HYDRODYNAMICS SECTION SEVEN

Concerning the motion of water through submerged vessels, where it is shown by examples, either how significantly useful the principle of the conservation of living forces shall be, or as in these cases, in which a certain amount is agreed to be lost from these continually.

[This section lacks in some details initially; it is an extension of the last section, as here an empty tube with a hole in the base is either plunged into a large tank filled with water, and the oscillations of the water entering via the opening are observed; or the tube immersed in the tank is filled to a certain level above the outer water level initially, and the oscillations of the water within the tube are observed. In this case the mass of water oscillating is not constant.]

FIRST PART.

Concerning the descent of water.

§. 1. Imagine a cylinder filled with water, the bottom of which shall have a hole in it, and that submerged to a certain depth of still water, as if of infinite depth, and you understand easily the surface of the water contained in the cylinder to begin falling, and below a certain depth below the surface of the external water, then again it will begin to rise, and thus repeating itself. However these oscillations are completely different from the oscillations considered in the previous section, in which evidently the reciprocal motions are always in the reverse direction with the same motions which were preceding. Moreover, who could presume the reflux of the water here or the ascent to be the same as it was in the descent? [*i.e.* the mass of water oscillating is not constant, as in the last section.] If anyone should consider that to be the case, it is certainly quite wrong, even if the motion either were not diminished by the adhesion of the water to the sides of the vessel or otherwise by other kinds of obstructions, and the rules of the motion which prevail for soft bodies are not very different from those for elastic bodies, as far as in each case the bodies may be considered to be moving freely. I use this analogy, because it illustrates our argument particularly well: For indeed just as the rules of motion with soft bodies are determined correctly, if after a collision that part of the *vis viva* may be considered to be lost, which had depended on the compression of the bodies (for neither is this progressively restored to the motion as in elastic bodies); thus the ascent of the fluid will be defined no less correctly, if it may be examined carefully, how much of the *vis viva* [*i.e.* kinetic energy] may be communicated internally to the motion of the particles of the water at individual instants, at no time going to be returned to the progressive motion which we are discussing.
[This would appear to be the first time anyone had the idea that the progressive motion of a body, or its kinetic energy in modern terms, can be transformed into the random motion of the individual particles of the body, or heat energy.]

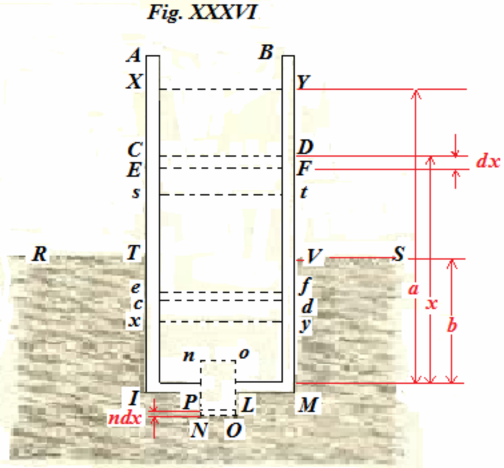
§. 2. And thus since this matter may be deduced from that condition, so that it may be investigated, how much of the *vis viva* may be lost continually in this reciprocal motion, we will begin the investigation from this point.

Initially, moreover it is apparent all the *vis viva* present in the particles flowing out and passing to the external water, does not in any way advance the subsequent ascent or inflow of external water into the tube : This hypothesis is more apparent than to be in need of a further explanation : moreover it is concerned with the outflow of water, and in this case that alone is required to be considered. Now the other case arises, which pertains to the inflow of the water.

Secondly, therefore it is of greater concern to me, when the water rushing in through the opening has a greater speed, than what the water present requires to ascend inside [the vessel], that excess may again invoke a certain internal motion [in addition] to the same motion within [the vessel], but bringing little or nothing to the ascent [potential]. Thus, if this shall be the case, and the cross-section of the opening $=1$, with the cross-section of the cylinder $=n$, the *ascent potential* of the droplet bursting in shall be $=nnv$, and its velocity $=n\sqrt{v}$, this particle will retain that part from its motion since the water [within the vessel] which has the common velocity $\sqrt{v}$, and thence the *ascent potential* v will be conserved ; but the remainder of the *ascent potential,* evidently $nnv-v$, is agreed to have gone over into the internal motion of the particles. This hypothesis, whatever the physical situation and however close to the truth it may be, yet has a great use for determining the motion of fluids without notable error, as often as either it may be disrupted continually into a uniform vessel, which at this stage was assumed, or as when water is forced to pass through several openings; certainly I might have believed a single case, with the aid of which motion of this kind would be able to explain correctly wonderful phenomena. On this account I wish to think about this correctly, before the reader is diverted to other matters.
[Note that from the continuity equation, an incremental change in the height of the water in the vessel dx corresponds to a change in the volume ndx in the vessel; this water flows out through an opening of cross-section $\frac{1}{n}^{\text{th}}$ of the upper vessel, and thus must travel with a speed n times as great; hence the ascent potential, corresponding to the square of the speed, becomes n^2 as great as in the vessel, or n^2dv for this droplet of water released.]

§. 3. Now therefore we will consider that question, by beginning from the descent of the water. We may consider the cylinder *AIMB* (Fig. 36), full of water as far as *XY* and submerged in the boundless water *RTVS* , thus so that its length shall be in the vertical position ; its base may have the hole *PL,* through which water from the vessel shall be able to flow out into the water flowing around. The velocity of the internal water is sought, after its surface has fallen through a given distance *XC* or *YD*, by putting *MY* or $IX=a,\ MV=b,\ MD=x$, with the cross-section of the opening $=1$, and finally with the cross-section of the cylinder $=n$.

Fig. XXXVI



The solution will be the same as for the similar question we gave in section three, but this solution certainly more general: yet it may be observed, because with a particle of the water *CDFE* taken infinitely small to equal the droplet *PLON* ejected from that for that time, the *actual descent* now shall be required to be estimated by the height *DV* or *CT*, as in the other case [discussed below] it was required to be defined from the whole height *DM*.

