A General Method :

*For determining the curvature of a string extended by the forces observed to be acting between them, according to some law, together with the solution of certain new problems pertaining to that***.**

Author

Daniel Bernoulli I.F.I.

I. Since the string is extended either by forces acting vertically or by acting normally to the curve, a certain uniform law is observed easily between the applied forces and the sines of certain angles, thus so that whatever law these forces may have, the curvature of the string shall be defined always by a differential equation of the second order, which itself often can be reduced to a simpler differential : Thus catenaries, awnings, sails, and other elastic curves [can be described] which formerly were elicited from sums by Geometers. But if at each one point of the string several different forces shall be applied in various directions, that uniformity fails, and we cannot determine the special nature of the curve other than by a differential equation of the third order, if we remain [considering] in general terms : So that my method, which indeed is for this general procedure, thus may be seen more clearly, I may imagine as many as two forces to be applied at each point of the string ; the one perpendicular to the curve, the other forming a given constant angle with the axis, which law thus still extends widely, so that all the curves considered by Geometers regarding this matter to the present are here, and so that the method may deal with the most specialized examples under that law. Afterwards I will show the solution of certain problems considered by me.

2. *Lemma* 1. The string (fig. I.) ABCD shall be fixed at the points A and D and separated into three equal parts AB, BC, CD. The forces BK and CI may be understood to be applied at the points B and C, the directions of which will bisect equally the angles ABC and BCD: and also so that the forces BH and CL, the directions of which are parallel to each other, there will be with the lines produced and with the perpendiculars dropped which the figure shows, and with the forces BK, CI, BH and CL designated by A, B, C and D, I say there will be :

$$\frac{\textit{sin. ang}.\mathrm{ABR}}{\textit{sin. ang}.\mathrm{AB}m}\mathrm{A}+\frac{\textit{sin. ang}.\mathrm{ABM}}{\textit{sin. ang}.\mathrm{AB}m}C=\frac{\textit{sin. ang}.\mathrm{BCS}}{\textit{sin. ang}.\mathrm{BC}n}\mathrm{B}+\frac{\textit{sin. ang}.\mathrm{DCX}}{\textit{sin. ang}.\mathrm{BC}n}\mathrm{D}$$

[In the diagram below, we have relabeled the angles and inserted the forces in red, giving

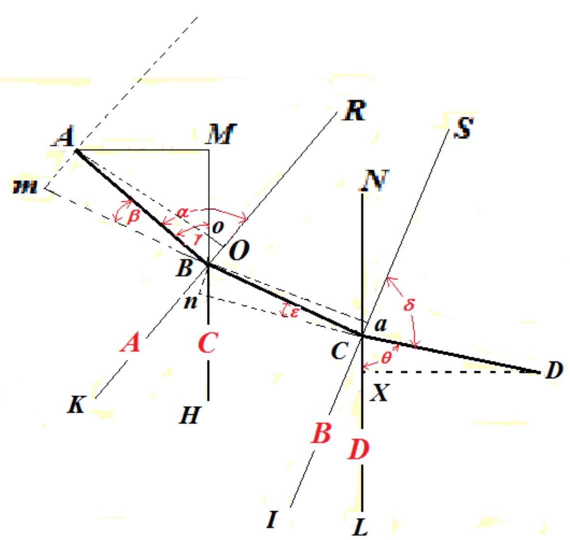


Fig. I

the corresponding equation :

$$\frac{sin\,\alpha}{sin\,\beta}\mathrm{A}+\frac{sin\,\gamma}{sin\,\beta}C=\frac{sin\,\delta}{sin\,\varepsilon}\mathrm{B}+\frac{sin\,\theta}{sin\,\varepsilon}\mathrm{D}.]$$

or on taking $Ao=\mathrm{BC}=\mathrm{CD}$ for the whole sine there will be

$$\frac{\mathrm{A}o}{\mathrm{A}m}\mathrm{A}+\frac{\mathrm{AM}}{\mathrm{A}m}C=\frac{\mathrm{B}a}{\mathrm{B}n}\mathrm{B}+\frac{\mathrm{DX}}{\mathrm{B}n}\mathrm{D}.$$

The demonstration agrees with mechanics and is deduced easily from the composition of the forces, with the aid of which the forces pulling towards B and C at some point taken on BC are determined, which must be equal to each other.

[At the point B, the two forces A and C are decomposed into components along the directions BC and AB, while likewise at C, the two forces B and D are decomposed into forces along the directions BC and CD : Thus, in triangle *BAm*, the side *m*A is taken for the force A, in which case, from the triangle of forces, the side *m*B is taken for the force $=\dfrac{A}{\sin\beta}\times\sin\alpha$; and likewise in triangle BA*o*, if B*o* is taken for the vertical force C, the force along AB is given, while along Ao the force is $=\frac{C}{\sin\beta}\times\sin\gamma.$ Thus the total force pulling BC to the left is the sum of these two components, and likewise for the force to the right from the two forces acting on the right hand end of BC.]

