The Screw of Archimedes

The Archimedes' screw, both on account of its ancient invention, as well as its most frequent use in the raising of water, and being known so well to the common populace, that scarcely any writer of hydraulic machines may be found who will not have explained its construction and use in great abundance. But if indeed we may inquire into the reason, why this machine shall be suitable for raising water and how its action may be resolved according to the principles of mechanics, clearly in the older work indeed we have found nothing which may contain within it perhaps a reasonable account, and truly the more recent authors either have ignored this investigation entirely, or at least perused it lightly and with less accuracy. Thus although this machine shall be the most noteworthy and its use the most frequent, yet we are forced to admit at this point, its theory to be especially obscure and both the manner, by which water is raised by that, as well as the forces required for its action, even now lie almost completely hidden. And therefore this must be seen to be the more wonderful, since other machines handed down to us from antiquity not only may be referred to the laws of mechanics with the happiest success, but also mechanical knowledge itself may be developed from that point, so that it may be seen to be sufficient for explaining [the workings of] machines of all kinds. Moreover and even with all the greatest enthusiasm, which may be consumed in the further improvements of mechanics from geometry with the aid of the most sublime analysis, that usually may be considered more subtle than useful.

Truly if we may consider an account of the screw of Archimedes more carefully, we may understand the common mechanical principles to be of the least sufficiency for explaining that : therefore because that evidently shall relate to the theory of the motion of water through moving tubes, which discussion at this stage has hardly been treated by anyone. Indeed since it pertains to the motion of water in general, certainly it is not a long time since this has been investigated thoroughly and has began to be considered according to the principles of mechanics, but concerning the motion of water through moving tubes scarcely anything is to be found, which anyone has addressed in the literature, or even considered. On account of which now certain principles, on which the motion of all water may depend, may have been explained well enough, I will give a work, so that I may apply that also to the motion of water where it is carried by the screw of Archimedes, and thus I shall explain all phenomena, which are required to be considered in this motion, clearly and distinctly. Therefore what I have considered concerning this matter, I have included in the following propositions; and because screws of Archimedes of two kinds are accustomed to be made, the former of which has its spirals wound around a cylinder, truly the latter around a cone, I will begin with the cylindrical screw and with its theory established it will be permitted to progress without difficulty to any cones requiring to be studied carefully.

PROBLEM 1.

1. *With a given motion, by which a cylinder is turned around and the water may be driven around quickly through the screw or cylindrical spiral, to determine the true motion of each particle of water, this is that motion, which is composed from the rotational motion of the cylinder and from the motion of the water progressing along the spiral.*

Solution

The circle *ACB* shall be the base of the cylinder (Fig.1 : [in three dimensions]), of which the spiral has been wound around the surface, the right line *CD* the perpendicular axis of the cylinder at the centre *C* for the base, about which the cylinder with the spiral is driven around in a circle. The radius of the base is put $CA = CB = a$, and *EZ* shall be a part of the spiral on the surface of the cylinder, which may make an angle with the circumference of the base $ZEY = \zeta$ and from some point *Z* of the spiral *ZY* may be drawn parallel to the axis of the base *;* and the arc may be called $EY = s$, there is $YZ = s\,\text{tang}\zeta$, since which will make the angle with the spiral $EZY = 90° - \zeta$ *;* and the length of the spiral will be $EZ = \dfrac{s}{\cos\zeta}$.

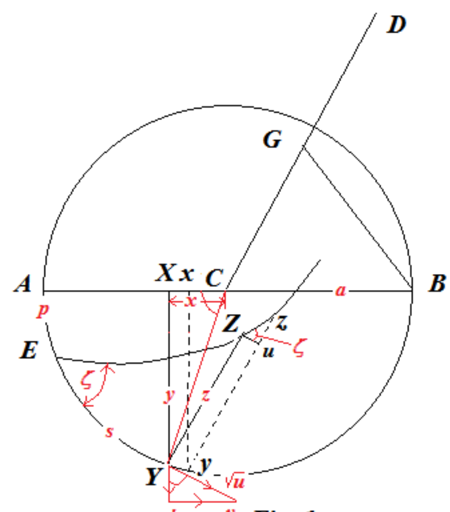


Fig. 1

Now the speed of the water flowing around the spiral shall be due to the height *v* [*i.e.* relative to the cylinder surface, and in accordance with $ht. = vel.^2$]*;* for I assume the width of the spiral *EZ* to be the same everywhere, thus so that in the same instant of time all the water in the spiral shall have the same speed $= \sqrt{v}$. Then because the whole spiral is rotating about the axis *CD*, the rotational speed of the point *E* about the point *C* must be due to the height *u* [*i.e.* the speed relative to the lab reference frame, later to become *k* if constant]. But the right line *AB* shall be fixed, because clearly it shall not be moved with the cylinder [*i.e.* it represents the horizontal plane in the lab frame]: and at the beginning the point *E* was indeed at *A*, but thence with the time elapsed $= t$ by the circular motion it will arrive at *E*, and the arc shall be $AE = p$, and on account of the angular motion there will be $dp = dt\sqrt{u}$.

Now at first the motion of the water through the spiral may be considered as if quiescent [*i.e.* at rest relative to the rotating spiral, or working in the steady-state arrangement], and the speed of a particle of water at *Z* will be $= \sqrt{v}$ the direction of which will be *Zz,* which motion may be resolved into two parts, of which the first shall be along the direction *YZ* [*i.e.* parallel to the axis *CD*], the second along *Zu* or *Yy* [*i.e.* in the *xy*-plane perpendicular to the radius *CY*] , and the speed of the first $YZ = \sqrt{v}\cdot\sin\zeta$, indeed the speed of the second along *Zu* or *Yy* will be $= \sqrt{v}\cdot\cos\zeta$.

Now the rotary motion must be added to this latter motion, which of course points in the same direction, from which the whole speed of the point *Z* emerges along the direction $Yy = \sqrt{u} + \sqrt{v}\cdot\cos\zeta$.

Truly because the direction *Yy* is variable, that may be reduced to constant directions ; which at the end the perpendicular *YX* may be drawn from *Y* to the fixed line *AB*, and the three coordinates determining the position of the point *Z* may be called :

$$CX = x,\ XY = y \text{ and } YZ = z\,,$$

in the first place $z = s\,\mathrm{tang}\zeta$ *;* Then truly on account of the arc $AY = p + s$, and the angle $ACY = \frac{p+s}{a}$, there will be

$$CX = x = a\cos\frac{p+s}{a} \text{ and } XY = y = a\sin\frac{p+s}{a}.$$

Then with *Zu* [or *Yy*; for looking straight down, *Z* coincides with *Y* and *u* with *y*, so that *Zu* coincides with *Yy*:] drawn parallel to the right line *AB*, the angle *Yyu* will be

$$= \frac{p+s}{a}.$$

[There appears to be some mix-up in the labeling, so I have added the small triangle below *Yyu'* in the *xy* plane, for which the angle *yYu'* gives the correct ratio.] Hence the motion along *Yy* may be resolved into two others, the first along *Yu* or *AC*, of which the speed

$$= \left(\sqrt{u} + \sqrt{v}\cdot\cos\zeta\right)\sin\frac{p+s}{a}\,,$$

truly the second along *XY* of which the speed

$$= \left(\sqrt{u} + \sqrt{v}\cdot\cos\zeta\right)\cos\frac{p+s}{a}\,;$$

with the speed present along $YZ = \sqrt{v}\cdot\sin\zeta$.

Whereby in place of the point *Z* by being reduced to the three fixed coordinates, which are:

$$CX = x = a\cos\frac{p+s}{a},\ XY = y = a\sin\frac{p+s}{a} \text{ and } YZ = z = s\,\mathrm{tang}\zeta\,,$$

the true turning motion of the particle *Z* equally may be resolved along these three fixed directions, and there will be :

The speed of the motion along $CX = -\left(\sqrt{u} + \sqrt{v}\cdot\cos\zeta\right)\sin\frac{p+s}{a}\,,$

The speed of the motion along $XY = +\left(\sqrt{u} + \sqrt{v} \cdot \cos\zeta\right)\cos\dfrac{p+s}{a}$,

The speed of the motion along $YZ = \sqrt{v} \cdot \sin\zeta$.

[Note the left and down positive *x* and *y* axis sign convention. Perplexing, the end of the spiral shown in Fig. 1 will not scoop up water, as the spiral is wound in the wrong curvature for it to work in the anticlockwise sense of rotation described, although later the clockwise sense is adopted. We should be aware of the basic principle used here : essentially water is forced up an inclined plane, usually at a steady rate, so that the pressure at the base just overcomes the weight of the vertical head of water at any instant, ignoring all frictional forces; winding the spiral on the cylinder reduces the effective slope of the incline from an angle around 60 degrees in everyday use to one of around 5 degrees, as Daniel Bernoulli asserts in Ch. IX of his *Hydrodynamicae* ; a smaller pressure would not raise the water far enough to flow out from the top, while a greater pressure would insure the water to emerge with some velocity, *i.e*. the water is now moving along the spiral, instead of simply being carried by its rotation. Several forms of the device can be made, but the simplest possible is to be considered here.]

Corollary
1.

2. Hence the true speed of a particle of water turning at Z [*i.e.* in the lab reference frame but rotating within the spiral] may now be found easily, for since these three directions shall be normal to each other, the true speed will be the equal to the square root from the sum of the squares of these three speeds, from which the true speed will be

$$= \sqrt{\left(u + v + 2\sqrt{uv} \cdot \cos\zeta\right)}.$$

Corollary 2

3. Since a particle of water at Z shall arrive at the point z on the spiral in the small time dt, with

$$Zz = \frac{ds}{\cos\zeta} \text{ and } Yy = Zu = ds,$$

present, but the speed in the spiral shall be $= \sqrt{v}$, there will be $Zz = \dfrac{ds}{\cos\zeta} = dt\sqrt{v}$, from which besides now truly we may see that there is $dp = dt\sqrt{u}$.

Corollary 3

4. The speeds of the particle of water Z along the three fixed directions may be expressed also through the differentials of the coordinates x, y, z according to the element of time dt applied.

Clearly there will be from the nature of the resolved motion:

Speed along $CX = \frac{dx}{dt} = -\left(\sqrt{u} + \sqrt{v} \cdot \cos\zeta\right)\sin\frac{p+s}{a}$

Speed along $XY = \frac{dy}{dt} = \left(\sqrt{u} + \sqrt{v} \cdot \cos\zeta\right)\cos\frac{p+s}{a}$

Speed along $YZ = \sqrt{v} \cdot \sin\zeta$.

The identity of which formulas is understood from the differential values $dp = dt\sqrt{u}$ and $ds = dt\sqrt{v} \cdot \cos\zeta$.

PROBLEM 2

5. *With both the speed given, by which the water may be moved forwards through the spiral, as well as the speed, with which the cylinder is driven with the spiral rotating around the axis CD, to find the forces, by which some particle of water Z is required to be acted on, so that it may be able to describe the motion.*

Solution

Let the speed by which the water at the present instant of time may be moving through the spiral EZ, $= \sqrt{v}$, and moreover the speed of the rotating cylinder $= \sqrt{u}$. Then the beginning of the spiral shall be at E, so that there shall be $AE = p$, and the particle of water, which we may examine at Z, so that with ZY drawn parallel to the axis CD, the arc shall be $EY = s$, with the angle of the spiral arising $YEZ = \zeta$. Again the position of the point Z may be reduced to three fixed coordinates

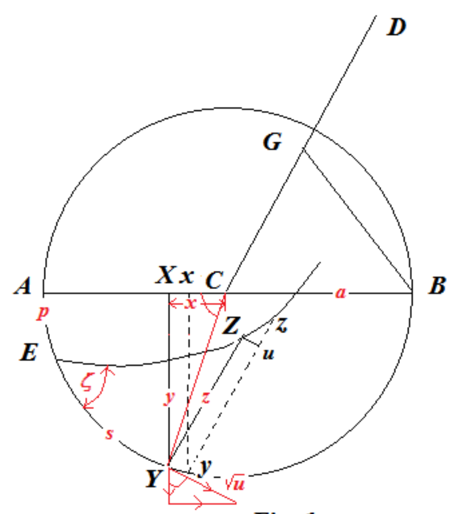


Fig. 1

$$CX = x,\ XY = y \text{ and } YZ = z;$$

there will be as we have seen:

$$x = a\cos\frac{p+s}{a},\ y = a\sin\frac{p+s}{a} \text{ and } z = s\,\mathrm{tang}\zeta$$

with a denoting the radius $CA = CB$ of the cylindrical base. Truly with the element of time put $= dt$, so that there shall be $dp = dt\sqrt{u}$ and $ds = dt\sqrt{v} \cdot \cos\zeta$, and with this differential dt assumed to be constant, it is agreed from the principles of mechanics that the particle of water at Z must be pressed upon by three accelerating forces, which shall be :

along $CX = \frac{2ddx}{dt^2}$, along $XY = \frac{2ddy}{dt^2}$, along $YZ = \frac{2ddz}{dt^2}$.

[The factor of 2 arises when one sets the acceleration due to gravity $= \frac{1}{2}$, leading to the simple height vs. speed squared *vis viva* relation above, but the mass in Newton's second law must then be doubled, if we are to equate mass with weight as the unit of force, as was the unfortunate state of affairs regarding units at this time; later in this paper, Euler confuses the issue by asserting that $\mathrm{g} = 1$; this leads one to suspect that the paper was started by Euler while Daniel Bernoulli was composing the text for his *Hydrodynamicae* at St. Petersbugh, which contains a dissertation on the Archimedes screw, but languished for some 30 years before its completion, due to differences of opinion which arose between the two scientists.]

Truly since from above there was shown to be :

$$\frac{dx}{dt} = -\left(\sqrt{u} + \sqrt{v}\cdot\cos\zeta\right)\sin\frac{p+s}{a},\ \frac{dy}{dt} = \left(\sqrt{u} + \sqrt{v}\cdot\cos\zeta\right)\cos\frac{p+s}{a},$$

and

$$\frac{dz}{dt} = \sqrt{v}\cdot\sin\zeta,$$

and by differentiation again

$$\frac{ddx}{dt^2} = -\left(\frac{du}{2dt\sqrt{u}} + \frac{dv\cos\zeta}{2dt\sqrt{v}}\right)\sin\frac{p+s}{a} - \frac{1}{a}\left(\sqrt{u} + \sqrt{v}\cdot\cos\zeta\right)^2\cos\frac{p+s}{a},$$

$$\frac{ddy}{dt^2} = -\left(\frac{du}{2dt\sqrt{u}} + \frac{dv\cos\zeta}{2dt\sqrt{v}}\right)\cos\frac{p+s}{a} - \frac{1}{a}\left(\sqrt{u} + \sqrt{v}\cdot\cos\zeta\right)^2\sin\frac{p+s}{a},$$

$$\frac{ddz}{dt^2} = \frac{dv}{2dt\sqrt{v}}\sin\zeta.$$

Therefore the three accelerations sought are

I. along $CX = -\frac{1}{dt}\left(\frac{du}{\sqrt{u}} + \frac{dv\cos\zeta}{\sqrt{v}}\right)\sin\frac{p+s}{a} - \frac{2}{a}\left(\sqrt{u} + \sqrt{v}\cdot\cos\zeta\right)^2\cos\frac{p+s}{a}$,

II. along $XY = +\frac{1}{dt}\left(\frac{du}{\sqrt{u}} + \frac{dv\cos\zeta}{\sqrt{v}}\right)\cos\frac{p+s}{a} - \frac{2}{a}\left(\sqrt{u} + \sqrt{v}\cdot\cos\zeta\right)^2\sin\frac{p+s}{a}$,

III. along $YZ = \frac{dv}{dt\sqrt{v}}\sin\zeta$.

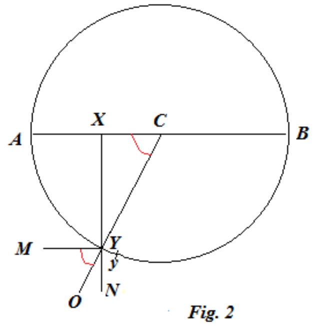


Fig. 2

Corollary 1

6. The first two forces may be transferred to the point *Y* (Fig. 2), thus so that this point may be pressed on by two accelerating forces, along the directions *YM* and *YN*, which are

$$\text{Force along } YM = -\frac{1}{dt}\left(\frac{du}{\sqrt{u}}+\frac{dv\cos\zeta}{\sqrt{v}}\right)\sin\frac{p+s}{a}-\frac{2}{a}\left(\sqrt{u}+\sqrt{v}\cdot\cos\zeta\right)^{2}\cos\frac{p+s}{a},$$

$$\text{Force along } YN = +\frac{1}{dt}\left(\frac{du}{\sqrt{u}}+\frac{dv\cos\zeta}{\sqrt{v}}\right)\cos\frac{p+s}{a}-\frac{2}{a}\left(\sqrt{u}+\sqrt{v}\cdot\cos\zeta\right)^{2}\sin\frac{p+s}{a}.$$

Corollary 2

7. Now these two forces are able to be transformed into these two forces, which act along the directions *Yy* and *YO*, the latter of which shall be normal to the surface of the cylinder: and on account of the angle $MYO = ACY = \frac{p+s}{2}$, from these two forces there will result :

$$\text{I. Force along } Yy = \text{Force } YN\cdot\cos\frac{p+s}{2}-\text{Force } YM\cdot\sin\frac{p+s}{2},$$

$$\text{II. Force along } YO = \text{Force } YN\cdot\sin\frac{p+s}{2}+\text{Force } YM\cdot\cos\frac{p+s}{2}.$$

Corollary 4

8. Hence therefore in place of the two forces, which were acting along the directions *CX* and *XY*, or *YM* and *YN*, these two other forces may be introduced into the calculation along the directions *Yy* and *YO*, which will be

$$\text{Force along [tangent] } Yy = +\frac{1}{dt}\left(\frac{du}{\sqrt{u}}+\frac{dv\cos\zeta}{\sqrt{v}}\right),$$

$$\text{Force along [normal] } YO = -\frac{2}{a}\left(\sqrt{u}+\sqrt{v}\cdot\cos\zeta\right)^{2},$$

and thus the angle $p+s$ is no longer found in the calculation.

PROBLEM 3

9. *To reduce the three forces found before to three other forces, one of which shall be along the direction of the spiral Zz, and truly the other two shall be normal to the spiral itself.*

Solution

Zz shall be (Fig. 3) an element of the spiral, where a particle of water now may be turning, which the forces found may sustain : and *Zo* not only shall be normal to the spiral *Zz*, but also to the surface of the cylinder itself at *Z*, then there shall be the right line *Zr* situated on the surface of the cylinder itself, and normal to *Zz*. Therefore the three forces found must be reduced to three other forces, which act on the particle of water along the directions *Zz*, *Zo* and *Zr* [*i.e.*, *Zz* along the spiral on the cylinder, *Zo* normal to the spiral and cylinder, and *Zr* normal to the spiral but on the cylinder surface.] And indeed the force found acting at first along *YZ*

$$=\frac{dv}{dt\sqrt{v}}\sin\zeta,$$

on account of the angle of the spiral $YEZ=\zeta$, will give

I. force along $\quad Zr=-\dfrac{dv}{dt\sqrt{v}}\sin\zeta\cos\zeta,$

II. force along $\quad Zz=+\dfrac{dv}{dt\sqrt{v}}\sin\zeta\sin\zeta,$

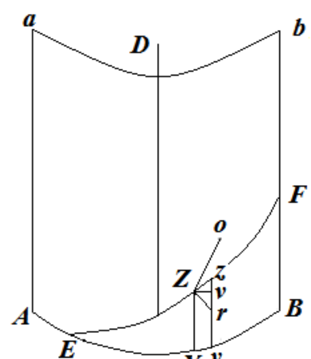


Fig. 3

From that the force, which was found to act along the direction *Yy* or *Zv* is

$$=\frac{1}{dt}\left(\frac{du}{\sqrt{v}}+\frac{dv\cos\zeta}{\sqrt{v}}\right),$$

will give the forces

I. along $\quad Zz=\dfrac{du}{dt\sqrt{u}}\cos\zeta+\dfrac{dv}{dt\sqrt{v}}\cos^2\zeta,$

II. along $\quad Zr=\dfrac{du}{dt\sqrt{u}}\sin\zeta+\dfrac{dv}{dt\sqrt{v}}\sin\zeta\cos\zeta.$

The third force, which acts along the direction *YO* has been found, now will give the force alone along $Zo = -\frac{2}{a}\left(\sqrt{u} + \sqrt{v} \cdot \cos\zeta\right)^2$.

Whereby the three accelerative forces, by which the particle of water must be disturbed at *Z*, so that it may pursue the proposed motion, will be :

I. following the direction Zz $\quad = \frac{du}{dt\sqrt{u}}\cos\zeta + \frac{dv}{dt\sqrt{v}}$,

II. following the direction Zr $\quad = \frac{du}{dt\sqrt{u}}\sin\zeta$,

III. following the direction Zo $\quad = -\frac{2}{a}\left(\sqrt{u} + \sqrt{v} \cdot \cos\zeta\right)^2$.

[The results for *Zz* and *Zr* can be arrived at from Fig. 1, by resolving the velocity $\sqrt{u}$ due to the base rotating into components $\sqrt{u}\cos\zeta$ and $\sqrt{u}\sin\zeta$ parallel and normal to the spiral and subsequently differentiated, and adding the force along *Zz* due to the spiral rotation, in the plane of the spiral; the last term is of course the centripetal force .]

Scholium 10.

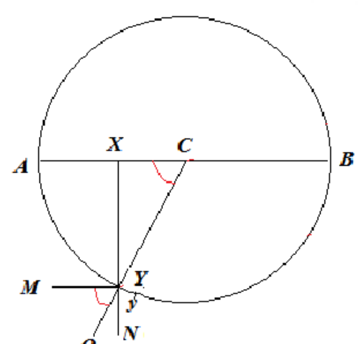


Fig. 2

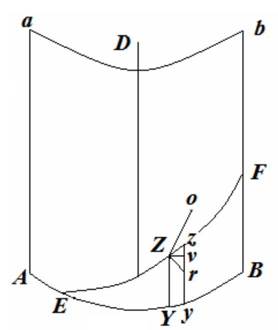


Fig. 3

Therefore we have the forces, by which the individual particles of water must be disturbed, so that the motion, which we have assumed, may be able to be supported. But these forces themselves thus I have referred here to the three directions *Zz*, *Zr* et *Zo*, so that they may be compared more easily with the forces, by which the water in the tube may actually be disturbed; indeed so that the quantities *v* and *u* may show the true motion of the water and the cylinder, it is necessary, that these three forces found may agree with the forces, by which the water may actually be pressed [*i.e.* put under pressure]. But these forces are in the first place the state of compression of the water in the tube, then the pressure of the water on the sides of the tube, which is usually shown along both the directions *Zr* and *Zo* normal to the direction of the tube. Truly the third being the weight, by which individual particles of water are pressed downwards, which is subjected in the first place to an examination, which we will put in place in the following problem.

