HYDRODYNAMICS SECTION EIGHT.

Concerning the motion both of homogeneous as well as heterogeneous fluids through vessels of irregular construction divided up into several parts, where the individual phenomena of the trajectories of the fluids through a number of openings may be explained and a part of the motion may be absorbed continually from the theory of living forces; and with the general rules for the motions of the fluids defined everywhere.

§. 1. Up to the present we have not made use of other principles than as with these two in the last section, the *velocities of the fluids shall be everywhere inversely proportional to the cross-sections of the vessels,* with the aid of which the *ascent potential* of the whole water was found from the given *ascent potential* of any part; then because the *ascent potential* of all the water shall remain always equal to the *actual descent.* As often as both these principles have a place, there is no doubt, why by the method used by us the motion of fluids may not be defined correctly. Yet I will not deny, vessels of this kind are able to be constructed, in which fluids will be moving, so that neither of these principles will proceed correctly. The first principle indeed rarely or ever will depart noticeably from the truth, because generally no location will be found, where almost no motion of the water may usually be had, and they can be considered without sensible error indeed as being stagnant: Truly the other principle is to be considered otherwise, as will appear from the examples below, and these are able to provide the evidence to bring that matter to light, which we brought forwards in the above section concerning the reflux of water ; for yet it is lacking, that water falling from a given height in a submerged vessel, shall be able to go back to that height, just as they ought according to the strength of this principle, with all external impediments removed, but rather in general the ascent of these shall be scarcely noticeable besides the descent, which they had done before: indeed not even again can the water ascent above the water surface, to which the tube is immersed, to how great it was depressed below the same, except with the opening is the whole tube : thus indeed the surface is much less depressed than it was raised before. We have given an account of these in the above section: Because these are thus, I will now give two rules required to be defined for the motion of water everywhere, and these again I will illustrate with such examples, which no theory has been able to explain up to this time, but certainly agree with our theory in an outstanding way.

[Thus, in any experiment, the continuity equation is always valid, but the conservation of kinetic energy is valid only under certain circumstances.]

Rule 1.

§. 2. With the velocity of the fluid assumed or indeed known somewhere in the vessel, it is seen what the velocity shall become in the remaining parts of the fluid. Thus indeed the *ascent potential* of the whole fluid may be known, and its increment. Thus far we have considered fluids in infinite parallel layers or rather divided up everywhere perpendicular to the sides of the vessel everywhere, and we have put in place the velocities to be inversely proportional to these [layers]:

[In the original derivations in Section III, the layers are cross-sectional cylinders of infinitesimal length corresponding to the diameter of the vessel, assumed cylindrical locally at least, and placed vertically or horizontally; thus, the greater the area of a layer, the slower the speed of the fluid flowing across it proportionally, and vice-versa.]

Indeed it is easy to make a vessel, where the fluids are moved otherwise ; however I might have believed with these in place at no time would the fluid have a different motion, so that almost no perceptible error would be able to arise from that : yet the given rule used would be able to be of greater accuracy. Especially indeed here it may pertain to the contraction of the streams, when the fluids are forced to pass through openings in excessively thin plates, which is required to be taken into consideration in large applications: The effect of contractions of this kind is not at all bad, I think they are to be foreseen, when they will have been considered correctly, as I have advised about these in section four.

Rule 2.

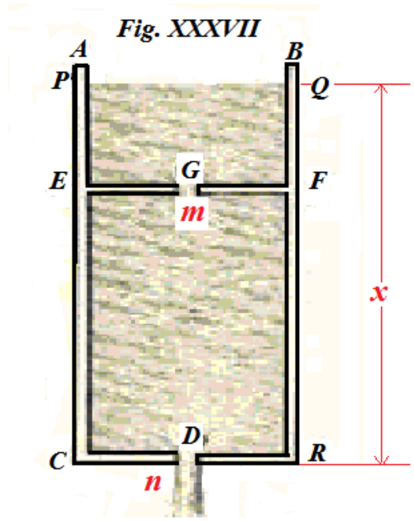
§. 3. It is required to see clearly that nothing is contributed at individual instants to the magnitude of the *vis viva,* or arising from the product of the *ascent potential* by the mass for a particular flow, of which the nature is sought. Indeed that product is required to be left again by any close inspection considered. Because if it arises thus, that an addition has been made to the *ascent potential* by the mass, which a particular motion involved, and the sum of all the products is required to have been made equal to the mass of all the water by its same *actual descent.*
[Thus, the sum of all products of the increases in the *ascent potential* by the associated masses is balanced by the product of the *actual descent* (or fall of the centre of gravity) by the total mass; meaning that there is no change overall, or the *vis viva* remains the same. Thus Daniel Bernoulli's ideas are almost the same as the modern idea of the conservation of potential plus kinetic energy; except that he has not used the concept of work done, rather the increase in the *vis viva* due to the mass falling - the *actual descent,* is balanced completely by its increased ability to rise higher - the *ascent potential*. It appears that Thomas Young, in his *Physics Lectures* at the Royal Institution [~ 1803], was the person who coined the word *energy*, and used this earlier concept of the *vis viva* principle expressed in terms of the work done, rather than by taking moments.]

This rule is certainly of great advantage, and as I think, almost the only one towards obtaining the measures of the motions, which occur in irregular vessels divided into several cavities communicating with each other, which I will show now by some examples.

Problem.

§. 4. The vessel *ACRB* (Fig. 37) were to be proposed as if of infinite cross-section everywhere, to the ratio of the opening to be discussed, and divided by some diaphragm *EF* into two distinct cavities communicating with each other, by the middle opening *G*: in addition the same vessel shall have in its lower part another opening *D*: then the vessel may be put full of water as far as to *PQ*, thus so that the lower cavity *CEFR* shall be filled completely with water, and the other part *PQFE* above the diaphragm. With these in place, and now with the fluid beginning to move, the velocity of the water flowing out into the air through the opening *D* or the generating height of this velocity, is sought.

Solution.



Let the height of the surface *PQ* above the opening $D = x$, the cross-section of the opening $D = n$, and of the other $G = m$. Moreover it is seen that the *ascent potential* of any drop flowing through *G* provides no increase to the efflux through *D*, and the whole of any motion to be expended in some internal excitation, which soon will be absorbed without other effect : therefore it is necessary that at individual moments a new motion may be produced in the particles passing across the opening *G*, and no less with the particles flowing out through *D*. But if the *ascent potential* of the drop flowing out through *D* may be called *v*, that is, if the water bursting forth through *D* may be considered with a velocity, of which *v* shall be the generating height, the height will be equal to its first for the same weight of the droplet but in the ratio $\frac{nnv}{mm}$, flowing out through *G* in the same time. With these *ascent potentials* multiplied by the mass, as they may be had equal, and which I will call *M*, the sum of the products will be $= Mv + \frac{Mnnv}{mm}$. And since on account of the infinite cross-section of the vessel no other motion may be generated, the before-mentioned sum will be agreed (by rule 2) to be made equal to the mass of all the water in the *actual descent* of the same. But indeed if the total mass of all the water is called μ, the *actual descent* will become (by §. 7 Sect. III) $= \frac{Mx}{\mu}$, while the droplet flows out, thus so that the common product shall be $= Mx$.

[Recall that moments are taken, in this case about the surface *PQ*, so that the moment of the droplet *Mx* is equal to the *actual descent* by the total mass μ, and this amount equals the total *ascent potential* by the mass.]

Therefore there will be had :

$$Mv+\frac{Mnnv}{mm}=Mx\ ,$$

or

$$v=\frac{mmx}{nn+mm}.$$

Q.E.F.

[In modern terms, we would consider the ascent potential at *G* to be due to a certain velocity V_G acquired in falling through some height H_G, where $V_G^2/2g=H_G$; in passing through the lower opening, we can apply the continuity equation to find the corresponding velocity V_D :

$V_G\times m=V_D\times n;$ or $V_D=\frac{V_G\times m}{n}$ and hence $\frac{V_D^2}{2g}=\frac{V_G^2\times m^2}{2gn^2}=v,$ and $\frac{V_G^2}{2g}=\frac{n^2}{m^2}v.$ Thus, on adding to find the total potential

$$x=\frac{V_D^2}{2g}+\frac{V_G^2}{2g}=\frac{V_D^2}{2g}\left(1+\frac{n^2}{m^2}\right)=\left(\frac{m^2+n^2}{m^2}\right)v \text{ on taking } 2g=1, \text{or } v=\left(\frac{m^2}{m^2+n^2}\right)x.\ ]$$

Scholium 1.

§. 5. It is evident from this example, the motion can be determined without the differential calculus, since the shape of the vessel everywhere is the widest and hence the motion cannot be changed. Meanwhile it would not be difficult likewise in the future, to define the flow, also from a consideration according to the cross-sections of the vessels used, and it is only by carefully striving after brevity that we have avoided and equally omitted that in the following, except perhaps the motion may be changed notably by the shape of the vessel, which can be shown well enough in wide tubes, but with these the longest, in which the fluid may be moving, especially if the motion shall be determined to be oscillatory. Indeed we have seen in the preceding section, if the oscillations are very small in the deepest submerged tubes, then so much [*vis viva*] may be absent, that only the opening of the base itself shall require attention, with the cross-sections ignored even if for a large enough size, but rather it shall be required generally to refer to these alone.

Scholium 2.

§. 6. Because in the calculation which we have put in place, the *vis viva* of any droplet flowing out through *G* from the water of the lower cavity ought to be absorbed, it is to be seen, the proposition is not required to be extended to these cases, which are not in agreement with the hypothesis, or as when the diaphragm *EF* is close to the base *CR* and likewise the openings themselves correspond directly to each other: thus indeed it is not an arduous task to foresee, that the motion will be quite different from that which the present theory indicates. But however, if the distance *DG* shall be great, and if likewise the openings shall be placed obliquely and the walls of the openings of the prevent the contraction of the water jets ; there is no doubt, why the theory may not rather correspond accurately with all the phenomena.

Corollary.

§. 7. If the opening *G* is certainly larger than the other, there shall be almost $v = x$, but this altitude *v*, to which without doubt the velocity of the water flowing out through *D* shall correspond, will decrease greatly, with the increase in the opening *D*, thus so that if it were for example twice the size of the opening *G*, there shall be $v = \frac{1}{5}x$ and yet it does not vanish completely, when the opening *G* is exceedingly small with respect to the opening *D*.

Thus with these found, whoever now will consider the true account of these motions, which Mariotte was the first to observe, and which he himself showed filled with admiration and to be delighted beyond measure, and likewise it will be understood, how far this most observant author had deviated from the path in these remaining investigations. I think it would not be inappropriate to append here Mariotte's observations.

§. 8. The vessel used, of such a kind as Fig. 38 represents, which does not differ from the former except in that, which in the lower part of the cylindrical tube *ABC* shall be put in place *MD* horizontally with the opening *D*, through which the water may leap put vertically: now the diaphragm *EF* has been perforated as before by the opening *G* : below that a little is the opening *K*, in order that the lower cavity may be filled easier with water, with what done the same may be stopped up, and the rest of the vessel was being refilled.

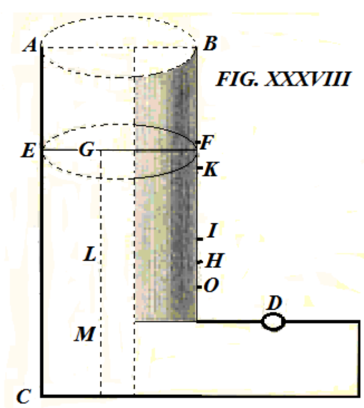


FIG. XXXVIII

Thus from these preparations, and with the efflux of the water through *D*, Mariotte observed, that soon it rose as far as to *I*, then with the velocity slowly diminished as far as to *H* and finally, with the depletion of the whole upper chamber *ABFE* imminent, as far as to *O*, and then immediately with new forces acting to leap up as far as to *F*. He also noted, if I remember well, the height of the initial throw therefore to be smaller, as the opening *G* shall be less on account of the other *D*. See his *Tract. de motu aquarum, part.* IV, *disc.* 1. But he thought it was possible to explain the changes of these motions by imagining a narrower tube *GLMD* to be connected to the widest tube *ABFE*, through which the water may flow. But indeed we have shown and everyday experience teaches, the motion of the water from the vessel *ABGLMD* certainly to be different from that, which was just indicated. Not less is it to be in error, if from that it might be thought the water to leap out with the same velocity through the opening *D*, as if that were put into the diaphragm *EF* ; for it can happen, that the initial height of the jet may be greater or less with the height *FB*. Nor finally that amount of water may flow out, as easily as one could expect, where in the same time it may flow out from the upper vessel simply with the part *EFDC* cut off, just as thus the same may be had approximately, when the opening *G* is exceedingly smaller than the opening *D*.

§. 9. However our equation, namely $v = \frac{mmx}{nn + mm}$, corresponds entirely correctly with all the phenomena : indeed it indicates the water soon after the first flow rises to a certain height, and with that to be so much smaller, to the same extent as the opening of the diaphragm compared with the other opening; then that height to be diminished gradually, while all the water has flowed out from the upper cavity, when at once from that moment it begins to increase, only it does not reach the whole height of water lying above, because then the outflow of water from a simple vessel and that from a vessel of infinite cross-section are required to be considered : yet for a little time even now the water is retarded by the passage of air through the opening *G*, and certainly noticeably delayed, when the upper opening is very small, about which argument we will say something soon, when the discussion will be about heterogeneous fluids. If Mariotte's figure must correspond to the argument put in place in proportion, it is required, that the opening *G* shall be made a little less than half the other.

§. 10. Again our formula indicates, because with many things perhaps still not understood, this theory may be able to be seen to satisfy the paradox : the position of the diaphragm *EF* either higher or lower in no way changes the force or velocity of the water flowing out; but the account of this phenomena is clear now from everything, I think.