Truly the velocity of the surface of the water *CD* shall be that which is due to the height *v*, and in the place infinitely nearby *EF* the same velocity will correspond to the height $v - dv$; and when the *ascent potential* of the water *CDMLPIC* shall be *v*, the *ascent potential* of the same nearby *EFMLONPIE* will be obtained, if the mass *EFMLPIE*, $(nx - ndx)$, may be multiplied by its *ascent potential* $(v - dv)$, as also the droplet *LONP*, (ndx), multiplied likewise by its *ascent potential nnv*, and the sum of the products may be divided by the sum of the masses (*nx*): and thus this *ascent potential* may be had

$$v = \frac{(nx - ndx) \times (v - dv) + ndx \times nnv}{nx}$$

or

$$\frac{xv - vdx - xdv + nnvdx}{x}.$$

Therefore the increment of the *ascent potential*

$$dv = \frac{-vdx - xdv + nnvdx}{x}$$

(cf. §. 6 Sect. III). Truly this increment is required equally with the infinitely small *actual descent*, which (by §. 7 Sect. III by the note just given) is $= \left[\frac{n(x-b)dx}{nx}\right] = \frac{(x-b)dx}{x}$. And thus such an equation will be had :

$$-vdx - xdv + nnvdx = (x - b)dx,$$

which integrated in the due manner is changed into this :

$$v = \frac{1}{nn-2} \times \left(x - \frac{x^{nn-1}}{a^{nn-2}} \right) - \frac{b}{nn-1} \times \left(1 - \frac{x^{nn-1}}{a^{nn-1}} \right).$$

$\left[i.e.: \ -vdx - xdv + nnvdx = (x-b)dx;\text{or } -xdv + (nn-1)vdx = (x-b)dx;\right.$

$\therefore xdv - (nn-1)vdx = -(x-b)dx;\text{or } x^{1-nn}dv - (nn-1)x^{-nn}vdx = -x^{-nn}(x-b)dx;$

$$d\left(\frac{v}{x^{nn-1}}\right) = -x^{-nn}(x-b)dx;\text{hence } \frac{v}{x^{nn-1}} = \int_a^x \left(bx^{-nn} - x^{1-nn}\right)dx$$

$$= \left[b\frac{x^{1-nn}}{1-nn} - \frac{x^{2-nn}}{2-nn}\right]_a^x \text{ hence } v = bx^{nn-1}\left(\frac{x^{1-nn}}{1-nn} - \frac{a^{1-nn}}{1-nn}\right) - x^{nn-1}\left(\frac{x^{2-nn}}{2-nn} - \frac{a^{2-nn}}{2-nn}\right)$$

$$\left.= \frac{1}{nn-2} \times \left(x - \frac{x^{nn-1}}{a^{nn-2}}\right) - \frac{b}{nn-1} \times \left(1 - \frac{x^{nn-1}}{a^{nn-1}}\right).\right]$$

Truly from such an equation such corollaries follow.

§. 4. Just as, if the cross-section of the cylinder were in an infinite ratio to the opening, and hence considering $v = \frac{x-b}{nn}$; that height corresponding to the speed of the water, while it flows out, is $= x - b$. From which it follows, the water is to flow out with a velocity, which a weight acquires by falling from some height of the internal water above the outer water, and from that until it has all flowed out, then both surfaces shall be on the level position, and then all motion will cease : and thus the water flows out by the same law, as if the base *IM* changed position with *TV*.

However since the opening cannot be considered to be indefinitely small, the surface of the water inside descends beyond the surface of the water outside; and since it may be known to what depth *xy* the surface *CD* is going to fall, by making $v = 0$, or

$$(nn-1)\left(a^{nn-1}x - x^{nn-1}a\right) = (nn-2) \times \left(a^{nn-1}b - x^{nn-1}b\right);$$

yet however at no time will the internal surface fall below the surface of the external water to the same extent it was above the same height ; this deficiency arises from the *ascent potential* of the water removed during the descent, to which it must be proportional.

§. 5. It is to be observed, since there because the water may fall deeper within the cylinder, as the descent from the start should be raised, and when the bottom is perforated with a larger opening, still at no time shall all the water be able to flow out from the cylinder however great the elevation was before the descent and the part of the cylinder submerged were as small as you please, and likewise either the opening or the whole base itself may be put in place to drain the water.

[*i.e.* Bernoulli claims that there is no over-damped or critically damped situation, where the oscillations die away exponentially.]

§. 6. The speed of the inner water surface is a maximum, when there is taken

$$x=\left(\frac{a^{nn-1}}{nna-nnb-a+2b}\right)^{1:(nn-2)}.$$

Hence if $n=1$, clearly with the opening present equal to the opening of the whole cylinder, there becomes $x=b$, and the speed is a maximum, since both surfaces are placed at the same height. Because truly there are many, which cannot be diagnosed into these two cases, clearly $nn=1$ & $nn=2$, and these have many particular features, the same that I will now mention briefly separately.

§. 7. In the first place there shall be $nn=1$, and there becomes $-xdv=(x-b)dx$ (by §. 3) or $-dv=dx-\frac{bdx}{x}$, which thus integrated, so that there becomes at the same time $v=0$ & $x=a$, gives

$$-v=x-a+b\log\frac{a}{x},$$

or

$$v=a-x-b\log\frac{a}{x}.$$

From thence such can be deduced.

I. So that the maximum descent may be obtained by making $a-x-b\log\frac{a}{x}=0$; but it is apparent from this equation, at no time does the letter *x* obtain a negative value, indeed neither can the whole vanish without contradiction, unless there may be put $\frac{a}{b}=\infty$, which shows it is not possible, as all the water shall flow out during the descent in that case and much less and with much less in what remains, as paragraph five confirms.

II. The maximum velocity is such, which is due to the height $a-b-b\log\frac{a}{b}$, and if the difference between *a* and *b*, which I may put $=c$, shall be very small, doubtless with the very small excursions of the fluid arising in the ratio of the length, to which the cylinder is submerged, the $\log\frac{a}{b}$ can be put $=\frac{c}{b}-\frac{cc}{2b}$, and therefore that height due to the maximum velocity, or $a-b-b\log\frac{a}{b}=\frac{cc}{2b}$, which motion is proven to be exceedingly slow.