PROBLEM 4

11. *If the cylinder were inclined somehow to the horizontal, to define the forces along the three preceding directions, by which the individual particle of water Z in the spiral may be acted on by gravity.*

Solution

The angle (Fig. 4) θ expresses the inclination of the base of the cylinder to the horizontal, and in the plane of the base the fixed point *A* the highest, truly the point *B* the lowest, thus so that the right line *AB* may be put in place with the axis of the cylinder *CD* in the vertical plane. In this plane through the centre of the base *C* the horizontal line may be drawn *CH,* and the angle will be given $ACH = \theta$, or if from the point *B* the vertical right line *BG* may be erected intersecting the axis in *G*, also the angle $BGC = \theta$, Fig. 4 and on account of the weight the individual particles of water will be disturbed downwards along directions parallel to *GB* itself, and this acceleration force everywhere will be $=1$. [Recall that the acceleration of gravity has now been take as 1 at this stage, indicating perhaps the point where Euler resumed work on this paper; and we may assume a mass of 1 for the element of volume.] Now in the first figure, also the right line *BG* may be drawn making the angle $BGC = \theta$ with the axis *CD*, and a particle of water at *Z* will be pressed by an accelerating force $=1$ following the direction parallel to the right line *BG*. This force may be resolved along the directions *GC* and *CB,* and there will be produced

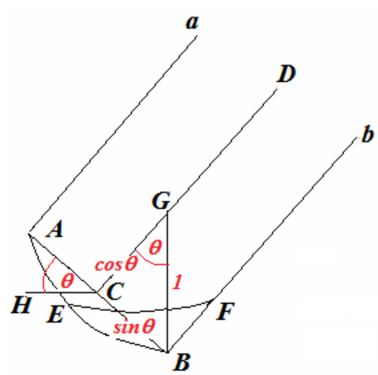


Fig.4

$$\text{force along } GC = 1 \cdot \cos\theta\,,$$

and

$$\text{force along } CB = 1 \cdot \sin\theta.$$

From the former, for the particle of water *Z* (Fig. 2) we will have the force along $ZY = \cos\theta$, truly from the latter the force along $YM = -\sin\theta$, from which on account of the angle

$$MYO = \frac{p+s}{a}$$

there arises

the force along YO $\quad = -\sin\theta\cos\dfrac{p+s}{a}$

and

the force along Yy $\quad = +\sin\theta\sin\dfrac{p+s}{a}.$

Hence therefore (Fig. 1 & Fig. 3) the point Z will be acted on by these three accelerative forces :

I. force [down the axis] along the direction $ZY = \cos\theta$,

II. force along the direction Zo [normal to surface] $= -\sin\theta \cos\dfrac{p+s}{a}$,

III. force along the direction Zv [parallel to base] $= +\sin\theta \sin\dfrac{p+s}{a}$.

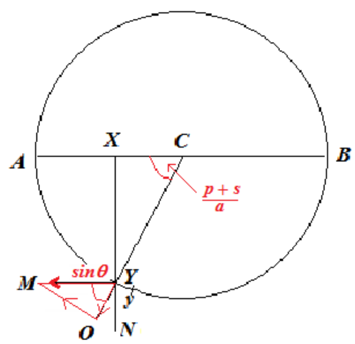


Fig. 2

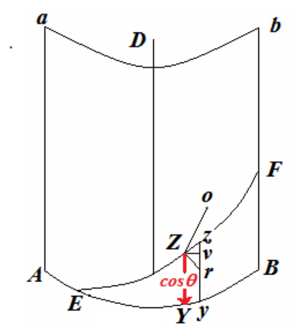


Fig. 3

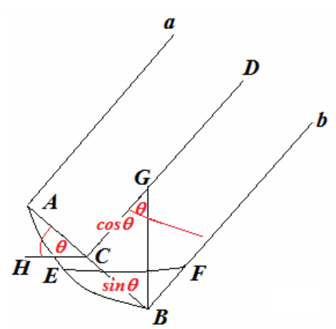


Fig.4

[I. is apparent at once; however to understand II. and III. it is required to consider point on the spiral at the angle $\dfrac{p+s}{a}$ to the initial starting point on the cylinder, as in Fig. 3 above, (which may be at the highest or lowest point of the base circle); there is the constant force ZY acting on the spiral parallel to the axis of cylinder, corresponding to $\cos\theta$, and a normal force Zv given by $\sin\theta$ whose direction depends on the angle of the spiral at that point: at the lowest point, the weight of a droplet in the spiral acts away from the surface of the cylinder, while at the highest point a similar force acts into the surface of the cylinder: hence the negative sign appearing for II; at the points corresponding to the horizontal, the whole force $\sin\theta$ acts along the tangent to the cylinder, and so the normal component is zero.]

Again from these on account of the angle $zZv = \zeta$ there will arise:
First the force along $Zz =$ force $Zv \cdot \cos\zeta -$ force $ZY \cdot \sin\zeta$.
Then the force along $Zr =$ force $Zv \cdot \sin\zeta +$ force $ZY \cdot \cos\zeta$.
Whereby for the three directions Zz, Zr and Zo we will obtain the following accelerative forces arising from gravity:

I. Force along $Zz = \cos\zeta \sin\theta \sin\frac{p+s}{a} - \sin\zeta \cos\theta$,

II. Force along $Zr = \sin\zeta \sin\theta \sin\frac{p+s}{a} + \cos\zeta \cos\theta$,

III. Force along $Zo = -\sin\theta \cos\frac{p+s}{a}$.

PROBLEM 5

12. *With* (Fig. 3) *given, in order that at this point in time, to define the state of compression of the water at the individual points of the spiral, both for the motion of the cylinder as well as of the water moving through the spiral.*

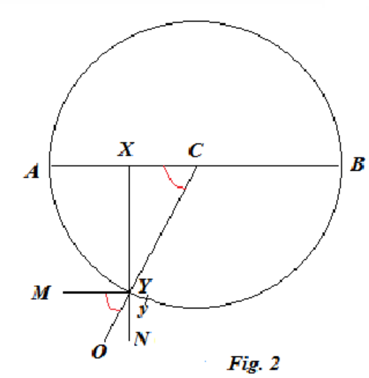


Fig. 2

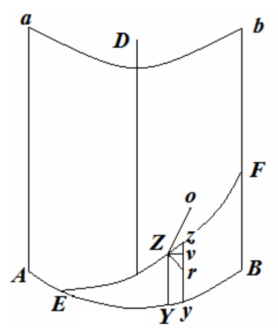


Fig. 3

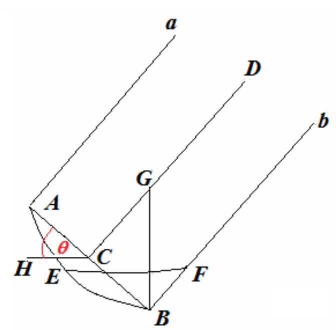


Fig.4

Solution

At the present instant of time, when the start of the spiral is at *E*, with the arc arising $AE = p$, we may consider the point *Z* of the helix, so that there shall be

$$EY = s \text{ and } YZ = s\,\text{tang}\,\zeta$$

with the angle of the spiral arising $YEZ = \zeta$, and the state of compression of the water at the point Z shall be equal to *q* [*i.e.* the pressure], or *q* may denote the depth, according to which water may exist in an equal state of compression at rest, and *q* will be some function of *s* for this amount, and at some nearby point *z*, with $Yy = ds$ arising, the state of compression will be $= q + dq$ .Now the cross-section of the spiral shall be $= hh$, the [volume or mass for unit density of the] particle of water contained in the small portion *Zz* will be

$$= \frac{hhds}{\cos\zeta};$$

which therefore may be propelled at *Z* by the motive force $=hhq$, at *z* therefore it may be repelled by a force

$$=hh(q+dq);$$

from which the repelling motive force arises, or acting along *Zz* $=hhdq$, which provides the accelerative force [*i.e.* force per unit mass]

$$=\frac{dq\cos\zeta}{ds}.$$

[*i.e.* $\frac{hhds}{\cos\zeta}\times acc.=hhdq \therefore acc.=\frac{dq\cos\zeta}{ds}:$ the acceleration down the slope due to the water pressure along the spiral.]

Whereby on account of the state of compression, a particle of water contained in the element of the spiral *Zz* is acted on by a force along the direction *Zz* with the acceleration

$$=-\frac{dq\cos\zeta}{ds}.$$

Truly in addition on account of the weight, as we have seen, the same particle is acted on by an accelerative force along *Zz*

$$=\cos\zeta\sin\theta\sin\frac{p+s}{a}-\sin\zeta\cos\theta,$$

from which jointly both on account of gravity, as well as on account of the state of compression of the water, the particle of water contained in the spiral at the point *Z* will be forced along the direction *Zz* by an accelerative force, which will be

$$=\cos\zeta\sin\theta\sin\frac{p+s}{a}-\sin\zeta\cos\theta-\frac{dq\cos\zeta}{ds},$$

and this is the force, by which this particle itself actually is pushed along in the direction *Zz* ; from which it is necessary, as it shall be equal to that force, by which the above point *Z* must be acted on to conserve the motion we have found along the same direction *Zz.* Since which shall be found

$$=\frac{du}{dt\sqrt{u}}\cos\zeta+\frac{dv}{dt\sqrt{v}},$$

we will have this equation:

$$dq\cos\zeta=ds\cos\zeta\sin\theta\sin\frac{p+s}{a}-ds\sin\zeta\cos\theta-\frac{du}{dt\sqrt{u}}ds\cos\zeta-\frac{dv}{dt\sqrt{v}}ds,$$

where, because at present we consider only an instant of time, the amounts depending on the time *t*, which are $p, u, v,$ and likewise just as $\frac{du}{dt}$ and $\frac{dv}{dt}$ are considered as constants,

from which with the integration put in place we will have

$$q\cos\zeta = C - s\cos\zeta\sin\theta\cos\frac{p+s}{a} - s\sin\zeta\cos\theta - \frac{sdu\cos\zeta}{dt\sqrt{u}} - \frac{sdv}{dt\sqrt{v}},$$

from which the state of compression of the water becomes known at the individual points of the spiral at the present instant of time.

PROBLEM 6

13. *If some given amount of water may be found in the spiral and the cylinder holding a given inclination to the horizontal is driven in some rotational motion, to find the motion, by which that portion of water will be moved through the spiral.*

Solution

The radius of the base of the cylinder (Fig. 5 : see the Latin text for the diagram free of added red lines and angles, etc.) shall be $CA = CB = a$, and the angle, which the spiral *EF* makes with the base of the cylinder $BEF = \zeta$. Moreover the axis of the cylinder *PQ* shall make an angle $PQR = \theta$ to the vertical *QR*, by which same angle the base of the cylinder

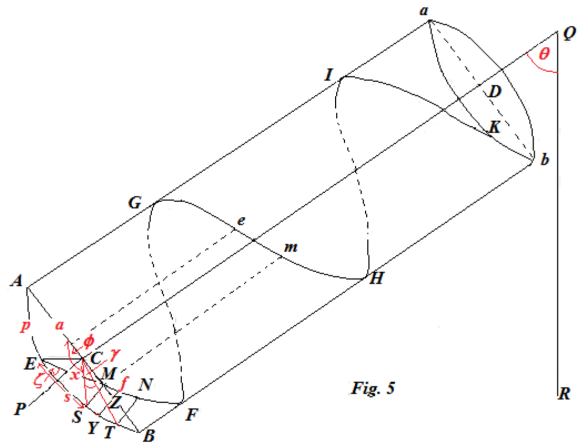


Fig. 5

will be inclined to the horizontal. Again, *A* shall be the uppermost point and *B* the lowest point in the base. But at the present instant of time the start of the spiral shall be at *E*, with its separation or arc $AE = p$ arising from the uppermost point *A* : and the cylinder shall be turning in the direction *AEB* [*i.e.* anticlockwise], thus so that the speed of the point E shall be equal to $=\sqrt{u}$, and there will be $dp = dt\sqrt{u}$. Now a part of the water held in the spiral occupies the interval *MN*, whose length shall be $MN = f$, and with the parallels *MS* and *NT* drawn to the axis the distance of the water from the start of the spiral shall be $EM = x$, there will be

$$EN = x + f,\quad ES = x\cos\zeta \text{ and } ET = (x+f)\cos\zeta;$$

truly the speed, by which this part of the water at the present moment may be moving through the spiral, $=\sqrt{u}$. With these in place, if in the portion of the water *MN* a certain mid-point may be considered *Z* and the arc *EY* may be put $=s$, the state of the compression of the water at *Z*, which may be expressed by the height *q*, has been elicited as in the previous problem:

$$q\cos\zeta = C - a\cos\zeta\sin\theta\cos\frac{p+s}{a} - s\sin\zeta\cos\theta - \frac{sdu\cos\zeta}{dt\sqrt{u}} - \frac{sdv}{dt\sqrt{v}}.$$

Now truly it is agreed that at each end *M* and *N* the state of the compression must vanish ; therefore to make $q=0$ there may be put either $s = x\cos\zeta$ or $s=(x+f)\cos\zeta$: from which the two-fold equations arises :

$$0 = C - a\cos\zeta\sin\theta\cos\frac{p+x\cos\zeta}{a} - x\sin\zeta\cos\theta$$
$$- x\cos\zeta\left(\frac{du\cos\zeta}{dt\sqrt{u}} + \frac{dv}{dt\sqrt{v}}\right),$$

$$0 = C - a\cos\zeta\sin\theta\cos\frac{p+(x+f)\cos\zeta}{a}$$
$$-(x+f)\cos\zeta\left(\sin\zeta\cos\theta + \frac{du\cos\zeta}{dt\sqrt{u}} + \frac{dv}{dt\sqrt{v}}\right),$$

from which, with the constant *C* being eliminated, this equation will be obtained, on dividing by $\cos\zeta$:

$$a\sin\theta\cos\frac{p+x\cos\zeta}{a} = a\sin\theta\cos\frac{p+(x+f)\cos\zeta}{a} + f\sin\zeta\cos\theta + \frac{fdu\cos\zeta}{dt\sqrt{u}} + \frac{fdv}{dt\sqrt{v}},$$

from which the motion of the water along the spiral must be defined, as indeed there is $dp = dt\sqrt{u}$, thus there will be $dx = dt\sqrt{v}$.

This equation may be multiplied by

$$dp + dx\cos\zeta = dt\sqrt{u} + dt\cos\zeta\cdot\sqrt{v},$$

and by integrating it becomes :

$$a^2\sin\theta\sin\frac{p+x\cos\zeta}{a} = a^2\sin\theta\sin\frac{p+(x+f)\cos\zeta}{a} + f\left(p+x\cos\zeta\right)\sin\zeta\cos\theta$$
$$+ f\int\left(\frac{du\cos\zeta}{\sqrt{u}} + \frac{dv}{\sqrt{v}}\right)\left(\sqrt{u}+\cos\zeta\cdot\sqrt{v}\right).$$

Corollary 1

14. Therefore if the rotational motion of the cylinder were uniform, or *u* constant, there may be put $u = k$, on account of $du = 0$ there will be

$$a^2 \sin\theta \sin\frac{p + x\cos\zeta}{a} = a^2 \sin\theta \sin\frac{p + (x + f)\cos\zeta}{a} + f\,(p + x\cos\zeta)\sin\zeta\cos\theta$$
$$+ 2f\sqrt{kv} + fv\cos\zeta + \text{Const.}$$

where there is $p = t\sqrt{k}$, thus so that this equation on account of $\sqrt{v} = dx$ may involve only the two variables *t* and *x*. But the constant must be defined from the initial state.

Corollary 2

15. If the part of the water in the tube *MN* were infinitely small $f = 0$, there will be

$$\sin\frac{p + (x + f)\cos\zeta}{a} = \sin\frac{p + x\cos\zeta}{a} + \frac{f\cos\zeta}{a}\cos\frac{p + (x + f)\cos\zeta}{a}$$

therefore in this case the motion may be defined by this equation :

$$\text{Const.} = a\cos\zeta\sin\theta\cos\frac{p + x\cos\zeta}{a} + (p + x\cos\zeta)\sin\zeta\cos\theta + 2\sqrt{kv} + v\cos\zeta.$$

Because if therefore this particle initially were at rest at *E*, and the point *E* were at *A*, thus on putting $x = 0$ there shall be $p = 0$ and $v = 0$, there will be

$$a\cos\zeta\sin\theta\left(1 - \frac{\cos p + x\cos\zeta}{a}\right) = (p + x\cos\zeta)\sin\zeta\cos\theta + 2\sqrt{kv} + v\cos\zeta.$$

Corollary 3

16. If in this case of the preceding corollary the angle may be put

$\dfrac{p + x\cos\zeta}{a} = \varphi,$ so that there shall be $dt = \dfrac{ad\varphi}{\sqrt{k} + \cos\varphi\cdot\sqrt{v}}$, on account of $dp = dt\sqrt{k}$ and $dx = dt\sqrt{v}$, the relation between φ and *v* may be expressed by this equation:

$$a\cos\zeta\sin\theta(1 - \cos\varphi) = a\varphi\sin\zeta\cos\theta + 2\sqrt{kv} + v\cos\zeta$$

from which there becomes [from the quadratic in $\sqrt{v}$] :

$$\sqrt{k}+\cos\zeta\cdot\sqrt{v}=\sqrt{\left(k-a\,\varphi\sin\zeta\cos\zeta\cos\theta+a\cos^2\zeta\sin\theta\left(1-\cos\varphi\right)\right)}$$

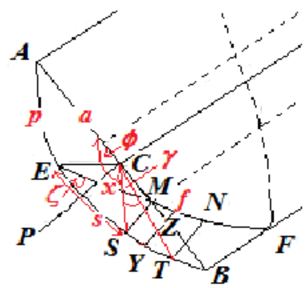
and thus

$$dt=\frac{ad\varphi}{\sqrt{\left(k-a\varphi\sin\zeta\cos\zeta\cos\theta+a\cos^2\zeta\sin\theta\left(1-\cos\varphi\right)\right)}}.$$

Corollary 4

17. If generally in a similar manner, on still considering the rotational motion to be constant or $u=k$, there may be put

$$\frac{p+x\cos\zeta}{a}=\varphi \text{ and } \frac{f\cos\zeta}{a}=\gamma,$$



there will be also

$$dt=\frac{ad\varphi}{\sqrt{k}+\cos\zeta\cdot\sqrt{v}}$$

and

$$\frac{a\cos\xi\sin\theta}{\gamma}\sin\varphi=\frac{a\cos\xi\sin\theta}{\gamma}\sin\left(\gamma+\varphi\right)+a\varphi\sin\zeta\cos\theta+2\sqrt{kv}+v\cos\zeta+C$$

and thus

$$\sqrt{k}+\cos\zeta\cdot\sqrt{v}=\sqrt{\left(C+\frac{a}{\gamma}\cos^2\zeta\sin\theta\left(\sin\varphi-\sin\left(\gamma+\varphi\right)\right)-a\varphi\sin\zeta\cos\zeta\cos\theta\right)},$$

from which there becomes

$$dt=\frac{ad\varphi}{\sqrt{\left(C+\frac{a}{\gamma}\cos^2\zeta\sin\theta\left(\sin\varphi-\sin\left(\gamma+\varphi\right)\right)-a\varphi\sin\zeta\cos\zeta\cos\theta\right)}},$$

where φ indicates the angle *ACS* and γ the angle *SCT*, which is constant.

Corollary 5

18. If the cylinder may be driven around in the opposite direction with a speed $=\sqrt{k}$ [*i.e.* clockwise with a constant speed as seen from the base, so that any water present falls through the spiral], for $\sqrt{k}$ there must be written $-\sqrt{k}$, and the arc p will be required to be taken negative, thus so that there shall be [there is a misprint here in the original, $\varphi = x \cdot \cos\zeta$ rather than :]

$$\frac{p - x\cos\zeta}{a} = \varphi .$$

Whereby since there shall be

$$p > x \cdot \cos\zeta ,$$

also the angle φ may be taken negative, therefore we will have for this motion:

$$\varphi = \frac{p - x\cos\zeta}{a} \text{ and } dt = \frac{ad\varphi}{\sqrt{k} - \cos\zeta \cdot \sqrt{v}}$$

and

$$\sqrt{k} - \cos\zeta \cdot \sqrt{v} = \sqrt{\left(C - \frac{a}{\gamma}\cos^2\zeta \sin\theta \left(\sin\varphi - \sin\ (\varphi - \gamma) \right) + a\varphi \sin\zeta \cos\zeta \cos\theta \right)} .$$

Corollary 6

19. If in this case initially $t = 0$, so that there was $p = 0$ and $\sqrt{v} = 0$, there will be $x = EM = g$, and thus $\varphi = -\frac{g\cos\zeta}{a}$, we may put this initial angle $ECS = \varepsilon$, as initially there would be $\varphi = -\varepsilon$, there will be

$$\sqrt{k} - \cos\zeta \cdot \sqrt{v} =$$

$$\sqrt{\left(k + \frac{a}{\gamma}\cos^2\zeta \sin\theta \left(\sin\ (\varepsilon + \gamma) - \sin\varepsilon - \sin\varphi + \sin(\varphi - \gamma) \right) + a(\varepsilon + \varphi) \sin\zeta \cos\zeta \cos\theta \right)} .$$

PROBLEM 7

20. *If, while the cylinder is turning with a given speed uniformly in the direction BEA, a particle or globule of water may be inserted into the spiral at E, which then may be moved away by the motion of the cylinder, to determine the motion of the globule through the spiral.*

Solution

The speed, by which the point *E* of the cylinder is driven to rotate in the sense *EA* shall be $\sqrt{k}$ [*i.e.* clockwise], and therefore at that instant, when the globule is inserted into the opening of the spiral *E*, the angle $ACE = \alpha$ and $t = 0$. But it cannot happen, that the initial speed of the globule shall be $= 0$; for if its speed with respect to the tube along *EM* may be put $= \sqrt{v}$, its true speed will be

$$\sqrt{(k + v - 2\cos\zeta \cdot \sqrt{kv})},$$

which cannot vanish. Therefore we may put this initial speed to be a minimum, and we find $\sqrt{v} = \cos\zeta \cdot \sqrt{k}$, thus so that the true speed will be as $= \sin\xi \cdot \sqrt{k}$, the direction of which shall be normal to the direction *EM*. Now with the time elapsed *t*, there shall be as above $AE = p$; truly the globule may be found at *M* with $EM = x$ arising, the speed of which relative to the tube shall be $= \sqrt{v}$ along the tube *MN*, there will be

$$dp = -dt\sqrt{k} \quad \text{and} \quad dx = dt\sqrt{v}$$

and by paragraph 15 the motion may be defined by this equation, evidently with the speed $\sqrt{k}$ taken negative :

$$\text{Const.} = a\cos\zeta\sin\theta\cos\frac{p + x\cos\zeta}{a} + \left(p + x\cos\zeta\right)\sin\zeta\cos\theta - 2\sqrt{kv} + v\cos\zeta.$$

Moreover the constant is required to be defined thus, so that on putting $t = 0$ or $\frac{p}{a} = \alpha$ there becomes

$$x = 0 \text{ and } \sqrt{v} = \cos\zeta \cdot \sqrt{k},$$

and thus there will be

$$\text{Const} = a\cos\zeta\sin\theta\cos\alpha + \alpha a\sin\zeta\cos\theta - 2k\cos\zeta + k\cos^3\zeta .$$

The angle $ACS = \varphi$ may be put in place, there will be

$$\varphi = \frac{p + x\cos\zeta}{a} \quad \text{and} \quad d\varphi = \frac{-dt\sqrt{k} + dt\cos\zeta \cdot \sqrt{v}}{a} .$$

Moreover, the cylinder may be set in motion with the angle $=\omega$ at the time *t* in the direction *BEA*, there will be

$$d\omega=\frac{dt\sqrt{k}}{a} \text{ and } \omega=\frac{t\sqrt{k}}{a},$$

we may introduce that angle in place of the time as its just measure, there will be

$$\frac{p}{a}=\alpha-\omega,\ \varphi=\alpha-\omega+\frac{x\cos\zeta}{a};$$

and on account of

$$dx=dt\sqrt{v}=\frac{ad\omega\sqrt{v}}{\sqrt{k}}$$

we will have

$$d\varphi=-d\omega+\frac{d\omega\cos\zeta\cdot\sqrt{v}}{\sqrt{k}}$$

or

$$d\omega=\frac{d\varphi\sqrt{k}}{-\sqrt{k}+\cos\zeta\cdot\sqrt{v}}.$$

Moreover our equation will be

$$\begin{aligned}&a\cos\zeta\sin\theta\cos\alpha+\alpha a\sin\zeta\cos\theta-2k\cos\zeta+k\cos^3\zeta\\=&a\cos\zeta\sin\theta\cos\varphi+a\varphi\sin\zeta\cos\theta-2\sqrt{kv}+v\cos\zeta,\end{aligned}$$

from which we obtain:

$$\cos\zeta\cdot\sqrt{v}-\sqrt{k}=-\sqrt{\left(k\sin^4\zeta+a\cos^2\zeta\sin\theta\left(\cos\alpha-\cos\varphi\right)+a\left(\alpha-\varphi\right)\sin\zeta\cos\zeta\cos\theta\right)}$$

from which for the given value of φ we may elicit the value of $\sqrt{v}$ itself, from which there will be found

$$d\omega=\frac{-d\varphi\sqrt{k}}{\sqrt{\left(k\sin^4\zeta+a\cos^2\zeta\sin\theta\left(\cos\alpha-\cos\varphi\right)+a\left(\alpha-\varphi\right)\sin\zeta\cos\zeta\cos\theta\right)}},$$

whose integral must be taken thus, so that by putting $\omega=0$ there becomes $\varphi=\alpha$. Therefore from this integral equation in turn according to the angle expressed in the give time ω, the angle φ may be found and from that again the position of the globule on the spiral or the part

$$EM = x = \frac{a(\varphi - \alpha + \omega)}{\cos\zeta}$$

and of which in addition the speed relative to the spiral $\sqrt{v}$ is clearly :

$$\sqrt{v} = \frac{\sqrt{k}}{\cos\zeta} - \sqrt{\left(k\sin^2\zeta\tan^2\zeta + a\sin\theta(\cos\alpha - \cos\varphi) + a(\alpha - \varphi)\tan\zeta\cos\theta\right)}.$$

Corollary 1

21. The expression $\cos\zeta\sqrt{v} - \sqrt{k}$ indicates the true speed of the point *S* [relative] to the base, which corresponds to the globe at *M*. For since the globule can progress with the velocity $\sqrt{v}$ along *MN* in the spiral, its rotational speed about the axis $= \cos\zeta \cdot \sqrt{v}$, with respect to the spiral ; but because the spiral itself nevertheless is turning in the opposite direction with the speed $\sqrt{k}$ [in this relative sense], the true rotational speed of the globule , or the motion by which the point *S* recedes from the highest point *A* will be $= cos\zeta \cdot \sqrt{v} - \sqrt{k}$.