3. *Lemma* 2. If the angles AB*m* and *o*C*n* shall be infinitely small with negligible parts being ignored, there will be :

$$\mathrm{A}o=\mathrm{AB},\ \mathrm{A}m=2\mathrm{B}o;\ \mathrm{BQ}=\mathrm{BC}=\mathrm{AB}\ \text{ et }\ \mathrm{B}n=2\mathrm{C}a,$$

if besides the radius of the circle passing through the three points A , B, C may be put $=\mathrm{R}$ and the radius of the circle passing through the other three points B, C , D $=\mathrm{S}$, there will be [recalling the formula for the sagitta BO, which is approximately equal to the semi-chord squared on twice the radius]

$$2\mathrm{BO}=\frac{\mathrm{AB}^2}{\mathrm{R}}\ \text{ and }\ \mathrm{C}a=\frac{\mathrm{BC}^2}{\mathrm{S}},$$

from which with these values substituted the equation from the preceding lemma will be changed into this :

$$\frac{\mathrm{R.A}}{\mathrm{AB}}+\frac{\mathrm{AM.R.C}}{\mathrm{AB}^2}=\frac{\mathrm{S.B}}{\mathrm{AB}}+\frac{\mathrm{DX.S.D}}{\mathrm{AB}^2}.$$

4. *Problem.* To find the curvature of the string, the individual points of which are pulled by any two given forces, the one normal to the curve, and the other everywhere parallel to a given line.

Solution. Three elements of the line sought shall be the equal line increments AB, BC et CD ; and with the significances retained for the same the abscissas x may be taken on an axis perpendicular to the direction of the parallel forces : the applied lines y shall be perpendicular to the axis, the radius of osculation at the point B shall be $=\mathrm{R}$; the element of the curve $AB=ds$ *;* thus there will be

$$AM=dx,\ DX=dx+2ddx,\ \mathrm{B}=\mathrm{A}+d\mathrm{A},\ D=C+dC,\ \mathrm{S}=\mathrm{R}+d\mathrm{R}:$$

and thus with the application made of the second lemma to the present case, there will be had :

$$\frac{\mathrm{R.A}}{ds}+\frac{dx.\mathrm{R.C}}{ds^2}=\frac{(\mathrm{R}+d\mathrm{R})\times(\mathrm{A}+d\mathrm{A})}{ds}+\frac{(dx+2ddx)\times(\mathrm{R}+d\mathrm{R})\times(\mathrm{C}+d\mathrm{C})}{ds^2}:$$

from which equation, correctly handled, there shall be :

$$-\mathrm{A}d\mathrm{R}ds-\mathrm{R}d\mathrm{A}ds=\mathrm{R}d\mathrm{C}dx+2\mathrm{CR}ddx+\mathrm{C}dxd\mathrm{R}\ ;$$

or

$$-\mathrm{AR}ds-\mathrm{CR}dx=\int\mathrm{CR}ddx,$$

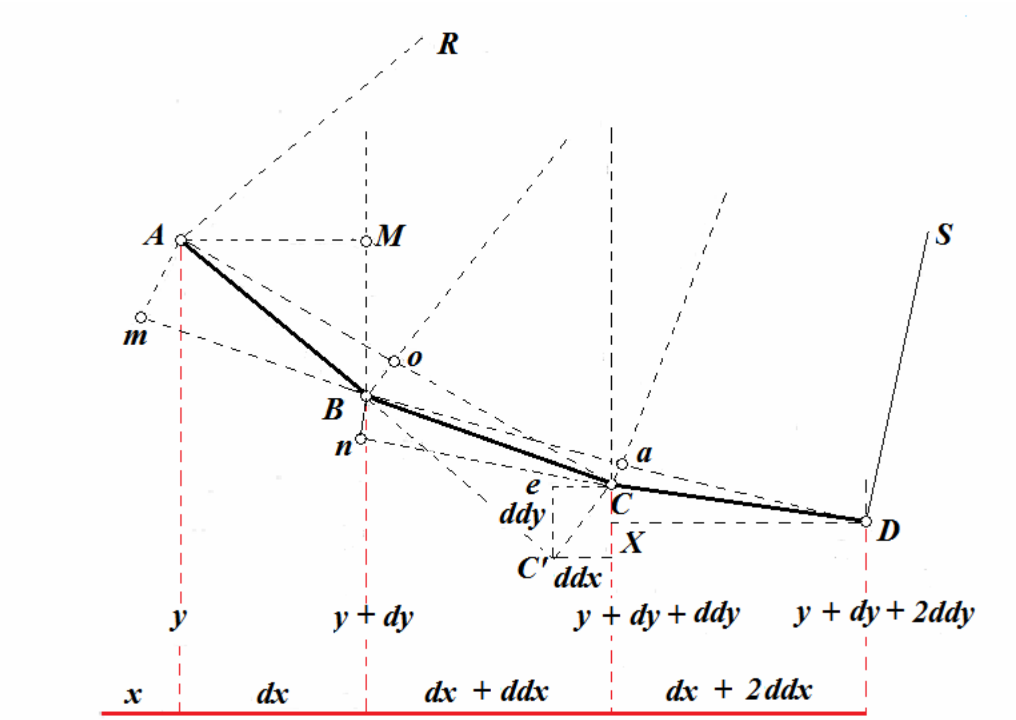


Fig. Ia

or (on putting *dyds* in place of R*ddx*)

$$-\mathrm{AR}ds-\mathrm{CR}dx=ds\int \mathrm{C}dy.$$

Q.E.I.