But I will demonstrate in the following the whole motion remains the same with everything else equal, when cylinders are considered to be submerged indefinitely, the base perforated with whatever opening, thus so that the motion of the inner water may not be retarded by a diminished opening ; because it may seem at first inspection certainly to be a unexpected, yet however the physical account of this will not be able to escape the mind more attentive to such matters. Clearly in this case it arises, because the *vis viva*, which may be produced in the tube, shall be as we may say infinite besides the *vis viva* of the water passing through the opening, and thus no consideration of its opening shall make the computation different.

We will demonstrate also the motions resemble reciprocal motions and the oscillations both large and small to be isochronous amongst themselves, and for these we will determine the length of the simple tautochronous pendulum.

§. 8. Now there will be $nn = 2$; thus truly there will be had by §. 3:
$vdx - xdv = (x-b)dx$, or

$$\frac{xdv - vdx}{xx} = \frac{(b-x)dx}{xx},$$

which integrated correctly will go into this

$$v = \frac{bx}{a} - b + x\log\frac{a}{x}.$$

[*i.e.* $\int \frac{xdv - vdx}{xx} = \int \frac{(b-x)dx}{xx}$; $\int \frac{dv}{x} - \int \frac{vdx}{x^2} = -\frac{b}{x} - \ln x$; $\therefore \int d\left(\frac{v}{x}\right) = -\frac{b}{x} - \ln x$;
$\therefore \frac{v}{x} + \text{const.} = -\frac{b}{x} - \ln x$; when $x = a, v = 0$; $\therefore \text{const.} = -\frac{b}{a} - \ln a$, and $\frac{v}{x} = \frac{b}{a} + \ln a - \frac{b}{x} - \ln x$.]

If there is made $\frac{bx}{a} - b + x\log\frac{a}{x} = 0$, x will give the place of maximum descent ;

moreover the place of maximum velocity will be had, by making $x = c^{\frac{b-a}{a}} \cdot a$, where by c the number is understood, of which the logarithm is one.

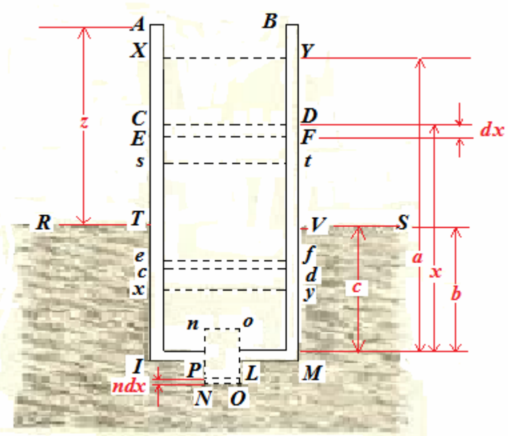
Thus after we have tightened up the various cases for different magnitudes of holes, it remains that we consider how it may succeed in cases with different heights a and b.

§. 9. And indeed in the first place if b may be placed as zero besides a, which will happen when the base of the cylinder only touches the surface of the external water, then the equation will produce

$$v = \frac{1}{nn-2}\left(x - \frac{x^{nn-1}}{a^{nn-2}}\right),$$

which equation indeed does not differ from that form, which was given in §.14 Sect. III for that case, where water put in place was ejected from the cylinder into the air. And often also I have found the cylinder to be emptied at the same time, either water squirts into the air, or the base may be submerged a little into the still water. This experiment teaches us that the air offers little or no resistance to the external outflow, since the air resistance cannot exert an effect notably greater than eight hundred times [*i.e.* the resistance arising from the density of water in comparison]. And thus because this case provides nothing in particular which was not mentioned in the place cited, we will not tarry further with this : Rather we will investigate, what must happen, when the height of the internal water is above the external; how great the descent is from the start, is taken to be very small and ignored before the immersion of the cylinder; it satisfies this hypothesis, when the excess of the height a above the height b is exceedingly small (which excess again we will call c (as in §. 7)).

Fig. XXXVI



§.10. And thus when $a-b=c$ is put in place, also there is required to be $a-x=z$, and while each quantity, truly c and z, were negligible besides the quantities a and b, but if $a-x=z$, there will be $x=a-z$ and

$$x^{nn-1}=(a-z)^{nn-1}$$
$$=a^{nn-1}-(nn-1)a^{nn-2}z+\left(\frac{\overline{nn-1}\cdot\overline{nn-2}}{2}\right)a^{nn-3}zz$$
$$-\left(\frac{\overline{nn-1}\cdot\overline{nn-2}\cdot\overline{nn-3}}{2\cdot 3}\right)a^{nn-4}z^{3}+\text{etc.}$$

This series is required to be continued as far as it suffices for our conditions ; but is will suffice up to three terms. Therefore in the integrated equation that we deduced in §. 3 we may put , $x=a-z$ &

$$x^{nn-1}=a^{nn-1}$$
$$=a^{nn-1}-(nn-1)a^{nn-2}z+\left(\frac{\overline{nn-1}\cdot\overline{nn-2}}{2}\right)a^{nn-3}zz$$
$$-\left(\frac{\overline{nn-1}\cdot\overline{nn-2}\cdot\overline{nn-3}}{2\cdot 3}\right)a^{nn-4}z^{3}+\text{etc.}$$

thus there will be :

$$v=\frac{1}{nn-1}\left(a-z-a+(nn-1)z-\left(\frac{\overline{nn-1}\cdot\overline{nn-2}}{2}\right)\frac{zz}{a}\right)$$
$$-\frac{b}{nn-1}\left(1-1+(nn-1)\frac{z}{a}-\left(\frac{\overline{nn-1}\cdot\overline{nn-2}}{2}\right)\frac{zz}{aa}\right).$$

In which equation if the canceling terms may be deleted, and $a-c$ is put for *b*, and the term may be rejected which affects the quantity $\frac{czz}{aa}$, the simpler equation is produced :

$$v=\frac{2cz-zz}{2a},$$

from which formula, since the letter *n* has disappeared, we have an indication that no magnitude of the opening pertains to the motion of the inner water, the origin of this matter I have just indicated above (§. 7).