Corollary 2

22. But for the initial motion itself, for which $\sqrt{v} = \cos\zeta \cdot \sqrt{k}$; this speed was negative, clearly

$$= \left(\cos^2\zeta - 1\right)\sqrt{k} = -\sin^2\zeta \cdot \sqrt{k},$$

therefore from the beginning at once even now it will be negative or the angle $ACS = \varphi$ will be diminished, which is the reason, why the calculation for $cos\zeta\cdot\sqrt{v} - \sqrt{k}$ will have provided a negative value [*i.e.* the walls of the spiral are coming towards the globule in this relative motion]

$$\cos\zeta \cdot \sqrt{v} - \sqrt{k} = -\sqrt{\left(k\sin^4\zeta + a\cos^2\zeta\sin\theta(\cos\alpha - \cos\varphi) + a(\alpha - \varphi)\sin\zeta\cos\zeta\cos\theta\right)}$$

Therefore here the value is unable to be taken to change into positive or the angle $ACS = \varphi$ cannot be taken to increase until after the value of that root were $= 0$. But after this comes about, then the value of the sign of that root will be required to be taken as positive

Corollary 3

23. But since from the beginning the angle φ decreases to such an extent with time, while the value of the magnitude of the root diminishes, and at that point φ may be diminished beyond α or there shall be $\varphi < \alpha$. Therefore there may be put $\varphi = \alpha - \psi$, so that there shall be

$$\cos\zeta \cdot \sqrt{v} - \sqrt{k} = -\sqrt{\left(k\sin^4\zeta + a\cos^2\zeta\sin\theta\left(\cos\alpha - \cos\left(\alpha - \psi\right)\right) + a\psi\sin\zeta\cos\zeta\cos\theta\right)}$$

and thus as long as by increasing the value of ψ itself, that value of the root holds its real value, just as long the globule may be dragged along by the motion of the cylinder in the direction *BEA* nor earlier changed its motion into the opposite direction, as where ψ will have increased at that point, so that there shall be

$$k\sin^4\zeta + a\cos^2\zeta\sin\theta\left(\cos\alpha - \cos\left(\alpha - \psi\right)\right) + a\psi\sin\zeta\cos\zeta\cos\theta = 0.$$

Corollary 4

24. But because by increasing ψ the furthest end continually increases, truly the middle part which only is negative:

$$-a\cos^2\zeta\sin\theta\left(\cos(\alpha - \psi) - \cos\alpha\right)$$

while it increases to where there becomes $\psi = \alpha$ or $\varphi = 0$, it is evident, unless that formula may be changed into nothing, before there arises $\psi = \alpha$,that globule at no point is going to be vanishing and is going to be continually carried along more quickly following the motion of the rotating cylinder. Therefore in this case the point *S* will be continually turning more quickly in the direction *BEA*.

Corollary 5

25. Therefore if this magnitude of the root may be put $= V$, so that there becomes

$$cos\zeta \cdot \sqrt{v} - \sqrt{k} = -V \text{ or } \sqrt{v} = \frac{\sqrt{k} - V}{cos\zeta},$$

on account of the value of *V* in this case the speed of the globule continually increases in the spiral along its direction *EMN* finally may vanish, and after that it may become negative, because when it happens, the globule will change direction through the spiral and emerge again through the opening *E*; if indeed the cylinder were long, so that the globule may not emerge through the upper end of the spiral *K*, before it may revert.

[It may be helpful to put the following in place :

V is the speed of the globule relative to the spiral across a normal cross-section of the cylinder; it may be positive or negative or zero, depending on whether the globule is moving out or in along the spiral, or being carried along by the spiral's motion.

$\sqrt{v}$ is the speed associated with the motion of some salient feature such as the uppermost point traveling along the spiral parallel to the axis ; each point fixed on the spiral executes a simple harmonic motion in the plane normal to the axis according to the rate the handle is turned, with constant phase differences between neighbouring points on the spiral, the pitch being the distance along the axis for which the phase difference is one revolution, giving the illusion of a wave translation when the handle is turned: thus the motion of the spiral appears like a wave moving along it, which can of course carry along a free particle, which is continually being pushed up and rolling along.
Note however, as we have seen, that both clockwise and anticlockwise rotations of the spiral are allowed, and indeed a spiral can be used both in the clockwise and anticlockwise sense, depending on which end is lower and used to introduce the water.

$\sqrt{k}$ is the constant speed of a point along the tangent of the circular normal cross-section of the cylinder relative to the lab frame.]

Scholium

26. Since on putting $\varphi = \alpha - \psi$ and

$$V = \sqrt{\left(k\sin^4\zeta - a\cos^2\zeta\sin\theta\left(\cos(\alpha-\psi) - \cos\alpha\right) + a\psi\sin\zeta\cos\zeta\cos\theta\right)}$$

the quantity V shall be negative as long as there may be taken or had

$$cos\zeta \cdot \sqrt{v} - \sqrt{k} = -V,$$

as long as by increasing the angle ψ the quantity V maintains a real value; but immediately this real quantity avoids V, thence the angle ψ again decreases, and the opposite sign must be attributed to V itself, thus so that there shall be

$$cos\zeta \cdot \sqrt{v} - \sqrt{k} = +V;$$

we will have these two principle cases requiring to be set out, of which the one by increasing ψ anywhere from the beginning will make $V = 0$, truly with the other this never comes about. But immediately from the start there may arise $V = 0$, if here shall be either $k = 0$ or $\zeta = 0$: Then at some time after the start we may consider this to happen, and then truly at no time ; from which we may establish with care the following cases.

CASE I

27. Therefore in the first place we will consider the rotational motion of the cylinder to vanish completely or to be $k = 0$. Therefore since from that itself there becomes $V = 0$, at once from the beginning V must be attributed with a negative sign, so that there shall be

$$cos\zeta \cdot \sqrt{v} - \sqrt{k} = +V \text{ or } \sqrt{v} = \frac{V}{cos\,\zeta}$$

and the angle ψ now thence will be negative or the angle φ increases continually, so that there shall be

$$V = \sqrt{\left(a\cos^2\zeta\sin\theta\left(\cos\alpha - \cos\,\varphi\right) + a\left(\alpha - \varphi\right)\sin\zeta\cos\zeta\cos\theta\right)}$$

and

$$dt = \frac{ad\omega}{\sqrt{k}} = \frac{ad\varphi}{V} \text{ and } EM = x = \frac{a\left(\varphi - \alpha\right)}{\cos\zeta}.$$

Indeed because initially there was $\varphi = \alpha$ and $\sqrt{v} = 0$, we may put $\varphi = \alpha + \psi$ to be for the elapsed time *t*, so that there shall be

$$V = \sqrt{\left(a\cos^2\zeta\sin\theta\left(\cos\alpha - \cos\left(\alpha + \psi\right)\right) - a\psi\sin\zeta\cos\zeta\cos\theta\right)}$$

and

$$x = \frac{a\psi}{\cos\zeta} \text{ and } dt = \frac{ad\psi}{V}.$$

Now it is evident this cannot happen entirely, that the angle ψ may increase continually, unless there shall be $\sin\zeta\cos\zeta\cos\theta = 0$, which case it will be convenient to consider separately. But if truly ψ may cease to increase, which may arise, when $V = 0$, where the globule is reduced to a state of rest and may begin to go backwards along the spiral, at which instant therefore the value of V must be taken to be negative and the angle ψ decreases again, while it becomes $\psi = 0$, and at that time the body returns to *E,* just as it was initially, it will stay a little; from which the same motion anew will begin.

But it can happen, that this reversion of the globule in itself may as if fall on the start of the motion and indeed the angle ψ may be able to increase while not a minimum, since the angle φ may remain zero or thus be made negative. The first case will have a place, if by putting ψ infinitely small the value of V nevertheless may remain $= 0$; as that usually may come about if

$$a\cos^2\zeta\sin\theta\sin\alpha = a\sin\zeta\cos\zeta\cos\theta \text{ or } \sin\alpha = \frac{\text{tang}\zeta}{\text{tang}\theta}$$

and then the body remains at rest for ever at the point E ; for here the direction of the spiral will be horizontal [as $\theta = \frac{\pi}{2}$].

But the latter case has a place, if

$$\sin\alpha < \frac{\text{tang}\zeta}{\text{tang}\theta},$$

where the globule indeed does not enter into the spiral, but at once thence may fall away ; or if the cylinder might be made continuous downwards, the globule moved in the direction through the interior part of the continued spiral ; thus so that the angle ψ then may becomes negative and therefore the value of x and of V itself.

But these cases will not find a place, unless there shall be $\theta > \zeta$, or the inclination of the base of the cylinder to the horizontal shall be greater than the angle *BEF*, which the spiral makes with the base of the cylinder [*i.e.* the angle of the pitch of the screw]. But I will not pursue this motion in the resting spiral in more detail, since nothing of difficulty may be had.

CASE II

28. We may consider the motion of the rotations of the cylinder to be propagated thus, so that the motion of the rotation of the globule about the axis, which may be indicated by the angle ψ and from the beginning were placed in the same direction with the motion of the rotating cylinder, but after some time may be reflected in the opposite direction.

Therefore the angle ψ can be increased so far there, that there may become

$$k\sin^4\zeta = a\cos^2\zeta\sin\theta\big(\cos(\alpha-\psi)-\cos\alpha\big)-a\psi\sin\zeta\cos\zeta\cos\theta$$

or $V = 0$; but this cannot come about, unless there shall be

$$\cos\big(\alpha-\psi\big)-\cos\alpha > \frac{\text{tang}\zeta}{\text{tang}\theta}\cdot\psi;$$

since therefore from the beginning there were $\psi = 0$, it is necessary that on putting ψ to be vanishing, there shall be

$$\sin\alpha > \frac{\text{tang}\zeta}{\text{tang}\theta}.$$

Then the value of

$$\cos\big(\alpha-\psi\big)-\cos\alpha-\frac{\text{tang}\zeta}{\text{tang}\theta}\cdot\psi$$

will be a maximum, if

$$\sin\alpha = \frac{\text{tang}\zeta}{\text{tang}\theta}$$

We may consider this with according to the value substituted for ψ to become

$$\cos(\alpha-\psi)-\cos\alpha-\frac{\operatorname{tang}\zeta}{\operatorname{tang}\theta}\cdot\psi=M$$

and so that the value of *V* from increasing ψ may be able to vanish finally, it is necessary that there shall be

$$k\sin^4\zeta<aM\cos^2\zeta\sin\theta.$$

Whereby since this case shall be able to have a place, the three following conditions are required :

I. So that there shall be $\operatorname{tang}\theta>\operatorname{tang}\zeta$ or $\theta>\zeta$; thus so that the fraction $\frac{\operatorname{tang}\zeta}{\operatorname{tang}\theta}$ may not exceed unity,

II. so that there shall be $\sin\alpha>\frac{\operatorname{tang}\zeta}{\operatorname{tang}\theta}$ and finally

III. so that there shall be $k<aM\frac{\cos\zeta^2\sin\theta}{\sin\zeta^4}$.

Therefore whenever these three conditions have been satisfied, the globule will be carried around in the spiral in the sense *BEA* about the axis of the cylinder, while the angle ψ will have been described, so that there may come about :

$$V=\sqrt{\left[k\sin^4\zeta-a\cos^2\zeta\sin\theta\left(\cos(\alpha-\psi)-\cos\alpha\right)+a\psi\sin\zeta\cos\zeta\cos\theta\right]}=0,$$

moreover there will be [at this stage]

$$cos\zeta\cdot\sqrt{v}-\sqrt{k}=0,$$

or the relative speed of the globule through the spiral will be $\sqrt{v}=\frac{\sqrt{k}}{cos\zeta}$; since before it may arrive at that place, there shall be

$$\sqrt{v}=\frac{\sqrt{k}-V}{cos\zeta},$$

with

$$x=\frac{a\omega-a\psi}{\cos\zeta}\text{ and }d\omega=\frac{dt\sqrt{k}}{a}=\frac{d\psi\sqrt{k}}{V}$$

present.

But after this position has been reached, the angle ψ will decrease continually, or the angular motion of the globule will become opposite to the motion of the cylinder, and by attributing the opposite sign of V there will be had [the globule with speed V is now meeting the pipe traveling with speed $\sqrt{k}$ in the opposite direction],

$$\sqrt{v}=\frac{\sqrt{k}+V}{cos\zeta},$$

and when there becomes $\psi=0$, there will be $V=\sin^2\zeta\sqrt{k}$, and hence

$$\sqrt{v}=\frac{\left(1+\sin^2\zeta\right)}{cos\zeta}\sqrt{k} \text{ and } x=\frac{a\omega}{cos\zeta}.$$

Thereafter ψ will become negative, and the distance x will increase more, while on putting ψ negative there will become $x=\dfrac{a\omega+\psi}{cos\zeta}$, then there may become

$$V=\sqrt{\left(k\sin^4\zeta-a\cos^2\zeta\sin\theta\left(\cos(\alpha-\psi)-\cos\alpha\right)+a\psi\sin\zeta\cos\zeta\cos\theta\right)}=0,$$

and up to that point there will be $\sqrt{v}=\dfrac{\sqrt{k}+V}{cos\zeta}$: but where there shall be $V=0$, there arises $\sqrt{v}=\dfrac{\sqrt{k}}{cos\zeta}$; which therefore shall be the value before this maximum time, where there was

$$\sin\left(\alpha+\psi\right)=\frac{\text{tang}\zeta}{\text{tang}\theta}.$$

But after there was $V=0$, the angle ψ will decrease again, finally also the distance x will take smaller increments and there will be $\sqrt{v}=\dfrac{\sqrt{k}-V}{cos\zeta}$, then there may arise $\psi=0$, and then there will be $V=\sin^2\zeta\cdot\sqrt{k}$ and $\sqrt{v}=\cos\zeta\cdot\sqrt{k}$, and $x=\dfrac{a\omega}{\cos\zeta}$. Therefore at this time the speed $\sqrt{v}$ will be the same as it was initially, and then the motion will be propagated in a like manner.

Therefore the motion along the spiral will be continually progressive, if there were always $V<\sqrt{k}$ *:* but if between these parts of the motion, in which $\sqrt{v}=\dfrac{\sqrt{k}-V}{cos\zeta}$ may arise, so that there may become $V>\sqrt{k}$, then the globule there may be moving backwards relative to the spiral, while $\sqrt{v}$ again may become positive. But positive values will prevail; for we have seen after the first period, in which the speed has returned to the initial state, the globular separation to be resolved on the spiral $x = \frac{a\omega}{\cos\zeta}$, and after *n* periods of this kind it will have moved through a distance of the spiral $x = \frac{na\omega}{\cos\zeta}$, and thus the globule will be continually raised higher, while finally it may be ejected through the upper opening *K*.

CASE III

29. We may consider the motion to be prepared thus, so that after the beginning the angle $\psi = \alpha - \varphi$ begins to increase, at no time will there emerge

$$V = \sqrt{\left(k\sin^4\zeta - a\cos^2\zeta\sin\theta\left(\cos(\alpha-\psi)\right) - \cos\alpha\right) + a\psi\sin\zeta\cos\zeta\cos\theta\Big)} = 0,$$

from which this angle ψ will continually be made greater and the value of *V* will increase. But then $\sqrt{v} = \frac{\sqrt{k} - V}{cos\zeta}$ will be produced from which it follows the speed $\sqrt{v}$ only can vanish and the globule thence will return to the bottom part of the cylinder, while it will again be released from *E*. This also is understood from the formula

$$x = \frac{a(\omega - \psi)}{\cos\zeta};$$

indeed the distance *x* may be diminished, if there were

$$d\omega < d\psi \quad \text{or} \quad \frac{d\psi\sqrt{k}}{V} < d\psi\,,$$

which certainly will arise, when $\sqrt{k} < V$ or $\sqrt{v}$ is negative.

Therefore this case, where the globule is not allowed to progress beyond a certain limit in the spiral, is treated in the following cases:

1°. If $\mathrm{tang}\,\theta <\ \mathrm{tang}\,\zeta$ or the angle $PQR < BEF$, in whatever manner the remaining quantities may be considered.

2°. If there were $\sin\alpha < \frac{\mathrm{tang}\,\zeta}{\mathrm{tang}\,\theta}$, thus so that even if there shall be $\zeta < \theta$, still in this case the globule may return in the spiral.

3°. Even if there shall be

$$\zeta < \theta \text{ and } \sin\alpha > \frac{\mathrm{tang}\,\zeta}{\mathrm{tang}\,\theta},$$

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the third case still finds a place, if there were

$$k > \frac{a\cos^2\zeta \sin\theta}{\sin^4\zeta}\cdot M$$

with *M* denoting the maximum value, which the expression

$$\cos(\alpha-\psi)-\cos\alpha-\frac{\tan g\zeta}{\tan g\theta}\cdot\psi$$

prevails to adopt.

Hence it is apparent therefore, an exceedingly swift motion of rotation is not suitable for raising the globule to some given height, since a slower motion may be able to perform this effect. Therefore it can happen, so that on account of an excessive rotational velocity we may be disappointed by the effect, which yet we may be able to follow with a slower motion.

Example

30. Let $\frac{\tan g\zeta}{\tan g\theta}=\frac{1}{2}$ and the initial angle *ACE* shall be right or $\alpha = 90°$, and $\psi = 90° - \varphi$, and thus ψ will denote the angle, by which the globule has been translated about the axis towards the highest point *A* from *E* in the time *t*, in which the cylinder has turned through the angle $=\omega$, thus so that there shall be $d\omega = \frac{dt\sqrt{k}}{a}$. Therefore we will have

$$V = \sqrt{\left(k\sin^4\zeta - a\cos^2\zeta\sin\theta\left(\sin\psi - \tfrac{1}{2}\psi\right)\right)}$$

and

$$d\omega = \frac{dt\sqrt{k}}{a} \quad\text{and}\quad \sqrt{v} = \frac{\sqrt{k}-V}{cos\zeta} \quad\text{and}\quad x = \frac{a(\omega-\psi)}{cos\zeta}.$$

Therefore while the rotary motion of the globule is directed in the sense *BEA*, the value of *V* in these formulas must be taken as positive, in the opposite sense truly negative.

Therefore from the beginning with ψ increasing, the value of *V* decreases on account of $\sin\psi > \frac{1}{2}\psi$: while there remains

$$k\sin^4\zeta > a\cos^2\zeta\sin\theta\left(\sin\psi - \tfrac{1}{2}\psi\right).$$

Therefore since the maximum value, $\sin\psi = 60° = \frac{1}{3}\pi$, of $\sin\psi - \frac{1}{2}\psi$, with π the angle equal to two right angles, and this maximum value becomes $=\frac{1}{2}\sqrt{3}-\frac{1}{6}\pi = 0,3424266,$ but if therefore there were

$$k\sin^4\zeta < 0,3424266a\cos^2\zeta\sin\theta,$$

the following case needs to be considered, truly the third case, if

$$k\sin^4\zeta > 0,3424266a\cos^2\zeta\sin\theta.$$

Evidently there the globule by its motion at no time by its motion finally will turn back. Therefore for brevity there shall be

$$\frac{\cos^2\zeta\sin\theta}{\sin^4\zeta} = n,$$

so that there shall be

$$V = \sin^2\zeta\sqrt{\left(k - na\left(\sin\psi - \tfrac{1}{2}\psi\right)\right)}.$$

1st And we may put initially to be $k > 0,3424266na$; and the angle ψ will increase continually, but the initial value of V will decrease initially, then there becomes $\psi = 60°$, where the value of V will be a minimum, evidently

$$= \sin^2\zeta\sqrt{(k - 0,3424266na)},$$

and thus is the maximum progressive speed of the globule along the spiral. Thence truly the value of V itself again will be increased, and finally, when $\sin\psi = \frac{1}{2}\psi$, which arises, if

$$\psi = 108°\ 36'13''45'''28'''',$$

there becomes

$$V = \sin^2\zeta\cdot\sqrt{k} \text{ and } \sqrt{v} = \cos\zeta\cdot\sqrt{k},$$

which is equal to the initial speed. Truly afterwards with the angle ψ increasing further, the value of V will be increased more, and at last the speed will become either

$$V = \sqrt{k} \text{ or } k(1-\sin^4\zeta) = a\cos^2\zeta\sin\theta\left(\tfrac{1}{2}\psi - \sin\psi\right)$$

or

$$\tfrac{1}{2}\psi - \sin\psi = \frac{k(1+\sin^2\zeta)}{a\sin\theta};$$

and here the speed of the globule in the spiral may vanish, from which by reverting ω it may begin, and indeed with an accelerated motion because, with ψ increasing beyond this term, the magnitude V may be taken to increase more. But by defining ψ from the equation

$$\tfrac{1}{2}\psi - \sin\psi = \frac{k(1+\sin^2\zeta)}{a\sin\theta};$$

the quantity $x = \frac{a(\omega - \psi)}{cos\zeta}$ will give the distance into the spiral, to which the globule will have penetrated and from which successively it will return. But the time, when it reaches this point, or the angle ω conferred by the rotational motion of the cylinder meanwhile, may be defined by this equation :

$$\omega = \int \frac{d\psi\sqrt{k}}{\sin^2\zeta \cdot \sqrt{\left(k - na\sin\psi + \frac{1}{2}na\psi\right)}},$$

with which found, likewise the true path x traversed in the spiral becomes known.

II. We may consider there to be $k < 0{,}3424266na$, and the angle ψ as far as there it may increase, at last becomes $k = na\left(\sin\psi - \frac{1}{2}\psi\right)$, which happens, before there may emerge $\psi = 60°$; and then there will be $\sqrt{v} = \frac{\sqrt{k}}{cos\zeta}$ on account of $V = 0,$ therefore at this point the speed $\sqrt{v}$ has grown by being increased; and in these circumstances we may establish the first part of the motion of the globule through the spiral.

2nd But from this moment the angle ψ may be diminished again, and the value

$$V = \sin^2\zeta \cdot \sqrt{\left(k - na\left(\sin\psi - \frac{1}{2}\psi\right)\right)}$$

must be taken negative, so that there shall be

$$\sqrt{v} = \frac{\sqrt{k} + V}{cos\zeta}$$

and thus by slipping with time the value of V again will increase, while arriving at $\psi = 0$ there becomes

$$V = \sin^2\zeta \cdot \sqrt{k} \text{ and } \sqrt{v} = \frac{\left(1 + \sin^2\zeta\right)\sqrt{k}}{\cos\zeta}:$$

and here we will terminate the second part of the motion, at the end of which $\psi = 0$, and the speed of the globule $\sqrt{v}$ becomes greater, than it was hitherto.

3rd Now therefore the angle ψ begins to be negative ; therefore on putting $-\psi$ in place of ψ, we will have

$$V = \sin^2\zeta \cdot \sqrt{\left(k + na\left(\sin\psi - \frac{1}{2}\psi\right)\right)},$$

with

$$\sqrt{v} = \frac{\sqrt{k} + V}{cos\zeta} \text{ remaining :}$$

and because $\sin\psi - \frac{1}{2}\psi$ grows, as long as ψ is $< 60°$, thus as far as to the end $\psi = 60°$ the value of V, and hence the speed $\sqrt{v}$ will be increased; and by making $\psi = 60°$, the speed of the globule will progress to a maximum, evidently

$$\sqrt{v} = \frac{\sqrt{k} + \sin^2\zeta \cdot \sqrt{(k - 0{,}3424266na)}}{\cos\zeta}.$$

4th From thence by further increase in this angle ψ, which now is $= \varphi - \alpha$, the value of V decreases again, and when it becomes

$$\varphi = 108°\ 36'13''45'''28'''',$$

there will be

$$V = \sin^2\zeta \cdot \sqrt{k} \text{ and } \sqrt{v} = \frac{\left(1 + \sin^2\zeta\right)\sqrt{k}}{\cos\zeta}.$$

5th But the angle ψ will go on to increase beyond this limit, and because then $\frac{1}{2}\psi > \sin\psi$, there will be

$$V = \sin^2\zeta \cdot \sqrt{\left(k - na\left(\sin\psi - \frac{1}{2}\psi\right)\right)},$$

and

$$\sqrt{v} = \frac{\sqrt{k} + V}{cos\zeta}.$$

Therefore this value of V continually becomes less, and thus also the speed $\sqrt{v}$, while there becomes

$$\frac{1}{2}\psi - \sin\psi = \frac{k}{na},$$

in which case there will be $\sqrt{v} = \dfrac{\sqrt{k} - V}{cos\zeta}$.

6th But at that time this angle ψ, which is greater than 108° 36', will decrease again, and there becomes $\sqrt{v} = \dfrac{\sqrt{k} - V}{cos\zeta}$ with

$$V = \sin^2\zeta \cdot \sqrt{\left(k - na\left(\frac{1}{2}\psi - \sin\psi\right)\right)};$$

and thus the speed $\sqrt{v}$ will decrease, and when there becomes $\psi = 108^\circ\ 36'$, there will be produced $V = \sin^2\zeta \cdot \sqrt{k}$ and $\sqrt{v} = \cos\zeta \cdot f\sqrt{k}$, which is equal to the initial speed.