5. *Corollary* 1. If $\mathrm{C}=0$, that is, if forces shall be present pressing on the curve normally, there shall be $\mathrm{AR}=$constant quantity, as that indicates always and everywhere in curves of this kind the radius of osculation to be inversely proportional to the force, such as for elastic strings, awning, sails and a great many others.

6. *Coroll.* 2. Truly if $A=0$, that is, if only the forces parallel to each other may pull on the string, this equation can be used

$$-CRdx=ds\int Cdy:$$

to reducing which I take the differentials:

$$-\mathrm{CR}ddx-\mathrm{C}dxd\mathrm{R}-\mathrm{R}dxd\mathrm{C}=\mathrm{C}dyds=\mathrm{CR}dx \text{ or } -\mathrm{C}dxd\mathrm{R}-\mathrm{R}dxd\mathrm{C}=2\mathrm{CR}ddx$$

and dividendo per CR*dx* there arises :

$$-\frac{d\mathrm{R}}{\mathrm{R}}-\frac{d\mathrm{C}}{\mathrm{C}}=\frac{2ddx}{dx},$$

and on integrating :

$$-\log.\mathrm{CR} = 2\log.dx - \log.gds^2,$$

with the numbers taken [i.e. antilogs, and where the element *ds* is constant, and used in the constant of integration gds^2]:

$$\frac{1}{\mathrm{CR}} = \frac{dx^2}{gds^2}\text{ , or }\mathrm{CR}dx^2 = gds^2,$$

from which equation it is understood, to be everywhere a force composed in the inverse ratio of the radius of oscillation and the square of the sine of the angle that the force makes with the axis: and the equation will be used for determining the curvature of catenaries of any unequal thicknesses. And thus in these two corollaries all the curves may be present, which until now have been considered by the Geometers concerning this matter ; now it pleases to add certain other new curves to which nature aspires.

7. *Problem.* To find the curve of a heavy sail filled with water.

Solution. It is assumed the sail to be of uniform thickness and likewise to have a rectangular figure, of which the two opposite sides shall be bound firmly by two firm sticks; and thus it happens that the individual threads may be placed together according to the same curve, the nature of which is now to be investigated: moreover it is apparent the individual threads are pulled by two forces, of which the one has arisen from the pressure of the fluid resting on the sail above and is perpendicular to the curve and themselves taken proportional to the depth of the fluid with *ds* constant, truly with the other, which must be due to the weight of the sail, is vertical everywhere and constant. Thus if the axis of the curve may be taken on the surface of the fluid , and the abscissas taken on the axis are called *x*, the applied vertical lines *y*, there will be on substituting in the final equation §.4. *myds* for A and *nnds* for C , and thus there will be had

$$-myRds^2 - nnRdxds = ds\int nndsdy = nnyds^2 + a^3ds^2$$

(by a^3ds^2 I understand a constant quantity being taken arbitrarily): in place of R the value of $\frac{dyds}{ddx}$ itself is put in place,

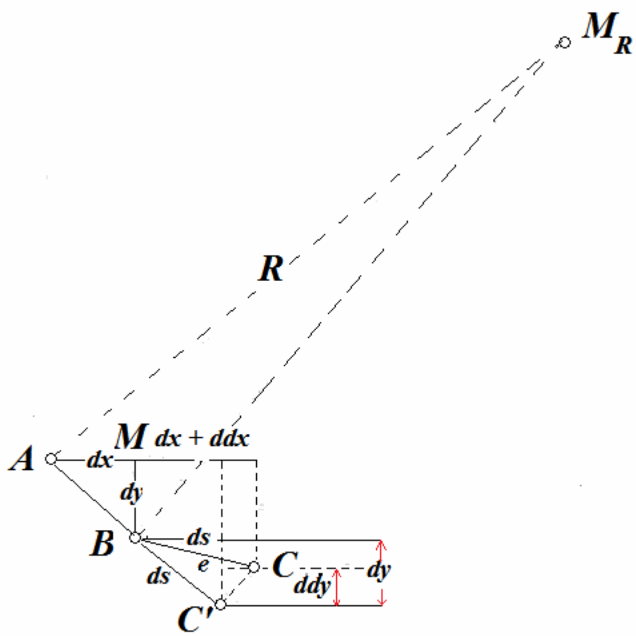


Fig. Ib

[Fig. *Ib*, adapted from that in KF, Section II. p.9, shows two similar figures, AM_RBMA and C'BCeC', from which the values of the radius of curvature used by Bernoulli can be derived from proportion, initially found by James Bernoulli, who also derived the customary formula $R = \frac{\left(1+y'^2\right)^{\frac{3}{2}}}{y''}$: namely,

$$\mathrm{AM_R} : dx = ds : ddy \text{ from which } \mathrm{R} = \frac{dsdx}{ddy},$$

and

$$\mathrm{BM_R} : dy = ds : ddx \text{ from which } \mathrm{R} = \frac{dsdy}{ddx}.]$$

and with the terms handled correctly:

$$-mydyds = nnyddx + nndydx + a^3 ddx\,,$$

which integrated with the addition of the constant $b^3 ds$ gives

$$b^3 ds - \tfrac{1}{2} myyds = nnydx + a^3 dx\,,$$

and from which reduced equation thence there arises :

$$dx = \left(b^3 - \tfrac{1}{2} myy\right) dy : \sqrt{\left(nny + a^3\right)^2 - \left(b^3 - \tfrac{1}{2} myy\right)^2}\,.$$