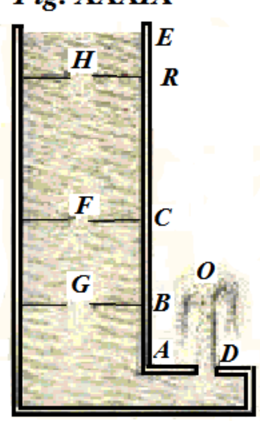
§. 11. Indeed now we will examine the above motion of the water, when there shall be several diaphragms with holes bored, through which water is forced to pass through, so that the efflux may happen through the opening *D*. That will be able to be resolved in the same manner, as we have used in §. 4. Moreover thus with the calculation put in place correctly and with the same denominations used it will appear to be :

$$v = x : \left(1 + \frac{nn}{\alpha\alpha} + \frac{nn}{\beta\beta} + \frac{nn}{\gamma\gamma} + \&\text{c.}\right),$$

where by $\alpha,\ \beta,\ \gamma$ &c. are understood the cross-sections of the openings, which are in the diaphragms, while *n* expresses as before the cross-section of the opening *D*, through which the water flows out.

§.12. If hence in place of a single diaphragm, several diaphragm such as *B, C, R* &c. shall be in the same vessel, such as Fig. 39 may represent, through which water may pass through, while it flows out by the lowest opening *D*, the velocity of the water will be changed and increased at once, as often as the cavity may be filled in some manner : but such can be the proportion between the heights *AB, BC, CR, RE* &c. and the cross-sections of the openings *D, G, F, H* &c.

Fig. XXXIX



so that always, whenever a new compartment starts to be drain, the jet flowing out rises to the same height *O*, or may flow out with the same velocity. Indeed that prevails (with the cross-sections of the openings *D, G, F, H* &c. designated by $n,\ \alpha,\ \beta,\ \gamma$, &c.) by making $BC = \frac{nn}{\alpha\alpha}AB,\ CR = \frac{nn}{\beta\beta}AB,\ RE = \frac{nn}{\gamma\gamma}AB$ &c., thus so that with the openings placed equally amongst themselves, equally the lines *AB*, *BC*,*CR*, *RE* &c. are being made equal to each other. It will be easy too in a cylindrical vessel to reconcile that magnitude with the openings, so that the surface of the fluid in the same time may fall from one diaphragm to the following whatever, and since these diaphragms in turn are equally distant from each other and from the base, the construction of a uniform water clock can be devised.

§. 13. However if all the diaphragms shall be put highest, there will be a pleasant hydraulic game, the jet *DO* is seen to jump up, which in equal increments of the height and time increases by leaps when each is able to happen.

§.14. Now the proposition to be investigated shall be the motion of the fluid bursting forth, when different fluids flow out through the different individual openings. But it is apparent the lighter fluid continually are being placed, so that they have been put higher, lest the motion may be disturbed, which happens when in the same time lower fluid rises, with the upper fluid falling, through a common orifice. In this way it will become known what the motion shall be in water flowing out from a vessel with both sides closed besides some small opening present above, which it is agreed passes to the air. Truly we will retain the hypothesis of the cylindrical vessel of infinite cross-section, and again we will designate by *A* the specific gravity of the fluid leaping through, and of that which flows through *G* we will indicate by the letter *B*, and similarly the specific gravities of the fluids flowing through the openings *F*, *H*, &c. we will indicate by the letters *C, D,* &c., respectively. Finally since also to be considered here shall be the heights of the diverse fluids, of which indeed, on account of the shape of the vessel, only the height of the lowest outflow changes, we will call *x* the height of the lowest fluid above the opening *D*; and of the remaining fluids, there in the order in which they lie above, we will designate the heights [due, according to the *vis viva* principle] by *b, c, d* &c., respectively, and we will retain the denominations of paragraph eleven ; thus with which prepared the computation may be put in place, as has been done in §. 4 ; indeed neither is anything else required to be noted in addition, as the masses of the drops passing through the different openings in the same time intervals are to be judged not only from the amount, but also from the specific gravity: but the *actual descent* for the individual fluids will be required to be take for each: From which steps put in place, such a first equation is found :

$$Av + \frac{nn}{\alpha\alpha}Bv + \frac{nn}{\beta\beta}Cv + \frac{nn}{\gamma\gamma}Dv + \&\text{c.} = Ax + Bb + Cc + Dd + \&\text{c.},$$

which reduced gives *v* :

$$v=\left(Ax+Bb+Cc+Dd+\&c.\right):\left(A+\frac{nn}{\alpha\alpha}Bv+\frac{nn}{\beta\beta}Cv+\frac{nn}{\gamma\gamma}Dv+\&c.\right).$$

§. 15. If there shall be two fluids, there will be two terms requiring to be taken in the numerator as well as in the denominator; and three terms when there shall be three liquids, and thus henceforth: If therefore the liquid flowing shall be for example mercury, and itself may be resting with water and the specific gravities of these may be liquids may be put in place as 14 to 1, there becomes :

$$v=\frac{14x+b}{14+\frac{nn}{\alpha\alpha}};$$

and if the ratio of the openings *D* and *G* were for example as 3 to 1, there becomes

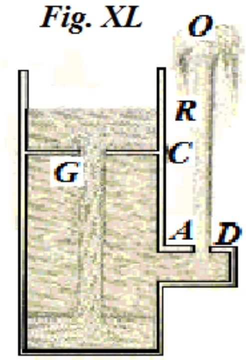
$$v=\frac{14x+b}{23}.$$

§. 16. It is apparent also that account does not exclude these cases, in which the upper fluids are of lesser specific gravities, but only the lower fluids may not ascent through the same holes, through which the upper fluids descend : nor indeed will I assume that in the future (nor yet do I confirm), since in place of the simple opening there shall be a tubule of somewhat exceedingly small height, through which the upper liquid may descent into the lower cavity, just as in Fig. 40, where indeed only two liquids are being considered.

But here the height *CR* is variable, and the height *AC* constant; yet meanwhile for the sake of uniformity of the letters we will call the height $AC=x$, the other $CR=b$; the specific gravity of the fluid erupting through *D* we may make again $=A$, and of the other fluid passing through *G* $=B$, and the height will be *DO* or

$$v=\frac{Ax+Bb}{A+\frac{nn}{\alpha\alpha}B}.$$

Fig. XL



Therefore if the respective fluids water and mercury may flow through the holes *D* and *G* respective, now there will be

$$v=\frac{x+14b}{1+\frac{14nn}{\alpha\alpha}}.$$

§. 17. As again the motion of a single fluid from the vessel on top may be observed, by a little opening admitting air, it being noted that here the height [due] can never be *b;*

because by Section VIII, the air can be considered to press upon each opening to the same height, hence there will be

$$v=\frac{Ax}{\mathrm{A}+\frac{nn}{\alpha\alpha}B},$$

and if there were $\frac{A}{B}=850$, which is accustomed to be the approximate proportion between the specific gravity of water and air, there will be

$$v=\frac{850x}{850+\frac{nn}{\alpha\alpha}}.$$

§. 18. All these principles, which we have used so far, as I have said, can now be extended easily to vessels, which have finite cross-sections for the ratio of the openings ; but I have shown the truth of these also by another certainly different way, as I will show when I arrive at static-hydraulics, because by that other way the pressures of fluids at individual parts of the vessel are made more evident; but the rules of static fluids are completely different from the rules, which are due to stagnant fluids.

These other rules have this very use in correctly understanding hydraulic machines ; nor indeed should craftsmen seem to be enough to attend to this : but the occasion will be given to discuss these more abundantly in the following section, where we put in place the calculation, how much force of the water used may be lost in propelling the water by its passage through several openings, I will show also the solution to be used, so that that decrement of the forces, however great is shall be, may be diminished. However first we will consider certain other composite in this section, so that we can get down to these.

§. 19. Finally it can arise, that vessels placed next to each other can receive water the one from the other and finally flowing out from the last. Truly we will illustrate these motions now with an example.

The proposed vessel was of whichever shape *AGMB* (Fig. 41), so that with a new in-pouring of water it may be constantly filled as far as to *AB*. From the same vessel meanwhile fluid may be understood to pass through the opening *M* into another nearby vessel *BMNC* and from this again into another *CNRD* through the opening *N* and thus again, then finally water may be ejected into the air, and the locations of the surfaces *HL*, *PQ* &c. may be sought after they have been reduced to a *permanent* state. Moreover the question will be resolved thus.

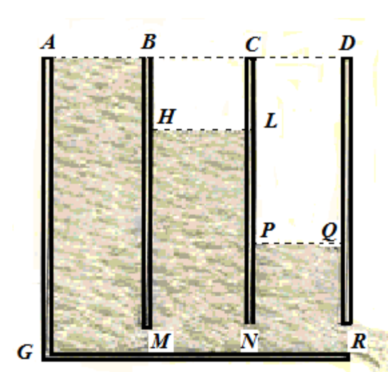


Fig. XLI

Clearly it is seen from that, since the surfaces *AB*, *HL*, *PQ* &c. remain in the same place, the water from these passes through the openings *M*, *N*, *R* with velocities, which are due to the heights *BH*, *LP*, *QR*, but only if the transition of the water through one opening may not accelerate the flow of the same through the nearest opening, which certainly will not happen, unless by a need expressly given, so that it may happen a little. However in addition it is required to consider, the velocities of the water flowing across are to be inversely proportional with the openings, because in the state of *permanency*, in the same time the same amounts of water are transferred through the individual openings. From these it is understood, with the cross-sections of the openings *M, N, R* designated by *m, n, p,* to be

$LP = \frac{mm}{nn} \times BH;\ QR = \frac{mm}{pp} \times BH$: However $BH + LP + QR$ is equal to the height of the surface *AB* above the final opening *R*, or *DR;* therefore there will be

$$BH + \frac{mm}{nn} \times BH + \frac{mm}{pp} \times BH = DR,$$

and hence

$$BH = DR : \left(1 + \frac{mm}{nn} + \frac{mm}{pp}\right);$$

and equally

$$LP = \frac{mm}{nn} \times DR : \left(1 + \frac{mm}{nn} + \frac{mm}{pp}\right)$$

and

$$QR = \frac{mm}{pp} \times DR : \left(1 + \frac{mm}{nn} + \frac{mm}{pp}\right)$$

or

$$BH = DR : \left(1 + \frac{mm}{nn} + \frac{mm}{pp}\right);$$
$$LP = DR : \left(1 + \frac{nn}{mm} + \frac{nn}{pp}\right)$$
$$QR = DR : \left(1 + \frac{pp}{nn} + \frac{pp}{mm}\right),$$

and thus the invariable states of the surfaces *HL*, *PQ*, &c may be determined. But in what time that may happen, if these surfaces shall be in place otherwise and how the amount of water may flow through the individual openings between these, we will examine below together with other questions relevant to that : Now however we may deduce particular consequences from the values advanced of the heights *BH, LP, QR* &c..

[KF has given an extended analysis of this section regarding connected channels, on p. 81 onwards, in modern terms, which is quite straightforward.]

§. 20. I. When the individual openings are of equal cross-section to each other, there will be $BH = LP = QR$ &c. and any number of these heights will be contained in the height *DR*, as often as the vessel may be replicated.

II. However if a few of the openings shall be infinitely small be reason of the others, all the surfaces will be, which shall be placed on this side of the opening, in the same ratio with the first surface *AB:* but the rest will be approximately with the base *GR*.

III. If the channel may be imagined continuous passing through the individual openings *M, N, R* &c., it is understood, water must flow through the openings of the channel with a velocity, which is due to the total height *DR.* In our case that velocity indeed corresponds only to the height *QR*, the reckoning and origin of this matter is, because the *ascent potential* of the individual droplets flowing across the openings may be absorbed, with the single opening efflux excepted. Therefore the *living force* which is lost at individual instants is to the *living force* which may be generated at individual times, as *DQ* to *DR.* However the heights *BH, LP,* &c. represent the *respective* living force, which is removed continually by the droplets flowing across the openings *M, N* separately. Yet I think that if the openings were almost equal, and their centres placed on a straight line, and finally the walls *BM, CN, DR* shall not be excessively distant from each other in turn, it can happen, that a certain amount could burst out with a greater velocity of the water, than the theory thus indicates : In the remaining cases I do not doubt the accuracy of the same, ignoring the impediments often indicated.

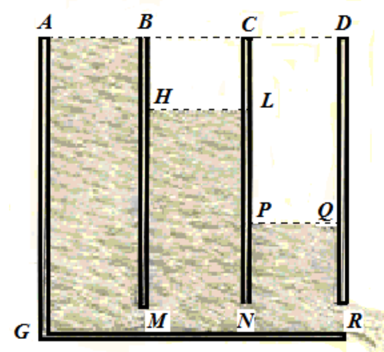


Fig. XLI

IV. Finally it is evident, whenever the surfaces of the water *HL, PQ* &c. change their positions, either one alone or several, soon all the surfaces will be changing positions, then therefore they will have been restored to equilibrium, from what was just said. But these changes generally equally are full of difficulties and involve extensive calculations, unless the vessels may be considered rectangular and as if to be of infinite cross-section in the reckoning of the apertures, so that evidently the increments of the *ascent potentials* of the water *ML, NQ* &c. which change position, shall be ignored on account of the *ascent potentials*, which are generated by the droplets perpetually flowing through *M, N, R*. Nor with the restriction removed should this affect us, since now we will have seen everywhere in vessels either half or completely full, the increments of the internal motions of the masses rejected, are to be without sensible error in the calculation. Therefore I shall omit the general solution, which is by me on account of its excessive extent, and so that in this section at this point I have put in place, vessels certainly infinitely wide and indeed for the greater elegance I have put the shapes in the calculations to be prismatic.

Moreover I shall begin with a vessel divided into two parts.

§. 21. A two-part vessel of this kind is shown (Fig. 42), of which the part *AM* is put filled with water, the other *BN* perhaps as far as to *HL*, now when the flow begins through each of the openings *M* and *N* : and as the water is running out in *AB*, so that the vessel may be kept constantly full, thus moreover it may come about, that the water in *BN* may rise (or also descend according to the circumstances); as since it shall be thus, we will seek the velocity of the surface of the water, when it will arrive at the position *hl*.