7th Again the angle ψ will decrease below this limit, and on account of $\sin\psi > \frac{1}{2}\psi$, there will be

$$V = \sin^2\zeta\sqrt{\left(k + na\left(\sin\psi - \tfrac{1}{2}\psi\right)\right)} \text{ and } \sqrt{v} = \frac{\sqrt{k} - V}{cos\zeta}:$$

and when there becomes $\psi = 60^\circ$, in which case the value of V itself will be a maximum

$$= \sin^2\zeta \cdot \sqrt{(k + 0,3424266na)},$$

and the speed of the globule a minimum

$$\sqrt{v} = \frac{\sqrt{k} - \sin^2\zeta \cdot \sqrt{(k + 0,3424266na)}}{\cos\zeta}.$$

Therefore unless there shall be

$$\sqrt{k} > \sin^2\zeta\sqrt{(k + 0,3424266na)}.$$

or

$$k > \frac{0,3424266a\sin\theta}{1 + \sin^2\zeta},$$

when there shall be

$$k < \frac{0,3424266a\cos^2\zeta\sin\theta}{\sin^4\zeta},$$

the globule, before it arrives at that limit, will turn round on the spiral, therefore because its speed $\sqrt{v}$ shall be negative. Therefore the globule turns back, if there shall be

$$k < \frac{0,3424266a\sin\theta}{1 + \sin^2\zeta};$$

but it will not revert, but continually goes on to progress through the spiral, if there shall be

$$k > \frac{0,3424266a\sin\theta}{1 + \sin^2\zeta}.$$

But because here must be

$$k < \frac{0,3424266a\cos^2\zeta\sin\theta}{\sin^4\zeta},$$

it is evident that this case cannot be considered, unless there shall be

$$1 > 2\sin^4\zeta \text{ or } \sin\zeta < \sqrt[4]{\tfrac{1}{2}};$$

that is unless the angle of the spiral ζ shall be less than 57° 14'.

8th But after the angle ψ were diminished beyond 60°, also it will decrease further, and at this point there will be

$$V = \sin^2\zeta\cdot\sqrt{\left(k + na\left(\sin\psi - \tfrac{1}{2}\psi\right)\right)} \text{ and } \sqrt{v} = \frac{\sqrt{k} - V}{cos\zeta}$$

and

$$\sqrt{v} = \frac{\sqrt{k} + V}{cos\zeta}.$$

and the value of V continually becomes less, and the speed $\sqrt{v}$ will be returned positive soon, and by making $\psi = 0$ and that will be returned, as there was initially, to $\sqrt{v} = cos\zeta\cdot\sqrt{k}.$

When the globule arrives at this point, the angle ψ again becomes negative, or the angular motion of the globule will follow the motion of the cylinder, or now there will be $\varphi < \alpha$ or $\varphi < 90°$; or the globule will be raised above the central part of the cylinder, since it must turn three times in the lower part: and now it will pursue its motion in a like manner, and it will be made from the beginning; thus so that now the parts of the motion shall be reverting to the same, as we have described. Truly because it pertains the time, in which some part of this motion may be absolved, that is unable to be defined except by quadrature with the aid of the formula $d\omega = \frac{d\psi\sqrt{k}}{V}$; indeed its integration cannot be shown.

PROBLEM 8

31. *If one whole winding of the spiral EFGe were filled with water and the cylinder suddenly began to be driven around with a uniform speed, which at the point E shall be* $=\sqrt{k}$ *, and that in the sense of the spiral contrary to BEA , to find the motion, by which that same part of the water will be moved forwards through the spiral.*

Solution

With the base of the cylinder put in place with the radius $CA = a$, with the angle of the

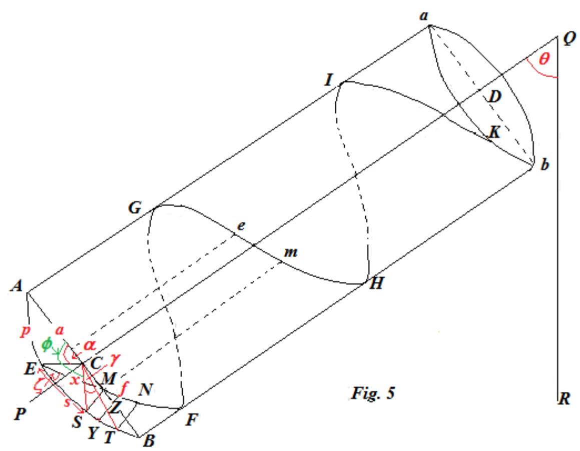

Fig. 5

spiral $BEF = \zeta$, with the angle, which the axis of the cylinder *PQ* makes with the vertical, $PQR = \theta$*;* the initial angle for this motion shall be $ACE = \alpha$; at which time the water will occupy the distance along the spiral $EFGe = f$, which shall be equal to one whole revolution, on putting $\frac{f\cos\zeta}{a} = \gamma$, where γ shall be an angle equal to four right angles, or by denoting $1:\pi$ to the ratio of the radius to the circumference, there will be $\gamma = 2\pi$ and $f = \frac{2\pi a}{\cos\zeta}$ and the amount of the water itself $= \frac{2\pi ahh}{\cos\zeta}$, if indeed *hh* may designated the cross-section of the spiral.

Now in the elapsed time *t,* in which the cylinder itself will have turned about the axis by the angle $=\omega$, so that there shall be $d\omega = \frac{dt\sqrt{k}}{a}$ or $\omega = \frac{t\sqrt{k}}{a}$ and thus $t = \frac{a\omega}{\sqrt{k}}$ the water in the spiral will have arrived at the position *MFGem;* therefore the distance may be put $EM = x$ and the speed, by which the water may be moved forwards through the spiral $=\sqrt{v}$ *;* so that there shall be

$$dx = dt\sqrt{v} = \frac{ad\omega\sqrt{v}}{\sqrt{k}}.$$

Putting the angle $ACS=\varphi$, and on account of the angle $ECS=\frac{x\cos\zeta}{a}$ because the point E has approached towards A by the angle ω, there will be

$$\varphi=\alpha-\omega+\frac{x\cos\zeta}{a} \text{ and thus } \frac{x\cos\zeta}{a}=\omega+\varphi-\alpha$$

and hence

$$\frac{dx\cos\zeta}{a}=\frac{d\omega\cos\zeta\cdot\sqrt{v}}{\sqrt{k}}=d\omega+d\varphi,$$

and thus there shall be

$$d\omega=\frac{d\varphi\cdot\sqrt{k}}{\cos\zeta\cdot\sqrt{v}-\sqrt{k}}.$$

But from paragraph 17 this equation will be had on account of

$$\gamma=2\pi \text{ and } \sin(\gamma+\varphi)=\sin\varphi:$$

$$\cos\zeta\cdot\sqrt{v}-\sqrt{k}=\sqrt{(C-a\varphi\sin\zeta\cos\zeta\cos\theta)}.$$

But a motion of this kind is expressed on the initial motion of the water in the spiral, so that there shall be $\sqrt{v}=\cos\zeta\cdot\sqrt{k}$, in which case since there shall be $\varphi=\alpha$, the equation becomes :

$$\cos\zeta\cdot\sqrt{v}-\sqrt{k}=-\sqrt{(k\sin^4\zeta+a(\alpha-\varphi)\sin\zeta\cos\zeta\cos\theta)}.$$

Therefore from the beginning the angle φ, which itself at the start was $=\alpha$, decreases, or the boundary of the water M is raised nearer the upper line Aa, than it was at the start. We may put this approach made in the time t to be through the angle ψ, so that there shall be $\varphi=\alpha-\psi$, there will be

$$\frac{x\cos\zeta}{a}=\omega-\psi \text{ and } x=\frac{a(\omega-\psi)}{\cos\zeta},$$

then truly

$$\cos\zeta\cdot\sqrt{v}-\sqrt{k}=-\sqrt{(k\sin^4\zeta+a\psi\sin\zeta\cos\zeta\cos\theta)}.$$

or

$$\sqrt{v}=\frac{\sqrt{k}-\sqrt{(k\sin^4\zeta+a\psi\sin\zeta\cos\zeta\cos\theta)}}{\cos\zeta},$$

and there will be

$$d\omega = \frac{d\psi\sqrt{k}}{\sqrt{(k\sin^4\zeta + a\psi\sin\zeta\cos\zeta\cos\theta)}}.$$

Hence since in the beginning when $\omega = 0$, also there shall be $\psi = 0$, by integrating there will be :

$$\frac{a\omega\sin\zeta\cos\zeta\cos\theta}{2\sqrt{k}} = \sqrt{(k\sin^4\zeta + a\psi\sin\zeta\cos\zeta\cos\theta)} - \sin^2\zeta\cdot\sqrt{k}$$

and hence again:

$$\psi = \omega\sin^2\zeta + \frac{a\omega\omega}{4k}\sin\zeta\cos\zeta\cos\theta.$$

From which we obtain for the time through the angle ω indicated:

$$\sqrt{v} = \cos\zeta\cdot\sqrt{k} - \frac{a\omega\sin\zeta\cos\theta}{2\sqrt{k}}$$

and

$$x = a\omega\cos\zeta - \frac{aa\omega\omega}{4k}\sin\zeta\cos\theta;$$

but in the elapsed time t there is $\omega = \frac{t\sqrt{k}}{a}$; thus so that there shall be

$$\sqrt{v} = \cos\zeta\cdot\sqrt{k} - \tfrac{1}{2}t\sin\zeta\cos\theta$$

and

$$x = t\cos\zeta\cdot\sqrt{k} - \tfrac{1}{4}tt\sin\zeta\cos\theta;$$

therefore the distance SM, through which the water now will be advanced along the direction of the axis of the cylinder, will be

$$x\sin\zeta = t\sin\zeta\cos\zeta\cdot\sqrt{k} - \tfrac{1}{4}tt\sin^2\zeta\cos\theta$$

from which the distance, through which the water will have been raised vertically is concluded to be:

$$x\sin\zeta\cos\theta = t\sin\zeta\cos\zeta\cos\theta\cdot\sqrt{k} - \tfrac{1}{4}tt\sin^2\zeta\cos^2\theta$$

Corollary 1

32. Clearly if the cylinder may not be set in motion, but may be left at rest, so that there shall be $k = 0$, then in the elapsed time t there shall be

$$\sqrt{v} = -\tfrac{1}{2}t\sin\zeta\cos\theta \text{ and } x = -\tfrac{1}{4}tt\sin\zeta\cos\theta.$$

Therefore the water, if indeed the spiral were to be continued upwards beyond *E*, would descend along the cylinder with a uniform acceleration, and its motion would be similar to the descent of a body on an inclined plane, of which the angle of inclination to the horizontal sine would be $= \sin^2\zeta \cos\theta$.

Corollary 2

33. But with the cylinder set in motion in the sense *BEA* with the speed $\sqrt{k}$, water indeed rises along the cylinder from the initial motion, provided there were

$$k > \tfrac{1}{2} a\omega \operatorname{tang}\zeta \cos\theta \text{ or } k > \tfrac{1}{2} t \operatorname{tang}\zeta \cos\theta.$$

Moreover in the elapsed time

$$t = \frac{2\sqrt{k}}{\operatorname{tang}\zeta \cos\theta},$$

the ascending motion will cease, and afterwards the water in the cylinder will begin to descend.

Corollary 3

34. Therefore on putting

$$t = \frac{2\sqrt{k}}{\operatorname{tang}\zeta \cos\theta},$$

the greatest distance *x* through which the water was raised, will be

$$x = \frac{k\cos^2\zeta}{\sin\zeta \cos\theta};$$

and therefore the distance completed along the length of the cylinder

$$x\sin\zeta = \frac{k\cos^2\zeta}{\cos\theta};$$

and perpendicularly it will be found raised to the height $x\sin\zeta \cos\theta = k\cos^2\zeta$.

Corollary 4

35. Therefore the portion of water, which the whole turn of the spiral holds, with the aid of the Archimedian spiral is unable to raise water to a greater height, than what shall be $= k\cos^2\zeta$. Therefore when the cylinder is made to rotate faster, there this portion of

water can be raised higher, and indeed this height will be proportional to the square of the speed of rotation.

Corollary 5

36. Let the height, to which water with the aid of the Archimedian spiral should be able to be raised, $=c$, and this will have performed in the time *t*, so that there becomes

$$c=t\sin\zeta\cos\zeta\cos\theta\cdot\sqrt{k}-\tfrac{1}{4}tt\sin^2\zeta\cos^2\theta$$

or

$$t=\frac{2\cos\zeta\cdot\sqrt{k}-2\sqrt{\left(k\cos^2\zeta-c\right)}}{\sin\zeta\cos\theta}.$$

Now so that this time generally shall be a minimum, the angle ζ must be prepared thus, so that there shall be

$$\operatorname{tang}^2\zeta=1-\frac{c}{k}\text{ or }\operatorname{tang}\zeta=\sqrt{\left(1-\frac{c}{k}\right)}.$$

[For: $\tfrac{1}{2}\cos\theta\cdot t=\dfrac{\cos\zeta\cdot\sqrt{k}-\sqrt{\left(k\cos^2\zeta-c\right)}}{\sin\zeta}=\cot\zeta\cdot\sqrt{k}-\dfrac{\sqrt{\left(k\cos^2\zeta-c\right)}}{\sin\zeta}$

$$\frac{d\,\tfrac{1}{2}\cos\theta\cdot t}{d\zeta}=-\frac{1}{\sin^2\zeta}\cdot\sqrt{k}+\frac{\sqrt{\left(k\cos^2\zeta-c\right)}\cos\zeta}{\sin^2\zeta}+\frac{k\cos\zeta}{\sqrt{\left(k\cos^2\zeta-c\right)}}=0$$

$$\left(k\cos^2\zeta-c\right)\cos\zeta+k\cos\zeta\cdot\sin^2\zeta=\sqrt{k}\cdot\sqrt{\left(k\cos^2\zeta-c\right)}$$

$$k\cos^3\zeta-c\cos\zeta+k\cos\zeta-k\cos^3\zeta=\sqrt{k}\cdot\sqrt{\left(k\cos^2\zeta-c\right)}\therefore(k-c)\cos\zeta=\sqrt{k}\cdot\sqrt{\left(k\cos^2\zeta-c\right)}$$

$$\therefore(k-c)^2\cos^2\zeta=k^2\cos^2\zeta-kc\therefore\cos^2\zeta\left(k^2-(k-c)^2\right)=kc\therefore\cos^2\zeta=\frac{k}{(2k-c)};$$

$$\therefore\cos\zeta=\sqrt{\frac{k}{(2k-c)}}.\therefore\sin\zeta=\sqrt{1-\frac{k}{2k-c}}=\sqrt{\frac{k-c}{2k-c}}\text{ and }\tan\zeta=\sqrt{\frac{k-c}{2k-c}}\times\sqrt{\frac{(2k-c)}{k}}=\sqrt{1-\frac{c}{k}}.]$$

Corollary 6

37. But on putting

$$\operatorname{tang}\zeta = \sqrt[4]{\left(1-\frac{c}{k}\right)}.$$

[Euler has erred in his calculation above, for which I have set out the calculation; however, we will carry his value in this paragraph.]
the time will be that minimum, in which the water is raised through the height *c*:

$$t=\frac{2\sqrt{k}}{\cos\theta}\left(\cos\zeta-\operatorname{tang}\zeta\right)=\frac{2\sqrt{k}-2\sqrt{(k-c)}}{\cos\theta\cdot\sqrt[4]{\left(1-\frac{c}{k}\right)}},$$

which shall be infinite, if $k=c$, but truly zero if *k* is infinite. So that the greater the speed of rotation may be taken $\sqrt{k}$ and when likewise the angle $PQR=\theta$ may be taken smaller, there the water will be raised to the height *c* in a shorter time.

[Here the correct value becomes $t=\dfrac{2c}{\cos\theta\cdot\sqrt{k-c}}$.]

Corollary 7

38. Therefore it is apparent, even if the Archimedes' screw may be placed vertical, yet with its help water can be raised to whatever height, provided the spiral may be driven in a circle fast enough. But in this case on account of $\theta=0$ thus , either the whole turn of the spiral may be filled with water, or otherwise. And indeed the time of raising in this case will be less, than if the cylinder were inclined to the horizontal.

Scholium 39.

Therefore there is a significant difference between the raising of water by the Archimedes' screw, exactly as the water being raised may fill the whole spiral, or may occupy only a small portion of this; indeed if the water fills up the whole spiral, that cannot be raised beyond a certain height, however quickly the spiral may be driven in a circle ; but we have seen on the other hand, if only a small amount of water may be sent into the spiral, it can happen, that it can be raised to any height, and this indeed by a rotational motion not excessively fast: for from the preceding it is evident exceedingly fast motion to be advised against for the ascent and water to be carried up again, which yet by a slower motion might have gone on. For so that a particle of water sent into the tube initially at *E* may go on rising continually, first it is required that there shall be

$$\theta>\zeta \text{ or the angle } PQR > \text{the angle } BEF.$$

Then so that there shall be

Novi commentarii academiae scientiarum Petropolitanae (1754/5), 1760, p. 259-29
Translated by Ian Bruce 2014

$$\sin\alpha \text{ or } \sin ACE > \frac{\tang\zeta}{\tang\theta}:$$

moreover in the third place, so that, with M denoting the maximum positive value, which the expression

$$\cos(\alpha-\psi)-\cos\alpha-\frac{\tang\zeta}{\tang\theta}\cdot\psi$$

is able to receive, which happens in the case

$$\sin(\alpha-\psi)=\frac{\tang\zeta}{\tang\theta},$$

there shall be

$$k < aM\cdot\frac{\cos^2\zeta\sin\theta}{\sin^4\zeta}.$$

Therefore if the height corresponding to the height of the rotational motion k shall exceed this amount, the water, after it may have arrived at a certain height, may fall down again. Truly neither of these cases has a place in common practice, where the spiral of Archimedes may be used in raising water : because indeed if the whole base of the lower cylinder AB is submerged in water, the whole spiral is full of water always, from which the question, with how great a speed and to how great a height shall the water driven in the rotating spiral be going to be raised, is completely different from these two effects which we have examined, because therefore water continually flows in at E, truly at K is flowing out again. Therefore this most difficult question I shall try to make clear in the following problem.

PROBLEM 9

40. *If the whole base of the cylinder shall be submerged in water, and that to be driven round in a uniform motion, to define the motion of the water through the spiral.*

Solution

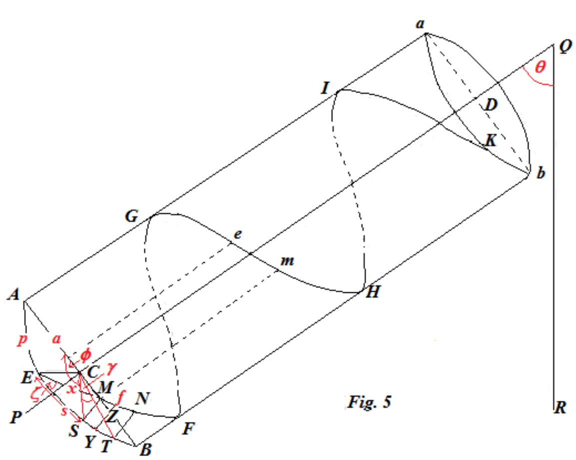

Fig. 5

Putting in place, as up to the present, the radius of the base $CA = a$, the angle of the spiral $BEF = \zeta$ and the angle of the inclination $PQR = \theta$: the height of the water above the centre of the base $C = c$, the length of the whole cylinder $Aa = Bb = b$, and *EFGHIK* may represent the whole spiral, of which the length therefore is $= \frac{b}{\sin\zeta}$ [by a trivial integration]; and if its cross-section may be called $= hh$, the amount of water contained in the spiral will be $= \frac{bhh}{\sin\zeta}$; then truly the sum of the spirals will be present related to the base in the circumference of that arc $= \frac{b\cos\zeta}{\sin\zeta}$. Evidently if from a some point of the spiral *Z* a right line may be drawn to the base parallel to the axis *ZY* and the arc *EY* may be put $= s$, on putting $s = 0$ the lower end *E* of the spiral will be had, on putting $s = \frac{b\cos\zeta}{\sin\zeta}$ the upper end *K* of the spiral will be produced. Now the cylinder may be turned in the sense *BEA*, thus so that the speed of the point *E* shall be $= \sqrt{k}$: and by putting the arc $EA = p$, with the elapsed time *dt* there will be $dp = -dt\sqrt{k}$. Moreover for the present moment of time the velocity of the ascent of the water through the spiral shall be $= \sqrt{v}$ *:* but if now the state of the compression of the water in the spiral at some place *Z* may be put $= q$, with the arc present $EY = s$, we have found this equation above :

$$q\cos\zeta = C - a\cos\zeta\sin\theta\cos\frac{p+s}{a} - s\sin\zeta\cos\theta - \frac{sdv}{dt\sqrt{v}}.$$

But when the water flows freely at *K*, by putting $s = \frac{b\cos\zeta}{\sin\zeta}$ the state of compression at *K* must vanish, therefore there will be :

$$C = a\cos\zeta\sin\theta\cos\frac{p\sin\zeta + b\cos\zeta}{a\sin\zeta} + b\cos\zeta\cos\theta + \frac{bdv\cos\zeta}{dt\sin\zeta\cdot\sqrt{v}}.$$

g may express the state of compression at the other end *E*, where $s = 0$, it will be given by :

$$g\cos\zeta = a\cos\zeta\sin\theta\cos\frac{p\sin\zeta + b\cos\zeta}{a\sin\zeta} + b\cos\zeta\cos\theta$$
$$+\frac{bdv\cos\zeta}{dt\sin\zeta\cdot\sqrt{v}} - a\cos\zeta\sin\theta\cos\frac{p}{a}.$$

or on dividing by $\cos\zeta$:

$$g = a\sin\theta\cos\frac{p\sin\zeta + b\cos\zeta}{a\sin\zeta} - a\sin\theta\cos\frac{p}{a} + b\cos\theta + \frac{bdv}{dt\sin\zeta\cdot\sqrt{v}}.$$

Therefore the whole matter is reduced here, so that the state of compression of the water at the end may be defined, which shall depend on the depth of the opening *E* below the water surface, the height of the point *E* above the centre *C*

$$=a\cos\frac{p}{a}\sin\theta,$$

and thus the depth of the opening *E* below the surface will be

$$=c-a\sin\theta\cos\frac{p}{a}.$$

Therefore since the speed of the water flowing in the spiral shall correspond to the height *v*, the state of compression of the water at *E* should be estimated by the height

$$=c-a\sin\theta\cos\frac{p}{a}-v,$$

from which we have :

$$c=a\sin\theta\cos\frac{p\sin\zeta+b\cos\zeta}{a\sin\zeta}+b\cos\theta+\frac{bdv}{dt\sin\zeta\cdot\sqrt{v}}+v.$$

The angle may be put in place $ACE=\varphi$, so that there shall be

$$p=a\varphi \text{ and } dt=\frac{-ad\varphi}{\sqrt{k}},$$

then truly there shall be the angle :

$$\frac{b\cos\zeta}{a\sin\zeta}=\gamma \text{ or } b=a\gamma\operatorname{tang}\zeta,$$

$$c=a\sin\theta\cos\left(\varphi+\gamma\right)+a\gamma\operatorname{tang}\zeta\cos\theta-\frac{\gamma dv\sqrt{k}}{d\varphi\cos\zeta\cdot\sqrt{v}}+v.$$

We may put $2\sqrt{kv}=z$, so that there shall be

$v=\frac{zz}{4k}$ [and $zdz=2kdv+2vdk$; also $dz=\frac{kdv+vdk}{\sqrt{kv}}=\sqrt{k}\frac{dv}{\sqrt{v}}+\sqrt{v}\frac{dk}{\sqrt{k}}$,]

and we shall have:

$$-\gamma dz+\frac{zzd\varphi\cos\zeta}{4k}+ad\varphi\cos\zeta\sin\theta\cos\left(\varphi+\gamma\right)=d\varphi(c\cos\zeta-a\gamma\sin\zeta\cos\theta).$$

From which equation the value of *z* itself ought to be defined.

But what may be attributed to the pressure of the water on the sides of the tube, as much as thence it offers resistance to the motion of the rotation, above we have considered the following force to arise from the weight of the water *Zr* [*i.e.* acting down the slope of the spiral. Recall that Euler's forces actually are forces per unit mass, so that they are essentially accelerations, where the acceleration of gravity is taken as 1; although as mentioned at the start, he initially took $g=\frac{1}{2}$ to agree with the *vis viva* principle; this corrupted state of affairs seems to have gone through the whole exercise.]

$$=\sin\zeta\sin\theta\sin\frac{p+s}{a}+\cos\zeta\cos\theta,$$

from which the following force arises

$$Zv=\sin^2\zeta\sin\theta\sin\frac{p+s}{a}+\sin\zeta\,\cos\zeta\cos\theta,$$

which multiplied by the element of the water $=\dfrac{hhds}{\cos\zeta}$ and the radius a gives the moment, for the element opposing the motion, from which the whole moment will be [on integrating along the curve] :

$$ahh\left(b\cos\zeta\cos\theta+\frac{a\sin^2\zeta\sin\theta}{\cos\zeta}\left(\cos\varphi-\cos\ \left(\varphi+\gamma\right)\right)\right);$$

so great a moment therefore must be overcome by the rotating force.

[Thus, Euler seems to be saying, this resistive moment, or something proportional to it to account for viscosity, which does not get a mention, must be overcome by applying a greater torque to the handle, and the rest of the calculation can continue unhindered by friction; the whole *vis viva* idea rests on the conservation of energy in its kinetic and gravitational forms.]