Q. E. I.

8. *Coroll.* 1. If the weight of the sail vanished in comparison with the weight of the fluid superimposed the curve first considered by my uncle James Bernoulli arises, for in this case there is $n = 0$ and it shall be :

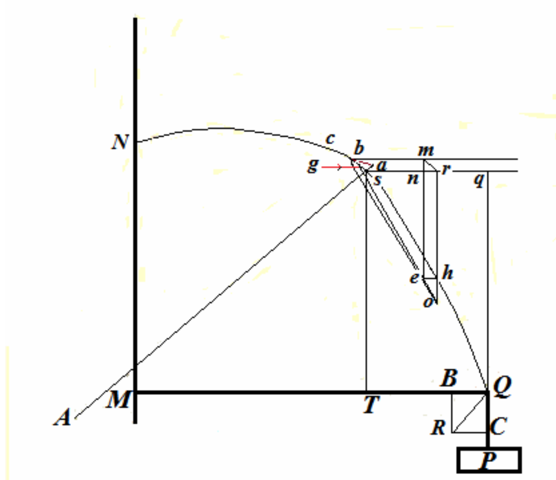
$$dx = \left(b^3 - \tfrac{1}{2}myy\right)dy : \sqrt{a^6 - \left(b^3 - \tfrac{1}{2}myy\right)^2}.$$

9. *Coroll.* 2. Truly if the weight of the superimposed fluid were put equal to zero, thus so that the sail may be curved by its weight alone, the ropes or catenaries arise first proposed to the Geometers by my father, then namely there shall be $m = 0$, and such an equation will be had :

$$dx = b^3 dy : \sqrt{\left(nny + a^3\right)^2 - b^6}.$$

10. *Problem.* To find the curvature of an elastic lamina, curved partially by its own weight, and partially on account of an attached weight.

Solution. I shall assume the lamina to be of an equal structure through its whole length, although the problem will not made more difficult, if the lamina shall be of an unequal structure. Truly the length of the whole lamina shall be *l*, is weight *p* , the attached weight P, and the lamina *NsQ*. (fig. 2.) inserted into the vertical wall NM at some angle, and to its extremity *Q* to be attached the weight P. Now we may consider the element *bs* of the curve, and draw the horizontal *sq* : The centre of gravity of the arc *sQ* shall be at *o* : the vertical *or* is drawn, and to that the parallel line Q*q* ; and thus it is apparent the weight of the arc *sQ* to act in the lever *sr*, and the suspended weight P to act in the lever *sq:* but these two weights arrange so that the element of the curve (which otherwise would lie in line with the nearby element *cb*, and will have *ba* in place), may bend into *bs*, again it is



agreed that the line element *sa* (perpendicular to the curve) be proportional to the force [*i.e.* moment] bending the element *bs* ; from which it remains that the bending moment be sought. Truly the arc $\text{Q}s = s$, the horizontal abscissa QT $= x, sr = q$; thus the weight of the arc $\text{Q}s = \frac{sp}{l}$, which taken by the lever arm *sr* gives the absolute moment $= \frac{sqp}{l}$, to which if $x\text{P}$ is added, the total bending moment on the element bs shall be $\frac{sqp}{l} + x\text{P}$, which is itself proportional to *as* or inversely proportional to the radius of osculation *sA*, which I will call R; from which such an equation will be had :

$$\frac{sqp}{l} + xP = \frac{m}{R}.$$

But truly since *q* may be found in this equation, this quantity is required to be eliminated ; towards this end it is required, that an equation may be found between *q, s* et *x*, which may be obtained in the following manner: The element *sb* may be divided into two equal parts at the point *g*, and *ro* is drawn; then if such a proportionality may be made [for taking moments (approximately) about *b* along *bo*, and where the increment of arc *bs* in the diagram has become the differential *ds* :

$bg \times bs + bo \times s = be \times (s + bs) = be \times s + be \times bs \therefore eo \times s = ge \times bs$ or $eo = \frac{ge \times bs}{s} \to \frac{ge \times ds}{s}$]

$s.ds :: og.oe$; $oe = \frac{ds.og}{s}$, *e* will be the centre of gravity of the arc $s + ds$, if in the next place *eh* may be drawn parallel to *bm* itself , there will be [again on taking moments, this time about the line *mr* :] $eh = qds : s = nr$ and the derivative of the point *r*, or

$$dq = bm - sr = dx - nr = (\text{because } nr = \frac{qds}{s}:)\ dx - \frac{qds}{s} \text{ or } sdq + qds = sdx$$

or $qs = \int sdx$; which value on being substituted into the equation for the curve, there arises $\frac{p}{l}\int sdx + Px = \frac{m}{R}$. Q.E.I.