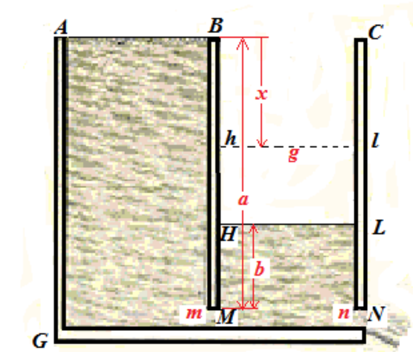


Fig. XLII

This agreed on, we express the cross-section of the opening *M* by *m*, of the opening *N* by *n* and the cross-section *hl* (which indeed is put the same everywhere) by *g*. Then we will put $BM = a,\ HM = b,\ Bh = x$, and thus $hM = a - x$. Thus indeed it appears from the position of the almost infinite cross-sections of the vessels *AM* & *BN*, when the variable surface of the water is at *hl*, the height due to the velocity of the water flowing through $M = Bh = x$, and that same velocity $= \sqrt{x}$, and likewise the height in the account of the opening $N = hM = a - x$, and the velocity of the water flowing through $N = \sqrt{a-x}$; therefore the amount flowing into the vessel *BN* through *M* in a given time to the amount for the same time flowing out from the vessel shall be as $m\sqrt{x}$ to $n\sqrt{a-x}$, and of these the difference of the quantities divided by the cross-section *g* gives the velocity of the surface *hl*, therefore which velocity, which we will call *v*, may be expressed by this equation,

$$v = \frac{m\sqrt{x} - n\sqrt{a-x}}{g}.$$

[Note carefully that until now, *v* has expressed the height corresponding to the velocity squared for the *vis viva*.]

§. 22. Now in order that the time may be known, in which the surface of the fluid comes from *HL* to *hl*, we will call that time *t:* but because there shall be $dt = \frac{-dx}{v}$, there will be, with the value just found for *v*,

$$dt = \frac{-gdx}{m\sqrt{x} - n\sqrt{a-x}}.$$

Indeed this formula can become rational by putting $x = \frac{4aqq}{(1+qq)^2}$, and then to be arranged in the due manner : Truly this method is a little longer than the other, with which the quantity requiring to be reduced is divided into two members themselves to be integrated, evidently the equation presented does not differ from this :

$$dt = \frac{mgdx\sqrt{x}}{nna-(mm+nn)x} + \frac{ngdx\sqrt{a-x}}{nna-(mm+nn)x}:$$

But there becomes :

$$\int \frac{mgdx\sqrt{x}}{nna-(mm+nn)x} = -\frac{2mg}{mm+nn}\sqrt{x} + \frac{mng\sqrt{a}}{(mm+nn)\sqrt{(mm+nn)}} \times \log\frac{n\sqrt{a}+\sqrt{(mm+nn)}\sqrt{x}}{n\sqrt{a}-\sqrt{(mm+nn)}\sqrt{x}};$$

[Following KF: putting

$y = nna-(mm+nn)x;$ then there will be $dy = -(mm+nn)dx$ or $dx = \frac{-dy}{mm+nn}$, and

$x = \frac{nna-y}{mm+nn}$.

With this substitution in place, the above integral becomes :

$$\frac{-mg}{(mm+nn)\times\sqrt{mm+nn}} = \int \frac{nna-y}{y\sqrt{nna-y}}dy.$$

Further putting

$$\sqrt{nna-y} = u, \text{ then } \frac{-dy}{\sqrt{nna-y}} = 2du \text{ and } y = nna-uu.$$

into the above integral, we find on integrating :

$$\frac{-mg}{(mm+nn)\sqrt{(mm+nn)}}\left[-2nna\frac{1}{2\sqrt{nna}}\times\log\frac{n\sqrt{a}+\sqrt{nna-y}}{n\sqrt{a}-\sqrt{nn-y}} + \sqrt{nna-y}\right]$$

or

$$\int \frac{mgdx\sqrt{x}}{nna-(mm+nn)x} = -\frac{2mg}{mm+nn}\sqrt{x} + \frac{mng\sqrt{a}}{(mm+nn)\sqrt{(mm+nn)}} \times \log\frac{n\sqrt{a}+\sqrt{(mm+nn)}\sqrt{x}}{n\sqrt{a}-\sqrt{(mm+nn)}\sqrt{x}}; \quad]$$

and the other member to be integrated, evidently

$$\int \frac{ngdx\sqrt{a-x}}{nna-(mm+nn)x},$$

becomes

$$= -\frac{2ng}{mm+nn}\sqrt{(a-x)} + \frac{mng\sqrt{a}}{(mm+nn)\times\sqrt{(mm+nn)}} \times \log\frac{m\sqrt{a}+\sqrt{(mm+nn)}\sqrt{a-x}}{m\sqrt{a}-\sqrt{(mm+nn)}\sqrt{a-x}};$$

it is apparent from that with the constant added to become :

$$t=\frac{2mg\sqrt{a-b}-2mg\sqrt{x}+2ng\sqrt{b}-2ng\sqrt{a-x}}{mm+nn}+$$

$$\frac{mng\sqrt{a}}{(mm+nn)\times\sqrt{(mm+nn)}}$$

$$\times\log\frac{mna+(mm+nn)\times\sqrt{(ax-xx)}+m\sqrt{(mm+nn)}\sqrt{ax}+n\sqrt{(mm+nn)}\times\sqrt{(aa-ax)}}{mna+(mm+nn)\times\sqrt{(ax-xx)}-m\sqrt{(mm+nn)}\sqrt{ax}-n\sqrt{(mm+nn)}\times\sqrt{(aa-ax)}}$$

$$-\frac{mng\sqrt{a}}{(mm+nn)\times\sqrt{(mm+nn)}}$$

$$\times\log\frac{mna+(mm+nn)\times\sqrt{(ax-bb)}+m\sqrt{(mm+nn)}\sqrt{aa-bb}+n\sqrt{(mm+nn)}\sqrt{ab}}{mna+(mm+nn)\times\sqrt{(ab-bb)}-m\sqrt{(mm+nn)}\sqrt{aa-ab}-n\sqrt{(mm+nn)}\sqrt{ab}}.$$

§. 23. From paragraph 19 [or by setting $v=0$ above] it is clear the surface *hl* remains in its place when there is

$$Bh(=x)=\frac{nna}{mm+nn}.$$

But if however in the integrated equation in the preceding paragraph there may be put $x=\frac{nna}{mm+nn}$, the denominator in the logarithmic quantity becomes $=0$, and hence with that infinite quantity: therefore the time of the whole motion is infinitely great, as from whatever ratio of the cross-sections.

But so that we may determine another case above, we will see in what time the surface of the water may rise from the bottom in place *MN* (clearly by putting $b=0$) by the amount $\frac{1}{2}a$, by putting $m:n=4:\ 3$; moreover there becomes

$$t=\frac{8g\sqrt{a}-14g\sqrt{\tfrac{1}{2}a}}{25}+\frac{12g\sqrt{a}}{125}\log\left(\frac{49+35\sqrt{2}}{49-35\sqrt{2}}\right)-\frac{12g\sqrt{a}}{125}\log 4,$$

or

$$t=\frac{8g\sqrt{a}-7g\sqrt{2a}}{25}+\frac{12g\sqrt{a}}{125}\log\left(\frac{49+35\sqrt{2}}{140\sqrt{2}-196}\right),$$

there is approximately $t=\frac{15g}{100}\times 2\sqrt{a}$, which indicates, to be that time to the time with which a weight falls freely through the height *BM* as approximately 15*g* to 100 [on the assumption that $m=4$]: The time of descent can be found equally, if from the initial surface *hl* there was a further position of equilibrium in place. For example if there were some vessel completely full of water, but the openings *M* and *N* how may have the ratio which is between 3 and 4, and the time shall be required to be determined, in which the surface of the water may fall from *B* through half the distance *BM:* these hypotheses make $m = 3$; $n = 4$; $b = a$, and $x = \frac{1}{2}a$, thus indeed there shall be:

$$t = \frac{8g\sqrt{a} - 7g\sqrt{2a}}{25} + \frac{12g\sqrt{a}}{125}\log\left(\frac{49+35\sqrt{2}}{49-35\sqrt{2}}\right) - \frac{12g\sqrt{a}}{125}\log 4.$$

[on the assumption that $m = 3$.]
From which it appears in each case the time to be the same.

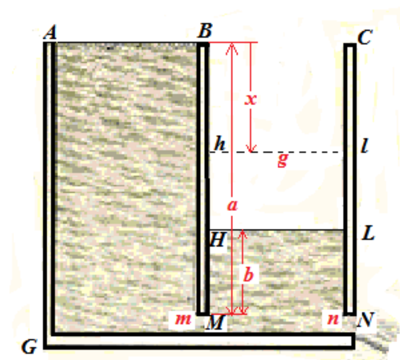


Fig. XLII

§. 24. Before we shall descend to vessels with many chambers it will be appropriate to have investigated, what amount of water may flow through each of the openings *M* and *N*, while the surface of the water arrives at *hl* from the position *HL*. And indeed in the first place, which pertains to the opening *M*, it is clear the amount of water in a given element of time (*dt*) flowing through that to be proportional to the velocity $\left(\sqrt{x}\right)$ multiplied by the magnitude of the opening (*m*) and in that element of time *dt*, thus so that this amount shall be (on account of $dt = \frac{-gdx}{m\sqrt{x} - n\sqrt{a-x}}$ given by §. 22)

$$= \frac{-mgdx\sqrt{x}}{m\sqrt{x} - n\sqrt{a-x}},$$

and therefore the whole amount that will flow in from the start

$$= -\int \frac{mgdx\sqrt{x}}{m\sqrt{x} - n\sqrt{a-x}}.$$

but there shall be

$$-\int \frac{mgdx\sqrt{x}}{m\sqrt{x} - n\sqrt{a-x}} = \frac{mnga}{(m+n)^2}\log\left(\frac{ma-mb-nb}{mx+nx-na}\right) + \frac{mg}{m+n}\times(a-b-x).$$

In the same manner the amount of water meanwhile flowing out through the opening *N* (which clearly is $= -\int \frac{ngdx\sqrt{a-x}}{m\sqrt{x} - n\sqrt{a-x}}$.)

$$= \frac{mnga}{(m+n)^2}\log\left(\frac{ma-mb-nb}{mx+nx-na}\right) - \frac{ng}{m+n}\times(a-b-x).$$

[These two formulae are incorrect, see the note on p.265 of volume 5 of Daniel Bernoulli's *Werke,* as Leonhard Euler pointed out in his letter to Daniel Bernoulli of 16 May 1739, and comments are made by Euler, translated on p.59-61 of the same. These however differ in some terms from these presented in Euler's letter, which is not available to me, apart from that given in the introduction. As only differences of the integrals are used – which differ by one term only, no real harm has been done. The corrected integrations supplied by Euler, according to G.K.Mikhailov, are :

$$-\int \frac{mg\sqrt{x}dx}{m\sqrt{x}-\ n\sqrt{a-x}} = \text{const.} + \frac{mga}{\left(m^2+n^2\right)^2}\left(n\left(m^2-n^2\right)\arctan\sqrt{\frac{a-x}{x}} + 2mn^2\log\frac{a\sqrt{a}}{m\sqrt{x}-\ n\sqrt{a-x}}\right)$$
$$-\frac{m^2gx}{m^2+n^2} - \frac{mng\sqrt{x(a-x)}}{m^2+n^2}.$$

and

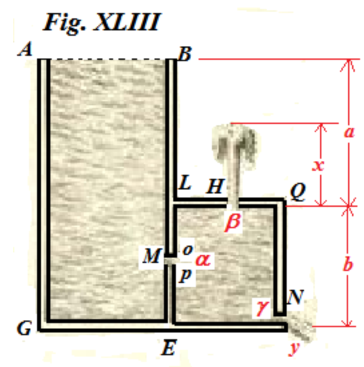
$$-\int \frac{ng\sqrt{a-x}dx}{m\sqrt{x}-\ n\sqrt{a-x}} = \frac{mga}{\left(m^2+n^2\right)^2}\left(n\left(m^2-n^2\right)\arctan\sqrt{\frac{a-x}{x}} + 2mn^2\log\frac{a\sqrt{a}}{m\sqrt{x}-\ n\sqrt{a-x}}\right)$$
$$+\frac{n^2gx}{m^2+n^2} - \frac{mng\sqrt{x(a-x)}}{m^2+n^2} + \text{const.}]$$

And thence also the amount of water is known, which is poured into *AB*, and neither indeed does it differ from that, which flows through *M*: finally the water gathered in the *BN* is expressed by $g(a-b-x)$, and since the difference is taken of the water flowing through *M* and *N*, this same amount $g(a-b-x)$ arises.

§. 25. Just as in §. 21 we have determined the velocity of the continually moving position of the surface for a two-fold vessel, thus now we will define the velocities of the individual surfaces in vessels with many compartments. Cleary the height of the uppermost surface above the nearest surface $=x$, the height of this above the following $=y$, then $=z$, and again the height for the nearest $=s$, and thus so forth. Truly the cross-sections of the openings may be designated by *m, n, p, q* &c.; the cross-sections of the second, third, fourth, etc. vessels shall be *M, N, P* &c. Thus it is apparent the velocity of the second surface shall be $=\frac{m\sqrt{x}-n\sqrt{y}}{M}$, the velocity of the third surface $=\frac{n\sqrt{y}-p\sqrt{z}}{N}$, the velocity of the fourth surface $=\frac{p\sqrt{z}-q\sqrt{s}}{P}$, &c.

Again since the small intervals ran through in the same times by the surfaces shall be as the velocities, it is evident thus with the positions of these surfaces to be determined at individual moments, whatever equations arising shall be almost intractable. From that it is apparent from these, if even a single surface beyond the position of equilibrium defined above in §.19 were put in place, to be so that all the remaining may be disturbed by the reciprocal motions, then after an infinite time likewise to be returned to the first position.

§. 26. Again let a vessel be formed thus, as shown in Fig. 43, known to be divided into the two parts *ABEG* and *LQNE* communicating between each other by the middle opening *M*; and in addition they shall have openings *H* and *N* through which water springs forth, while into *AB* just as much flows in as out. But these cross-sections in each vessel shall be as infinitely great to the ratio of the openings *M*, *H* & *N;* and with these in place it shall be proposed to find the velocities, with which the water may be ejected both through both *H* and *N*, or the heights corresponding to these velocities. But the velocities will be unchanging, because the vessel is kept full of water, and likewise the cross-sections of the vessel with respect to the openings are considered boundless.