Corollary 1

41. Therefore the determination of the motion of the water through Archimedes' screw depends on the resolution of this differential equation:

$$-\gamma dz+\frac{zzd\varphi\cos\zeta}{4k}+ad\varphi\cos\zeta\sin\theta\cos\left(\varphi+\gamma\right)=d\varphi(c\cos\zeta-a\gamma\sin\zeta\cos\theta)$$

or on account of $\gamma=\dfrac{b\cos\zeta}{a\sin\zeta}$ of this equation :

$$-\frac{bdz}{a\sin\zeta}+\frac{zzd\varphi}{4k}+ad\varphi\sin\theta\cos\left(\varphi+\gamma\right)=d\varphi(c-b\cos\theta),$$

which shall be subject to many difficulties, it is apparent the theory of Archimedes' screw to be especially hard.

Corollary 2

42. If the screw is left at rest, so that there shall be $k=0$, in place of the element $d\varphi$ removed, with the element of time *dt* relinquished in the calculation, and on account of the constant angle φ there will be had:

$$\frac{bdv}{dt\sin\zeta\cdot\sqrt{v}}+v=c-b\cos\theta-a\sin\theta\cos(\varphi+\gamma),$$

[recall that $dt=\frac{-ad\varphi}{\sqrt{k}}$ and $v=\frac{zz}{4k}$; hence

$$-\frac{bdz}{ad\varphi\sin\zeta}=\frac{b}{dt\sqrt{k}\sin\zeta}\left(\sqrt{k}\frac{dv}{\sqrt{v}}+\sqrt{v}\frac{dk}{\sqrt{k}}\right)=\frac{bdv}{dt\sqrt{v}\sin\zeta}+\frac{b\sqrt{v}dk}{kdt\sin\zeta}=\frac{bdv}{dt\sin\zeta},]$$

from which the uniform motion arises soon

$$v=c-b\cos\theta-a\sin\theta\cos(\varphi+\gamma),$$

by which the water will flow through of the spiral, if indeed there shall be $c>b\cos\theta+a\sin\theta\cos(\varphi+\gamma)$.

Corollary 3

43. If the cylinder shall be placed in the vertical position, on account of $\theta=0$ there will be

$$-\frac{bdz}{\sin\zeta}+\frac{zzd\varphi}{4k}=d\varphi(c-b);$$

from which there arises

$$d\varphi=\frac{4bkdz}{(4k(b-c)+zz)a\sin\zeta}$$

and by integration

$$\frac{a\varphi}{4bk}\sqrt{4k(b-c)}=\text{A tang}\frac{z}{\sqrt{4k(b-c)}}$$

where there is

$$\sqrt{v}=\frac{z}{\sqrt{4k}}.$$

But since, if at the start there were $\varphi = 0$ and $z = 0$, by sliding with time the angle φ may emerge negative, it is evident the value of z itself to emerge negative; and thus in this case the water does not rise, but falls, which by itself is indeed evident.

Corollary 4

44. But in the case of the preceding corollary, when $b > c$, a constant of this kind is required to be added, so that by putting $\varphi = 0$ there becomes

$$\sqrt{v} = \frac{z}{\sqrt{4k}} = \cos\zeta \cdot \sqrt{k},$$

and thus there will be

$$\frac{\sqrt{v}}{\sqrt{(b-c)}} = \operatorname{tang}\left(A \operatorname{tang} \frac{\cos\zeta \cdot \sqrt{k}}{\sqrt{(b-c)}} + \frac{a\varphi \sin\zeta}{2b}\sqrt{\frac{b-c}{k}} \right);$$

but with time progressing φ shall become and thus the ascent stops completely, when there becomes

$$-\varphi = \frac{2bk\cos\zeta}{a(b-c)\sin\zeta}.$$

Scholium

45. I have assumed in the integration of this case initially the spiral was filled with water and to have began to rotate suddenly; thus indeed everywhere the speed of the water at first progressing through the screw becomes $= \cos\zeta \cdot \sqrt{k}$. But if thus the initial state may be considered, so that the screw may be driven with the lower opening closed, then truly to be opened again at once, it is necessary that the water at this moment itself shall have adjusted itself to the motion of the tube, thus to that the initial motion is going to become $v = 0$. Therefore on account of this matter the water at once will begin to fall, unless any drop may be ejected above, if indeed there shall be $b > c$. Nevertheless in this case, when $\theta = 0$, happily it is allowed to occur, yet for the situation of the inclined screw, certainly nothing is allowed to be freed from the equation found, but the nature of the motion of the water in these cases remains hidden from us, therefore since this equation is unable to be treated by referring to the formula of Riccati. From which example we acknowledge a significant failing of the analysis, because for a machine especially well known in the most frequent use the effect may depend on the resolution of an equation of this kind, for which the analytical skills at this stage may not be sufficient to uncover, which case by me is considered thus to be wonderful, so that, even if the goal in this investigation, which I had proposed for myself, I did not reach, yet this I have brought into being the most dignified argument, so that I may inspire the strengths of the Geometers to extricate that completely, by which labor not only will the greatest mechanical conveniences be rewarded in excess, but also the boundaries of analysis will be advanced not a little.

DE COCHLEA ARCHIMEDIS

Cochlea ARCHIMEDIS cum ob inventionis antiquitatem, Tum ob eius frequentissimum usum in aquis hauriendis tantopere celebrata atque in vulgus cognita, ut vix ullus Hydraulicarum Machinarum scriptor reperiatur, qui eius constructionem atque utilitatem non abunde explicuerit. Quodsi vero ad causam spectemus, cur haec machina ad aquam elevandam sit apta et quomodo eius actio secundum mechanica principia absolvatur, apud vetustiores quidem auctores nihil plane invenimus, quod rationem saltem probabilem in se contineat, recentiores vero hanc investigationem vel prorsus praeterierunt, vel leviter saltem ac minus accurate sunt persecuti. Ita quamvis haec machina sit notissima eiusque praxis frequentissima, tamen fateri cogimur, eius Theoriam maxime adhuc esse absconditam atque tam modum, quo aqua per eam elevatur, quam vires ad eius actionem requisitas etiam nunc fere penitus latere. Atque hoc eo magis mirum videri debet, cum non solum ceterae Machinae ab antiquitate ad nos transmissae felici cum successu ad leges mechanicas sint revocatae, sed etiam ipsa scientia mechanica eousque exculta sit, ut ad omnis generis machinas explicandas sufficiens videatur. Quin etiam a plerisque omne studium, quod a Geometris ope Analyseos sublimioris in Mechanica ulterius excolenda consumitur, subtile magis quam utile censeri solet.

Verum si rationem cochleae ARCHIMEDIS diligentius contemplemur, vulgaria mechanicae principia ei explicandae minime sufficientia deprehendemus: propterea quod ea manifeste ad Theoriam motus aquae per tubas mobiles pertineat, quod argumentum a nemine fere adhuc est tractatum. Quod enim ad motum aquae in genere attinet, non dudum admodum est, ex quo is studiosius investigari atque ad principia mechanica investigari est coeptus, de motu autem aquae per tubas mobiles vix quisquam reperitur, qui aliquid in medium attulerit, vel tantum cogitaverit. Quamobrem cum nunc quidem principia, quibus omnis aquae motus innititur, satis sint evoluta, operam dabo, ut ea quoque ad motum aquae, quo per cochleam hanc ARCHIMEDIS fertur, accommodem, indeque omnia phaenomena, quae in hoc motu consideranda occurrant, clare ac distincte explanem. Quae igitur hac de re sum meditatus, sequentibus propositionibus sum complexurus; et quoniam cochleae ARCHIMEDEAE duplicis generis construi solent, quarum alterae helicis suas circa cylindrum, alterae vero circa conum habent circumvolutas, a cochlea cylindrica exordiar eiusque Theoria stabilita ad cochleas quoque conicas perscrutandas non difficulter progredi licebit.

PROBLEMA 1.

1. *Dato motu, quo cylindrus circumagitur et aquae celeritate per cochleam seu helicem cylindro circumductam, determinare verum cuiusque aquae particulae motum, hoc est eum motum, qui ex motu gyratorio cylindri et motu aquae progressivo per helicem componitur.*

Solutio

Sit (Fig.1) circulus *ACB* basis cylindri, cuius superficiei helix est circumducta, recta *CD* ad basin in centro *C* perpendicularis axis cylindri, circa quem cylindrus cum helice in gyrum agitur. Ponatur basis semidiameter $CA = CB = a$, et sit *EZ* portio helicis in superficie cylindri, quae cum peripheria basis faciat angulum $ZEY = \zeta$ et a puncto helicis quocunque *Z* ad basin ducatur axi parallela *ZY;* voceturque arcus $EY = s$, est $YZ = s\,\mathrm{tang}\zeta$, quae cum helice faciet angulum $EZY = 90° - \zeta$; et longitudo helicis erit $EZ = \dfrac{s}{\cos\zeta}$.

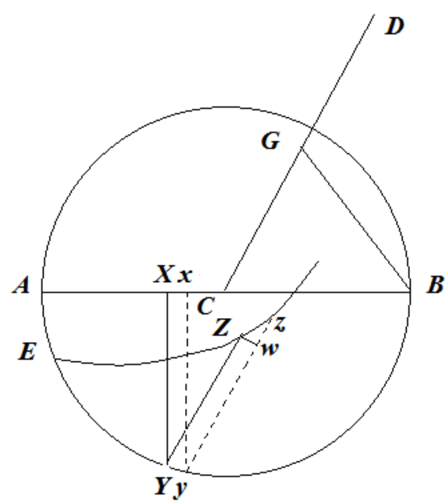

Fig. 1

Iam aquae per helicem transfluentis celeritas sit debita altitudini *v;* helicem enim *EZ* ubique eiusdem amplitudinis assumo, ita ut eodem temporis instanti omnis aquae in helice contentae eadem sit celeritas $= \sqrt{v}$. Deinde quia tota helix circa axem *CD* gyratur, sit puncti *E* celeritas gyratoria circa punctum *C* debita altitudini *u*. Recta autem *AB* sit fixa, quae scilicet non cum cylindro moveatur: atque initio quidem punctum *E* fuerit in *A*, inde autem tempore elapso $= t$ motu angulari pervenerit in *E*, sitque arcus $AE = p$, erit ob motum angularem $dp = dt\sqrt{u}$.

Nunc consideretur primo motus aquae per helicem quasi quiescentem, ac celeritas particulae aquae in *Z* erit $= \sqrt{v}$ eiusque directio erit *Zz*, qui motus resolvatur in duos, quorum alterius directio sit secundum *YZ*, alterius secundum *Zv* seu *Yy*, atque celeritas secundum *YZ* erit $= \sqrt{v}\cdot\sin\zeta$, celeritas vero secundum *Zv* seu *Yy* erit $= \sqrt{v}\cdot\cos\zeta$.

Ad hunc posteriorem motum adiungi nunc debet motus gyratorius, quippe qui in eandem directionem tendit, ex quo prodit tota celeritas puncti *Z* secundum directionem *Yy* $= \sqrt{u} + \sqrt{v}\cdot\cos\zeta$.

Quoniam vero directio *Yy* est variabilis, reducatur ea ad directiones constantes; quem in finem ex *Y* ad rectam fixam *AB* ducatur perpendicularis *YX*, ac vocentur tres coordinatae locum puncti *Z* determinantes

$$CX = x,\ XY = y \text{ et } YZ = z,$$

erit prima $z = s\operatorname{tang}\zeta$; tum vero ob arcum $AY = p + s$, et angulum $ACY = \frac{p+s}{a}$ erit

$$CX = x = a\cos\frac{p+s}{a} \text{ et } XY = y = a\sin\frac{p+s}{a}.$$

Tum ducta *Yu* rectae *AB* parallela erit

$$\text{angulus } Yyu = \frac{p+s}{a}.$$

Hinc motus secundum *Yy* resolvetur in binos alios, alterum secundum *Yu* seu *AC* cuius celeritas

$$= \left(\sqrt{u} + \sqrt{v}\cdot\cos\zeta\right)\sin\frac{p+s}{a},$$

alterum vero secundum *XY* cuius celeritas

$$= \left(\sqrt{u} + \sqrt{v}\cdot\cos\zeta\right)\cos\frac{p+s}{a};$$

celeritate secundum *YZ* existente $= \sqrt{v}\cdot\sin\zeta$.

Quare loco puncti *Z* ad ternas coordinatas fixas reducto, quae sunt:

$$CX = x = a\cos\frac{p+s}{a}, XY = y = a\sin\frac{p+s}{a} \text{ et } YZ = z = s\operatorname{tang}\zeta,$$

verus particulae in *Z* versantis motus pariter secundum has ternas directiones fixas resolvetur, eritque

Celeritas motus secundum $CX = -\left(\sqrt{u} + \sqrt{v}\cdot\cos\zeta\right)\sin\frac{p+s}{a}$,

Celeritas motus secundum $XY = +\left(\sqrt{u} + \sqrt{v}\cdot\cos\zeta\right)\cos\frac{p+s}{a}$,

Celeritas motus secundum $YZ = \sqrt{v}\cdot\sin\zeta$.

Corollarium 1.

2. Hinc iam facile reperitur vera celeritas particulae aquae in *Z* versantis, cum enim hae ternae directiones sint inter se normales, erit vera celeritas aequalis radici quadratae ex summa quadratorum harum trium celeritatum, ex quo vera celeritas erit

$$= \sqrt{\left(u + v + 2\sqrt{uv}\cdot\cos\zeta\right)}.$$

Corollarium 2

3. Cum particula aquae in *Z* tempuscolo *dt* perveniat in helicis punctum *z*, existente

$$Zz = \frac{ds}{\cos\zeta} \text{ et } Yy = Zv = ds,$$

celeritas autem in helice sit $=\sqrt{v}$, erit $Zz = \frac{ds}{\cos\zeta} = dt\sqrt{v}$, unde fit

praeterea vero iam vidimus esse $dp = dt\sqrt{u}$.

Corollarium 3

4. Celeritates quoque particulae aquae *Z* secundum ternas directiones fixas exprimentur per differentialia coordinatarum *x, y, z* ad elementum temporis *dt* applicata.

Erit scilicet ex natura resolutionis motus:

Celeritas secundum $CX = \frac{dx}{dt} = -\left(\sqrt{u} + \sqrt{v}\cdot\cos\zeta\right)\sin\frac{p+s}{a}$

Celeritas secundum $XY = \frac{dy}{dt} = \left(\sqrt{u} + \sqrt{v}\cdot\cos\zeta\right)\cos\frac{p+s}{a}$

Celeritas secundum $YZ = \sqrt{v}\cdot\sin\zeta$.

Quarum formularum identitas intelligitur ex valoribus differentialibus $dp = dt\sqrt{u}$ et $ds = dt\sqrt{v}\cdot\cos\zeta$.

PROBLEMA 2

5. *Datis tam celeritate, qua aqua per helicem promovetur, quam celeritate, qua cylindrus cum helice circa axem CD in gyrum agitur, invenire vires, quibus quamque aquae particulam Z sollicitari oportet, ut hunc motum prosequi queat.*

Solutio

Sit celeritas qua aqua praesenti temporis momento per helicem *EZ* promovetur, $=\sqrt{v}$, celeritas autem gyratoria cylindri $=\sqrt{u}$. Tum initium helicis iam sit in *E*, ut sit $AE = p$, et particula aquae, quam consideramus, in *Z*, ut ducta *ZY* axi *CD* parallela, sit arcus $EY = s$, existente angulo helicis $YEZ = \zeta$. Porro locus puncti *Z* reducatur ad ternas coordinatas fixas

$$CX = x,\ XY = y \text{ et } YZ = z;$$

erit uti vidimus:

$$x = a\cos\frac{p+s}{a},\ y = a\sin\frac{p+s}{a} \text{ et } z = s\operatorname{tang}\zeta$$

denotante *a* semidiametrum $CA = CB$ basis cylindri. Posito vero elemento temporis $= dt$, ut sit $dp = dt\sqrt{u}$ et $ds = dt\sqrt{v}\cdot\cos\zeta$, sumtoque hoc differentiali *dt* constanti, ex principiis mechanicis constat particulam aquae in *Z* a tribus viribus acceleratricibus urgeri debere, quae sint:

secundum $CX = \frac{2ddx}{dt^2}$, secundum $XY = \frac{2ddy}{dt^2}$, secundum $YZ = \frac{2ddz}{dt^2}$.

Verum cum ex supra ostensis sit

$$\frac{dx}{dt} = -\left(\sqrt{u}+\sqrt{v}\cdot\cos\zeta\right)\sin\frac{p+s}{a},\ \frac{dy}{dt} = \left(\sqrt{u}+\sqrt{v}\cdot\cos\zeta\right)\cos\frac{p+s}{a},$$

et

$$\frac{dz}{dt} = \sqrt{v}\cdot\sin\zeta,$$

et erit denuo differentiando

$$\frac{ddx}{dt^2} = -\left(\frac{du}{2dt\sqrt{u}}+\frac{dv\cos\zeta}{2dt\sqrt{v}}\right)\sin\frac{p+s}{a}-\frac{1}{a}\left(\sqrt{u}+\sqrt{v}\cdot\cos\zeta\right)^2\cos\frac{p+s}{a},$$

$$\frac{ddy}{dt^2} = -\left(\frac{du}{2dt\sqrt{u}}+\frac{dv\cos\zeta}{2dt\sqrt{v}}\right)\cos\frac{p+s}{a}-\frac{1}{a}\left(\sqrt{u}+\sqrt{v}\cdot\cos\zeta\right)^2\sin\frac{p+s}{a},$$

$$\frac{ddz}{dt^2} = \frac{dv}{2dt\sqrt{v}}\sin\zeta.$$

Tres ergo vires acceleratrices quaesitae sunt

$$\text{I. sec. } CX = -\frac{1}{dt}\left(\frac{du}{\sqrt{u}}+\frac{dv\cos\zeta}{\sqrt{v}}\right)\sin\frac{p+s}{a}-\frac{2}{a}\left(\sqrt{u}+\sqrt{v}\cdot\cos\zeta\right)^2\cos\frac{p+s}{a},$$

$$\text{II. sec. } XY = +\frac{1}{dt}\left(\frac{du}{\sqrt{u}}+\frac{dv\cos\zeta}{\sqrt{v}}\right)\cos\frac{p+s}{a}-\frac{2}{a}\left(\sqrt{u}+\sqrt{v}\cdot\cos\zeta\right)^2\sin\frac{p+s}{a},$$

$$\text{III. sec. } YZ = \frac{dv}{dt\sqrt{v}}\sin\zeta.$$

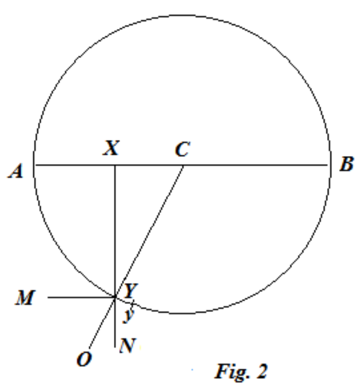

Fig. 2

Corollarium 1

6. Transferantur (Fig. 2) duae priores vires primum in punctum *Y*, ita ut hoc punctum a duabus viribus acceleratricibus urgeatur, secundum directiones *YM* et *YN*, quae sunt

$$\text{Vis sec. } YM = -\frac{1}{dt}\left(\frac{du}{\sqrt{u}}+\frac{dv\cos\zeta}{\sqrt{v}}\right)\sin\frac{p+s}{a}-\frac{2}{a}\left(\sqrt{u}+\sqrt{v}\cdot\cos\zeta\right)^2\cos\frac{p+s}{a},$$

$$\text{Vis sec. } YN = +\frac{1}{dt}\left(\frac{du}{\sqrt{u}}+\frac{dv\cos\zeta}{\sqrt{v}}\right)\cos\frac{p+s}{a}-\frac{2}{a}\left(\sqrt{u}+\sqrt{v}\cdot\cos\zeta\right)^2\sin\frac{p+s}{a}.$$

Corollarium 2

7. Nunc hae duae vires in duas alias transformari poterunt, quae agant secundum directiones *Yy* et *YO*, quarum haec sit ad superficiem cylindri normalis: atque ob angulum $MYO = ACY = \frac{p+s}{2}$, ex his duabus viribus resultabit

$$\text{I. Vis secundum } Yy = \text{ Vis } YN\cdot\cos\frac{p+s}{2}-\text{Vis } YM\cdot\sin\frac{p+s}{2},$$

$$\text{II. Vis secundum } YO = \text{ Vis } YN\cdot\sin\frac{p+s}{2}+\text{Vis } YM\cdot\cos\frac{p+s}{2}.$$

Corollarium 4

8. Hinc ergo loco duarum virium, quae sollicitabant secundum directiones *CX* et *XY*, vel *YM* et *YN*, in calculum introducentur duae hae aliae secundum directiones *Yy* et *YO*, quae erunt

$$\text{Vis secundum } Yy = +\frac{1}{dt}\left(\frac{du}{\sqrt{u}}+\frac{dv\cos\zeta}{\sqrt{v}}\right),$$

$$\text{Vis secundum } YO = -\frac{2}{a}\left(\sqrt{u}+\sqrt{v}\cdot\cos\zeta\right)^2,$$

sicque angulus $p+s$ non amplius in calculo reperitur.

PROBLEMA 3

9. *Tres vires ante inventas ad tres alias reducere, quarum una sit secundum directionem helicis Z z directa, duae reliquae vero sint ad ipsam helicem normales.*

Solutio

Sit (Fig. 3) *Zz* elementum helicis, ubi nunc particula aquae, quae vires inventas sustinet, versatur: sitque *Zo* non solum ad helicem *Zz*, sed etiam ad ipsius cylindri superficiem in *Z* normalis, deinde sit recta *Zr* in ipsa superficie cylindri sita, atque ad *Zz* normalis. Tres igitur vires inventae ad tres alias reduci debent, quae particulam aquae sollicitent secundum directiones *Zz*, *Zo* et *Zr.* Ac primo quidem vis inventa secundum *YZ* agens

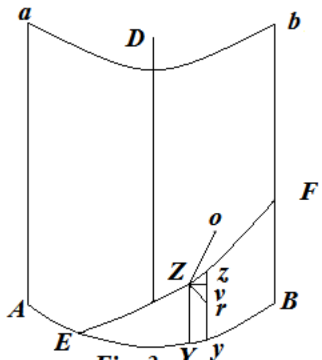

Fig. 3

$$=\frac{dv}{dt\sqrt{v}}\sin\zeta,$$

ob angulum helicis $YEZ=\zeta$, dabit

I. vim secundum $\quad Zr=-\dfrac{dv}{dt\sqrt{v}}\sin\zeta\cos\zeta,$

II. vim secundum $\quad Zz=+\dfrac{dv}{dt\sqrt{v}}\sin\zeta\sin\zeta,$

Deinde vis, quae secundum directionem *Yy* seu *Zv* agere inventa est

$$=\frac{1}{dt}\left(\frac{du}{\sqrt{v}}+\frac{dv\cos\zeta}{\sqrt{v}}\right),$$

dabit vires

I. secundum $\quad Zz=\dfrac{du}{dt\sqrt{u}}\cos\zeta+\dfrac{dv}{dt\sqrt{v}}\cos^2\zeta,$

II. secundum $\quad Zr=\dfrac{du}{dt\sqrt{u}}\sin\zeta+\dfrac{dv}{dt\sqrt{v}}\sin\zeta\cos\zeta.$

Tertio vis, quae secundum directionem *YO* agere est inventa, dabit nunc sola vim secundum $\;Zo=-\dfrac{2}{a}\left(\sqrt{u}+\sqrt{v}\cdot\cos\zeta\right)^2.$

Quare tres vires acceleratrices, quibus particula aquae in *Z* sollicitari debet, ut motum propositum persequatur, erunt:

I. secundum directionem *Zz* $\quad =\dfrac{du}{dt\sqrt{u}}\cos\zeta+\dfrac{dv}{dt\sqrt{v}},$

II. secundum directionem *Zr* $\quad =\dfrac{du}{dt\sqrt{u}}\sin\zeta,$

III. secundum directionem *Zo* $\quad =-\dfrac{2}{a}\left(\sqrt{u}+\sqrt{v}\cdot\cos\zeta\right)^2.$

Scholion 10.

Habemus ergo vires, quibus singulae aquae particulae sollicitatae esse debent, ut motus, quem assumsimus, subsistere possit. Istas autem vires hic ideo ad tres directiones *Zz*, *Zr* et *Zo* revocavi, quo facilius cum viribus, quibus aqua in tubo actu sollicitatur, comparari possint; ut enim quantitates *v* et *u* verum aquae et cylindri motum exhibeant, necesse est, ut tres illae vires inventae conveniant cum viribus, quibus aqua revera urgetur. Hae autem vires sunt primo status compressionis aquae in tubo, deinde appressio aquae ad latera tubi, quae secundum ambas directiones *Zr* et *Zo* ad directionem tubi normales exhiberi solet. Tertio vero gravitas, qua singulae aquae particulae deorsum nituntur, imprimis examini est subiicienda, quod sequenti problemate instituemus.