11. *Coroll.* If $p = 0$, common elasticity arises ; and if $P = 0$ new elasticity arises , by which the lamina composes itself according to its own weight : and thus is of such a property that $\int sdx$ shall be inversely proportional to the radius of curvature : further reduction I have not yet tested enough, I still doubt whether a further reduced equation shall be possible.

12. *Problem.* To find the curvature of a rope endowed both with elasticity as well as weight.

Solution. This problem, which was proposed by Geometers several years ago, but by nobody as far as I know has been solved publicly, now it is solved easily according to the method of the preceding problem. Only that in the above it is required to consider the point Q not to weigh vertically downwards but to pull in some other direction QR. Again we may put the force QR to be expressed by P by retaining the same names that we have put in place just now ; but by resolving the force QR into QB and QC, see Fig.II. Truly BQ, QC, QR, shall be as $m, n,$ g ; therefore the force $\text{QB} = \frac{m\text{P}}{g}$, the force $\text{QC} = \frac{n}{g}\text{P}$; moreover the force QB acts on the lever arm Qq or $Ts, = y$; from which it moment is $= \frac{m\text{P}y}{g}$ and the force QC acts on the lever arm $sq = \text{TQ} = x$, therefore its moment is $= \frac{n\text{P}x}{g}$ and the moment of the weight of the arc sQ is again $\frac{sqp}{l} = \frac{p\int sdx}{l}$;

therefore $as = \frac{m}{R} = \frac{mPy + nPx}{g} + \frac{p\int sdx}{l}$, which is the equation for a heavy elastic rope fixed to a wall at N at some angle N, and pulled by any two forces which are either positive or negative. Q E. I.

13. *Scholium.* Now the matter is set out most easily and generally : Surely α may express the weight of the arc Qs, β the force from the side at s , and the remaining denominations may be retained as before ; such an equation will be had :
$\int \alpha dx + \frac{mPy + nPx}{g} = \frac{\beta}{R}$; within which is contained whatever can be thought out about the kind of curve present, which our own illustrious Euler correctly observed, who himself thus had proposed this problem to be solved, so that nothing can be added.

METHODUS UNIVERSALIS

Determinandae curvaturae fili a potentiis quancunque legem inter se observantibus extensi , una cum solutione problematum quorundam novorum eo pertinentium.

Autore

Daniele Bernoulli I.F.I.

I. Cum filum extenditur a solis potentiis verticalibus aut solis normalibus ad curvam, facile lex quaedam uniformis inter potentias applicatas et angulorum quorundam funis perspicitur , ita ut quamcunque habeant legem hae potentias possit curvatura fili semper aequatione differentiali secundi ordinis definiri , quae saepe ad simpliciter differentialem se reduci patitur : Ita catenaria, velaria, lintearia, elastica aliaeque curvae olim a summis Geometris fuerunt erutae. At si in unoquoque fili puncto plures diversae potentiae sub variis directionibus applicatae sint , deficit illa uniformitas , neque aliter quam quantitatibus differentialibus tertii ordinis curvae proprietatem determinare possumus, si in generalibus subsistamus : Ut methodus mea , quae quidem pro hoc negotio generalis est, eo clarius perspiciatur, fingam in unoquoque fili puncto duas saltem potentias applicari; alteram ad curvam perpendicularem, alteram datum constantem angulum cum axe formantem, quae tamen lex ita late se extendit, ut omnes curvas hactenus a Geometris hac in re consideratas , ut specialissima exempla sub se comprehendat. Postmodum problematum quorundam mihi primo consideratorum solutionem ostendam.

2. *Lemma* I. Sit filum (fig. I.) ABCD affixum punctis A et D divisimque in tres partes aequales AB, BC, CD. Intelligantur in punctis B et C applicatae potentiae BK et CI quarum directiones bisecent aequaliter angulos ABC et BCD: ut et potentiae BH et CL, quarum directiones sunt inter se parallelae , erit productis lineis demissisque perpendicularibus quas figura ostendit atque dcsignatis potentiis BK, CI, BH et CL per A, B, C et D, erit inquam

$$\frac{sin.\ ang.\mathrm{ABR}}{sin.\ ang.\mathrm{AB}m}\mathrm{A}+\frac{sin.\ ang.\mathrm{ABM}}{sin.\ ang.\mathrm{AB}m}C=\frac{sin.\ ang.\mathrm{BC}d}{sin.\ ang.\mathrm{BC}n}\mathrm{B}+\frac{sin.\ ang.\mathrm{BC}X}{sin.\ ang.\mathrm{BC}n}\mathrm{D}$$