The solution of this problem may be deduced easily from the preceding, but only if the opening *M* may be considered to be divided into two parts *o* and *p*, of which the one *o* sends water to the opening *H*, and the other *p* to the opening *N*: moreover the parts *o* and *p* will have on that account (because the water flows through each with the same velocity), so that relative to each other, they have the same amount of water flowing out through *H* and *N* in the same time, that is, the ratio composed from the ratio of the cross-section *H* to the cross-section *N* and of the velocity at *H* to the velocity at *N.* With which advised it is evident, if the cross-sections of the openings *M*, *H* and *N* may be indicated by α, β, γ, the heights corresponding to the velocities at *H* and *N* may be designated by *x* and *y*, and thence from these the velocities designated by $\sqrt{x}$ and $\sqrt{y}$, with the cross-sections to become

$$o=\frac{\beta\sqrt{x}}{\beta\sqrt{x}+\gamma\sqrt{y}}\alpha \text{ and } p=\frac{\gamma\sqrt{y}}{\beta\sqrt{x}+\gamma\sqrt{y}}\alpha.$$

Now the height of the surface *AB* above the opening H is put $=a$, and *x* will be found, as was shown in §.4, if the square of the opening *o* may be divided by the sum of the squares of the openings *o* and *H* and what arises may be multiplied by *a;* therefore thus it comes about :

$$x=\frac{\alpha\alpha ax}{\alpha\alpha x+\left(\beta\sqrt{x}+\gamma\sqrt{y}\right)^2},$$

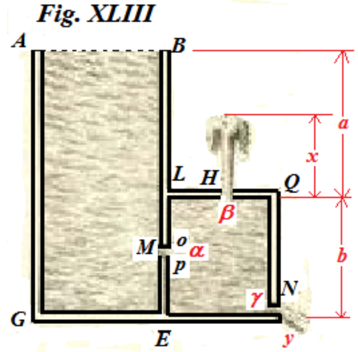
$$\left[i.e.\frac{o^2}{o^2+\beta^2}=\frac{\beta^2 x}{\left(\beta\sqrt{x}+\gamma\sqrt{y}\right)^2}\alpha\alpha\div\left(\frac{\beta^2 x}{\left(\beta\sqrt{x}+\gamma\sqrt{y}\right)^2}\alpha\alpha+\beta^2\right)\right.$$

$=\dfrac{x\alpha\alpha}{x\alpha\alpha+\left(\beta\sqrt{x}+\gamma\sqrt{y}\right)^2}$. This, if you like, is the fraction of the total *ascent potential*.

The ascent potential x then is found from $\dfrac{o^2}{o^2+\beta^2}\times a=x.\Big]$

from which this equation arises :

Fig. XLIII



(A) $$\alpha\alpha x+\left(\beta\sqrt{x}+\gamma\sqrt{y}\right)^2=\alpha\alpha a.$$

[Thus, writing this as $\left(\dfrac{\beta\sqrt{x}+\gamma\sqrt{y}}{\alpha}\right)^2=a-x$,

the decrease in the *ascent potential*, or in this case the head of water $a-x$, is equal to the *vis viva* or proportional to the kinetic energy of the water lost passing through *M*.]

In the same manner from the ratio of the openings *p* and *N*, with the height *AB* above $N=a+b$, this second equation is found :

(B) $$\alpha\alpha y+\left(\beta\sqrt{x}+\gamma\sqrt{y}\right)^2=\alpha\alpha\times(a+b).$$

With equation (B) taken from equation (A) the equation is produced $y=x+b$, from which it follows, if both jets may be directed upwards, each will leap up to the same position. Then if in equation (A) its value for *y* may be substituted $x+b$, there will be

(C) $$\alpha\alpha x+\left(\beta\sqrt{x}+\gamma\sqrt{x+b}\right)^2=\alpha\alpha a,$$

from which the value of *x* itself is deduced from the quadratic equation.

§. 27. From the equations of the preceding paragraph the following conditions follow.

I . Because the velocity of the water flowing through *M* is equal to $\dfrac{\beta\sqrt{x}+\gamma\sqrt{y}}{\alpha}$, the

height generating this velocity will be $\left(\dfrac{\beta\sqrt{x}+\gamma\sqrt{y}}{\alpha}\right)^2$; but if the equations (A) and (B)

may be added there shall become:

$$\left(\frac{\beta\sqrt{x}+\gamma\sqrt{y}}{\alpha}\right)^2=\frac{2a+b-x-y}{2}=a-x,\text{(on account of } y=x+b).$$

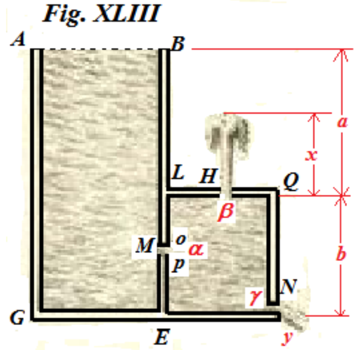
II. If the opening *H* shall be very small in the ratio of the openings *M* and *N*, that is, if β shall be considered to be zero in the reckoning α and γ, the equation (C) will go into

$$\alpha\alpha x+\gamma\gamma x+\gamma\gamma b=\alpha\alpha a,$$

or

$$x=\frac{\alpha\alpha a-\gamma\gamma b}{\alpha\alpha+\gamma\gamma};$$

Fig. XLIII



that truly agrees exceptionally with paragraph nineteen, since it shall be evident if the water leaps through a very small opening to the same height, as the water has, if this plane *LQ* may press downwards so much, as much as it may be pressed upwards by the water inside ; indeed this before-mentioned height in paragraph 19 is $\frac{\alpha\alpha a-\gamma\gamma b}{\alpha\alpha+\gamma\gamma}$.
Again according to this hypothesis the height of the velocity of the water at *N* or

$$x+b=\frac{\alpha\alpha a+\alpha\alpha b}{\alpha\alpha+\gamma\gamma}$$

and finally the height corresponding to the velocity of the water at *M*, or

$$a-x=\frac{\gamma\gamma a+\gamma\gamma b}{\alpha\alpha+\gamma\gamma};$$

which latter equations in this particular case equally shall be able to be deduced at once or predicted from §.19.

III. However if now the other opening *N* may be put exceedingly small relative to the rest, there will be on putting $\gamma=0$

$$x=\frac{\alpha\alpha a+\gamma\gamma b}{\alpha\alpha+\gamma\gamma};$$

then

$$x+b=\frac{\alpha\alpha a+\alpha\alpha b+\beta\beta b}{\alpha\alpha+\beta\beta},$$

and
$$a-x=\frac{\beta\beta a}{\alpha\alpha+\beta\beta}.$$

IV. If $\gamma\gamma b=\alpha\alpha a$, [in II] there becomes $x=0$. Therefore in this case the parts of the plate *LQ* sustain no pressure: indeed it is pressed inwards, if $\gamma\gamma$ shall be greater than $\frac{\alpha\alpha a}{b}$, and the plate shall be without perforation.

All these similarly are deduced easily from §.19.

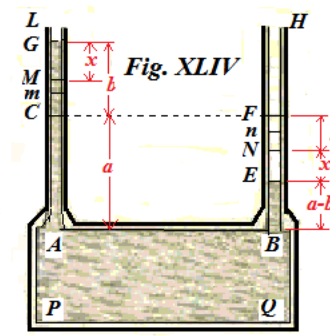
V. Thus also with the aid of this paragraph it may be seen without a new calculation, what must happen, when with the openings *H* and *N* placed at the same height the sum of the openings of these, can be considered as if with a single cross-section $\beta+\gamma$: Clearly both §.19 as well as §. 26 indicate to be
$$x=\frac{\alpha\alpha a}{\alpha\alpha+(\beta+\gamma)^2}.$$

VI. Also it can be noted, when the value of *x* is made imaginary, from that it comes about, because the water not only may flow out in several cases through *H*, but because also the surface *LQ* may descend; from which it can happen, that it may fall below the opening *M*, from which contrary to the hypothesis of the proposition it is freed from the adjoining water. But if the value of *x* is real, then it may be expressed in a two-fold way, but the other value is to be considered without use; thus being warned lest the root proposed shall be assumed to be useful.

VII. Finally so that we may touch on the most special case, we may put all the openings equal to each other, and there will be produced
$$5xx+(2b-6a)x=-aa+2ab-bb,$$
or
$$x=\frac{3a-b-2\sqrt{(aa+ab-bb)}}{5};$$
[*i.e.* $\alpha\alpha x+\left(\beta\sqrt{x}+\gamma\sqrt{x+b}\right)^2=\alpha\alpha a$ gives $\sqrt{x}+\sqrt{x+b}=\pm\sqrt{a-x}$ and eventually $5x^2+(2b-6a)x=-a^2+2ab-b^2$.]
and if in addition there were $a=3b$, there will be approximately $x=\frac{4}{15}b$, for which the height corresponds to the velocity at the opening *N* or $x+b=\frac{19}{15}b$ and the height corresponding to the velocity at *M* must be $a-x$ or $=\frac{41}{15}b$. And thus these velocities or also, because the openings are equal, the flowing out of quantities of water in the same time through the openings *M*, *H* and *N* shall be as $\sqrt{41}$, 2 and $\sqrt{19}$.

§. 28. From all these a method is apparent of determining the motion of fluids, moreover even when the amount of the living forces may not be conserved ; and the computation may be resolved in a similar way, just as often as by the nature of the subject of the question to be presumed, so much of the living forces required for determining the motion (as could be done accurately in the questions of this section) may vanish by individual useless movements. Nor indeed are the cases unique we have examined up to this point : and thus it is pleasing to add another, which considers the oscillations of the fluid, so that thence how great the decrease may become known taken by the displacements of fluids.

Two cylindrical tubes of equal cross-section AL and BH (Fig. 44, [shown at the instant of release of the head of water]) shall be inserted vertically into a vessel of the widest horizontal cross-section *ABOP*. This vessel shall be filled completely with water : moreover the tubes may hold water as far as to *C* and *F;* then with the equilibrium destroyed one surface may stay at *G* and the other at *E;* and soon the water itself left may begin to move. With these in place the surface *G* must fall below the position *C*, and the other surface *E* to rise above *F*, as much as the height *GC* or *EF*, if all the *vis viva* may be conserved (we have now considered to be free of all the impediments from friction and from all similar effects): However it is apparent, the *living force* of all the water flowing through *A* into the horizontal vessel to be spent without any effect from the water standing in that position, hence it follows the descent of the surface *G* and the ascent of the other to be less, than has just been said: we will now examine that decrease.



Towards this end the surface may be considered to have arrived at *M* from *G*, and there may be put $GM = x,\ GC = b,\ CA = a$: there will be $BE = a - b,\ EN = x,\ MC = FN = b - x$; then the height corresponding to the velocity of the surface at $M = v$, and at a nearby place $m = v + dv$; the increment of the *vis viva* of the water (while the surfaces run through the elements *Mm, Nn,* or $dx) = 2adv$, to which is required to be added the *vis viva* of the drops, which are absorbed from the water of the horizontal vessel, evidently *vdx*, and the sum $2adv + vdx$ will be equal to the actual descent of the water multiplied by the mass of the water, which product is equal to the *actual descent* of the drop *dx*, multiplied by $2b - 2x$. Therefore there is

$$2adv + vdx = 2bdx - 2xdx.$$

[Recall that Bernoulli's method can be considered from the equality of the moments of two opposing masses, one due to the potential ascent of all the masses, and the other due to the actual descent of the total mass, and their increments, in setting up his differential equation :

In the first place, the misnoma of the living force or *vis vivans*, is in fact the kinetic energy here of a droplet or a small incremental mass or volume of the fluid, taken at any relevant point; constant quantities of course omitted in the resulting d.e., on assuming a constant density of fluid, as well as a uniform tube, the height *v* corresponding to this speed *V* squared, where natural units are used in which on taking gravity $g = \frac{1}{2}$, giving $v = V^2$ for unit mass, and the sum of the moments of the incremental masses at all relevant potential heights is taken, to give the total moment of the *vis vivans*; if this quantity is divided by the mass involved, then the *ascent potential* results, equated to the movement of the centre of mass, called the *actual descent*.These ideas have been discussed in the notes to section III, and also can be found in the introduction to Daniel Bernoulli's *Hydrodynamicae*, *Werke*, Vol.5. Note that physically, this is not an actual rise, necessarily, but the ability or potential to rise to this height due to the velocity the fluid possesses. In this case, the contributions to the moments of the *vis vivans* when the surfaces arrive at *M* and *N* are $\left(a+b-x\right)v+\left(a-b+x\right)v=2av$; (the velocity of the fluid is constant throughout the tubes of length 2*a* at this point, and that of the large reservoir below is taken as zero always, so that $2av = 2aV^2$, proportional to the kinetic energy of the water in the tubes in modern units, where 2*a*, etc, are 'mass' terms, and *v* is a distance term).

The differential change in this quantity for all the moving water is hence $2adv$; to this is added the moment of the left hand increment itself for the added droplet, vdx (*i.e.* V^2dx the increase in the kinetic energy), and the right hand increment is zero as the corresponding droplet has gone into the reservoir and its kinetic energy or *vis viva* is lost]; hence, the total incremental change in the moment is $vdx + 2adv$.

Balancing this incremental moment is the opposing incremental moment of the actual descent of the whole moving fluid; this is composed of the moments of the two increments or droplets, one up and decreasing and the other down and increasing towards the equilibrium position *CF*, in the change in the centre of mass: $\left(a+b-x\right)dx-\left(a-b+x\right)dx=2\left(b-x\right)dx$. These moments are considered equal and opposite, leading to : $vdx+2adv=2\left(b-x\right)dx$]

Indeed this equation integrated correctly will go into this form :

$$v = 4a + 2b - 2x - c^{\frac{-x}{2a}} \times (2b + 4a);$$

from which if there is put $4a + 2b - 2x - c^{\frac{-x}{2a}} \times (2b + 4a) = 0$, the value *x* of the whole displacement will be given, if the water *b* may be taken away, the rest will indicate a descent below the point of equilibrium *C*.

§. 29. However so that it may become apparent by a certain example, how much the oscillations may be diminished by this reasoning, we may put $a = b$, evidently by making $CA = GC$ and $BE = 0$. Thus there becomes

$$3a - x = c^{\frac{-x}{2a}} \times (3a)$$

or

$$c^{\frac{x}{2a}} = \frac{3a}{3a-x}$$

or

$$x = 2a\log\frac{3a}{3a-x},$$

to which equation the value $x = \frac{7}{4}a$ certainly satisfies closely. Therefore the decrease of the displacement or $2b- \ x =$ fourth part of the height of the fluid above the mid-point : if a greater decrease may be observed by experiment, the rest will be required to be attributed to the adhesion of the water to the tube.