PROBLEMA 4

11. *Si cylindrus fuerit utcunque ad horizontem inclinatus, definire vires secundum ternas praedictas directiones, quibus singulae aquae particulae Z in helice ob gravitatem sollicitantur.*

Solutio

Exprimat (Fig. 4) angulus θ inclinationem basis cylindri ad horizontem, sitque in plano basis punctum fixum *A* summum, punctum *B* vero imum, ita ut recta *AB* cum axe cylindri *CD* in plano verticali sit constituta. In hoc plano per centrum basis *C* ducatur horizontalis *CH*, eritque angulus $ACH = \theta$, seu si ex puncto *B* erigatur recta verticalis *B G* axem in *G* intersecans, erit quoque angulus $BGC = \theta$, Fig. 4 atque ob gravitatem singulae aquae particulae sollicitabuntur deorsum secundum directiones ipsi *GB* parallelas, et vis acceleratrix haec ubique erit $=1$. Iam in prima figura ducatur quoque recta *BG* cum axe *CD* constituens angulum $BGC = \theta$, ac particula aquae in *Z* urgebitur vi acceleratrice $=1$ secundum directionem rectae *BG* parallelam. Resolvatur haec vis secundum directiones *GC* et *CB*, prodibitque

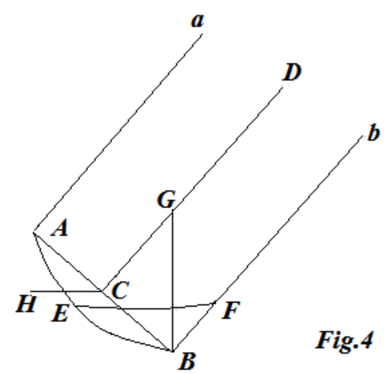

Fig.4

$$\text{vis secundum } GC = 1\cdot cos\,\theta\,,$$

et

$$\text{vis secundum } CB = 1\cdot \sin\theta.$$

Ex priori habebimus pro particula aquae *Z* (Fig. 2) vim secundum $ZY = cos\,\theta$, ex posteriori vero vim secundum $YM = -\sin\theta$, unde ob angulum

$$MYO = \frac{p+s}{a}$$

oritur

vis secundum YO $= -\sin\theta\cos\frac{p+s}{a}$

et

vis secundum Yy $= +\sin\theta\sin\frac{p+s}{a}.$

Hinc ergo (Fig. 3) punctum Z sollicitabitur ab his tribus viribus acceleratricibus:

I. secundum directionem ZY vi $= \cos\theta$,

II. secundum directionem Zo vi $= -\sin\theta\cos\frac{p+s}{a}$,

III. secundum directionem Zv vi $= +\sin\theta\sin\frac{p+s}{a}$.

Ex his porro ob angulum $zZv = \zeta$ orientur:
Prima vis secundum $Zz =$ vi $Zv \cdot \cos\zeta -$ vi $ZY \cdot \sin\zeta$.
Tum vis secundum $Zr =$ vi $Zv \cdot \sin\zeta +$ vi $ZY \cdot \cos\zeta$.
Quare pro tribus directionibus Zz, Zr et Zo obtinebimus sequentes vires acceleratrices ex gravitate oriundas:

I. Vim secundum $Zz = \cos\zeta\sin\theta\sin\frac{p+s}{a} - \sin\zeta\cos\theta$,

II. Vim secundum $Zr = \sin\zeta\sin\theta\sin\frac{p+s}{a} + \cos\zeta\cos\theta$,

III. Vim secundum $Zo = -\sin\theta\cos\frac{p+s}{a}$.

PROBLEMA 5

12. *Dato* (Fig. 3), *ut hactenus, tam cylindri, quam aquae per helicem motu, definire statum compressionis aquae in singulis helicis punctis.*

Solutio

Praesenti temporis instanti, quo initium helicis est in E, existente arcu $AE = p$, consideremus helicis punctum Z, ut sit

$$EY = s \text{ et } YZ = s\,\mathrm{tang}\,\zeta$$

existente helicis angulo $YEZ = \zeta$, sitque status compressionis aquae in puncta $Z = q$, seu denotet q profunditatem, ad quam aqua quiescens in pari statu compressionis existat, eritque pro hoc momento q functio quaepiam ipsius s, et in puncta proximo z, existente $Yy = ds$, status compressionis erit $= q + dq$. Sit iam amplitudo helicis $= hh$, erit particula aquae in portiuncula Zz contenta

$$=\frac{hhds}{\cos\zeta};$$

quae ergo in Z propelletur vi motrice $=hhq$, in z vero repelletur vi

$$=hh(q+dq);$$

unde existit vis motrix repellens seu secundum zZ urgens praebet vim acceleratricem $=hhdq$, quae

$$=\frac{dq\cos\zeta}{ds}.$$

Quare ob statum compressionis particula aquae in elemento helicis Zz contenta secundum directionem Zz sollicitabitur vi acceleratrice

$$=-\frac{dq\cos\zeta}{ds}.$$

Praeterea vero ob gravitatem eadem particula, uti vidimus, sollicitatur secundum Zz vi acceleratrice

$$=\cos\zeta\sin\theta\sin\frac{p+s}{a}-\sin\zeta\cos\theta,$$

unde coniunctim tam ob gravitatem, quam ob statum compressionis aquae, particula aquae in helicis puncto Z contenta urgebitur secundum directionem Zz vi acceleratrice, quae erit

$$=\cos\zeta\sin\theta\sin\frac{p+s}{a}-\sin\zeta\cos\theta-\frac{dq\cos\zeta}{ds},$$

haecque est vis, qua ista particula actu urgetur secundum directionem Zz ; ex quo necesse est, ut ea aequalis sit illi vi, qua supra punctum Z ad motus conservationem sollicitari debere invenimus secundum eandem directionem $Z\,z$. Quae cum sit inventa

$$=\frac{du}{dt\sqrt{u}}\cos\zeta+\frac{dv}{dt\sqrt{v}},$$

habebimus hanc aequationem:

$$dq\cos\zeta=ds\cos\zeta\sin\theta\sin\frac{p+s}{a}-ds\sin\zeta\cos\theta-\frac{du}{dt\sqrt{u}}ds\cos\zeta-\frac{dv}{dt\sqrt{v}}ds,$$

ubi, quoniam ad praesens tantum temporis momentum respicimus, quantitates a tempore t pendentes, quae sunt p, u, v, itemque $\frac{du}{dt}$ et $\frac{dv}{dt}$, tanquam constantes sunt spectandae, ex quo integratione instituta habebimus

$$q\cos\zeta=C-a\cos\zeta\sin\theta\cos\frac{p+s}{a}-s\sin\zeta\cos\theta-\frac{sdu\cos\zeta}{dt\sqrt{u}}-\frac{sdv}{dt\sqrt{v}},$$

unde status compressionis aquae in singulis helicis punctis pro praesenti temporis momento innotescit.

PROBLEMA 6

13. *Si data aquae portio in helice reperiatur atque cylindrus datam ad horizontem inclinationem tenens motu quocunque in gyrum agatur, invenire motum, quo ista aquae portio per helicem promovebitur.*

Solutio

Sit (Fig. 5) basis cylindri semidiameter $CA = CB = a$, et angulus, quem helix *EF* cum basi cylindri constituit $BEF = \zeta$, et angulus. Axis autem cylindri *PQ* cum recta verticali *QR* constituat angulum $PQR = \theta$, quo eodem angulo basis cylindri ad horizontem erit inclinata. In basi autem sit *A* punctum summum et *B* infimum. Praesenti autem temporis

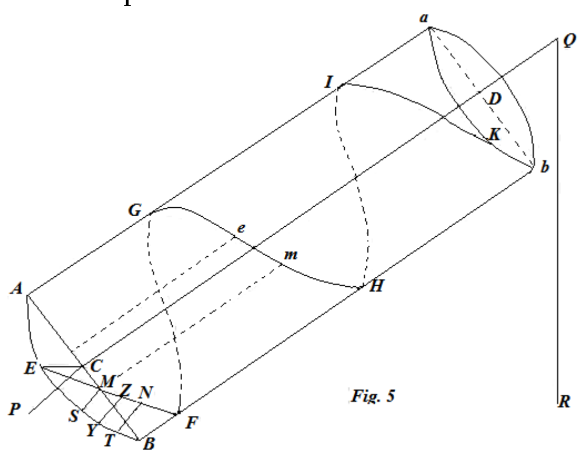

Fig. 5

momento sit initium helicis in *E*, existente eius a puncto summo intervallo seu arcu $AE = p$: et cylindrus in plagam *AEB* gyretur, ita ut puncti *E* celeritas sit $= \sqrt{u}$, erit $dp = dt\sqrt{u}$. Occupet nunc portio aquae in helice contenta spatium *MN*, cuius longitudo sit $MN = f$, ac ductis axi parallelis *MS* et *NT* sit aquae ab initio helicis distantia $EM = x$, erit

$$EN = x + f \text{ et } ES = x\cos\zeta \text{ atque } ET = (x + f)\cos\zeta;$$

celeritas vero, qua haec aquae portio praesenti momento per helicem promovetur, $= \sqrt{u}$. His positis, si in portione aquae *MN* punctum quodpiam medium *Z* consideretur et arcus *EY* ponatur $= s$, erit status compressionis aquae in *Z*, qui per altitudinem *q* exprimatur, uti in problemate praecedente est erutus:

$$q\cos\zeta = C - a\cos\zeta\sin\theta\cos\frac{p+s}{a} - s\sin\zeta\cos\theta - \frac{sdu\cos\zeta}{dt\sqrt{u}} - \frac{sdv}{dt\sqrt{v}}.$$

Iam vero constat in utroque termino *M* et *N* statum compressionis evanescere debere; sive ergo ponatur $s = x\cos\zeta$ sive $s = (x + f)\cos\zeta$, fieri debet $q = 0$: unde duplex nascitur aequatio

$$0 = C - a\cos\zeta\sin\theta\cos\frac{p + x\cos\zeta}{a} - x\sin\zeta\cos\theta$$
$$- x\cos\zeta\left(\frac{du\cos\zeta}{dt\sqrt{u}} + \frac{dv}{dt\sqrt{v}}\right),$$
$$0 = C - a\cos\zeta\sin\theta\cos\frac{p + (x+f)\cos\zeta}{a}$$
$$-(x+f)\cos\zeta\left(\sin\zeta\cos\theta + \frac{du\cos\zeta}{dt\sqrt{u}} + \frac{dv}{dt\sqrt{v}}\right),$$

unde, constantem C eliminando, obtinebitur, dividendo per $\cos\zeta$, haec aequatio
$$a\sin\theta\cos\frac{p + x\cos\zeta}{a} = a\sin\theta\cos\frac{p + (x+f)\cos\zeta}{a} + f\sin\zeta\cos\theta + \frac{fdu\cos\zeta}{dt\sqrt{u}} + \frac{fdv}{dt\sqrt{v}},$$

unde motus aquae per helicem definiri debet, uti enim est $dp = dt\sqrt{u}$, ita erit $dx = dt\sqrt{v}$.
Multiplicetur ergo haec aequatio per
$$dp + dx\cos\zeta = dt\sqrt{u} + dt\cos\zeta\cdot\sqrt{v},$$
eritque integrando
$$a^2\sin\theta\sin\frac{p + x\cos\zeta}{a} = a^2\sin\theta\sin\frac{p + (x+f)\cos\zeta}{a} + f(p + x\cos\zeta)\sin\zeta\cos\theta$$
$$+ f\int\left(\frac{du\cos\zeta}{\sqrt{u}} + \frac{dv}{\sqrt{v}}\right)\left(\sqrt{u} + \cos\zeta\cdot\sqrt{v}\right).$$

Corollarium 1

14. Si igitur motus gyrationis cylindri fuerit uniformis, seu u constans, ponatur $u = k$, ob $du = 0$ erit
$$a^2\sin\theta\sin\frac{p + x\cos\zeta}{a} = a^2\sin\theta\sin\frac{p + (x+f)\cos\zeta}{a} + f(p + x\cos\zeta)\sin\zeta\cos\theta$$
$$+ 2f\sqrt{kv} + fv\cos\zeta + \text{Const.}$$
ubi est $p = t\sqrt{k}$, ita ut haec aequatio ob $\sqrt{v} = dx$ duas tantum variabiles t et x involvat. Constans autem ex statu initiali debet definiri.

Corollarium 2

15. Si portio aquae in tubo MN fuerit infinite parva seu $f = 0$, erit
$$\sin\frac{p + (x+f)\cos\zeta}{a} = \sin\frac{p + x\cos\zeta}{a} + \frac{f\cos\zeta}{a}\cos\frac{p + (x+f)\cos\zeta}{a}$$

hoc ergo casu motus definietur hac aequatione :

$$\text{Const.} = a\cos\zeta\sin\theta\cos\frac{p+x\cos\zeta}{a} + (p+x\cos\zeta)\sin\zeta\cos\theta + 2\sqrt{kv} + v\cos\zeta.$$

Quodsi ergo haec particula initio quieverit in *E*, punctumque *E* fuerit in *A*, ita ut posito $x=0$ sit $p=0$ et $v=0$, erit

$$a\cos\zeta\sin\theta\left(1-\frac{\cos p + x\cos\zeta}{a}\right) = (p+x\cos\zeta)\sin\zeta\cos\theta + 2\sqrt{kv} + v\cos\zeta.$$

Corollarium 3

16. Si in casu corollarii praecedentis ponatur

$$\text{angulus } \frac{p+x\cos\zeta}{a} = \varphi, \text{ut sit } dt = \frac{ad\varphi}{\sqrt{k}+\cos\varphi\cdot\sqrt{v}},$$

ob $dp = dt\sqrt{k}$ et $dx = dt\sqrt{v}$, relatio inter φ et v hac exprimetur aequatione:

$$a\cos\zeta\sin\theta(1-\cos\varphi) = a\varphi\sin\zeta\cos\theta + 2\sqrt{kv} + v\cos\zeta$$

ex qua fit

$$\sqrt{k}+\cos\zeta\cdot\sqrt{v} = \sqrt{\left(k - a\varphi\sin\zeta\cos\zeta\cos\theta + a\cos^2\zeta\sin\theta(1-\cos\varphi)\right)}$$

ideoque

$$dt = \frac{ad\varphi}{\sqrt{\left(k - a\varphi\sin\zeta\cos\zeta\cos\theta + a\cos^2\zeta\sin\theta(1-\cos\varphi)\right)}}$$

Corollarium 4

17. Simili modo si generaliter, posito tamen motu gyratorio constante seu $u=k$, ponatur

$$\frac{p+x\cos\zeta}{a} = \varphi \text{ et } \frac{f\cos\zeta}{a} = \gamma,$$

erit quoque

$$dt = \frac{ad\varphi}{\sqrt{k}+\cos\zeta\cdot\sqrt{v}}$$

et

$$\frac{a\cos\xi\sin\theta}{\gamma}\sin\varphi = \frac{a\cos\xi\sin\theta}{\gamma}\sin(\gamma+\varphi) + a\varphi\sin\zeta\cos\theta + 2\sqrt{kv} + v\cos\zeta + C$$

ideoque

$$\sqrt{k}+\cos\zeta\cdot\sqrt{v}=\sqrt{\left(C+\frac{a}{\gamma}\cos^2\zeta\sin\theta\left(\sin\varphi-\sin\ (\gamma+\varphi)\right)-a\varphi\sin\zeta\cos\zeta\cos\theta\right)},$$

unde fit

$$dt=\frac{ad\varphi}{\sqrt{\left(C+\frac{a}{\gamma}\cos^2\zeta\sin\theta\left(\sin\varphi-\sin\ (\gamma+\varphi)\right)-a\varphi\sin\zeta\cos\zeta\cos\theta\right)}},$$

ubi φ denotat angulum ACS et γ angulum *SCT*, qui est constans.

Corollarium 5

18. Si cylindrus in partem contrariam celeritate $=\sqrt{k}$ circumagatur, pro $\sqrt{k}$ scribi debet $-\sqrt{k}$, arcusque *p* negative erit accipiendus, ita ut sit

$$\varphi=x\cdot\cos\zeta\ .$$

Quare cum sit

$$p>x\cdot\cos\zeta\ ,$$

etiam angulus φ negative accipiatur, habebimus ergo pro hoc motu:

$$\varphi=\frac{p-x\cos\zeta}{a}\quad\text{et}\quad dt=\frac{ad\varphi}{\sqrt{k}-\ \cos\zeta\cdot\sqrt{v}}$$

atque

$$\sqrt{k}-\cos\zeta\cdot\sqrt{v}=\sqrt{\left(C-\frac{a}{\gamma}\cos^2\zeta\sin\theta\left(\sin\varphi-\sin\ (\varphi-\gamma)\right)+a\varphi\sin\zeta\cos\zeta\cos\theta\right)}\ .$$

Corollarium 6

19. Si hoc casu initio $t=0$, quo erat $p=0$ et $\sqrt{v}=0$, fuerit $x=EM=g$, ideoque $\varphi=-\frac{g\cos\zeta}{a}$, ponamus hunc angulum initialem $ECS=\varepsilon$, ut fuerit initio $\varphi=-\varepsilon$, erit

$$\sqrt{k}-\cos\zeta\cdot\sqrt{v}=$$

$$\sqrt{\left(k+\frac{a}{\gamma}\cos^2\zeta\sin\theta\left(\sin\ (\varepsilon+\gamma)-\sin\varepsilon-\sin\varphi+\sin(\varphi-\gamma)\right)+a(\varepsilon+\varphi)\sin\zeta\cos\zeta\cos\theta\right)}\ .$$

PROBLEMA 7

20. *Si dum cylindrus data celeritate uniformiter in plagam BEA gyratur, helici in E particula aquae seu globulus inseratur, qui deinde a motu cylindri abripiatur, determinare motum globuli per helicem.*

Solutio

Sit $\sqrt{k}$ celeritas, qua punctum cylindri *E* in gyrum agitur in sensum *EA*, fueritque eo momento, quo globulus in orificium helicis *E* immittitur, angulus $ACE = \alpha$ et $t = 0$. Fieri autem nequit, ut celeritas globuli initialis sit $= 0$; si enim celeritas eius respectu tubi secundum *EM* ponatur $= \sqrt{v}$, eius celeritas vera erit

$$\sqrt{(k + v - 2\cos\zeta \cdot \sqrt{kv})},$$

quae non potest evanescere. Ponamus ergo hanc celeritatem initio fuisse minimam, ac reperimus $\sqrt{v} = \cos\zeta \cdot \sqrt{k}$, ita ut celeritas vera fuerit $= \sin\xi \cdot \sqrt{k}$, cuius directio ad *EM* erit normalis. Iam elapso tempore *t*, sit ut supra $AE = p$; globulus vero reperiatur in *M* existente $EM = x$, cuius celeritas relativa in tubo secundum *MN* sit $= \sqrt{v}$, erit

$$dp = -dt\sqrt{k} \quad \text{et} \quad dx = dt\sqrt{v}$$

et per paragraphum 15 motus definietur hac aequatione, sumta scilicet celeritate $\sqrt{k}$ negativa :

$$\text{Const.} = a\cos\zeta\sin\theta\cos\frac{p + x\cos\zeta}{a} + \left(p + x\cos\zeta\right)\sin\zeta\cos\theta - 2\sqrt{kv} + v\cos\zeta.$$

Constans autem ita est definienda, ut posito $t = 0$ seu $\frac{p}{a} = \alpha$ fiat

$$x = 0 \quad \text{et} \quad \sqrt{v} = \cos\zeta \cdot \sqrt{k},$$

sicque erit

$$\text{Const} = a\cos\zeta\sin\theta\cos\alpha + \alpha a\sin\zeta\cos\theta - 2k\cos\zeta + k\cos^3\zeta.$$

Ponatur angulus $ACS = \varphi$, erit

$$\varphi = \frac{p + x\cos\zeta}{a} \quad \text{et} \quad d\varphi = \frac{-dt\sqrt{k} + dt\cos\zeta \cdot \sqrt{v}}{a}.$$

Confecerit autem cylindrus motu angulari tempore *t* angulum $= \omega$ in plagam *BEA*, erit

$$d\omega = \frac{dt\sqrt{k}}{a} \quad \text{et} \quad \omega = \frac{t\sqrt{k}}{a},$$

quem angulum loco temporis, tamquam eius mensuram, in calculum introducamus, erit

$$\frac{p}{a} = \alpha - \omega, \quad \varphi = \alpha - \omega + \frac{x\cos\zeta}{a};$$

et ob

$$dx = dt\sqrt{v} = \frac{ad\omega\sqrt{v}}{\sqrt{k}}$$

habebimus

$$d\varphi = -d\omega + \frac{d\omega \cos\zeta \cdot \sqrt{v}}{\sqrt{k}}$$

seu

$$d\omega = \frac{d\varphi\sqrt{k}}{-\sqrt{k} + \cos\zeta \cdot \sqrt{v}}.$$

Nostra autem aequatio erit

$$\begin{aligned} &a\cos\zeta\sin\theta\cos\alpha + \alpha a\sin\zeta\cos\theta - 2k\cos\zeta + k\cos^3\zeta \\ &= a\cos\zeta\sin\theta\cos\varphi + a\,\varphi\sin\zeta\cos\theta - 2\sqrt{kv} + v\cos\zeta, \end{aligned}$$

ex qua obtinemus:

$$\cos\zeta \cdot \sqrt{v} - \sqrt{k} = -\sqrt{\left(k\sin^4\zeta + a\cos^2\zeta\sin\theta\left(\cos\alpha - \cos\ \varphi\right) + a\left(\alpha - \varphi\right)\ \sin\zeta\cos\zeta\cos\theta\right)}$$

unde ad datum valorem ipsius φ elicimus valorem ipsius $\sqrt{v}$, quo invento erit

$$d\omega = \frac{-d\varphi\sqrt{k}}{\sqrt{\left(k\sin^4\zeta + a\cos^2\zeta\sin\theta\left(\cos\alpha - \cos\ \varphi\right) + a\left(\alpha - \varphi\right)\sin\zeta\cos\zeta\cos\theta\right)}},$$

cuius integrale ita debet capi, ut posito $\omega = 0$ fiat $\varphi = \alpha$. Ex hac ergo aequatione integrali vicissim ad datum tempus angulo ω expressum, reperitur angulus φ ex eoque porro locus globuli in helice seu portio

$$EM = x = \frac{a\left(\varphi - \alpha + \omega\right)}{\cos\zeta}$$

eiusque insuper celeritas relativa in helice $\sqrt{v}$ scilicet

$$\sqrt{v} = \frac{\sqrt{k}}{\cos\zeta} - \sqrt{\left(k\sin^2\zeta\,\operatorname{tang}^2\zeta + a\sin\theta\left(\cos\alpha - \cos\ \varphi\right) + a\left(\alpha - \varphi\right)\operatorname{tang}\zeta\cos\theta\right)}.$$

Corollarium 1

21. Expressio $\cos\zeta\sqrt{v} - \sqrt{k}$ designat celeritatem veram puncti S in basi, quod globulo in M respondet. Cum enim globulus velocitate $\sqrt{v}$ in helice secundum MN progredi ponatur, erit eius celeritas angularis circa axem $= \cos\zeta \cdot \sqrt{v}$, respectu helicis ; quia autem helix ipsa in plagam opposi tam convertitur celeritate $\sqrt{k}$, erit vera globuli celeritas rotatoria, seu motus quo punctum S a summitate A recedit $= cos\zeta \cdot \sqrt{v} - \sqrt{k}$..

Corollarium 2

22. Ipso autem motus initio, quo $\sqrt{v}=\cos\zeta\cdot\sqrt{k}$; haec celeritas era negativa, scilicet

$$=\left(\cos^2\zeta-1\right)\sqrt{k}=-\sin^2\zeta\cdot\sqrt{k},$$

statim ergo ab initio etiam nunc erit negativa seu angulus $ACS=\varphi$ diminuetur, quae est ratio, cur calculus pro $cos\zeta\cdot\sqrt{v}-\sqrt{k}$ valorem praebuerit negativum

$$\cos\zeta\cdot\sqrt{v}-\sqrt{k}=-\sqrt{\left(k\sin^4\zeta+a\cos^2\zeta\sin\theta\left(\cos\alpha-\cos\varphi\right)+a\left(\alpha-\varphi\right)\sin\zeta\cos\zeta\cos\theta\right)}$$

Hic ergo valor in affirmativum abire seu angulus $ACS=\varphi$ augmenta capere nequit, nisi postquam fuerit quantitas illa radicalis $=0$. Postquam autem hoc evenerit, tum signi illius radicalis valor affirmative erit accipiendus

Corollarium 3

23. Quoniam autem ab initio angulus φ decrescit tam diu, donec valor quantitatis illius radicalis evanescit, eousque φ ultra α diminuetur seu erit $\varphi<\alpha$. Ponatur ergo $\varphi=\alpha-\psi$, ut sit

$$\cos\zeta\cdot\sqrt{v}-\sqrt{k}=-\sqrt{\left(k\sin^4\zeta+a\cos^2\zeta\sin\theta\left(\cos\alpha-\cos\left(\alpha-\psi\right)\right)+a\psi\sin\zeta\cos\zeta\cos\theta\right)}$$

sicque quamdiu augendo valorem ipsius ψ, ista quantitas radicalis realem retinet valorem, tamdiu globulus a motu cylindri in plagam *BEA* abripietur neque prius in plagam contrariam motum suum vertet, quam ubi ψ eousque increverit, ut sit

$$k\sin^4\zeta+a\cos^2\zeta\sin\theta\left(\cos\alpha-\cos\left(\alpha-\psi\right)\right)+a\psi\sin\zeta\cos\zeta\cos\theta=0.$$

Corollarium 4

24. Quia autem augendo ψ extremus terminus continuo ccrescit, medius vero qui est negativus

$$-a\cos^2\zeta\sin\theta\left(\cos(\alpha-\psi)-\cos\alpha\right)$$

tamdiu tantum ccrescit, quoad fiat $\psi=\alpha$ seu $\varphi=0$, manifestum est, nisi formula illa in nihilum abeat, antequam fiat $\psi=\alpha$, eam nunquam esse evanituram globulumque continuo celerius secundum motum cylindri gyratorium abreptum iri. Hoc ergo casu punctum *S* continuo celerius in plagam *BEA* convertetur.