Moreover in the following we will demonstrate, this motion does not differ from the subsequent reflux motion, and hence the oscillations become tautochrones. However before I go on to other matters I have been led to be reminded, in this calculation the quantities $\frac{c}{a}$ and $\frac{z}{a}$ not only are very small besides unity, but also besides $\frac{1}{nn}$ in which case they should be considered infinitely small, which considerations we bear in mind on setting up experiments; certainly it is allowed to call the theorem to account by experiment by diminishing certain infinitely small quantities without sensible error, which in the theory were considered as infinitely small, but it is required to make everything subject to this law in the experiment. Thus for example, if in a cylinder any base was absent, on putting $n=1$, and that submerged may be put to a depth of thirty five inches, the experiment will be supposed accurate enough, when the water before the oscillations had risen only to a height of one inch above the surface of the surrounding water; while neither will the error be noteworthy, either if the lower opening may be blocked off to half then with $\frac{c}{a}$ to $\frac{1}{nn}$ present as 1 to 9, which ratio in our experiment at this point can be ignored without risk
[i.e. here $a=36$ and $c=1$ and $\frac{c}{a}=\frac{1}{36}$; in the first case $n=\frac{1}{2}$ and $nn=\frac{1}{4}$, hence $\frac{c}{a}$ to $\frac{1}{nn}=\frac{1}{36}$ to $4=\frac{1}{9}$; note that Bernoulli is using diameters here and below]:

but if now you put the diameter of the tube to be twice the diameter of the opening, with three quarter parts of the whole aperture closed off, now there becomes
$n=4$ and $\frac{c}{a}$ to $\frac{1}{nn}$ as 4 to 9, which ratio will no longer be small enough, that the experiment may be able to be confirmed with sufficient precision with the conditions of the theory.

And thus here again it will be convenient to inquire, which of these cases it shall be required to put in place, in which $\frac{c}{a}$ to $\frac{1}{nn}$ may have a noteworthy ratio between each other, truly each quantity certainly shall be small, which happens without doubt when the cylinder is submerged the deepest, but likewise the base with a very small hole bored through.

§. 11. But that case which we have just considered, is better to be deduced from the differential equation of paragraph three, than from the integral, as done above: moreover it is possible under these circumstances to reject the term $-vdx$ before $nnvdx$, and thus to assume,

$$-xdv+nnvdx=\left(x-b\right)dx,$$

in which again if there is put $a-b=c$ and $a-x=z$, the equation is produced

$$adv-zdv+nnvdz=\left(c-z\right)dz,$$

the second term of which *zdv* again can be ignored before the first term, thus so that it becomes

$$adv-nnvdz=\left(c-z\right)dz.$$

Here there is put (by taking α for the number, of which the hyperbolic logarithm is one) $v=\frac{1}{nn}\alpha^{\frac{-nnz}{a}}q$; in this manner the last equation will be changed into this :

$$\alpha^{\frac{-nnz}{a}}adq=nn\left(c-z\right)dz,$$

or

$$adq=nn\alpha^{\frac{nnz}{a}}\times\left(c-z\right)dz.$$

However this is required to be integrated, so that *z* and *v* or also *z* and *q* vanish together ; therefore there will be had :

$$q=\left(c+\frac{a}{nn}-z\right)\alpha^{\frac{nnz}{a}}-c-\frac{a}{nn},$$

or finally

$$v=\frac{1}{nn}\left(c+\frac{a}{nn}-z\right)-\frac{1}{nn}\left(c+\frac{a}{nn}\right)\alpha^{\frac{-nnz}{a}}.$$

Truly from that equation there can be deduced :

I. It was found to arise again, by the other method as in paragraph ten, $v=\frac{2cz-zz}{2a}$,

namely if again the number $\frac{nnz}{a}$ may be put very small. Truly so that it may become apparent, it is required to resolve the quantity of the exponential $\alpha^{\frac{-nnz}{a}}$ into a series, which itself is equal to,

$$1-\frac{nnz}{a}+\frac{n^4z}{2aa}-\frac{n^6z^3}{2\cdot 3a^3}+\text{ etc.},$$

from which for our search the first three terms suffice ; but there with the value substituted and with the term rejected as required to be rejected, so that it is found as I said :

$$v=\frac{2cz-zz}{2a}.$$

II. But if in turn $\frac{nn}{1}$ may be put infinitely greater than $\frac{a}{z}$ or $\frac{a}{c}$, because then $\alpha^{\frac{-nnz}{a}}=0$, and so that $\frac{a}{nn}=0$, it is understood to become $v=\frac{c-z}{nn}$, or $v=\frac{x-b}{nn}$, as in § 4.

III. However neither of the previously presented formulas appear to have a place without a notable error, when the number $\frac{nnc}{a}$ lies in the middle, clearly neither infinitely large nor indefinitely small, and yet each quantity $\frac{nn}{1}$ and $\frac{a}{c}$ is infinite.

For example, were the height indicated by *c* of one inch, with the immersion of the cylinder *b*, 80 inches, and *a* itself 81 inches; then the diameter of the tube may be put to be three diameters of the opening, that is, $nn=81$, there will be $v=\frac{2-z-2\alpha^{-z}}{nn}$, and if again there may be put $z=c=1$, so that the height corresponding to the velocity may be found, when each surface is placed on the level, it will be $v=\frac{\alpha-2}{nn\alpha}$, that is, approximately $v=\frac{1}{307}\text{inch.}$, since following paragraph ten there ought to arise $v=\frac{1}{162}\text{inch.}$ and following paragraph four $v=0$. In the same example the whole interval arises, which the surface travels through, not entirely the eight-fifth part of one inch, and the position of the maximum velocity is about the sixty-ninth part of the same measure below the initial height.

§. 12. It shall not be more difficult to extend to the shapes of all vessels, which have been mentioned so far, indeed also to finite volumes, by which the external water may be determined : but the formulas generally thus become prolix, so that I will consider more prudently to pass by the same in silence, and perhaps to show the method by some particular example, so that the theory shall be applied to any others requiring to be elicited.