§. 30. Nor should this account of the decrease of the displacements be removed, I suspect, even if the horizontal tube is made with the same cross-section as the vertical, on account of the changed direction of the fluid at the points *A* and *B.* Infinitely many other cases can be imagined to be resolved by the same principles, just as if the nature of the oscillations shall be required to be investigated in vessel Fig. 44, when that is separated into two parts by a horizontal diaphragm with a single opening, so that the diaphragm may have communications between themselves and others of this kind. But I think this will now suffice, so that anyone himself shall be able to form the general rules easily required for solving questions of this kind.

Experiments relating to section eight.

Experiment 1.

The fourth paragraph, where the height of the velocity of the water flowing out through the opening *D* (Fig. 37) is said to be $\frac{mmx}{nn+mm}$, I have confirmed in this manner, so that each opening *G* and *D* will have a rim if the form of a little raised belt, there shall be a position of the contraction of the jet, and it shall be able to judge safely from the amount of water in a given time for the velocities. Then with the measures taken accurately, and with the time observed in which the surface may fall through a given interval *AP*, I have seen that time to correspond correctly with the velocities defined in the said paragraph: also I have observed with nothing to be changed from the raising or lowering of the diaphragm. The rest pertaining to the experiment have disappeared from memory, nor have these been noted down on paper: moreover I have thought the experiment superfluous to repeat, because each is easy to perform: but it is the foundation for the rest, which therefore scarcely have a need for experimental inquiry : yet I wished still to test the following in addition.

Experiment 2.

I used a vessel, almost of such a kind Mariotte was using *(see* Fig. 38) and again I have confirmed our equation in this manner : I arranged so that the water flowed out horizontally through the opening *D*, and then I took measurements of the height of the opening *D* above the floor and the distance of the place, where the jet landed on the floor from the point on the same floor, to which the opening *D* was overhanging vertically; from that I knew the height corresponding to the velocity of the water flowing out from *D*: but I had found this same height approximately by experiment, which the theory of this section shows in section §. 4. I may put in place similar experiments at the end of the experiments pertaining to section twelve, which likewise will confirm our static-hydraulic theory.

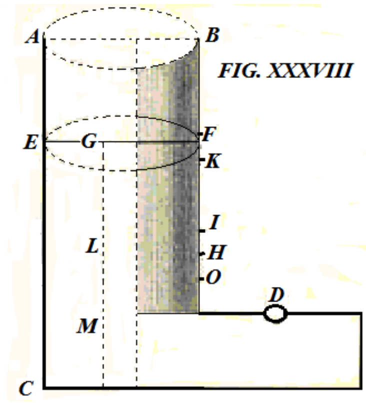


FIG. XXXVIII

Finally since there shall be many things in §§. 26 and 27 which were elicited by individual calculations, there will be a need for these experiments also to be performed, especially since other experiments likewise have the same need to be performed, which will be reviewed in Sect. XII, if a vessel is set up, such as in Fig. 43, that happens to attend to this end.

Furthermore, this theory also may be confirmed by the experiments reviewed in Section seven, which I have taken concerned with the oscillations of fluids in tubes from the inflow through openings.

HYDRODYNAMICAE SECTIO OCTAVA.

De motu fluidorum cum homogeneorum tum heterogeneorum per vasa irregularis & *praeruptae structurae, ubi ex theoria virium vivarum, quarum pars continue absorbeatur, explicantur praecipue phaenomena singularia fluidorum per plurima foramina trajectorum, praemissis regulis generalibus pro motibus fluidorum ubique definiendis.*

§. 1. Aliis adhuc principiis praeter quam in sectione proxime praecedente usi non sumus, quam hisce duobus *quod velocitates fluidorum sint ubique reciproce proportionales amplitudinibus vasorum,* cujus ope invenitur *ascensus potentialis* totius aquae ex dato *ascensu potentiali* cujusvis particulae; tum quod *ascensus pot.* totius aquae perpetuo aequalis maneat *descensui actuali.* Quoties ambo haec principia locum habent, minime dubitandum est. quin methodo a nobis adhibita motus fluidorum recte definiatur. Non diffitebor tamen, hujusmodi fieri posse structurae vasa, in quibus fluida moventur, ut neutrum istorum principiorum recte procedat. Prius equidem raro aut nunquam notabiliter a vera abducit, quia ubicunque locum non habet, ibi nullum fere aquae habere solent motum, possuntque sine sensibili errore ceu stagnantes considerari: Longe vero aliter comparatum est alterum principium, quod apparebit ex inferioribus exemplis, & cujus rei luculentum esse possunt testimonium ea, quae in superiori sectione protulimus circa refluxum aquarum; tantum enim abest, ut aquae in vase submerso ex data altitudine delapsae, ad hanc altitudinem regredi possint, prouti vi istius principii deberent, sublatis impedimentis extrinsecis, quin potius plerunque vix sensibilis sit earum ascensus prae descensu, quem antea fecerunt: imo nequidem ascendere superficies aquae potest tantum supra aquam, cui tubus immergitur, quantum infra eandem depressa fuerat, nisi cum tubus totus est apertus: ista vero superficies multo minus deprimitur quam antea fuerat elevata. Horum rationem dedimus in superiori sectione: Haec quia ita sunt, regulas nunc dabo duas pro motu aquarum ubique definiendo, easque porro exemplis illustrabo talibus, quae nulla adhuc theoria explicari potuerunt, cum nostra autem egregie admodum conveniunt.

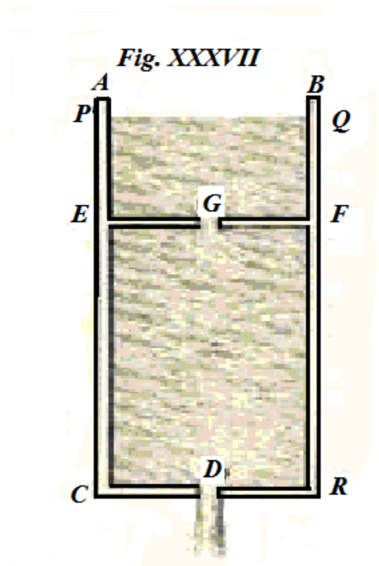
Regula 1.

§. 2. Dispiciendum est, assumta alicubi in vase proposito velocitate fluidi ceu cognita, quaenam reliquis fluidi partibus futura sit velocitas. Ita enim cognoscetur *ascensus potentialis* totius fluidi ejusque incrementum. Hactenus consideravimus fluida in infinita strata parallela vel potius ad latera vasis ubique perpendicularia divisa, statuimusque velocitates hisce stratis reciproce proportionales: Facile quidem est vasa effingere, ubi aliter moventur fluida; crediderim autem his in locis motum notabilem nunquam habere fluida ita, ut error ex ista hypothesi sensibilis nasci fere non possit: poterit tamen majoris accurationis ergo praefata regula adhiberi. Praesertim vero huc pertinet contractio venarum, cum fluida per foramina in tenuibus admodum laminis facta transire coguntur, qua in re magna est adhibenda circumspectio: Effectus hujusmodi contractionum haud male, puto, praevidebuntur, cum recte perpensa fuerint, quae in sectione quarta de illis monui.

Regula 2.

§. 3. Singulis momentis dispiciendum est, quantum *vis vivae*, seu quodnam productum ex *ascensu potentiali* in massam oriatur ad fluxum praecipuum, cujus natura quaeritur, nihil conferens. Id vero rursus uniuscujusque circumspectae aestimationi relinquendum est. Quod sic oritur, addendum est facto ex *ascensu potentiali,* quem motus praecipuus involvit, in massam, aggregatumque productorum demum aequale censendum est facto ex massa omnis aquae in ejusdem *descensum actualem.*

Magni profecto est momenti haec regula, & ut puto, fere unica ad motuum mensuras obtinendas, qui in vasis irregularibus, pluribusque cavitatibus inter se communicantibus divisis fiunt, quod nunc pluribus illustrabo exemplis.



Problema.

§. 4. Propositum fuerit vas *ACRB* (Fig. 37) infinitae quasi ratione foraminum mox dicendorum ubique amplitudinis & diapbragmate aliquo *EF* in duas distinctum cavitates inter se communicantes, mediante foramine G: habeat praeterea vas istud in infima sui parte aliud foramen D: deinde ponatur vas aqua plenum usque in *PQ*, sic ut cavitas inferior *CEFR* tota sit humido repleta, atque insuper diaphragmati superjaceat pars altera *PQFE.* His positis, fluidoque jam moveri incipiente, quaeritur velocitas aquae per foramen *D* in aerem effluentis vel altitudo genitrix hujus velocitatis.

Solutio.

Fuerit altitudo superficiei *PQ* supra foramen $D = x$, amplitudo foraminis $D = n$, alteriusque $G = m$. Perspicuum autem est *ascensum potentialem* cujusvis guttae per *G* transfluentis nihil promovere effluxum per *D,* totumque impendi in motum aliquem excitandum intestinum, qui mox absorbetur sine alio effectu: necesse igitur est ut singulis momentis motus generetur novus in particulis foramen *G* transeuntibus, non minus atque in particulis per *D* effluentibus. Sed si *ascensus potentialis* guttulae per *D* effluentis dicatur *v,* id est, si aqua exilire ponatur per *D* velocitate, cujus altitudo genitrix sit *v*, erit similis altitudo ratione guttulae mole sua priori aequalis, per *G* eodem tempore transfluentis, $\frac{nnv}{mm}$. Multiplicatis istis *ascensibus potentialibus* per massam, quam aequalem habent, quamque vocabo *M,* erit aggregatum productorum $= Mv + \frac{Mnnv}{mm}$. Et cum ob infinitam amplitudinem vasis alius motus non generetur, erit praefatum aggregatum (per *reg.* 2) censendum aequale facto ex massa onmis aquae in ejusdem *descensum actualem.* At vero si massa omnis aquae dicatur μ, erit (per§. 7 Sect. III) *descensus actualis,* qui fit dum guttula $= \frac{Mx}{\mu}$, effluit, ita ut productum commune sit $= Mx$. Igitur habetur

$$Mv + \frac{Mnnv}{mm} = Mx,$$

sive

$$v = \frac{mmx}{nn + mm}.$$

Q.E.F.

Scholium 1.

§. 5. Apparet ex isto exemplo, motum sine calculo differentiali determinari posse, cum figura vasis ubique amplissimi motum hunc mutare non potest. Interim difficile futurum non fuisset, consideratione quoque habita ad amplitudines vasis, fluxum definire, & solo brevitatis studio id vitavimus pariterque omittemus in sequentibus, nisi fortasse motus notabiliter a figura vasis varia mutetur, quod fieri potest in tubis satis amplis, sed iis longissimis, in quibus fluidum movetur, praesertim si motus determinandi sint oscillatorii. Imo vidimus in praecedente Sectione, si oscillationes sint valde parvae in tubis profundissime submersis, tunc tantum abesse, ut ad solum foramen fundi sit attendendum, neglectis amplitudinibus etiamsi satis magnis, quin potius ad has solas fere sit respiciendum.

Scholium 2.

§. 6. Quia in calculo, quem posuimus, *vis viva* cujusvis guttulae per *G* transfluentis ab aqua cavitatis inferioris absorberi debet, perspicuum est, propositionem non esse extendendam ad illos casus, qui hypothesi repugnent, veluti cum diaphragma *EF* fundo *CR* proximum est simulque foramina sibi directe respondent: ita enim non arduum est providere, motum longe diversum fore ab eo, quem praesens theoria indicat. At vero, si distantia *DG* magna sit, sique simul foraminum situs sit obliquus & latera foraminum venis aqueis negent contractionem; dubium nullum est, quin theoria accurate omnibus phaenomenis respondeat.

Corollarium.

§. 7. Si foramen *G* est admodum amplum prae altero, fit fere $v = x$, sed haec altitudo *v,* cui nimirum respondet velocitas aquae per *D* effluentis, non parum decrescit, crescente foramine *D,* ita ut si fuerit v. gr. duplum foraminis *G,* sit $v = \frac{1}{5}x$ & tantum non tota evanescat, cum foramen *G* est valde exiguum respectu foraminis *D.*

His ita inventis, jam quivis veram perspiciet rationem motuum illorum, quos Mariottus primus observavit, & quibus ceu valde *admirabilibus* testatur se supra modum fuisse delectatum, simulque intelliget, quam longe Auctor iste in reliquis perspicacissimus a via aberraverit in hisce disquisitionibus. Non abs re fore puto observata Mariotti hic apponere.

§. 8. Vas adhibuit, quale repraesentat Figura trigesima octava, quae non differt a priori nisi in eo, quod in ima parte cylindro *ABC* tubus horizontalis *MD* insertus sit perforatus lumine *D*, per quod aquae verticaliter exiliunt: Diaphragma vero *EF* in medio perforatum est lumine *G* ut antea: infra illud parvulum erat foramen *K*, ut facilius cavitas inferior aquis impleri posset, quo facto idem obturabatur, reliquumque vasis replebatur.

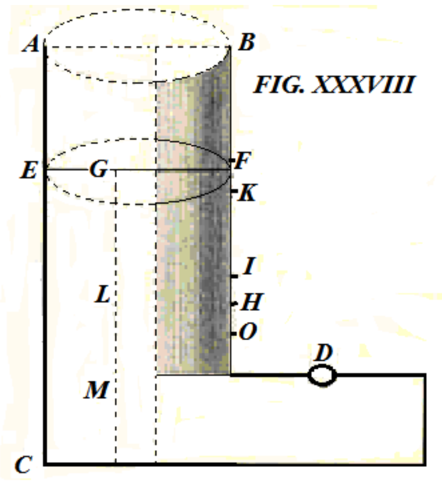


FIG. XXXVIII

His ita praeparatis, effluentibusque aquis per *D*, observavit Mariottus, mox illas ascendisse usque in *I*, deinde sensim imminuta velocitate usque in *H* & tandem, imminente depletione tota cavitatis superioris *ABFE*, usque in *O*, tuncque assumtis confestim novis viribus assilivisse fere usque in *F*. Animadvertit etiam, si bene memini, altitudinem jactus initialis eo minorem esse, quo minus sit foramen *G* ratione alterius *D*. Videatur ejus *Tract. de motu aquarum, part.* IV, *disc.* 1. Putat autem horum motuum mutationes explicari posse fingendo vasi *ABFE* amplissimo tubum strictiorem adhaerere *GLMD*, per quem aquae fluant. At vero demonstravimus & experientia quotidie docet, motum aquarum ex vase ABGLMD admodum diversum esse ab eo, qui modo indicatus fuit. Non minus falleretur si quis putaret aquam eadem velocitate exilire per foramen *D*, quasi illud in diaphragmate *EF* positum esset; nam fieri potest, ut altitudo jactus initialis sit major & minor altitudine *FB*. Nec denique ea effluent aquae quantitate, uti facile quis suspicari posset, qua eodem tempore effluerent ex vase superiori simplici rescissa parte *EFDC* quamvis ita proxime se res habeat, cum foramen *G* admodum minus est foramine *D*.