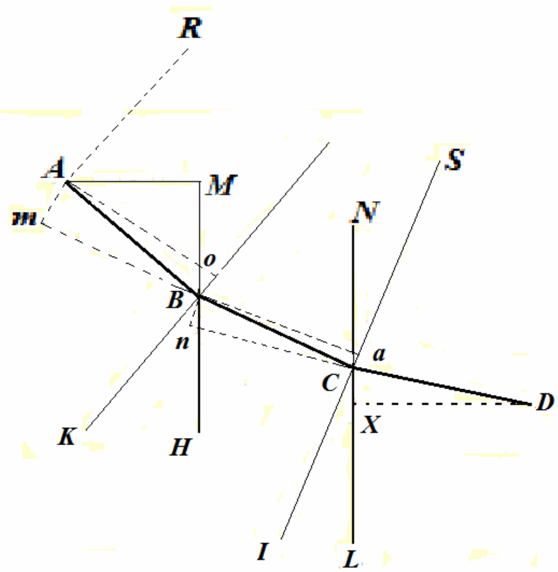


Fig. I

seu sumta $Ao=\mathrm{BC}=\mathrm{CD}$ pro sinu toto erit

$$\frac{\mathrm{AO}}{\mathrm{A}m}\mathrm{A}+\frac{\mathrm{AM}}{\mathrm{A}m}C=\frac{\mathrm{B}a}{\mathrm{B}n}\mathrm{B}+\frac{DX}{\mathrm{B}n}\mathrm{D}.$$

Demonstratio constat ex mechanicis atque facile deducitur ex compositione potentiarum, cuius ope determinantur potentiae punctum aliquod in BC assumtum versus B et G trahentes , quae inter se debent esse aequales.

3. *Lemma* 2. Si anguli *ABm* et OC*n* sint infinite parvi neglectis negligendis , erit $\mathrm{AO} = \mathrm{AB},\ \mathrm{A}m = 2\mathrm{B}o;\ \mathrm{BQ} = \mathrm{BC} = \mathrm{AB}$ et $\mathrm{B}n = 2\mathrm{C}a$, si praeterea radivs circuli per tria puncta A , B, C transeuntis ponatur $= \mathrm{R}$ et radivs circuli per tria alia puncta B, C , D transeuntis $= \mathrm{S}$, erit $2\mathrm{BO} = \frac{\mathrm{AB}^2}{R}$ et $\mathrm{C}a = \frac{\mathrm{BC}^2}{S}$, unde substitutis hisce valoribus mutabitur aequatio praecedentis lemmatis in hanc

$$\frac{\mathrm{R.A}}{\mathrm{AB}} + \frac{\mathrm{AM.R.C}}{\mathrm{AB}^2} = \frac{\mathrm{S.B}}{\mathrm{AB}} + \frac{\mathrm{DX.S.D}}{\mathrm{AB}^2}.$$

4. *Problema.* Invenire curvaturam fili , cuius singula puncta datis duabus quibuscunque potentiis , altera ad curvam normali , altera ad datam lineam ubique parallela trahuntur.

Solutio. Sint tria elementa curvae quaesitae tres lineolae aequales AB, BC et CD ; retentisque significationibus iisdem sumantur abscissae x in axe perpendiculari ad directionem potentiarum parallelarum : applicatae y sint ad axem perpendiculares, radivs osculi in puncto B sit $= \mathrm{R}$; elementum curvae $AB = ds$; ita erit $\mathrm{AM} = dx,\ \mathrm{DX} = dx + 2ddx,\ \mathrm{B} = \mathrm{A} + d\mathrm{A},\ \mathrm{D} = \mathrm{C} + d\mathrm{C},\ \mathrm{S} = \mathrm{R} + d\mathrm{R}$: facta itaque applicatione lemmatis secundi ad praesentem casum, habebitur

$$\frac{\mathrm{R.A}}{ds} + \frac{dx.\mathrm{R.C}}{ds^2} = \frac{(\mathrm{R}+d\mathrm{R})\times(\mathrm{A}+d\mathrm{A})}{ds} + \frac{(dx+2ddx)\times(\mathrm{R}+d\mathrm{R})\times(\mathrm{C}+d\mathrm{C})}{ds^2}:$$

qua aequatione recte pertractata sit

$$-\mathrm{A}d\mathrm{R}ds - \mathrm{R}d\mathrm{A}ds = \mathrm{R}d\mathrm{C}dx + 2\mathrm{CR}ddx + \mathrm{C}dxd\mathrm{R}\ ;$$

vel

$$-ARds - CRdx = \int CRddx,$$

vel (ponendo *dyds* loco R*ddx*)

$$-\mathrm{AR}ds - \mathrm{CR}dx = ds\int \mathrm{C}dy.$$

Q.E.I.

5. *Corollarium* 1. Si $\mathrm{C} = 0$, id est: , si solae potentiae ad curvam normaliter insistentes adsint, sit $\mathrm{AR} =$ quantitati constanti , id quod indicat esse in huiusmodi curvis semper et ubique radivm osculi reciproce proportionalem potentiae: estque haec proprietas pro elastica, velaria, lintearia aliisque infinitis.