More particular attention is deserved, which I have indicated about the motion of water in the greatly enlarged lower openings of the deepest tubes, because in these the motion of the oscillations as in pendulums is of constant duration [*i.e*. period], and the flow of the waves in the sea is shown by these. But I have judged to treat first generally the reflux of water in submerged cylinders, and showing according to this hypothesis the reflux not to differ from the preceding flow, as the whole oscillatory motion may be examined. Therefore we will comment now on this reflux, thereafter each motion shall be combined in different cases, lest it should be desired somehow in a proof.

PART TWO.

Concerning the ascent of water.

§. 13. After the water has descended in a submerged vessel, how great an amount is let through depends on the nature of the experiment, two matters chiefly offer themselves for consideration ; in the first place the excess of the height of the external surface above the internal height and in the second place the *vis viva* of the product formed from the *ascent potential* by the mass of its water, which was ejected from the cylinder during the descent into the water at rest around the cylinder : for this *vis viva*, which it is not possible to return to the water in the cylinder, makes it most possible that much of the water shall be absent, so that less, than which fell in the first place can reach a height in the reflux : either this ratio is not yet unity, even if no impediments of retention or of adhesion may stand in the way, or of any other kind : another ratio was indicated in §. 2. However the measure of this ratio is required to be deduced from the ascent itself, since the former pertained to the descent only, with extrinsic impediments not considered, the reason being, because no water may be raised further above the level of the external surface in the ascent, than as far as the same was depressed below [in the descent]. For it is required to be observed, either to be about to happen, or with the water either flowing out through the smallest opening, so that it could ascend with the same velocity, or as if with all the base missing, it could burst out through the whole opening, but only after the impact of the influx, which was made on the internal water, the whole would be forced out moving to the same height: Truly whichever way this may depend on correctly it is seen easily, generally almost all that same impetus to be expended on some inner motion [of the fluid particles], which may not advance the ascent; moreover I say it may be noted generally (which I wish to note well) because when the opening is excessively large, it may be foreseen without difficulty, impetus of the water flowing in thus becomes fittingly, so that the inner motion thence may be advanced hardly at all; but on the other hand when the opening is small, it is clear, the matter may be had otherwise. Therefore our hypothesis will be used correctly, either when the base is completely absent, or almost the whole is the opening (thus indeed the excess of the velocity of the water flowing in over the internal velocity of the water is almost nothing, or it is extremely small, and that impetus has no effect on this ascent) or also when the opening is a minimum, because thus all of the impetus is overcome. But if the opening had a ratio to the cross-section of the tube, such as 1 to $\sqrt{2}$, or as 1 to 2, or thereabouts, the motion will be a little greater than which follows from this hypothesis, because then remarkable the impetus makes the water rush in, nor is all lost by the nature of the happening.

Therefore it is easy to foresee the following in the water without putting a calculation in place, after it had fallen from a certain height, effected by the reflux.

I. Evidently there shall be no measurable reflux, if the opening shall be very small.

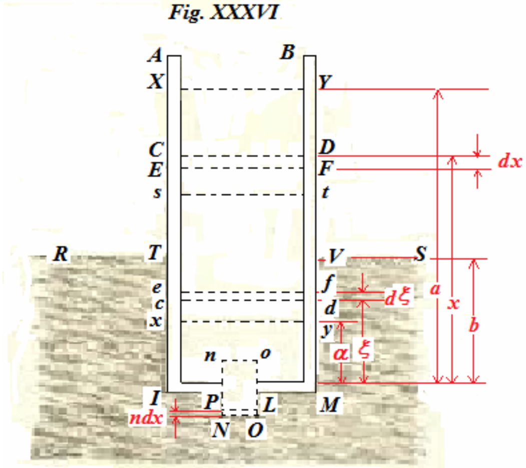
II. Since the submerged part of the cylinder shall remain unchanged, at no time shall the water in the reflux progress beyond a certain boundary, or if the water in a previous descent were elevated indefinitely: for at no time, from whatever height it may begin the descend, will all the water flow out of the cylinder, as we have seen in §§. 5 & 7.

III. When the descent may be understood to begin from the height *XY*, and subsequent ascent to be made as far as *CD*, the product of the *actual descent* of the mass of water *XYDC* as far as to *TV* by the mass, shall be a measure of the combined account of each, which, as §. 2 said, make the ascent to differ from the preceding descent, and since the account examined in the second place may vanish, if all the base *IM* may be taken away, then that product shall be equal to the *vi vivae* of all the water ejected during the descent, thus so that without any calculation, besides that now put in place so far, the ascent of the water in the whole open cylinder may be able to be defined.

IV. The ascent to be equal to the descent, when the cylinder is understood to be infinitely submerged, then with the aforementioned causes of diminution vanishing.

V. Hence therefore either the oscillations are to be without end, because the later oscillations always shall be, or as if infinitely small on account of the submerged height: but external impediments, of which we have given no account thus far, certainly acts so that all motion may stop quickly.

Fig. XXXVI



§. 14. From there general provisions, we will subject the problems to a more accurate calculation: moreover I will give a two-fold solution, the one according to the principles already put in place, the other to be of some different kind.

Therefore with both the diagram as well as the denominations retained as in §. 3, we will consider the water to be falling from the height *XY* as far as to *xy*, and from this terminus to begin its ascent; calling *My* or $Ix = \alpha$ and now after it has risen as far as to *cd* or *ef*, putting $Md = \xi,\ df = d\xi$. With these thus prepared for the calculation, and again with the height due to the velocity of the water designated by *v* at *cd*, and by $v + dv$ for the similar height at the nearby position *ef*, we may inquire into the *increment of the ascent potential* of the water approaching, while the droplet *LONP* enters the cylinder, and the surface rises from *cd* to *ef* ; but it is evident, since everywhere the *ascent potential* of the inner water multiplied by its mass may be expressed by $n\xi v$ (indeed without any attention paid to the internal motion [of the water particles]), to be the increment of the same product, $n\xi dv + nvd\xi$. However if besides we may consider the *ascent potential* $nnv - v$ (see §. 2), that the inflowing droplet loses $nd\xi$, and which equally is due to the *actual descent* of the water particles $nd\xi$ through the height $b-\xi$, it is apparent it is required to put

$$n\xi dv + nvd\xi + \left(nnv - v\right)nd\xi = (b-\xi)nd\xi ,$$

or,

$$\xi dv + nnvd\xi = (b-\xi)d\xi .$$

[*i.e*. the first equation is an expression for the conservation of potential plus kinetic energy of the droplet plus head of water, for the assumed speed of the droplet entering.]