§. 9. Nostra vero aequatio, nempe $v = \dfrac{mmx}{nn+mm}$, recte omnino respondet phaenomenis: indicat enim aquam mox ab initio fluxus ad certam ascendere altitudinem, eamque tanto minorem, quanto minus est foramen diaphragmatis prae foramine altero; dein istum ascensum sensim diminui, donec aqua omnis ex cavitate superiori effluxerit, quo ipso momento protinus augmentum capit, totamque aquae superincumbentis altitudinem tantum non attingit, quia tunc ex vase simplici eoque infinite amplo effluere censendae sunt aquae: paulisper tamen etiamnum retardantur aquae a transitu aeris per foramen *G*, & sane notabiliter retardantur, cum foramen superius valde parvum est, de quo argumento mox quaedam dicemus, cum de fluidis heterogeneis sermo erit. Si figura Mariotti debita proportione respondeat argumento instituto, oportet, ut foramen *G* alterius fecerit paullo plusquam dimidium.

§. 10. Indicat porro formula nostra, quod multis fortasse nondum perspecta hac theoria satis paradoxum videri potuisset, situm diaphragmatis *EF* sive altiorem sive humiliorem nullo modo mutare impetum sive velocitatem aquae effluentis; ratio autem istius phaenomeni omnibus nunc, puto, manifesta est.

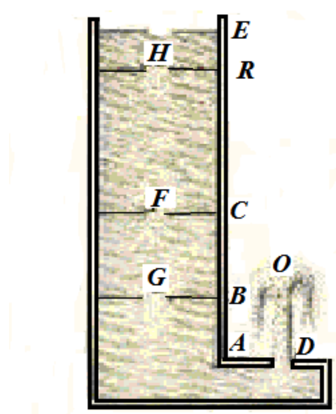
§. 11. Jam vero examinabimus insuper motum aquarum, cum plura sunt diaphragmata foraminibus pertusa, per quae aquae transire cogantur, ut effluxus per foramen *D* fieri possit. Poterit id eodem absolvi modo, quo usi sumus in problemate §. 4. Ita autem instituto recte calculo retentisque denominationibus ibidem adhibitis apparebit esse

$$v = x : \left(1 + \frac{nn}{\alpha\alpha} + \frac{nn}{\beta\beta} + \frac{nn}{\gamma\gamma} + \&\text{c.}\right),$$

ubi per $\alpha,\ \beta,\ \gamma\ \&\text{c.}$ intelliguntur amplitudines foraminum, quae sunt in diaphragmatibus, dum *n* exprimit ut antea amplitudinem foraminis *D*, per quod aquae effluunt.

§.12. Si proinde loco unius diaphragmatis sint in simili vase, quale Fig. 39 repraesentat, plura diaphragmata veluti in *B, C, R* &c. per quae aqua transfluat, dum per infimum foramen *D* effluit, mutabitur & augebitur confestim velocitas aquae effluentis, quoties aliqua cavitas depletur: talis autem esse potest proportio inter altitudines *AB, BC, CR, RE* &c. atque amplitudines foraminum *D, G, F, H* &c. ut semper, quoties nova depleri incipit concameratio, vena effluens ad eandem altitudinem *0* assurgat, seu eadem velocitate effluat. Id vero obtinetur (designatis amplitudinibus foraminum *D, G, F, H* &c. per $n,\ \alpha,\ \beta,\ \gamma$, &c.) faciendo $BC = \frac{nn}{\alpha\alpha}AB,\ CR = \frac{nn}{\beta\beta}AB,\ RE = \frac{nn}{\gamma\gamma}AB$ &c., ita ut positus foraminibus inter se aequalibus sint pariter lineae *AB, BC,CR, RE* &c. inter se aequales faciendae. Facile quoque erit in vase cylindrico eam concillare foraminibus magnitudinem, ut superficies fluidi eodem tempore ab uno diaphragmate ad subsequens quodcunque descendat, & cum haec diaphragmata aequaliter a se invicem & a fundo distant, uniformis clepsydrarum structura excogitari potest.

Fig. XXXIX



§. 13. Si vero omnia diaphragmata altissime posita sint, jucundus erit lusus hydraulicus, venam prosilientem *DO* videre, quae aequalibus incrementis aequalibusque temporum intervallis, quod utrumque fieri potest, subsultim crescat.

§.14. Propositum nunc sit motum fluidi exilientis indagare, cum per singula foramina alia atque alia fluida transfluunt. Fluida autem leviora continue ponenda esse apparet, quo sunt altius posita, ne motus turbetur, quod fit cum eodem tempore fluidum inferius ascendit, superiore descendente, per commune foramen. Innotescet hoc modo quisnam sit motus in aquis ex vase effluentibus undique clauso praeter foraminulum aliquod superne existens, quod aeri transitum concedit. Hypothesin vero infinitae vasis cylindrici amplitudinis ratione foraminum retinebimus, atque porro gravitatem specificam fluidi per

D exilientis designabimus per *A*, illiusque quod per *G* transfluit notabimus littera *B*, similiterque gravitates specificas fluidorum per foramina, *F*, *H*, &c. fluentium indicabimus respective litteris *C*, *D*, &c. Denique cum etiam considerandae hic sint altitudines diversorum fluidorum, quorum quidem, ob figuram vasis cylindricam, solum infimum effluens altitudinem mutat, vocabimus *x* altitudinem fluidi infimi supra foramen *D*, fluidorum reliquorum, eo quo sibi superincumbunt ordine, altitudines designabimus respective per *b*, *c*, *d* &c., reliquas denominationes paragraphi undecimi retinebimus; quibus ita praeparatis computus instituetur ut §. 4 factum est; neque enim quicquam aliud insuper observandum est, quam ut massae guttularum iisdem tempusculis per diversa foramina transeuntium non simpliciter ex mole, sed etiam ex gravitate specifica aestimentur: *descensus* autem *actualis* pro singulis fluidis erit seorsim sumendus: Hisce vestigiis insistendo reperitur talis primo aequatio

$$Av+\frac{nn}{\alpha\alpha}Bv+\frac{nn}{\beta\beta}Cv+\frac{nn}{\gamma\gamma}Dv+\&c.=Ax+Bb+Cc+Dd+\&c.,$$

quae reducta dat *v*

$$v=\left(Ax+Bb+Cc+Dd+\&c.\right):\left(A+\frac{nn}{\alpha\alpha}Bv+\frac{nn}{\beta\beta}Cv+\frac{nn}{\gamma\gamma}Dv+\&c.\right).$$

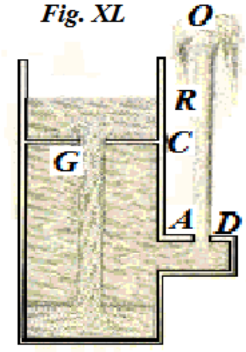
§. 15. Si duo sint liquores, erunt duo termini tam in numeratore quam in denominatore sumendi & tres termini cum tres fuerint liquores, atque sic porro: Si proinde liquor effluens sit v. gr. mercurius, ipsique superincumbat aqua statuanturque gravitates specificae horum liquorum ut 14 ad 1, fiet

$$v=\frac{14x+b}{14+\frac{nn}{\alpha\alpha}};$$

atque si ratio foraminum *D* & *G* fuerit ex. gr. ut 3 ad 1, fiet

$$v=\frac{14x+b}{23}.$$

Fig. XL



§. 16. Patet quoque ratiocinium istud non excludere eos casus, quibus fluida superiora sunt inferioribus specifice graviora, modo fluida inferiora non ascendant per eadem foramina, per quae superiora descendunt: neque vero id futurum esse praesumo (nec tamen affirmo), cum loco simplicis foraminis tubulus sit quamvis exiguae altitudinis, per quem liquor superior descendat in inferiorem cavitatem, veluti in Fig. 40, ubi quidem duo tantum liquores considerantur.

Hic autem altitudo *CR* variabilis est, & altitudo *AC* constans; interim tamen uniformitatis litterarum gratia vocabimus altitudinem $AC = x$, alteram $CR = b$; gravitatem specificam fluidi per *D* erumpentis faciemus rursus $= A$, alteriusque fluidi per *G* transeuntis $= B$, & erit altitudo *DO* seu

$$v = \frac{Ax + Bb}{\mathrm{A} + \frac{nn}{\alpha\alpha}B}.$$

Igitur si per foramina *D* & *G* respective fluant aqua & mercurius, erit nunc

$$v = \frac{x + 14b}{1 + \frac{14nn}{\alpha\alpha}}.$$

§. 17. Ut porro innotescat motus fluidi simplicis ex vase superne parvulo foramine aerem admittente, observandum est, nullam hic altitudinem esse *b;* quia aër utrique Sectio VIII orificio incumbere ad eandem altitudinem censeri potest, erit proinde

$$v = \frac{Ax}{\mathrm{A} + \frac{nn}{\alpha\alpha}B},$$

atque si fuerit $\frac{A}{B} = 850$, quae praeterpropter solet esse proportio inter gravitates specificas aquae & aëris, erit

$$v = \frac{850x}{850 + \frac{nn}{\alpha\alpha}}.$$

§. 18. Omnia haec principia, quae hactenus adhibuimus, facile ut jam dixi extenduntur ad vasa, quae finitam ratione foraminum habent amplitudinem; potest autem eorum veritas alio etiam modo admodum diverso evinci, uti ostendam, cum ad *hydraulico-staticam* pervenero, quia altero illo demonstrandi modo pressiones fluidorum in singulis vasis partibus magis fiunt perspicuae; differunt autem horum fluidorum regulae staticae vehementer a legibus, quae fluidis stagnantibus debentur.

Caeterum habent haec suam utilitatem ad machinas hydraulicas recte perspiciendas; neque enim satis ad haec attenti fuisse videntur artifices: dabitur autem occasio de iis uberius disserendi in sequenti sectione, ubi calculum ponemus, quantum vis in propellendis aquis adhibitae perdatur a transitu aquae per plura foramina, ostensuri simul remedia adhibenda, ut illud virium detrimentum, quantum fieri potest, diminuatur. Prius vero alia quaedam vasa composita in hac Sectione considerabimus, quam ad haec descendamus.

§. 19. Fit aliquando, ut vasa juxta se posita aquas unum ex altero recipiant effluxuras demum ex ultimo. Hosce vero motus jam exemplo illustrabimus.

Propositum fuerit vas cujuscunque formae *AGMB* (Fig. 41), quod nova aquarum affusione constanter plenum conservatur usque in *AB.* Ex eodem interim vase fluidum transire intelligatur per foramen *M* in aliud vas contiguum *BMNC* & ex hoc rursus in aliud *CNRD* per foramen *N* & sic porro, donec tandem aquae in aerem ejiciantur, quaeranturque loca superficierum *HL, PQ* &c. postquam fuerunt ad statum *permanentiae* reducta. Quaestio autem sic solvetur.

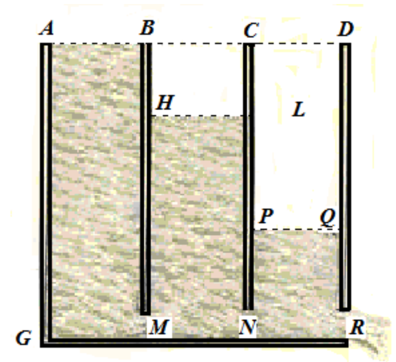


Fig. XLI

Perspicuum nempe est ex eo, quod superficies *AB, HL, PQ* &c. in eodem loco permanent, aquas iis transire per foramina *M, N, R* velocitatibus, quae debeantur altitudinibus *BH, LP, QR,* si modo transitus aquarum per unum foramen non acceleret earundem fluxum per foramen proximum, quod certe non fiet, nisi expresse opera detur, ut id aliquantum fiat. Praeterea vero considerandum est, velocitates aquarum per foramina transfluentium reciproce esse foraminibus proportionales, quia in statu *permanentiae* eadem tempore eaedem aquarum quantitates per singula foramina trajiciuntur. Ex istis intelligitur, designatis amplitudinibus foraminum *M, N, R* per *m, n, p,* fore $LP = \frac{mm}{nn} \times BH$; $QR = \frac{mm}{pp} \times BH$: Est vero $BH + LP + QR$ aequalis altitudini superficiei *AB* supra foramen ultimum *R* seu *DR;* erit igitur

$$BH + \frac{mm}{nn} \times BH + \frac{mm}{pp} \times BH = DR,$$

& proinde

$$BH = DR : \left(1 + \frac{mm}{nn} + \frac{mm}{pp}\right);$$

pariterque

$$LP = \frac{mm}{nn} \times DR : \left(1 + \frac{mm}{nn} + \frac{mm}{pp}\right)$$

atque

$$QR = \frac{mm}{pp} \times DR : \left(1 + \frac{mm}{nn} + \frac{mm}{pp}\right)$$

seu

$$BH = DR : \left(1 + \frac{mm}{nn} + \frac{mm}{pp}\right);$$

$$LP = DR : \left(1 + \frac{nn}{mm} + \frac{nn}{pp}\right)$$
$$QR = DR : \left(1 + \frac{pp}{nn} + \frac{pp}{mm}\right),$$

atque sic determinantur situs invariabiles superficierum *HL*, *PQ*, &c. At quanta tempore id fiat, si aliter superficies illae sint positae & quaenam interea aquae quantitas per singula foramina fluat, inferius examinabimus una cum aliis quaestionibus eo pertinentibus: Jam vero ex allatis valoribus altitudinum *BH*, *LP*, *QR* &c. praecipuas affectiones deducemus.

