(\frac{t}{a}+\text{etc.}\right)\end{array}\right\}.$$

Hinc ergo erit

$$v=\frac{2naf}{g}e^{-\frac{t}{g}}\int\frac{e^{\frac{t}{2b}}dt}{a+t}-2f\left(1-e^{-\frac{t}{g}}\right)$$

$$=\frac{2naf}{g}e^{\frac{-a-t}{g}}l\left(1+\frac{t}{a}\right)+\frac{2naf}{g}e^{\frac{-t}{g}}\left\{\begin{array}{l}\frac{a}{g}\left(\frac{t}{a}\right)-\frac{aa}{2gg}\left(\frac{t}{a}-\frac{tt}{2aa}\right)+\frac{a^3}{6g^3}\left(\frac{t}{a}-\frac{tt}{aa}+\frac{t^3}{3a^3}\right)\\ -\frac{a^4}{24g^4}\left(\frac{t}{a}-\frac{tt}{aa}+\frac{t^3}{3a^3}-\frac{t^4}{4a^4}+\text{etc.}\right)\end{array}\right\}$$
$$-2f+2fe^{\frac{-t}{g}};$$

at est

$$g=\frac{2A\left(1+\frac{2}{5}mm\right)}{\pi cc}.$$

Primum igitur longitudo tubi definiri potest, ut globus celeritate maxima explodatur; hinc autem fit

$$v=\frac{2naf}{a+f}-2f$$

Est vero

$$v=\frac{2(n-1)ft}{g}-\frac{(n-1)ftt}{1\cdot 2gg}+\frac{2(n-1)ft^3}{1\cdot 2\cdot 3g^3}-\frac{2(n-1)ft^4}{1\cdot 2\cdot 3\cdot 4g^4}+\frac{(n-1)ft^5}{1\cdot 2\cdot 3\cdot 4\cdot 5g^4}-\text{etc.}$$

$$-\frac{1\cdot 2nftt}{1\cdot 2ag}+\frac{1\cdot 2nft^3}{1\cdot 2\cdot 3agg}-\frac{1\cdot 2nft^4}{1\cdot 2\cdot 3\cdot 4ag^3}+\frac{1\cdot 2nft^5}{1\cdot 2\cdot 3\cdot 4\cdot 5ag^4}-\text{etc.}$$

$$+\frac{1\cdot 2\cdot 2nft^3}{1\cdot 2\cdot 3agg}-\frac{1\cdot 2\cdot 2nft^4}{1\cdot 2\cdot 3\cdot 4aagg}+\frac{1\cdot 2\cdot 2nft^5}{1\cdot 2\cdot 3\cdot 4\cdot 5aag^3}-\text{etc.}$$

$$-\frac{1\cdot 2\cdot 3\cdot 2nft^4}{1\cdot 2\cdot 3\cdot 4a^3g}+\frac{1\cdot 2\cdot 3\cdot 2nft^5}{1\cdot 2\cdot 3\cdot 4\cdot 5a^3gg}-\text{etc.}$$

$$+\frac{1\cdot 2\cdot 3\cdot 4\cdot 2nft^5}{1\cdot 2\cdot 3\cdot 4\cdot 5a^4gg}-\text{etc.}$$

$$-\text{etc.}$$

Gravitas specifica globi ad gravitatem specificam aëris ut r ad 1; erit

$$A=\tfrac{4}{3}\pi c^3 r$$

ergo

$$b=\tfrac{4}{3}rc(1+\tfrac{2}{5}mm)=\tfrac{1}{2}\mathrm{g},$$

$$\mathrm{g}=\tfrac{8}{3}rc(1+\tfrac{2}{5}mm).$$