However the same may be found thus. Clearly we may consider the velocity of the droplet *LONP* as if it were zero, before it may begin to flow in, and indeed to flow in and at once to acquire the ascent potential, which shall be $= nnv$ although soon after its influx (by the notes to sec. §. 2) it will be agreed the motion to continue with the common velocity $\sqrt{v}$. With which done thus the required reasoning will be: Before the influx of the drop the *ascent potential* of the water *cdMLPlc* (of which the mass $= n\xi$) $= v$, and the *ascent potential* of the drop *LONP* (of which the mass $= nd\xi$) $= 0$; therefore the *ascent potential* of all the water $cdMLONPIc = \dfrac{n\xi v}{n\xi + nd\xi} = \dfrac{\xi v}{\xi + d\xi}$.

But however after the droplet *LONP* flows in and *LonP* is put in place, it is assumed its *ascent potential* $= nnv$, but of the remaining water *efMLonPIe* (of which again indeed the mass $= n\xi$) the *ascent potential* is $= v + dv$ *;* therefore the *ascent potential* of all the water considered here after the influx of the droplet is :

$$\frac{nd\xi \times nnv + n\xi \times \left(v + dv\right)}{n\xi + nd\xi} = \frac{\xi v + \xi dv + nnvd\xi}{\xi + d\xi} ,$$

since before the same flux was $\dfrac{\xi v}{\xi + d\xi}$: therefore the increment is taken , $\dfrac{\xi dv + nnvd\xi}{\xi + d\xi}$ or more simply, $\dfrac{\xi dv + nnvd\xi}{\xi}$. Now this same increment is required to be equated to the *actual descent* which the water makes by changing its position *cdMLONPIc* to the situation *efMLPIe*, which descent is equal to the fourth proportion for the mass of the interior water $n\xi$, the drop $nd\xi$ and the height *Vf* or $b-\xi$, thus so that the aforementioned descent shall be $= \dfrac{(b-\xi)d\xi}{\xi}$: from which again such an equation is found :

$$\xi dv + nnvd\xi = (b-\xi)d\xi \;;$$

truly of which the integral after the addition of such a constant due becomes

$$v = \frac{b}{nn}\left(1-\left(\frac{\alpha}{\xi}\right)^{nn}\right) - \frac{1}{nn+1}\left(\xi - \left(\frac{\alpha}{\xi}\right)^{nn}\alpha\right),$$

as we may consider now according to the diverse circumstances of that.

§. 15. And indeed if the cross-section of the tube were infinitely greater than the cross-section of the opening, it is apparent the formula will become $v = \frac{b-\xi}{nn}$; and hence the water rushes in with a velocity which shall be due to the height of the external surface above the internal surface, nor then can it ascend above the external surface of the water.

However when the cross-section of the opening has a finite ratio to the cross-section of the tube, the ascent shall be beyond the surface *RS* or as far as to *st:* but *Vt* will be less always than *Vy*, except when all the base is missing, then indeed there will be $Vt = Vy$. Just as we have warned in §. 5, in the descent there shall be a proportional difference between *VY* and *Vy* and it must originate from the *ascent potential* of the water ejected during the descent; thus now it is possible to be observed in the ascent the difference between *Vy* and *Vt* to originate from the pushing of the droplet *LonP* against the mass of the water thrown up, which indeed may not be advanced by the force, but may be expended on the useless internal motion between the particles, as indicated in §. 2. Therefore when all the base *IM* is absent, the water enters the tube with the same velocity, which the water had before entering the tube and that shall happen without a collision, which is the reason why in that case the water may ascend just as far above the surface *RS*, as it would be depressed below that, which the equation indicates, as we shall see soon.

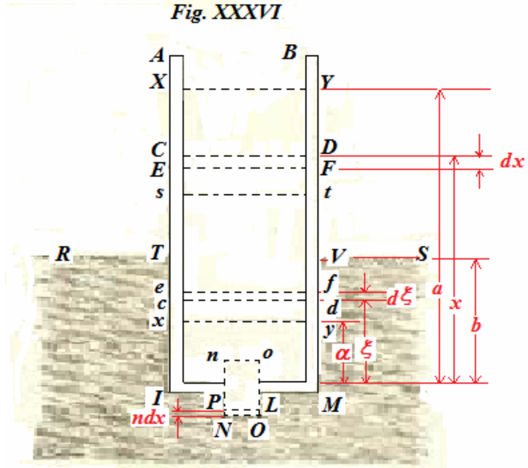
§. 16. The maximum ascent *st* will be found, by making $v = 0$. Therefore so that all the motion may be correctly defined, the formulas elicited in §§. 3 & 14 will be used alternately, so that now I will illustrate this by a single example, where $nn = 1$.

Hence if $nn = 1$, there becomes

$$v = b\left(1-\frac{\alpha}{\xi}\right) - \frac{1}{2}\left(\xi - \frac{\alpha\alpha}{\xi}\right),$$

and there will be $v = 0$, when there is taken $\xi = 2b - \alpha$, that is, when there is taken $Vt = Vy$. Therefore if, for the sake of an example, the tube *ABMI* filled with water and with the whole base missing were immersed as far as to the middle by the external water, and its whole length may be called *a*, thus the water

Fig. XXXVI



will be disturbed so that initially it may fall below *TV* by the interval 0,297*a*, then it will be raised by a similar interval above *TV*, and again depressed below that by the interval 0,240*a*,[actually 0,212*a* as noted by GKM] and that same line again crossed over, and so henceforth.