§. 20. I. Cum singula foramina sunt inter se aeque ampla, erit $BH = LP = QR$ &c. & quaevis istarum altitudinum toties continebitur in altitudine *DR*, quoties vasa replicantur.

II. Si vero aliquod foraminum sit infinite parvum ratione reliquorum, erunt omnes superficies, quae sunt cis foramen positae, in eadem altitudine cum prima superficie *AB:* reliquae autem fundo *GR* erunt proximae.

III. Si canalis fingatur continuus per singula foramina *M*, *N*, *R* &c. transiens, intelligitur, aquam per orificium canalis effluere debere velocitate, quae debeatur toti altitudini *DR*. In nostro vero casu ea velocitas respondet tantum altitudini *QR*, cujus rei ratio & origo est, quod *ascensus potentialis* singularum guttularum per foramina, excepto solo foramine effluxus, transfluentium absorbeatur. Igitur vis viva quae singulis momentis perditur est ad vim vivam quae singulis momentis generatur, ut *DQ* ad *DR*. Altitudines vero *BH*, *LP*, &c. repraesentant *respective* vim vivam, quae continue guttulis per foramina *M*, *N* transfluentibus separatim demitur. Puto tamen si foramina fuerint fere aequalia, eorumque centra in rectam lineam posita, ac denique parietes *BM*, *CN*, *DR* non admodum a se invicem remoti sint, fieri posse, ut aliquanto majori velocitate aquae erumpant, quam theoria ista indicat: In reliquis casibus non dubito de ejusdem accuratione, abstrahendo animum ab impedimentis saepe indicatis.

IV. Denique perspicuum est, quoties superficies aquae *HL*, *PQ* &c. situm suum mutant sive plures, sive una sola, mox omnes superficies loca mutaturas esse, donec eo quo dictum fuit modo fuerint ad aequilibrium repositae. Mutationes autem istas generaliter definire nodosi aeque ac prolixi est calculi, nisi vasa ponantur prismatica & infinitae quasi amplitudinis ratione foraminum, ut nempe incrementa *ascensuum potentialium* aquarum *ML*, *NQ* &c. quae locum mutant, negligi possint ratione *ascensuum potentialium*, qui in guttulis per *M*, *N*, *R* transfluentibus perpetuo generantur. Neque profecto restrictio haec afficere nos debet, cum passim jam viderimus in vasis vel mediocriter admodum amplis posse sine sensibili errore incrementa motus massarum internarum rejici in calculo. Omittam igitur solutionem generalem, quae mihi est, ob nimiam ejus prolixitatem, atque ut in hac sectione adhuc feci, vasa ceu infinite ampla & quidem ad majorem concinnitatem prismatica ponam. Incipiam autem a vase bifido.

§. 21. Repraesentatur hujusmodi vas bifidum (Fig. 42), cujus pars *AM* aquis plena, altera *BN* saltem usque ad *HL* repleta ponitur, cum jam fluxus per utrumque orificium *M* & *N* incipit: affundanturque aquae in *AB,* ut vas constanter plenum servetur, sic autem fiet, ut aquae in *BN* assurgant (aut etiam descendant pro rerum circumstantiis); quod cum ita sit, quaeremus velocitatem superficiei aqueae, cum perveniet in situm *hl.*

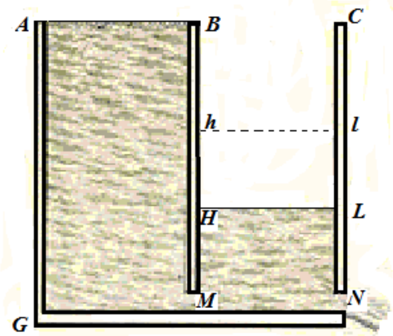


Fig.XLII

Hunc in finem exprimemus amplitudinem orificii *M* per *m,* orificii *N* per *n* & amplitudinem *hl* (quae quidem ubique eadem ponitur) per *g.* Deinde ponemus $BM = a,\ HM = b,\ Bh = x$, atque proinde $hM = a - x$. Sic vero patet ex positione infinitae veluti vasorum *AM* & *BN* amplitudinis, cum superficies aquae variabilis est in *hl,* fore altitudinem debitam velocitati aquae per *M* transfluentis $= Bh = x$, velocitatemque ipsam $= \sqrt{x}$, similemque altitudinem ratione orificii $N = hM = a - x$, atque velocitatem aquae per *N* transfluentis $= \sqrt{a-x}$; est igitur quantitas dato tempusculo per *M* in vas *BN* influentis ad quantitatem eadem tempusculo ex vase effluentis ut $m\sqrt{x}$ ad $n\sqrt{a-x}$, harumque quantitatum differentia divisa per amplitudinem *g* dat velocitatem superficiei *hl,* quae proinde velocitas, quam vocabimus *v,* exprimetur hac aequatione,

$$v = \frac{m\sqrt{x} - n\sqrt{a-x}}{g}.$$

§. 22. Ut jam innotescat tempus, quo superficies fluidi ex *HL* venit in *hl,* vocabimus illud tempus *t:* quia autem est $dt = \frac{-dx}{v}$, erit, posito pro *v* valore modo invento,

$$dt = \frac{-gdx}{m\sqrt{x} - n\sqrt{a-x}}.$$

Potest quidem haec formula immediate rationalis fieri ponendo $x = \frac{4aqq}{(1+qq)^2}$, atque deinde debito modo construi: Ista vero methodus paullo prolixior est hac altera, qua quantitas reducenda dividitur in duo membra seorsim integranda, nempe praemissa aequatio non differt ab hac:

$$dt = \frac{mgdx\sqrt{x}}{nna - (mm+nn)x} + \frac{ngdx\sqrt{a-x}}{nna - (mm+nn)x}:$$

Est autem

$$\int \frac{mgdx\sqrt{x}}{nna-(mm+nn)x} = -\frac{2mg}{mm+nn}\sqrt{x} + \frac{mng\sqrt{a}}{(mm+nn)\sqrt{(mm+nn)}} \times \log \frac{n\sqrt{a}+\sqrt{(mm+nn)}\sqrt{x}}{n\sqrt{a}-\sqrt{(mm+nn)}\sqrt{x}};$$

alteriusque membri integrate, nempe

$$\int \frac{ngdx\sqrt{a-x}}{nna-(mm+nn)x},$$

fit

$$= -\frac{2ng}{mm+nn}\sqrt{(a-x)} + \frac{mng\sqrt{a}}{(mm+nn)\times\sqrt{(mm+nn)}} \times \log \frac{m\sqrt{a}+\sqrt{(mm+nn)}\sqrt{a-x}}{m\sqrt{a}-\sqrt{(mm+nn)}\sqrt{a-x}};$$

patet exinde addita debita constante fore

$$t = \frac{2mg\sqrt{a-b}-2mg\sqrt{x}+2ng\sqrt{b}-2ng\sqrt{a-x}}{mm+nn} +$$
$$\frac{mng\sqrt{a}}{(mm+nn)\times\sqrt{(mm+nn)}}$$
$$\times \log \frac{mna+(mm+nn)\times\sqrt{(ax-xx)}+m\sqrt{(mm+nn)}\sqrt{ax}+n\sqrt{(mm+nn)}\times\sqrt{(aa-ax)}}{mna+(mm+nn)\times\sqrt{(ax-xx)}-m\sqrt{(mm+nn)}\sqrt{ax}-n\sqrt{(mm+nn)}\times\sqrt{(aa-ax)}}$$
$$-\frac{mng\sqrt{a}}{(mm+nn)\times\sqrt{(mm+nn)}}$$
$$\times \log \frac{mna+(mm+nn)\times\sqrt{(ax-bb)}+m\sqrt{(mm+nn)}\sqrt{aa-bb}+n\sqrt{(mm+nn)}\sqrt{ab}}{mna+(mm+nn)\times\sqrt{(ab-bb)}-m\sqrt{(mm+nn)}\sqrt{aa-ab}-n\sqrt{(mm+nn)}\sqrt{ab}}.$$

§. 23. Ex paragrapho 19 liquet superficiem *hl* in situ suo permanere cum est

$$Bh(=x) = \frac{nna}{mm+nn}.$$

At vero si in aequatione integrata praecedentis paragraphi ponitur $x = \frac{nna}{mm+nn}$, fit denominator in quantitate logarithmicali $=0$, ipsaque proinde quantitas infinita: tempus igitur totius motus infinities majus est, quam cujuscunque partis.

Sed ut alium insuper casum determinemus, videbimus quanto tempore superficies aquae ex infimo situ *MN* (posito nempe $b=0$) ascendat quantitate $\frac{1}{2}a$, posito $m:n = 4:\ 3$; fit autem

$$t=\frac{8g\sqrt{a}-14g\sqrt{\tfrac{1}{2}a}}{25}+\frac{12g\sqrt{a}}{125}\log\left(\frac{49+35\sqrt{2}}{49-35\sqrt{2}}\right)-\frac{12g\sqrt{a}}{125}\log 4,$$

seu

$$t=\frac{8g\sqrt{a}-7g\sqrt{2a}}{25}+\frac{12g\sqrt{a}}{125}\log\left(\frac{49+35\sqrt{2}}{140\sqrt{2}-196}\right),$$

est, proxime $t=\frac{15g}{100}\times 2\sqrt{a}$, quod indicat, esse tempus istud ad tempus quo grave libere cadit per altitudinem *BM* proxime ut 15g ad 100: Pariter tempus descensus invenitur, si ab initio superficies *hl* fuerit ultra situm aequilibrii posita. Fuerit v. gr. utrumque vas aquis totum repletum, orificia autem *M* & *N* rationem nunc habeant quae est inter 3 & 4, sitque tempus determinandum, quo superficies ex *B* descendat per dimidiam *BM:* hypotheses hae faciunt $m=3;\ n=4;\ b=a$, atque $x=\tfrac{1}{2}a$, ita vero fit

$$t=\frac{8g\sqrt{a}-7g\sqrt{2a}}{25}+\frac{12g\sqrt{a}}{125}\log\left(\frac{49+35\sqrt{2}}{49-25\sqrt{2}}\right)-\frac{12g\sqrt{a}}{125}\log 4.$$

Ex quo apparet in utroque exemplo idem esse tempus.

§. 24. Priusquam descendamus ad vasa multifida indagasse conveniet, quaenam aquae quantitas per utrumque orificium *M* & *N* fluat, dum superficies aquae ex situ *HL* venit in *hl*. Et primo quidem, quod ad orificium *M* pertinet, perspicuum est quantitatem aquae dato tempusculo (*dt*) per illud transfluentem proportionalem esse velocitati $\left(\sqrt{x}\right)$ ductae in magnitudinem orificii (*m*) ipsumque tempusculum *dt*, ita ut haec quantitas sit (ob $dt=\frac{-gdx}{m\sqrt{x}-\ n\sqrt{a-x}}$ per§. 22)

$$=\frac{-mgdx\sqrt{x}}{m\sqrt{x}-\ n\sqrt{a-x}},$$

atque proinde omnis quantitas quae ab initio effluxerit

$$=-\int\frac{mgdx\sqrt{x}}{m\sqrt{x}-\ n\sqrt{a-x}}.$$

Est autem

$$-\int\frac{mgdx\sqrt{x}}{m\sqrt{x}-\ n\sqrt{a-x}}=\frac{mnga}{(m+n)^2}\log\left(\frac{ma-mb-nb}{mx+nx-na}\right)+\frac{mg}{m+n}\times(a-b-x).$$

Eodem modo eruitur quantitas aquae interea per orificium N effluentis (quae scilicet est $= -\int \frac{ngdx\sqrt{a-x}}{m\sqrt{x} - n\sqrt{a-x}}$.)

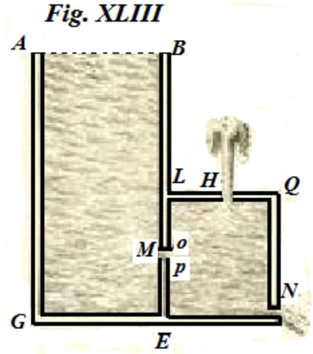
$$= \frac{mnga}{(m+n)^2}\log\left(\frac{ma-mb-nb}{mx+nx-na}\right) - \frac{ng}{m+n}\times(a-b-x).$$

Atque inde etiam innotescit quantitas aquae, quae in *AB* affunditur, neque enim differt ab illa, quae per *M* transfluit: aqua denique in vase *BN* collecta exprimitur per $g(a-b-x)$, & cum differentia sumitur aquarum per *M* & *N* transfluentium, oritur eadem ista quantitas $g(a-b-x)$.

§. 25. Prouti §. 21 velocitatem superficiei locum continue mutantis determinavimus pro vase bifido, ita nunc in vasis multifidis velocitates singularum superficierum definiemus. Fuerit nempe altitudo superficiei supremae supra proximam $= x$, altitudo hujus supra sequentem $= y$, deinde $= z$, rursusque altitudo proxima $= s$, & sic porro. Amplitudines vero orificiorum designentur per *m, n, p, q* &c.; amplitudines vasis secundi, tertii, quarti &c. sint *M, N, P* &c. Sic patet fore velocitatem superficiei secundae $= \frac{m\sqrt{x} - n\sqrt{y}}{M}$, veloc. superf. tert. $= \frac{n\sqrt{y} - p\sqrt{z}}{N}$, velocit superfic. quartae $= \frac{p\sqrt{z} - q\sqrt{s}}{P}$ &c.

Porro cum spatiola iisdem tempusculis a superficiebus percursa sint ut velocitates, apparet sic singulis momentis determinari situs istarum superficierum, quamvis aequationes sint intractabiles fere. Id ex se patet, si vel unica superficies extra situm aequilibrii supra §. 19 definiti posita fuerit, fore ut omnes reliquae motibus reciprocis agitentur, donec post tempus infinitum in pristinum situm redierint simul.