6. *Coroll.* 2. Si vero $A = 0$, id est, si solae potentiae inter se parallelae filum trahant, inserviet haec aequatio $-CRdx = ds\int Cdy$: ad quam reducendam sumo differentialia $-\mathrm{CR}ddx - \mathrm{C}dxd\mathrm{R} - \mathrm{R}dxd\mathrm{C} = \mathrm{C}dyds = \mathrm{CR}dx$ seu $-\mathrm{C}dxd\mathrm{R} - \mathrm{R}dxd\mathrm{C} = 2\mathrm{CR}ddx$ et dividendo per $\mathrm{CR}dx$ oritur $-\frac{d\mathrm{R}}{\mathrm{R}} - \frac{d\mathrm{C}}{\mathrm{C}} = \frac{2ddx}{dx}$, et integrando $-\log.\mathrm{CR} = 2\log.dx - \log.gds^2$, sumtisque numeris $\frac{1}{\mathrm{CR}} = \frac{dx^2}{gds^2}$, aut $\mathrm{CR}dx^2 = gds^2$, ex qua aequatione cognoscitur , esse ubique potentiam in ratione reciproca composita ex radio osculi et quadrato funis anguli , quem potentia facit cum axe : inservitque aequatio pro determinanda curvatura catenarum utcunque inaequaliter crassarum. Atque hisce duobus corollariis continentur omnes curvae, quae hactenus Geometris consideratae fuerunt circa hoc argumentum; lubet nunc alias quasdam superaddere nouas, quas natura affectat.

7. *Problema.* Invenire curvaturum lintei gravis aqua repleti.

Solut. Assumitur linteum uniformis esse crassitiei idemque explicatum habere figurum rectanguli, cuius latera opposita sint duobus baculis firmis alligata; itaque fiet ut singula fila ad eandem curvam se componant, cuius natura iam est indaganda: apparet autem singula fili puncta trahi a duabus potentiis, quarum altera a pressione fluidi linteo superincumbentis oriunda perpendicularis est ad curvam ipsique altitudini fluidi proportionalis sumtis *ds* constantibus , altero vero , quae gravitati lintei debetur , ubique est verticalis et constans. Si itaque axis curvae sumatur in superficie fluidi , dicanturque abscissae in axe sumtae *x* , applicatae verticales *y* , substituendum erit in ultima aequtione. §. 4. *myds* pro A et *nnds* pro C , et sic habebitur

$$-myRds^2 - nnRdxds = ds\int nndsdy = nnyds^2 + a^3ds^2$$

(per a^3ds^2 intelligo quantitatem constantem ad arbitrium sumendam): ponatur loco R valor ipsius $\frac{dyds}{ddx}$, et erit recte dispositis terminis

$$-mydyds = nnyddx + nndydx + a^3ddx\,,$$

quae integrata cum additione constantis b^3ds dat

$$b^3ds - \tfrac{1}{2}myyds = nnydx + a^3dx\,,$$

qua reducta aequatione oritur denique

$$dx = \left(b^3 - \tfrac{1}{2}myy\right)dy : \sqrt{\left(nny + a^3\right)^2 - \left(b^3 - \tfrac{1}{2}myy\right)^2}\,.$$

Q. E. I.

8. *Coroll*. 1. Si gravitas lintei evanescat prae pondere fluidi superincumbentis oritur curvatura lintei Patruo meo Jacobo Bernoulli prima considerta, nam hoc in casu est : $n = 0$ sitque

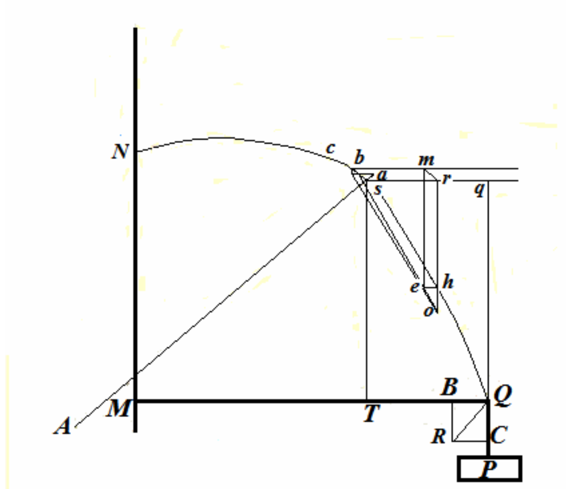
$$dx = \left(b^3 - \tfrac{1}{2}myy\right)dy : \sqrt{a^6 - \left(b^3 - \tfrac{1}{2}myy\right)^2}.$$

9. *Coroll.* 2. Si vero gravitas fluidi superincumbentis ponatur nulla , · ita ut linteum solo suo pondere incurvetur, oritur funicularia seu catenaria a Patre meo prima Geomctris proposita , tunc nempe est: $m = 0$, habeturque talis aequatio:

$$dx = b^3dy : \sqrt{\left(nny + a^3\right)^2 - b^6}.$$

10. *Problema.* Invenire curvaturam laminae elasiticae partim proprio pondere, partim pondere appenso incurvatae.