§. 17. It is apparent also when α is $=0$, evidently with the tube emptied of all water, [the above equation] to be generally either

$$v=\frac{b}{nn}-\frac{\xi}{nn+1}$$

and the whole ascent consequently to be $\frac{nn+1}{nn}b$, or the ascent above the exterior surface of the water $=\frac{b}{nn}$.

§. 18. I come now to tubes infinitely submerged, in which we have determined the descent and its conditions in §. 10. Moreover we have used the same clear method to define this case as we have used there: therefore the initial depression will be for us $Vy\ (=b-\alpha)=c$, thence the ascent made $yd(=\xi-\alpha)=z$. Thus there becomes $\xi=\alpha+z$ and $b=\alpha+c$, where the quantities z and c indeed are considered infinitely small on account of the quantity α. Hence there will be had :

$$\left(\frac{\alpha}{\xi}\right)^{nn}=\left(\frac{\alpha}{\alpha+z}\right)^{nn}=\left(1+\frac{z}{\alpha}\right)^{-nn}=$$

(by using the well known series expansion and taking the first three terms from that)

$$1-\frac{nnz}{\alpha}+\frac{nn\cdot\overline{nn+1}zz}{2\alpha\alpha}$$

With these values substituted for b, ξ and $\left(\frac{\alpha}{\xi}\right)^{nn}$, the final equation of paragraph fourteen

$$\text{[i.e. } v=\frac{b}{nn}\left(1-\left(\frac{\alpha}{\xi}\right)^{nn}\right)-\frac{1}{nn+1}\left(\xi-\left(\frac{\alpha}{\xi}\right)^{nn}\alpha\right),\text{]}$$

will be changed into this,

$$v=\frac{\alpha+c}{nn}\times\left(\frac{nnz}{\alpha}-\frac{nn\times\overline{nn+1}zz}{2\alpha\alpha}\right)-\frac{1}{nn+1}\times\left(\alpha+z-\alpha+nnz-\frac{nn\times\overline{nn+1}zz}{2\alpha}\right)$$
$$=(\alpha+c)\times\left(\frac{z}{\alpha}-\frac{\overline{nn+1}zz}{2\alpha\alpha}\right)-\left(z-\frac{nnzz}{2\alpha}\right)$$
$$=\frac{cz}{\alpha}-\frac{zz}{2\alpha}-\frac{\overline{nn+1}czz}{2\alpha\alpha}:$$

But it is possible to ignore this final term, and thus the equation becomes more simply :

$$v=\frac{2cz-zz}{2\alpha},$$

as n is no longer present in the equation: Nor is that different from the equation given for the descent in §.10, surely $v=\frac{2cz-zz}{2a}$, when indeed the quantities a and α only differ by the small quantity 2c.

Here also all the remaining are understood partially, concerning the same tube without undue obstruction, which have been discussed in §. 10.

§. 19. Therefore the descents and ascents are equal to each other ; for it is apparent from our equations, the liquid to be equally balanced on the other side of the surface of the external water. Then truly it follows chiefly from these formulas, the oscillations to be unequal among the isochrones themselves, but all shall be considered to be infinitely small on account of the submersion: Moreover the simple tautochronous pendulum to be of the same length as the submerged part of the tube.

This same theorem differs from that, which was cited in §. 4 Sect. VI concerning the oscillations in a cylindrical tube composed from two vertical legs, in that, because there all the oscillations shall be tautochronous without excluding oscillations of finite magnitude, since in the present case the finite oscillations shall be of unequal duration ; then because there the length of the pendulum shall be equal to half the length of the tube, as here it shall be equal to the whole length, although if the matter may be considered correctly, here there shall be agreement rather than disagreement being discussed on account of the tube, which in the former case is doubled

§. 20. Each of the oscillations generated is illustrated by the nature of the waves disturbed by the wind: nor indeed will they be moved, than because the water in these continually shall ascend and descend again. Thus it is apparent what Newton said, the times of he oscillations to be in the half ratio of the lengths of the waves [*i.e.* as the square root of the wavelength; now known to be true only for waves in very deep water], because he put the shape of the waves to be constantly similar and hence the lengths of these to be proportional to the depth, to what the water is disturbed. But it is plausible that to be the depth, which a simple pendulum tautochronous with the waves, certainly for example

$60\frac{1}{3}$ *Paris ft.*, if an ascent or descent of the waves of one at a time may be made in two second intervals.

§. 21. Although I may have been unwilling, to avoid the prolixity of the calculation, to follow this argument to its full extent, and on that account I have performed the calculations only for cylindrical vessels, yet because in the case of infinite submersion the pronouncements and theorems lose little of their great ingenuity, I may add on a general theorem generally for the oscillations of water in whatever kind of unequal tubes, with the demonstration only omitted, which will be remedied by what has been said elsewhere, especially from those matters set out in Sect. VI §§. 6, 7 etc, as far as § 20. But it is required to be made, that the upper part of the vessel in which the movements are made, shall be of a cylindrical construction.

§. 22. Therefore let *bd* be the length of the vessel submerged (Fig. 35*b*). *bF* may represent its cross-section at the position of the surface, and thus the vessel formed is put in place, so that the curve *FGH* shall be the scale of the cross-section: the line *bc* may be taken and the curve becomes *LMN*, of which the applied line everywhere *cM* shall be $=\frac{bF^2}{cG}$, and the length of the pendulum isochronous with the oscillations of the water surface shall be = volume *bdNL* divided by *bL*.

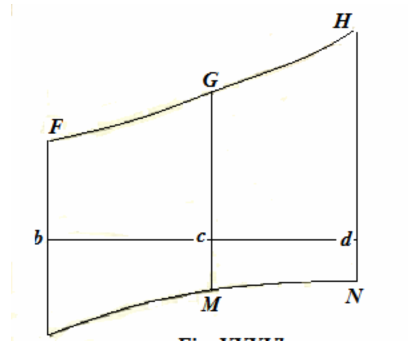


Fig. XXXVb

Corollary.