Corollarium 5

25. Si ergo quantitas ista radicalis ponatur $=V$, ut sit

$$cos\zeta\cdot\sqrt{v}-\sqrt{k}=-V \text{ seu } \sqrt{v}=\frac{\sqrt{k}-V}{cos\zeta},$$

ob valorem ipsius *V* hoc casu continuo crescentem celeritas globuli progressiva in helice secundum directionem eius *EMN* tandem evanescet, posteaque adeo fiet negativa, quod ubi acciderit, globulus per helicem revertetur ac per orificium *E* iterum erumpet; siquidem cylindrus fuerit longus, ut globulus in superiori helicis termino *K* non erumpat, antequam revertatur.

Scholion

26. Cum posito $\varphi=\alpha-\psi$ et

$$V=\sqrt{\left(k\sin^4\zeta-a\cos^2\zeta\sin\theta\left(\cos(\alpha-\psi)-\cos\alpha\right)+a\psi\sin\zeta\cos\zeta\cos\theta\right)}$$

quantitas *V* tamdiu negative sit accipienda seu habeatur

$$cos\zeta\cdot\sqrt{v}-\sqrt{k}=-V,$$

quamdiu augendo angulum ψ quantitas *V* realem obtinet valorem; statim autem atque haec quantitas *V* evaserit , inde angulus ψ iterum decrescat, signumque contrarium ipsi *V* tribui debeat, ut sit

$$cos\zeta\cdot\sqrt{v}-\sqrt{k}=+V;$$

duos habebimus casus principales evolvendos, quorum altero uspiam augendo ψ ab initio fit $V=0$, altero vero hoc nunquam evenit. Statim autem ab initio fiet $V=0$, si sit vel $k=0$ vel $\zeta=0$: tum aliquo tempore post initium hoc evenire ponamus, denique vero nunquam; unde sequentes casus diligentius evolvamus.

CASUS I

27. Ponamus ergo primo motum cylindri rotatorium penitus evanescere seu esse $k=0$. Cum igitur in ipso fiat $V=0$, statim ab initio ipsi *V* contrarium signo tribui debet, ut sit

$$cos\zeta\cdot\sqrt{v}-\sqrt{k}=+V \text{ seu } \sqrt{v}=\frac{V}{cos\,\zeta}$$

atque angulus ψ inde iam erit negativus seu angulus φ; continuo crescet, ut sit

$$V=\sqrt{\left(a\cos^2\zeta\sin\theta\left(\cos\alpha-\cos\varphi\right)+a\left(\alpha-\varphi\right)\sin\zeta\cos\zeta\cos\theta\right)}$$

et

$$dt = \frac{ad\omega}{\sqrt{k}} = \frac{ad\varphi}{V} \text{ atque } EM = x = \frac{a(\varphi - \alpha)}{\cos\zeta}.$$

Quia ergo initio erat $\varphi = \alpha$ et $\sqrt{v} = 0$, ponamus tempore elapso *t* esse $\varphi = \alpha + \psi$, ut sit

$$V = \sqrt{\left(a\cos^2\zeta\sin\theta\left(\cos\alpha - \cos(\alpha + \psi)\right) - a\psi\sin\zeta\cos\zeta\cos\theta\right)}$$

et

$$x = \frac{a\psi}{\cos\zeta} \text{ atque } dt = \frac{ad\psi}{V}.$$

Hic iam perspicuum est fieri omnino non posse, ut angulus ψ continuo crescat, nisi sit $\sin\zeta\cos\zeta\cos\theta = 0$, quem casum seorsim evolvere conveniet. Quodsi vero ψ crescere cesset, quo eveniet, ubi *V* 0, ibi globulus ad statum quietis redigetur ac in helice regredi incipiet, a quo ergo momento valor ipsius *V* negative capi debebit angulusque ψ iterum decrescet, donec fiat $\psi = 0$, et tum corpus rursus in *E*, sicuti initio, haerebit; unde eundem motum denuo inchoabit.

At evenire potest, ut haec globuli reversio in ipsum quasi initium motus incidat atque angulus ψ ne minimum quidem augeri queat, quin angulus φ maneat nullus vel adeo fiat negativus.

Prior casus locum habebit, si posito ψ infinite parvo valor ipsius *V* nihilominus maneat $= 0$; id quod usu veniet si

$$a\cos^2\zeta\sin\theta\sin\alpha = a\sin\zeta\cos\zeta\cos\theta \text{ seu } \sin\alpha = \frac{\operatorname{tang}\zeta}{\operatorname{tang}\theta}$$

ac tum corpus perpetuo in puncto *E* quiescet; hic enim directio helicis erit horizontalis.

Posterior casus autem locum habebit, si

$$\sin\alpha < \frac{\operatorname{tang}\zeta}{\operatorname{tang}\theta},$$

quo globulus ne in helicem quidem ingredietur, sed statim inde delabetur; vel si cylindrus deorsum esset continuotus, mutata directione globulus per helicis partem interiorem descensurus esset; ita ut angulus ψ tum fieret negativus perinde ac valor ipsius *x* et *V*.

Hi autem casus locum non inveniunt, nisi sit $\theta > \zeta$, seu inclinatio basis cylindri ad horizontem maior, quam angulus *BEF*, quem helix cum basi cylindri constituit. Hunc autem motum in helice quiescente fusius non persequor, cum nihil habeat difficultatis.

CASUS II

28. Ponamus motum gyratorium cylindri ita esse comparatum, ut motus gyratorius globuli circa axem, qui angulo ψ indicatur et initio cum motu gyratorio cylindri in eandem plagam fuerat directus, post aliquod tempus in plagam oppositam reflectatur.

Angulus ergo ψ eo usque augeri poterit, ut fiat

$$k\sin^4\zeta = a\cos^2\zeta\sin\theta\left(\cos(\alpha-\psi)-\cos\alpha\right)-a\psi\sin\zeta\cos\zeta\cos\theta$$

seu $V=0$; hoc autem fieri nequit, nisi sit

$$\cos\left(\alpha-\psi\right)-\cos\alpha > \frac{\operatorname{tang}\zeta}{\operatorname{tang}\theta}\cdot\psi;$$

cum igitur ab initio fuisset $\psi=0$, necesse est, ut posito ψ evanescente, sit

$$\sin\alpha > \frac{\operatorname{tang}\zeta}{\operatorname{tang}\theta}.$$

Deinde valor ipsius

$$\cos\left(\alpha-\psi\right)-\cos\alpha - \frac{\operatorname{tang}\zeta}{\operatorname{tang}\theta}\cdot\psi$$

erit maximus, si

$$\sin\alpha = \frac{\operatorname{tang}\zeta}{\operatorname{tang}\theta}$$

Concipiamus hoc pro ψ valore substituto fieri

$$\cos\left(\alpha-\psi\right)-\cos\alpha - \frac{\operatorname{tang}\zeta}{\operatorname{tang}\theta}\cdot\psi = M$$

atque ut valor ipsius V augendo ψ tandem evanescere queat, necesse est, ut sit

$$k\sin^4\zeta < aM\cos^2\zeta\sin\theta.$$

Quare ut hic casus locum habere possit, sequentes tres conditiones requiruntur:

I. ut sit $\operatorname{tang}\theta > \operatorname{tang}\zeta$ seu $\theta > \zeta$; ita ut fractio $\dfrac{\operatorname{tang}\zeta}{\operatorname{tang}\theta}$ unitatem non excedat,

II. ut sit $\sin\alpha > \dfrac{\operatorname{tang}\zeta}{\operatorname{tang}\theta}$ ac denique

III. ut sit $k < aM\,\dfrac{\cos\zeta^2\sin\theta}{\sin\zeta^4}$.

Quoties ergo hae tres conditiones locum inveniunt, globulus in helice in sensum *BEA* circa axem cylindri circumferetur, donec descripserit angulum ψ, ut fiat:

$$V=\sqrt{\left[k\sin^4\zeta - a\cos^2\zeta\sin\theta\left(\cos(\alpha-\psi)-\cos\alpha\right)+a\psi\sin\zeta\cos\zeta\cos\theta\right]}=0,$$

tumque erit

$$cos\zeta\cdot\sqrt{v}-\sqrt{k}=0,$$

seu globuli celeritas relativa per helicem $\sqrt{v} = \frac{\sqrt{k}}{cos\zeta}$; ; cum antequam ad hunc locum perveniat, sit

$$\sqrt{v} = \frac{\sqrt{k} - V}{cos\zeta},$$

existente

$$x = \frac{a\omega - a\psi}{\cos\zeta} \text{ et } d\omega = \frac{dt\sqrt{k}}{a} = \frac{d\psi\sqrt{k}}{V}.$$

Postquam autem hunc locum attigerit, angulus ψ continuo decrescet, seu motus angularis globuli fiet contrarius motui cylindri, et tribuendo ipsi V signum contrarium, habebitur

$$\sqrt{v} = \frac{\sqrt{k} + V}{cos\zeta},$$

et quando fiet $\psi = 0$, erit $V = \sin^2\zeta\sqrt{k}$, hincque

$$\sqrt{v} = \frac{\left(1+\sin^2\zeta\right)}{cos\zeta}\sqrt{k} \text{ et } x = \frac{a\omega}{cos\zeta}.$$

Inde fiet ψ negativum, et distantia x adhuc magis crescet, dum posito ψ negativo fiet $x = \frac{a\omega + \psi}{cos\zeta}$, donec fiat

$$V = \sqrt{\left(k\sin^4\zeta - a\cos^2\zeta\sin\theta\left(\cos(\alpha - \psi\right) - \cos\alpha) + a\psi\sin\zeta\cos\zeta\cos\theta\right)} = 0,$$

et eo usque erit $\sqrt{v} = \frac{\sqrt{k} + V}{cos\zeta}$: ubi autem fuerit $V = 0$, evadet $\sqrt{v} = \frac{\sqrt{k}}{cos\zeta}$; qui ergo valor ante hoc tempus maximus fuit, ubi erat

$$\sin\left(\alpha + \psi\right) = \frac{\text{tang}\zeta}{\text{tang}\theta}.$$

Postquam autem fuerit $V = 0$, angulus ψ iterum decrescet, indeque etiam distantia x minora capiet incrementa eritque $\sqrt{v} = \frac{\sqrt{k} - V}{cos\zeta}$, donec evadat $\psi = 0$, tumque erit $V = \sin^2\zeta \cdot \sqrt{k}$ et $\sqrt{v} =$ $\cos\zeta \cdot \sqrt{k}$, atque $x = \frac{a\omega}{\cos\zeta}$. Hoc ergo tempore celeritas $\sqrt{v}$ eadem erit, quae erat initio, indeque motus simili modo propagabitur.

Motus ergo per helicem continuo erit progressivus, si perpetuo fuerit $V < \sqrt{k}$ *:* sin autem inter eas motus partes, ubi $\sqrt{v} = \frac{\sqrt{k} - V}{cos\zeta}$ eveniat, ut fiat $V > \sqrt{k}$, tum globulus ibi per helicem regredietur, donec $\sqrt{v}$ iterum fiat affirmativum. Valores autem affirmativi praevalebunt; vidimus enim post primam periodum, qua celeritas ad initialem redit, globulum spatium absolvisse in helice $x = \frac{a\omega}{\cos\zeta}$, et post *n* huiusmodi periodos promovebitur *naw* per spatium helicis $x = \frac{na\omega}{\cos\zeta}$, sicque continuo altius elevabitur, donec tandem per superius orificium *K* eiiciatur.

CASUS III

29. Ponamus motum ita esse comparatum, ut postquam ab initio angulus $\psi = \alpha - \varphi$ increscere coepit, nunquam evadat

$$V = \sqrt{\left(k\sin^4\zeta - a\cos^2\zeta\sin\theta\left(\cos(\alpha-\psi)\right) - \cos\alpha\right) + a\psi\sin\zeta\cos\zeta\cos\theta\right)} = 0,$$

unde hic angulus ψ continuo magis augebitur valorque ipsius *V* increscet. Tum autem prodibit $\sqrt{v} = \frac{\sqrt{k} - V}{cos\zeta}$ ex quo sequitur celeritatem $\sqrt{v}$ tandem evanescere globulumque inde ad inferiorem cylindri partem reverti, donec in *E* iterum elabatur. Hoc etiam intelligitur ex formula

$$x = \frac{a(\omega - \psi)}{\cos\zeta};$$

distantia enim *x* diminuetur, si fuerit

$$d\omega < d\psi \ \text{ seu } \ \frac{d\psi\sqrt{k}}{V} < d\psi\,,$$

quod utique evenit, quando $\sqrt{k} < V$ seu $\sqrt{v}$ negativum. Hic ergo casus, quo globulum non ultra datum terminum in helice promovere licet, in sequentibus casibus locum habet:

1°. Si $\operatorname{tang}\theta < \operatorname{tang}\zeta$ seu angulus $PQR < BEF$, quomodocunque reliquae quantitates se habeant.

2°. Si fuerit $\sin\alpha < \frac{\operatorname{tang}\zeta}{\operatorname{tang}\theta}$, ita ut etiamsi sit $\zeta < \theta$, tamen hoc casu globulus revertatur in helice.

3°. Etiamsi sit

$$\zeta < \theta \text{ et } \sin\alpha > \frac{\operatorname{tang}\zeta}{\operatorname{tang}\theta},$$

tamen casus tertius locum invenit, si fuerit

$$k > \frac{a\cos^2\zeta\sin\theta}{\sin^4\zeta}\cdot M$$

denotante M maximum valorem, quem expressio

$$\cos(\alpha-\psi)-\cos\alpha-\frac{\mathrm{tang}\zeta}{\mathrm{tang}\theta}\cdot\psi$$

induere valet.

Hinc ergo patet, gyrationis motum nimis celerem non esse aptum ad globulum ad datam quamvis altitudinem elevandum, cum motus tardior hunc effectum praestare valeat. Fieri ergo potest, ut ob gyrationem nimium velocem effectu frustremur, quem tamen tardiore motu consequi possemus.

Exemplum

30. Sit $\frac{\mathrm{tang}\zeta}{\mathrm{tang}\theta}=\frac{1}{2}$ et angulus initialis *ACE* rectus seu $\alpha = 90°$, atque $\psi = 90° - \varphi$, sicque ψ denotabit angulum, quo globulus circa axem versus punctum summum *A* ab *E* est translatum tempore *t*, quo cylindrus per angulum $=\omega$ est conversus, ita ut sit $d\omega = \frac{dt\sqrt{k}}{a}$. Habebimus ergo

$$V = \sqrt{\left(k\sin^4\zeta - a\cos^2\zeta\sin\theta\left(\sin\psi - \tfrac{1}{2}\psi\right)\right)}$$

et

$$d\omega = \frac{dt\sqrt{k}}{a} \text{ atque } \sqrt{v} = \frac{\sqrt{k}-V}{cos\zeta} \text{ nec non } x = \frac{a(\omega-\psi)}{cos\zeta}.$$

Quamdiu ergo motus gyratorius globuli in sensum *BEA* dirigitur, valor ipsius *V* in his formulis affirmative accipi debet, contra vero negative.

Ab initio ergo crescente ψ, decrescit valor ipsius *V* ob $\sin\psi > \frac{1}{2}\psi$: quamdiu manet

$$k\sin^4\zeta > a\cos^2\zeta\sin\theta\left(\sin(\psi - \tfrac{1}{2}\psi\right).$$

Cum igitur ipsius $\sin\left(\psi - \frac{1}{2}\psi\right)$ valor maximus $\sin\psi = 60° = \frac{1}{3}\pi$, denotante π angulum duobus rectis aequalem, fiatque hic valor maximus $=\frac{1}{2}\sqrt{3}-\frac{1}{6}\pi = 0{,}3424266,$ quodsi ergo fuerit

$$k\sin^4\zeta < 0{,}3424266a\cos^2\zeta\sin\theta,$$

casus secundus locum habebit, casus vero tertius, si

$$k\sin^4\zeta > 0{,}3424266a\cos^2\zeta\sin\theta.$$

Illo scilicet globulus motu angulari tandem revertetur, hoc vero nunquam. Sit brevitatis ergo

$$\frac{\cos^2\zeta\sin\theta}{\sin^4\zeta}=n,$$

ut sit

$$V=\sin^2\zeta\sqrt{\left(k-na\left(\sin\psi-\tfrac{1}{2}\psi\right)\right)}.$$

1° Ac ponamus primo esse $k>0,3424266na$; atque angulus ψ continuo crescet, valor autem ipsius V initio decrescet, donec fiat $\psi=60°$, ubi valor ipsius V erit minimus, scilicet

$$=\sin^2\zeta\sqrt{(k-0,3424266na)},$$

ideoque celeritas globuli progressiva per helicem maxima. Inde vero valor ipsius V iterum augebitur, tandemque, quando $\sin\psi=\tfrac{1}{2}\psi$, quod evenit, si

$$\psi=108°\ 36'13''45'''28'''',$$

fiet

$$V=\sin^2\zeta\cdot\sqrt{k}\ \text{ et }\ \sqrt{v}=\cos\zeta\cdot\sqrt{k},$$

quae celeritati initiali est aequalis. Postea vero crescente ulterius angulo ψ, valor ipsius V magis augebitur, fietque tandem seu

$$V=\sqrt{k}\ \text{ seu }\ k(1-\sin^4\zeta)=a\cos^2\zeta\sin\theta\left(\tfrac{1}{2}\psi-\sin\psi\right)$$

seu

$$\tfrac{1}{2}\psi-\sin\psi=\frac{k(1+\sin^2\zeta)}{a\sin\theta};$$

hicque celeritas globuli in helice evanescet, ex quo ω reverti incipiet, et quidem motu accelerato, quoniam, crescente ψ ultra hunc terminum, quantitas V eo maiora capit augmenta. Definito autem ψ ex aequatione

$$\tfrac{1}{2}\psi-\sin\psi=\frac{k(1+\sin^2\zeta)}{a\sin\theta};$$

quantitas $x=\dfrac{a(\omega-\psi)}{cos\zeta}$ dabit spatium in helice, ad quod globulus penetraverit et unde deinceps revertitur. Tempus autem, quo huc usque pertingit, seu angulus ω, a cylindro interea motu gyratorio confectus, definitur hac aequatione:

$$\omega=\int\frac{d\psi\sqrt{k}}{\sin^2\zeta\cdot\sqrt{\left(k-na\sin\psi+\tfrac{1}{2}na\psi\right)}},$$

quo invento simul vera via x in helice percursa innotescit.

II. Ponamus esse $k < 0{,}3424266na$, angulusque ψ eo usque crescet, donec fiat $k = na\left(\sin\psi - \frac{1}{2}\psi\right)$, quod evenit, antequam evadet $\psi = 60°$; tumque erit $\sqrt{v} = \frac{\sqrt{k}}{cos\zeta}$ ob $V = 0,$ hactenus ergo celeritas $\sqrt{v}$ augendo increvit; hicque constituamus primam partem motus globuli per helicem.

2^{do} Ab hoc autem momento angulus ψ iterum diminuetur, et valor

$$V = \sin^2\zeta \cdot \sqrt{\left(k - na\left(\sin\psi - \tfrac{1}{2}\psi\right)\right)}$$

negative capi debet, ut sit

$$\sqrt{v} = \frac{\sqrt{k} + V}{cos\zeta}$$

sicque labente tempore valor ipsius *V* iterum increscet, donec evadente $\psi = 0$ fiat

$$V = \sin^2\zeta \cdot \sqrt{k} \text{ et } \sqrt{v} = \frac{\left(1 + \sin^2\zeta\right)\sqrt{k}}{\cos\zeta}:$$

hicque secundam motus partem terminemus, in cuius fine $\psi = 0$, et celeritas globuli $\sqrt{v}$ maior existit, quam adhuc fuit.

3° Nunc igitur angulus ψ negativus esse incipit; posito ergo $-\psi$ loco ψ, habebimus

$$V = \sin^2\zeta \cdot \sqrt{\left(k + na\left(\sin\psi - \tfrac{1}{2}\psi\right)\right)},$$

manente

$$\sqrt{v} = \frac{\sqrt{k} + V}{cos\zeta}:$$

et quia $\sin\psi - \frac{1}{2}\psi$ crescit, quamdiu ψ est $< 60°,$ ad hunc usque terminum $\psi = 60°$ valor ipsius *V*, hincque celeritas $\sqrt{v}$ augebitur; et facto $\psi = 60°$, celeritas globuli in helice progressiva erit maxima, scilicet

$$\sqrt{v} = \frac{\sqrt{k} + \sin^2\zeta \cdot \sqrt{\left(k - 0{,}3424266na\right)}}{\cos\zeta}.$$

4° Deinde ulterius crescente hoc angulo ψ, qui nunc est $= \varphi - \alpha$, valor ipsius *V* iterum decrescet, et quando fit

$$\varphi = 108°\ 36'13''45'''28'''',$$

erit

$$V=\sin^2\zeta\cdot\sqrt{k}\ \text{ et }\ \sqrt{v}=\frac{\left(1+\sin^2\zeta\right)\sqrt{k}}{\cos\zeta}.$$

5° Angulus autem ψ ultra hunc terminum crescere perget, et quia tum $\frac{1}{2}\psi>\sin\psi$, erit

$$V=\sin^2\zeta\cdot\sqrt{\left(k-na\left(\sin\psi-\tfrac{1}{2}\psi\right)\right)},$$

et

$$\sqrt{v}=\frac{\sqrt{k}+V}{cos\zeta}.$$

Valor ergo ipsius V continuo fiet minor, indeque etiam celeritas $\sqrt{v}$, donec fiat

$$\tfrac{1}{2}\psi-\sin\psi=\frac{k}{na},$$

quo casu erit $\sqrt{v}=\dfrac{\sqrt{k}-V}{cos\zeta}$.

6° Tum autem hic angulus ψ , qui maior est quam 108° 36', iterum decrescet, fietque $\sqrt{v}=\dfrac{\sqrt{k}-V}{cos\zeta}$ existente

$$V=\sin^2\zeta\cdot\sqrt{\left(k-na\left(\tfrac{1}{2}\psi-\sin\psi\right)\right)};$$

sicque celeritas $\sqrt{v}$ decrescet, et quando fit $\psi=108^\circ\ 36'$, prodibit

$V=\sin^2\zeta\cdot\sqrt{k}$ et $\sqrt{v}=\ \cos\zeta\cdot f\sqrt{k}$, quae aequalis est celeritati initiali.

7° Porro angulus ψ infra hunc terminum decrescet, et ob $\sin\psi>\frac{1}{2}\psi$, erit

$$V=\sin^2\zeta\sqrt{\left(k+na\left(\sin\psi-\tfrac{1}{2}\psi\right)\right)}\ \text{ et }\ \sqrt{v}=\frac{\sqrt{k}-V}{cos\zeta}:$$

et quando fit $\psi=60^\circ$, quo casu valor ipsius V erit maximus

$$=\sin^2\zeta\cdot\sqrt{\left(k+0,3424266na\right)},$$

et celeritas globuli minima

$$\sqrt{v}=\frac{\sqrt{k}-\sin^2\zeta\cdot\sqrt{\left(k+0,3424266na\right)}}{\cos\zeta}.$$

Nisi ergo sit

$$\sqrt{k} > \sin^2\zeta\sqrt{(k+0,3424266na)}.$$

seu

$$k > \frac{0,3424266a\sin\theta}{1+\sin^2\zeta},$$

cum sit

$$k < \frac{0,3424266a\cos^2\zeta\sin\theta}{\sin^4\zeta},$$

globulus, antequam ad hunc terminum pervenit, regredietur in helice, propterea quod eius celeritas $\sqrt{v}$ fit negativa. Revertitur ergo globulus, si sit

$$k < \frac{0,3424266a\sin\theta}{1+\sin^2\zeta};$$

non autem revertetur, sed perpetuo per cochleam progredi perget, si sit

$$k > \frac{0,3424266a\sin\theta}{1+\sin^2\zeta}.$$

Quia autem esse debet

$$k < \frac{0,3424266a\cos^2\zeta\sin\theta}{\sin^4\zeta},$$

manifestum est hunc casum locum obtinere non posse, nisi sit

$$1 > 2\sin^4\zeta \text{ seu } \sin\zeta < \sqrt[4]{\tfrac{1}{2}};$$

hoc est nisi angulus helicis ζ minor sit quam 57° 14'.