Solutio. Assumam laminam per totam suam longitudinem aequabilis esse structurae , utut problema non difficilius fiat, si lamina sit inaequalis structurae. Sit vero longitudo totius laminae *l*, pondus ipsius *p* , pondus appensum P, ponaturque laminam *NsQ.* (fig. 2.) parieti verticali NM insertam ad angulum quemcunque , eiusque extremitati *Q* appensum esse pondus P. Consideremus iam elementum curvae *bs* , ducamusque horizontalem *sq* : Sit centrum gravitatis arcus *sQ* in *o* : ducatur verticalis *or*, eique parallela Q*q* ; sicque patet pondus arcus *sQ* agere in vectem *sr*, pondusque appensum P agere in vectem *sq:* faciunt autem haec duo pondera, ut elementum curvae (quod alias in directum iaceret cum elemento proxime *cb* , situmque haberet in *ba*), inflectatur in *bs*, porro constat lineolam *sa* (perpendicularem ad curvam) proportionalem esse vi elementum *bs*



inflectenti; unde superest ut vis inflectens quaentur. Sit vero arcus Q$s = s$, abscissa QT horizontalis $= x, sr = q$; sic erit pondus arcus Q$s = \frac{sp}{l}$, quod ductum in vectem *sr* dat potentiam absolutam$=\frac{sqp}{l}$, cui si addatur xP , habebitur vis totalis elementum *bs* inflectens $\frac{sqp}{l}+x\mathrm{P}$, quod est proproportionale ipsi *as* seu reciproce proportionale radio osculi *sA*, quem vocabo R; unde habetur talis aequatio :

$$\frac{sqp}{l}+xp=\frac{m}{R}.$$

At vero cum in hac aequatione reperiatur q , haec quantitas eliminanda est; ad hoc requiratur , ut aequatio habeatur inter *q*, *s* et *x*, quae sequenti modo obtinetur. Dividatur bifariam elementum *sb* in puncto *g*, ducaturque *ro* ; tunc si fiat talis analogia :

$s.ds::og.oe=\frac{ds.og}{s}$ erit *e* centrum grauitatis arcus $s+ds$, si ducatur dein *eh* parallela ipsi *bm* , erit $eb=qds:s=nr$ et est fluxus puncti *r* seu

$$dq=bm-sr=dx-nr=(\text{quia } nr=\frac{qds}{s})\ dx-\frac{qds}{s} \text{ vel } sdq+qds=sdx$$

aut $qs=\int sdx$; quem valorem substituendo in aequatione ad curvam, oritur

$$\frac{p}{l}\int sdx+Px=\frac{m}{R}.$$

Q.E.I.

11. *Coroll.* Si $p=0$, oritur elastica ordinaria ; et si $P=0$ prodit elastica nova , ad quam se componit lamina proprio se pondere incurvans : cuius adeoque proprietas talis est ut $\int sdx$ sit reciproce proportionalis radio osculi: ulteriorem reductionem nondum satis tentavi , dubito tamen an ulterius reduci possit aequatio.

12. *Problema.* Invenire curvaturam funis tam elasticitate quam gravitate praediti.

Solutio. Problema hoc, quod ante plures annos Geometris propositum, sed a nemine quod sciam publice solutum fuit , facile nunc ad modum praecedentis problematis solvitur. Id solummodo insuper considerandum est , punctum Q non pondere verticaliter deorsum sed potentia alia sub quacunque directione QR trahi. Ponemus rursus potentiam QR exprimi per P reliquas denominationes retinendo easdem quas modo posuimus ; resolvenda autem est potentia QR in QB et QC. Sint vero BQ, QC, QR, ut, *m, n,* g ; erit ergo potentia $\mathrm{QB}=\frac{m\mathrm{P}}{g}$, potentia $\mathrm{QC}=\frac{n}{g}\mathrm{P}$; agit autem pot. QB in vectem *Qq* seu $Ts=y$; unde ipsius momentum $=\frac{m\mathrm{P}y}{g}$ et potentia QC agit in vectem $sq=\mathrm{TQ}=x$ ergo ipsius momentum $=\frac{n\mathrm{P}x}{g}$ et momentum ponderis arcus *sQ* est iterum $\frac{sqp}{l}=\frac{p\int sdx}{l}$; ergo

$as=\frac{m}{R}=\frac{mPy+nPx}{g}+\frac{p\int sdx}{l}$, quae est aequatio ad funem elasticum et gravem parieti verticali sub angulo quocunque infixi in N, et duabus potentiis quibuscunque sive affirmativis, sive negativis tracti. Q E. I.

13. *Scholium.* Facillimum nunc est generalissime rem expedire : Exprimat nempe α pondus arcus $\mathrm{Q}s, \beta$ vim elateris in puncto s , ceteraeque denominationes retineantur ut ante ; habebitur talis aequatio $\int \alpha dx+\frac{mPy+nPx}{g}=\frac{\beta}{R}$; sub qua continetur quicquid circa praesens curvarum genus excogitari potest, quod probe notavit clarissimus noster Eulerus, qui problemata haec a me sibi proposita ita soluit, ut nihil superaddi posse videatur.