Fig. XLIII



§. 26. Sit porro vas ita formatum, ut ostendit Fig. 43, divisum scilicet in duas partes *ABEG* & *LQNE* inter se mediante foramine *M* communicantes; sintque praeterea foramina *H* & *N* per quae aquae exiliant, dum in *AB* totidem affunduntur. Sint autem amplitudines in utroque vase veluti infinite amplae ratione foraminum *M, H* & *N;* hisque positis propositum sit velocitates invenire, quibus aquae tam per *H,* quam per *N* ejiciantur seu altitudines istis velocitatibus debitas. Erunt autem velocitates invariabiles, quia vas aquis plenum conservatur, simulque vasis amplitudines respectu foraminum infinitae censentur.
Solutio istius problematis ex praecedentibus facile colligetur, si modo concipiatur foramen *M* in duas divisum partes *o* & *p,* quarum altera *o* aquas foramini *H,* altera *p* foramini *N* mittat: partes autem *o* & *p* (quia per utramque eadem fluunt velocitate aquae) eam habebunt rationem, quam inter se habent quantitates aquarum eodem tempore per *H* & *N* effluentium, id est, rationem compositam ex ratione amplitudinis *H* ad amplitudinem *N* & velocitatis in *H* ad velocitatem in *N.* Quibus praemonitis perspicuum est, si

amplitudines foraminum *M, H* & *N* indicentur per $\alpha,\ \beta,\ \gamma$, altitudines autem velocitatibus in *H* & *N* debitae designentur per *x* & *y,* ipsaeque proinde velocitates per $\sqrt{x}$ & $\sqrt{y}$, fore amplitudinemo $o=\frac{\beta\sqrt{x}}{\beta\sqrt{x}+\gamma\sqrt{y}}\alpha$ & amplitudinem $p=\frac{\gamma\sqrt{y}}{\beta\sqrt{x}+\gamma\sqrt{y}}\alpha.$ Ponatur nunc altitudo superficiei *AB* supra orificium $H=a$, & habebitur *x,* ut demonstratum fuit §. 4, si quadratum foraminis *o* dividatur per summam quadratorum foraminum *o* & *H* & quod oritur multiplicetur per *a;* sic igitur fit

$$x=\frac{\alpha\alpha ax}{\alpha\alpha x+\left(\beta\sqrt{x}+\gamma\sqrt{y}\right)^{2}}$$

ex quo oritur haec aequatio

$$\text{(A)}\qquad \alpha\alpha x+\left(\beta\sqrt{x}+\gamma\sqrt{y}\right)^{2}=\alpha\alpha a.$$

Eodem modo ratione foraminum *p* & *N,* posita altitudine *AB* supra $N=a+b$, obtinetur haec altera aequatio:

$$\text{(B)}\qquad \alpha\alpha y+\left(\beta\sqrt{x}+\gamma\sqrt{y}\right)^{2}=\alpha\alpha\times(a+b).$$

Subtracta aequatione (B) ab aequatione (A) prodit $y=x+b$, ex quo sequitur, si venae ambae verticaliter sursum dirigantur, utramque ad eundem locum assilire. Deinde si in aequatione (A) substituatur pro *y* valor ejus $x+b$, erit

$$\text{(C)}\quad \alpha\alpha x+\left(\beta\sqrt{x}+\gamma\sqrt{x+b}\right)^{2}=\alpha\alpha a,$$

unde deducitur valor ipsius *x* aequatione quadrata.

§. 27. Ex praecedentis paragraphi aequationibus sequentes fluunt affectiones.

I . Quia velocitas aquae per *M* transfluentis est $=\frac{\beta\sqrt{x}+\gamma\sqrt{y}}{\alpha}$, erit altitude generans hanc velocitatem $=\left(\frac{\beta\sqrt{x}+\gamma\sqrt{y}}{\alpha}\right)^{2}$; sed si addantur aequationes (A) & (B) fit:

$$=\left(\frac{\beta\sqrt{x}+\gamma\sqrt{y}}{\alpha}\right)^{2}=\frac{2a+b-x-y}{2}=(\text{ob } y=x+b)\ a-x.$$

II. Si foramen *H* sit valde exiguum ratione foraminum *M* & *N*, id est, si β possit censeri nulla ratione α & γ, abit aequatio (C) in hanc

$$\alpha\alpha x + \gamma\gamma x + \gamma\gamma b = \alpha\alpha a,$$

seu

$$x = \frac{\alpha\alpha a - \gamma\gamma b}{\alpha\alpha + \gamma\gamma};$$

id vero egregie convenit cum paragrapho decimo nono, cum manifestum sit aquam per foramen valde exiguum ad eandem altitudinem assilire, quam haberet aqua, si haec laminam *LQ* tantum deorsum premat, quantum ab aqua interna sursum premitur; ista vero praefata altitudo paragraphi 19 est $\frac{\alpha\alpha a - \gamma\gamma b}{\alpha\alpha + \gamma\gamma}$. Est porro in ista hypothesi altitudo velocitatis aquarum in *N* seu

$$x + b = \frac{\alpha\alpha a + \alpha\alpha b}{\alpha\alpha + \gamma\gamma}$$

& denique altitudo velocitatis aquarum in *M*, seu

$$a - x = \frac{\gamma\gamma a + \gamma\gamma b}{\alpha\alpha + \gamma\gamma};$$

quae posteriores aequationes in isto casu particulari pariter ex §. 19 immediate colligi aut praevideri potuissent.

III Si vero nunc alterum foramen *N* admodum exiguum prae ambobus reliquis ponatur, erit facto $\gamma = 0$

$$x = \frac{\alpha\alpha a + \gamma\gamma b}{\alpha\alpha + \gamma\gamma};$$

deinde

$$x + b = \frac{\alpha\alpha a + \alpha\alpha b + \beta\beta b}{\alpha\alpha + \beta\beta},$$

&

$$a - x = \frac{\beta\beta a}{\alpha\alpha + \beta\beta}.$$

IV. Si $\gamma\gamma b = \alpha\alpha a$, fit $x = 0$. Nullam igitur in hoc casu pressionem sustinent partes laminae *LQ*: imo inferiora versus premitur, si $\gamma\gamma$ sit majus quam $\frac{\alpha\alpha a}{b}$, & lamina nullibi sit perforata.

Ista vero omnia similiter ex §.19 facile colliguntur.

V. Ita quoque ope ejusdem paragraphi sine calculo novo praevideri potuisset, quid fieri debeat, cum positis foraminibus *H* & *N* in eadem altitudine summa foraminum eorum, ceu unicum amplitudinis $\beta+\gamma$ considerari potest: Indicant nempe tam §. 19 quam §. 26 esse

$$x=\frac{\alpha\alpha a}{\alpha\alpha+(\beta+\gamma)^2}.$$

VI. Notari etiam potest, cum valor ipsius *x* fit imaginarius, id provenire ex eo, quod aquae non solum non effluant in aliquibus casibus per *H*, sed quod superficies *LQ* etiam descendat; unde fieri potest, ut infra orificium *M* descendat, quo ipso cessat aquarum contiguitas contra hypothesin propositionis. Si autem valor *x* est realis, tum dupliciter exprimitur, sed alter valor inutilis est reputandus; sic igitur cavendum ne praepostera radix ceu utilis assumatur.

VII. Denique ut casum specialissimum attingamus, ponemus omnia foramina inter se aequalia, & prodibit

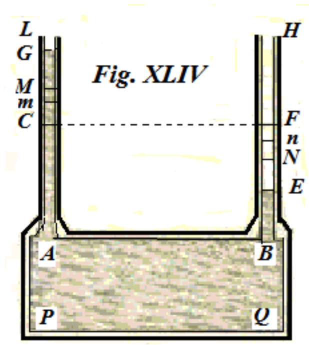
$$5xx+\ (2b-6a)x=-aa+2ab-bb,$$

seu

$$x=\frac{3a-b-2\sqrt{(aa+ab-bb)}}{5};$$

atque si fuerit praeterea $a=3b$, eritx = (proxime) $\frac{4}{15}b$, deinde altitudo velocitatis in foramine *N* seu $x+b=\frac{19}{15}b$ atque altitudo velocitati in *M* debita seu $a-x=\frac{41}{15}b$. Sunt itaque velocitates seu etiam, quia foramina aequalia sunt, quantitates aquarum iisdem temporibus per foramina *M, H* & *N* transfluentium proxime ut $\sqrt{41},\ 2\ \&\sqrt{19}$.

§. 28. Ex his omnibus patet methodus determinandi motum in fluidis tum etiam, cum quantitas *virium vivarum* non conservatur; & simili modo semper absolvetur computus, quoties ex natura subjectae quaestionis praesumi potest (uti in quaestionibus hujus sectionis accurate potuit) quantum *vis vivae* singulis momentis inutilis ad motum determinandum evanescat. Neque enim soli sunt casus, quos adhuc examinavimus: lubet itaque alium addere, qui oscillationes fluidorum spectat, ut innotescat quantum inde decrementum excursiones fluidi capiant.
Sint duo tubi amplitudine aequales & cylindrici *AL* & *BH* (Fig. 44) verticaliter inserti vasi amplissimo horizontali *ABOP*. Sit vas istud totum aqua repletum: tubi autem aquam habeant usque in *C* & *F;* deinde sublato aequilibrio haereat altera superficies in *G* altera in *E;* moxque aqua sibi relicta moveri incipiat. His positis tantum deberet superficies *G* descendere infra locum *C,* alteraque superficies *E* ascendere supra *F,* quanta est altitudo *GC* seu *EF,* si omnis *vis viva*



conservaretur (ab impedimentis frictionum aliisque similibus nunc animum abstrahimus): Verum patet, *vim vivam* omnis aquae per *A* in vas horizontale fluentis absumi sine alio effectu ab aqua ibidem stagnante, indeque sequitur descensum superficiei *G* alteriusque ascensum minorem fore, quam modo dictum fuit: id igitur decrementum nunc explorabimus.

Ponatur ad hunc finem superficiem ex *G* pervenisse in *M*, ponaturque $GM = x,\ GC = b,\ CA = a$: erit $BE = a - b,\ EN = x,\ MC = FN = b - x$; deinde fiat altitudo debita velocitati superficiei in $M = v$, in situ proximo $m = v + dv$; eritque incrementum *vis vivae* aquae (dum superficies percurrunt elementa *Mm, Nn,* seu $dx) = 2adv$, cui addenda est *vis viva* guttulae, quae ab aqua vasis horizontalis absumitur, nempe *vdx*, & erit summa $2adv + vdx$ aequalis *descensui actuali* aquae multiplicato per massam aquae, quod productum est aequale *descensui actuali* guttulae *dx*, multiplicato per $2b - 2x$. Est igitur

$$2adv + vdx = 2bdx - 2xdx.$$

Haec vero aequatio recte integrata abit in hanc

$$v = 4a + 2b - 2x - c^{\frac{-x}{2a}} \times (2b + 4a);$$

unde si ponatur $4a + 2b - 2x - c^{\frac{-x}{2a}} \times (2b + 4a) = 0$, dabit valor ipsius *x* totam excursionem, aqua si auferatur *b*, residuum indicabit descensum infra punctum aequilibrii C.

§. 29. Ut vero exemplo quodam appareat, quantum hac ratione oscillationes diminuantur, ponemus $a = b$, facta scilicet $CA = GC$ & $BE = 0$. Ita oritur

$$3a - x = c^{\frac{-x}{2a}} \times (3a)$$

sive

$$c^{\frac{x}{2a}} = \frac{3a}{3a - x}$$

vel

$$x = 2a \log \frac{3a}{3a - x},$$

cui aequationi prope admodum satisfacit valor $x = \frac{7}{4}a$. Est igitur decrementum excursionis seu $2b - x =$ quartae parti elevationis fluidi supra punctum medium: si majus observetur experimento, reliquum adhaesioni aquae ad latera tuborum tribuendum erit.

§. 30. Neque ista diminutarum excursionum ratio plane, ut suspicor, auferetur, si vel aequalis fiat amplitudinis tubus horizontalis cum verticalibus, ob mutatam fluidi directionem in punctis *A* & *B*. Caeterum infiniti alii fingi possent casus iisdem principiis solvendi, veluti si natura oscillationum indaganda sit in vase Fig. 44, cum id in parte horizontali diaphragmate in duas dispescitur partes solo lumine, quod diaphragma habeat, inter se communicantes & hujusmodi alii. Puto autem haec jam sufficere, ut quisque sibi facile regulas generales pro istiusmodi quaestionibus solvendis formare possit.

Experimenta ad Sectionem octavam pertinentia15•

Experimentum 1.

Paragraphum quartum, quo dicitur altitudinem velocitati aquae per orificium *D* effluentis (Fig. 37) esse $\frac{mmx}{nn+mm}$, eo confirmavi modo, ut utrumque orificium G & *D* limbum haberet instar zonulae paullulum elevatum, ne contractioni venarum locus esset, tutumque fieri posset judicium a quantitate aquae dato tempore effluentis ad velocitates. Deinde sumtis accurate mensuris, observatoque tempore quo superficies per datum spatium *AP* descenderet, vidi tempus istud recte respondere velocitatibus dicto paragrapho definitis: observavi etiam nihilo mutari motum ab elevatione aut depressione diaphragmatis. Reliqua ad experimentum pertinentia memoria exciderunt, neque ea in chartam conjeci: superfluum autem duxi experimentum repetere, quod unicuique facile erit imitari: fundamentum autem id est reliquis, quae adeoque ulteriori disquisitione experimentali vix opus habent: volui tamen sequentia praeterea tentare.

Experimentum 2.

Vase usus sum, quale fere adhibuit Mariottus *(vid.* Fig. 38) rursusque confirmavi aequationem nostram hunc in modum: feci ut aquae per orificium *D* horizontaliter effluerent, tuncque mensuras cepi altitudinis orificii *D* supra pavimentum & distantiam loci, ubi vena in pavimentum incidebat a puncto in eodem pavimento, cui orificium *D* verticaliter imminebat; inde cognovi altitudinem velocitati aquae in *D* effluentis debitam: eandem autem hanc altitudinem experimento proxime inveneram, quam theoria hujus sectionis indicat §. 4. Similia experimenta apponam in fine experimentorum ad sectionem duodecimam pertinentium, quae simul theoriam nostram *hydraulico-staticam* confirmabunt.

Denique cum multa sint in §§. 26 & 27 quae singulari calculo eruta fuerunt, operae pretium erit de illis quoque experimenta sumere, praesertim cum alia simul eadem opera sumi poterunt experimenta, quae in Sect. XII recensebuntur, si vas, quale Fig. 43 sistit, ad hunc finem fieri curetur.

Caeterum haec theoria etiam confirmatur experimentis in Sectione septima recensitis, quae de oscillationibus fluidorum in tubos per foramina influentium sumsi.