8° Postquam autem angulus ψ ultra 60° fuerit diminutus, etiam ulterius decrescet, eritque adhuc

$$V = \sin^2\zeta\cdot\sqrt{\left(k+na\left(\sin\psi-\tfrac{1}{2}\psi\right)\right)} \text{ et } \sqrt{v} = \frac{\sqrt{k}-V}{cos\zeta}$$

et

$$\sqrt{v} = \frac{\sqrt{k}+V}{cos\zeta}.$$

valorque ipsius V continuo fiet minor, et celeritas $\sqrt{v}$ mox affirmativa reddetur, et facto $\psi = 0$ redibit ea, uti erat initio, $\sqrt{v} = cos\zeta\cdot\sqrt{k}$.

Cum globulus huc pervenerit, angulus ψ iterum negativus evadet, seu motus angularis globuli motum cylindri sequetur, seu erit iam $\varphi < \alpha$ seu $\varphi < 90°$; vel globulus in superiorem cylindri medietatem elevabitur, cum a numero tertio in inferiore esset versatus: atque nunc pari modo motum suum prosequetur, atque ab initio fecerat; ita ut iam eaedem motus partes, quas descripsimus, sint rediturae. Quod vero ad tempora attinet, quibus quaeque motus huius pars absolvitur, ea nonnisi per quadraturas definiri poterunt ope formulae $d\omega = \frac{d\psi\sqrt{k}}{V}$; quippe cuius integratio exhiberi nequit.

PROBLEMA 8

31. *Si una integra helicis circumvolutio EFGe aqua fuerit repleta atque cylindrus subito in gyrum agi incipiat celeritate uniformi, quae in puncta E sit* $=\sqrt{k}$ *, idque in sensum helici contrarium BEA , invenire motum, quo ista aquae portio per helicem promovebitur.*

Solutio

Positis basis cylindri radio $CA = a$, angulo helicis $BEF = \zeta$, angulo, quem axis cylindri *PQ* cum verticali constituit, $PQR = \theta$*;* sit ipso motus initio angulus $ACE = \alpha$; quo tempore aqua in helice spatium $EFGe = f$ occupet, quod cum uni integrae revolutioni sit aequale, posito $\frac{f\cos\zeta}{a} = \gamma$, erit γ angulus quatuor rectis aequalis, seu denotante $1:\pi$ rationem radii ad semicircumferetiam, erit $\gamma = 2\pi$ et $f = \frac{2\pi a}{\cos\zeta}$ et ipsa aquae copia $= \frac{2\pi ahh}{\cos\zeta}$, siquidem *hh* designet amplitudinem helicis.

Iam elapso tempore *t*, quo ipse cylindrus circa axem conversus erit angulo $=\omega$, ut sit $d\omega = \frac{dt\sqrt{k}}{a}$ seu $\omega = \frac{t\sqrt{k}}{a}$ ideoque $t = \frac{a\omega}{\sqrt{k}}$ pervenerit aqua in helice in situm *MFGem;* ponatur ergo spatium $EM = x$ et celeritas, qua aqua per helicem promovetur $=\sqrt{v}$*;* ut sit

$$dx = dt\sqrt{v} = \frac{ad\omega\sqrt{v}}{\sqrt{k}}.$$

Ponatur angulus $ACS = \varphi$, et ob angulum $ECS = \frac{x\cos\zeta}{a}$ quia punctum *E* angulo ω ad *A* accessit, erit

$$\varphi = \alpha - \omega + \frac{x\cos\zeta}{a} \text{ ideoque } \frac{x\cos\zeta}{a} = \omega + \varphi - \alpha$$

et hinc

$$\frac{dx\cos\zeta}{a} = \frac{d\omega\cos\zeta\cdot\sqrt{v}}{\sqrt{k}} = d\omega + d\varphi,$$

ita ut sit

$$d\omega = \frac{d\varphi \cdot \sqrt{k}}{\cos\zeta \cdot \sqrt{v} - \sqrt{k}}.$$

At ex paragrapho 17 habebitur haec aequatio ob

$$\gamma = 2\pi \ \text{ et } \ \sin(\gamma + \varphi) = \sin\varphi :$$
$$\cos\zeta \cdot \sqrt{v} - \sqrt{k} = \sqrt{(C - a\varphi \sin\zeta \cos\zeta \cos\theta)}.$$

Ipso autem motus initio aquae in tubo helicis eiusmodi motus imprimitur, ut sit $\sqrt{v} = \cos\zeta \cdot \sqrt{k}$, quo casu cum sit $\varphi = \alpha$, erit

$$\cos\zeta \cdot \sqrt{v} - \sqrt{k} = -\sqrt{(k\sin^4\zeta + a(\alpha - \varphi)\sin\zeta \cos\zeta \cos\theta)}.$$

Ab initio ergo angulus φ, qui ipso initio erat $= \alpha$, decrescit, seu terminus aquae M propius ad lineam supremam Aa elevatur, quam fuerat initio. Ponamus tempore t hanc appropinquationem factam esse per angulum ψ, ut sit $\varphi = \alpha - \psi$, erit

$$\frac{x\cos\zeta}{a} = \omega - \psi \ \text{ et } \ x = \frac{a(\omega - \psi)}{\cos\zeta},$$

tum vero

$$\cos\zeta \cdot \sqrt{v} - \sqrt{k} = -\sqrt{(k\sin^4\zeta + a\psi \sin\zeta \cos\zeta \cos\theta)}.$$

seu

$$\sqrt{v} = \frac{\sqrt{k} - \sqrt{(k\sin^4\zeta + a\psi \sin\zeta \cos\zeta \cos\theta)}}{\cos\zeta},$$

eritque

$$d\omega = \frac{d\psi\sqrt{k}}{\sqrt{(k\sin^4\zeta + a\psi \sin\zeta \cos\zeta \cos\theta)}}.$$

Hinc cum initio quo $\omega = 0$, sit quoque $\psi = 0$, erit integrando :

$$\frac{a\omega \sin\zeta \cos\zeta \cos\theta}{2\sqrt{k}} = \sqrt{(k\sin^4\zeta + a\psi \sin\zeta \cos\zeta \cos\theta)} - \sin^2\zeta \cdot \sqrt{k}$$

hincque porro

$$\psi = \omega \sin^2\zeta + \frac{a\omega\omega}{4k} \sin\zeta \cos\zeta \cos\theta.$$

Ex quo obtinemus pro tempore per angulum ω indicato:

$$\sqrt{v} = \cos\zeta \cdot \sqrt{k} - \frac{a\omega \sin\zeta \cos\theta}{2\sqrt{k}}$$

et

$$x = a\omega\cos\zeta - \frac{aa\omega\omega}{4k}\sin\zeta \cos\theta;$$

elapso autem tempore *t* est $\omega = \frac{t\sqrt{k}}{a}$; ita ut sit

$$\sqrt{v} = \cos\zeta \cdot \sqrt{k} - \tfrac{1}{2}t\sin\zeta\cos\theta$$

et

$$x = t\cos\zeta \cdot \sqrt{k} - \tfrac{1}{4}tt\sin\zeta\cos\theta;$$

spatium ergo *SM*, per quod aqua iam secundum directionem axis cylindri erit promota, erit

$$x\sin\zeta = t\sin\zeta\cos\zeta \cdot \sqrt{k} - \tfrac{1}{4}tt\sin^2\zeta\cos\theta$$

unde spatium, per quod verticaliter iam erit elevata aqua concluditur

$$x\sin\zeta\cos\theta = t\sin\zeta\cos\zeta\cos\theta \cdot \sqrt{k} - \tfrac{1}{4}tt\sin^2\zeta\cos^2\theta$$

Corollarium 1

32. Si cylindrus plane non in gyrum ageretur, sed in quiete relinqueretur, ut esset $k = 0$, tunc elapso tempore *t* esset

$$\sqrt{v} = -\tfrac{1}{2}t\sin\zeta\cos\theta \quad \text{et} \quad x = -\tfrac{1}{4}tt\sin\zeta\cos\theta.$$

Aqua ergo, siquidem cochlea deorsum ultra *E* esset continuata, motu uniformiter accelerato per cylindrum descenderet, eiusque motus similis foret descensui corporis super plano inclinato, cuius anguli inclinationis ad horizontem sinus esset $= \sin^2\zeta\cos\theta$.

Corollarium 2

33. Cylindro autem in gyrum acto in sensum *BEA* celeritate $\sqrt{k}$, aqua quidem ab initio motus secundum cylindrum ascendet, quamdiu fuerit

$$k > \tfrac{1}{2}a\omega\,\mathrm{tang}\zeta\cos\theta \quad \text{seu} \quad k > \tfrac{1}{2}t\,\mathrm{tang}\zeta\cos\theta.$$

Elapso autem tempore

$$t = \frac{2\sqrt{k}}{\mathrm{tang}\zeta\cos\theta},$$

motus ascensus cessabit, posteaque aqua per cylindrum descendere incipiet.

Corollarium 3

34. Posito ergo

$$t = \frac{2\sqrt{k}}{\operatorname{tang}\zeta \cos\theta},$$

maximum spatium x per quod aqua in cochlea fuerit promota, erit

$$x = \frac{k\cos^2\zeta}{\sin\zeta\cos\theta};$$

ideoque secundum longitudinem cylindri confecit spatium

$$x\sin\zeta = \frac{k\cos^2\zeta}{\cos\theta};$$

et perpendiculariter reperietur elevata ad altitudinem $x\sin\zeta\cos\theta = k\cos^2\zeta$.

Corollarium 4

35. Portio ergo aquae, quae integram spiralis revolutionem implet, ope cochleae ARCHIMEDEAE ad maiorem altitudinem elevari nequit, quam quae sit $= k\cos^2\zeta$. Quo celerius ergo cylindrus in gyrum agitur, eo altius haec aquae portio elevari poterit, et haec quidem altitudo proportionalis erit quadrato celeritatis gyrationis.

Corollarium 5

36. Sit altitudo, ad quam aqua ope cochleae ARCHIMEDIS elevari debeat, $= c$ praestabiturque hoc tempore t, ut fit seu

$$c = t\sin\zeta\cos\zeta\cos\theta\cdot\sqrt{k} - \tfrac{1}{4}tt\sin^2\zeta\cos^2\theta$$

seu

$$t = \frac{2\cos\zeta\cdot\sqrt{k} - 2\sqrt{\left(k\cos^2\zeta - c\right)}}{\sin\zeta\cos\theta}.$$

Ut iam hoc tempus sit omnium minimum, angulus ζ ita esse debet comparatus, ut sit

$$\operatorname{tang}^2\zeta = 1 - \frac{c}{k} \text{ seu } \operatorname{tang}\zeta = \sqrt{\left(1 - \frac{c}{k}\right)}.$$

Corollarium 6

37. Posito autem

$$\operatorname{tang}\zeta = \sqrt[4]{\left(1 - \frac{c}{k}\right)}.$$

erit tempus illud minimum, quo aqua per altitudinem c elevatur:

$$t=\frac{2\sqrt{k}}{\cos\theta}\left(\cos\zeta-\operatorname{tang}\zeta\right)=\frac{2\sqrt{k}-2\sqrt{(k-c)}}{\cos\theta\cdot\sqrt[4]{\left(1-\frac{c}{k}\right)}},$$

quod fit infinitum, si $k=c$, at vero nullum, si k est infinitum. Quo maior ergo capiatur celeritas gyratoria $\sqrt{k}$ et quo minor simul statuatur angulus $PQR=\theta,$ eo breviori tempore aqua ad altitudinem c elevabitur.

Corollarium 7

38. Patet ergo, etiamsi cochlea ARCHIMEDIS situm obtineat verticalem, eius tamen ope aquam ad quamvis altitudinem elevari posse, dummodo cochlea satis celeriter in gyrum agatur. Hoc autem casu ob $\theta=0$ perinde est, sive aqua integram helicis revolutionem impleat, sive secus. Ac tempus quidem elevationis hoc casu minus erit, quam si cylindrus ad horizontem esset inclinatus.

Scholion 39.

Patet ergo insignem esse differentiam inter elevationem aquae per cochleam ARCHIMEDIS, prout aqua elevanda vel integram spirae revolutionem impleat, vel tantum minimam eius portionem occupet; si enim aqua integram spiram adimplet, ea non ultra certam altitudinem elevari potest, quantumvis celeriter cochlea in gyrum agatur; contra autem vidimus, si minima aquae portio tantum cochleae immittatur, fieri posse, ut ea ad quamvis altitudinem elevetur, atque hoc quidem motu gyrationis non admodum celeri: nam ex praecedentibus perspicitur motum nimis celerem ascensui adversari et aquam iterum deorsum ferre, quae tamen a motu tardiore continuo ascendere perrexisset. Ut enim particula aquae cochleae in E initio immissa continuo ascendere pergat, primum requiritur ut sit

$$\theta>\zeta \text{ seu ang } PQR>\text{ang } BEF.$$

Deinde ut sit

$$\sin\alpha \text{ seu } \sin ACE>\frac{\operatorname{tang}\zeta}{\operatorname{tang}\theta}:$$

tertio autem requiritur, ut, denotante M maximum valorem positivum, quem expressio

$$\cos\left(\alpha-\psi\right)-\cos\alpha-\frac{\operatorname{tang}\zeta}{\operatorname{tang}\theta}\cdot\psi$$

recipere valet, quod evenit casu

$$\sin\left(\alpha-\psi\right)=\frac{\operatorname{tang}\zeta}{\operatorname{tang}\theta},$$

sit

$$k<aM\cdot\frac{\cos^{2}\zeta\sin\theta}{\sin^{4}\zeta}.$$

Si ergo altitudo celeritati gyrationis debita *k* superaret hanc quantitatem, aqua, postquam ad certam altitudinem pervenisset, iterum delaberetur. Verum neuter horum casuum in praxi communi, ubi cochlea ARCHIMEDIS ad aquas elevandas adhibetur, locum habet: quodsi enim tota cylindri basis inferior *AB* aquae est submersa, tota helix semper est aqua repleta, unde quaestio, quanta celeritate et ad quantam altitudinem cochlea in gyrum acta aquam sit elevatura, ab his binis, quas tractavimus, penitus est diversa, propterea quod aqua in *E* continuo influit, in *K* vero iterum egeritur. Hanc igitur quaestionem difficillimam in sequente problemate enodare conabor.

PROBLEMA 9

40. *Si tota basis cylindri aquae sit submersa, isque motu uniformi in gyrum agatur, definire motum aquae per cochleam.*

Solutio

Positis, ut hactenus, radio basis $CA=a$, angulo helicis $BEF=\zeta$ et inclinationis $PQR=\theta$: sit altitudo aquae supra centrum basis $C=c$, longitudo totius cylindri $Aa=Bb=b$, et *EFGHIK* repraesentet totam *b* helicem, cuius propterea longitudo est $=\dfrac{b}{\sin\zeta}$; ac si eius amplitudo dicatur $=hh$, erit quantitas aquae in helice contentae $=\dfrac{bhh}{\sin\zeta}$; tum vero summa spirarum ad basin relatarum praebebit in eius peripheria arcum $=\dfrac{b\cos\zeta}{\sin\zeta}$. Scilicet si a puncto helicis quocunque *Z* ad basin ducatur recta axi parallela *ZY* arcusque *EY* ponatur $=s$, posito $s=0$ habebitur terminus helicis, inferior at posito $s=\dfrac{b\cos\zeta}{\sin\zeta}$ prodibit terminus helicis superior *K*. Gyretur nunc cylindrus in sensum *BEA*, ita ut celeritas puncti *E* sit $=\sqrt{k}$: positoque arcu $EA=p$, elapso tempusculo *dt* erit $dp=-dt\sqrt{k}$. Praesenti autem temporis momento sit aquae per helicem ascendentis celeritas $=\sqrt{v}$ *:* quodsi iam status compressionis aquae in helicis loco quocunque *Z* ponatur $=q$, existente arcu $EY=s$, hanc supra invenimus aequationem

$$q\cos\zeta=C-a\cos\zeta\sin\theta\cos\frac{p+s}{a}-s\sin\zeta\cos\theta-\frac{sdv}{dt\sqrt{v}}.$$

Quando autem aqua in *K* libere effluit, posito $s=\dfrac{b\cos\zeta}{\sin\zeta}$ status compressionis in *K* evanescere debet, erit ergo

$$C=a\cos\zeta\sin\theta\cos\frac{p\sin\zeta+b\cos\zeta}{a\sin\zeta}+b\cos\zeta\cos\theta+\frac{bdv\cos\zeta}{dt\sin\zeta\cdot\sqrt{v}}.$$

Exprimat *g* statum compressionis in altero termino *E,* ubi $s=0$, erit

$$g\cos\zeta = a\cos\zeta\sin\theta\cos\frac{p\sin\zeta + b\cos\zeta}{a\sin\zeta} + b\cos\zeta\cos\theta$$
$$+\frac{bdv\cos\zeta}{dt\sin\zeta\cdot\sqrt{v}} - a\cos\zeta\sin\theta\cos\frac{p}{a}.$$

sive per $\cos\zeta$ dividendo:

$$g = a\sin\theta\cos\frac{p\sin\zeta + b\cos\zeta}{a\sin\zeta} - a\sin\theta\cos\frac{p}{a} + b\cos\theta + \frac{bdv}{dt\sin\zeta\cdot\sqrt{v}}.$$

Totum ergo negotium huc redit, ut status compressionis aquae in termino definiatur, qui cum a profunditate orificii E sub aqua pendeat, reperitur puncti E altitudo super centro C

$$= a\cos\frac{p}{a}\sin\theta,$$

ideoque profunditas orificii E sub aqua erit

$$= c - a\sin\theta\cos\frac{p}{a}.$$

Cum igitur celeritas aquae in helicem influentis sit debita altitudini v, status compressionis aquae in E aestimari debet per altitudinem

$$= c - a\sin\theta\cos\frac{p}{a} - v,$$

unde habemus:

$$c = a\sin\theta\cos\frac{p\sin\zeta + b\cos\zeta}{a\sin\zeta} + b\cos\theta + \frac{bdv}{dt\sin\zeta\cdot\sqrt{v}} + v.$$

Ponatur angulus $ACE = \varphi$, ut sit

$$p = a\varphi \text{ et } dt = \frac{-ad\varphi}{\sqrt{k}},$$

tum vero sit angulus erit

$$\frac{b\cos\zeta}{a\sin\zeta} = \gamma \text{ seu } b = a\gamma\operatorname{tang}\zeta,$$

$$c = a\sin\theta\cos(\varphi+\gamma) + a\gamma\operatorname{tang}\zeta\cos\theta - \frac{\gamma dv\sqrt{k}}{d\varphi\cos\zeta\cdot\sqrt{v}} + v.$$

Ponamus $2\sqrt{kv} = z$, ut sit $v = \frac{zz}{4k}$, habemus:

$$-\gamma dz + \frac{zzd\varphi\cos\zeta}{4k} + ad\varphi\cos\zeta\sin\theta\cos(\varphi+\gamma) = d\varphi(c\cos\zeta - a\gamma\sin\zeta\cos\theta).$$

Ex qua aequatione valor ipsius z definiri debet.

Quod autem ad pressionem aquae ad latera tubi attinet, quatenus inde motui gyrationis resistitur, supra vidimus a gravitate aquae oriri vim secundum

$$Zr = \sin\zeta \sin\theta \sin\frac{p+s}{a} + \cos\zeta\cos\theta,$$

unde oritur vis secundum

$$Zv = \sin^2\zeta \sin\theta \sin\frac{p+s}{a} + \sin\zeta \cos\zeta\cos\theta,$$

quae per elementum aquae $=\dfrac{hhd}{\cos\zeta}$*:* et radium *a* multiplicata dat momentum, elementare motui resistens, unde totum momentum erit

$$ahh\left(b\cos\zeta\cos\theta + \frac{a\sin^2\zeta\sin\theta}{\cos\zeta}\left(\cos\varphi - \cos\ (\varphi+\gamma)\right)\right);$$

tantum ergo momentum a vi gyrante superari debet.

Corollarium 1

41. Pendet ergo determinatio motus aquae per cochleam ARCHIMEDIS a resolutione huius aequationis differentialis:

$$-\gamma dz + \frac{zzd\varphi\cos\zeta}{4k} + ad\varphi\cos\zeta\sin\theta\cos(\varphi+\gamma) = d\varphi(c\cos\zeta - a\gamma\sin\zeta\cos\theta)$$

vel ob $\gamma = \dfrac{b\cos\zeta}{a\sin\zeta}$ istius aequationis

$$-\frac{bdz}{a\sin\zeta} + \frac{zzd\varphi}{4k} + ad\varphi\sin\theta\cos(\varphi+\gamma) = d\varphi(c - b\cos\theta),$$

quae cum pluribus difficultatibus sit obnoxia, patet theoriam Cochleae ARCHIMEDIS maxime esse arduam.

Corollarium 2

42. Si cochlea in quiete relinquitur, ut sit $k=0$, loco elementi $d\varphi$ expedit in calculo relinqui elementum temporis *dt* et ob angulum φ constantem habebitur:

$$\frac{bdv}{dt\sin\zeta\cdot\sqrt{v}} + v = c - b\cos\theta - a\sin\theta\cos(\varphi+\gamma),$$

unde mox nascetur motus uniformis

$$v = c - b\cos\theta - a\sin\theta\cos(\varphi+\gamma),$$

quo aqua per cochleam fluet, siquidem sit $c > b\cos\theta + a\sin\theta\cos(\varphi+\gamma)$..

Corollarium 3

43. Si cylindrus in situ verticali sit positus, ob $\theta = 0$ erit

$$-\frac{bdz}{\sin\zeta} + \frac{zzd\varphi}{4k} = d\varphi\left(c-b\right);$$

unde fit

$$d\varphi = \frac{4bkdz}{\left(4k\left(b-c\right)+zz\right)a\sin\zeta}$$

et integrando

$$\frac{a\varphi}{4bk}\sqrt{4k\left(b-c\right)} = \text{A tang}\frac{z}{\sqrt{4k\left(b-c\right)}}$$

ubi est

$$\sqrt{v} = \frac{z}{\sqrt{4k}}.$$

Cum autem, si initio fuerit $\varphi = 0$ et $z = 0$, labente tempore angulus φ evadat negativus, perspicuum est valorem quoque ipsius z prodire negativum; ideoque hoc casu aqua non ascendet, sed descendet, quod quidem per se est evidens.

Corollarium 4

44. In casu autem corollarii praecedentis, quo $b > c$, eiusmodi constantem addi oportet, ut posito $\varphi = 0$ fiat

$$\sqrt{v} = \frac{z}{\sqrt{4k}} = \cos\zeta\cdot\sqrt{k},$$

sicque erit

$$\frac{\sqrt{v}}{\sqrt{(b-c)}} = \text{tang}\left(A\,\text{tang}\frac{\cos\zeta\cdot\sqrt{k}}{\sqrt{(b-c)}} + \frac{a\varphi\sin\zeta}{2b}\sqrt{\frac{b-c}{k}}\right);$$

progressu autem temporis fit φ negativum ideoque ascensus penitus cessat, cum fit

$$-\varphi = \frac{2bk\cos\zeta}{a(b-c)\sin\zeta}.$$

Scholion

45. Assumsi in huius casus integratione cochleam initio fuisse aqua repletam subitoque rotari incepisse; sic enim utique celeritas initialis aquae progressiva per cochleam fit $= \cos\zeta\cdot\sqrt{k}$. Sin autem status initialis ita concipiatur, ut obturato inferiori orificio cochlea in gyrum agatur, tum vero subito orificium iterum aperiri, aqua hoc momento

sese iam ad motum tubi accommodaverit, necesse est, ita ut tum pro motus initio futurum sit $v=0$. Hanc ergo ob rem aqua statim descendere incipiet, neque ulla eius gutta supra eiicietur, siquidem sit $b>c$. Quanquam autem hunc casum, quo $\theta=0$, feliciter expedire licuit, tamen pro situ cochleae inclinato, nihil admodum ex aequatione inventa elicere licet, sed natura motus aquae his casibus nobis abscondita manet, propterea quod haec aequatio ad formulam RICCATIANAM referenda commode tractari nequit. Ex quo insigne Analyseos defectus exemplum agnoscimus, quod machinae frequentissimo usu maxime pervulgatae effectus pendeat a resolutione huiusmodi aequationis, cui artificia in Analysi adhuc detecta non sufficiant, qui casus mihi adeo mirabilis est visus, ut, etiamsi in hac investigatione scopum, quem mihi proposueram, non attigerim, tamen hoc argumentum dignissimum existimaverim, quo Geometrarum vires ad id penitus expediendum incitarem, quo la bore non solum maxima commoda in Mechanicam redundabunt, sed etiam Analyseos limites haud mediocriter promovebuntur.