§. 23. It follows from the preceding paragraph, if the submerged tube were a cone, and it should have a cross-section in the neighborhood of the water surface, which shall be to the submerged opening as *m* to *n*, the length of the isochronous pendulum with the vibrating water to the length of the submerged tube to be, as $\sqrt{m}$ to $\sqrt{n}$, that is , as the roots of the aforementioned cross-sections, and if the tube may be placed in the correct manner likewise in the inverted manner only not totally submerged, the lengths of the isochronous pendulums to be in the contrary ratio of the submerged openings.

General Scholium.

§. 24. Because the matters encountered in this section generally depend on new hypothesis, there will be a greater need to test these by experiments. Indeed I have put in place different experiments, not trivial but individual ones that I had considered to carry out according to a plan: what I have done I will review below. Meanwhile so that a safer comparison can be brought to bear regarding the agreement of experiments with theory, initially it will be required to see clearly with regard to the circumstances of the properties, as well as to what extent the contraction of the flowing jet shall disturb the calculation (the nature of which I have established in Sect. IV) : because the greater part of the inconvenience can be removed, if it can be arranged that the sides of the lower opening may form some small cylinder, scarcely of half a line of height [i.e. $\frac{1}{12}^{\text{th}}$ of an inch], whereby regarding the matter, the fourth experiment pertaining to section four may be recalled. Then also it is required to extend our thoughts to the resistance arising from the adhesion of the water, which indeed slows down the motion a little, if you consider the times of the oscillations, moreover they will detract much from excursions of the water, especially if the tubes may be taken narrower and longer. Therefore there will be required to be more faith with the experiments, which were made near the times of the oscillations, because these times certainly were not diminished much by the excursions of the water. In the account of the first kind of experiments, where the excursions of the fluid in the tubes, both descending as well as ascending have come to be inquired into and to be observed, here I have used careful consideration, so that I would wind a thread around the tube in that place, where I was expecting the water either was going to descend or to ascend, likewise the thread after many repetitions of the experiment I have finally located thus, so that the surface of the oscillating fluid would extend neither nearer nor farther. Also the remaining places, which were observed in the tube, I noted equally by a thread wound around the tube. Because then it pertained to the time of the oscillation, because these decreased most quickly and became imperceptible and clearly were zero, that not to be able to be sought other than by finding after many runs of the experiment the length of the simple isochronous pendulum, which while it was swinging a finger was placed on the opening of the tube and that removed from there at the precise instant of time, so that the pendulum and the fluid would begin to oscillate at the same time.

Experiments referring to Section seven.

Experiment 1.

I used a cylindrical glass tube of almost four lines diameter, [equal to] the whole aperture below. With the water standing in the widest clear vessel, I submerged the tube to a depth of 44 *lines.* and I moved a finger over the top of the opening, lest by pulling out the tube a part should drop into that water : then I extracted to tube to a height of 22 *lines,* thus so that both a part of the tube should be submerged, as well as the height of the inner water should be 22 *lin.* above the external water, and soon with the finger removed I observed the descent of the surface in the tube, and that was seen to be $9\frac{1}{2}$ *lin.* below the surface of the still water.

But according to §.7 there should be a fall of thirteen lines ; the deficiency of three and a half may be considered to be attributed to the adhesion of the water to the wall of the tube.

I repeated the whole experiment with the descent observed, so that I could test the approximate ascent : Moreover this was apparent to me to be 8 *lin.,* which according to paragraph sixteen, by having with respect to the previous ascent, ought to have been $9\frac{1}{2}$ *lines.*, surely as great an amount as the preceding descent. However there the experiment was deficient by only one and a half lines, since in the first part of the experiment it had been deficient to the extent of three and a half lines, because doubtless the greater the displacement made and that with a greater velocity, thus so that the impediments which increase together with the velocity, certainly would be come upon to be greater.

Experiment 2.

I used with the same tube, but there strengthened by a plate, which had been bored with an opening in the ratio of cross-section $\sqrt{\frac{1}{2}}$ to the cross-section of the tube ; when the surface of water in the tube was raised eighteen lines above the surface of the still water, and submerged to just as many lines, I saw the surface of the water in the tube to be falling almost five lines below the still water. However paragraph eight argues a descent of $7\frac{1}{2}$ *lines;* the deficiency, which was more than $2\frac{1}{2}$ *lines*, again I consider to the adhesion of the water to the walls of the tube.

Then I sent the tube to a depth of 18 *lines* with the same plate attached with a finger placed on top, completely empty of water: with the finger removed the water rose by eight whole lines, with §.17 indicating nine lines for this case.

Because this difference certainly was smaller, than in the descent, on account of the reason I have ascribed, as I have shown at length in paragraph thirteen, since I may say the motion arising to be a little greater, since the cross-section of the opening with respect to the tube notably to be had as in the ratio $\sqrt{\frac{1}{2}}$ to 1, or almost to be had, as what follows from the hypothesis: and thus concerning this matter it certainly becomes clear, the tube used to be shorter and wider, so that the effect of almost all the other impediments can be removed, and I have seized upon the following experiment.

Experiment 3.

I used a tube the diameter of which was more than seven lines, which I had taken care to have made from iron, because a good enough glass cylinder was not at hand: its length was four inches and six and a half lines: its cross-section to the ratio of the opening was indicated by $n = 1{,}860$ and $nn = 3{,}458$. With this tube I undertake this experiment thus :

Evidently with the top end stopped up I tested repeatedly, to what depth it was required to be submerged in the greatest extent of still water, so that with the extended finger removed, which stopped up the opening, water would rise to precisely the edge of the same orifice, and would flow beyond no further. Truly I found this depth to be 3 inches and three lines*;* therefore the ascent above the external water was of one inch and three and a half lines, indeed since with all the impediments removed it ought to be able to ascent to a little beyond eleven lines according to paragraph 17. Therefore §.13 advised correctly, that the ascent can be a little greater in cases of this kind than the hypothesis postulates. Soon I attached another base to that tube; now there was $n = 3{,}68,$ *and* $nn = 13{,}54$: it was with difficulty to ascertain the success of this experiment correctly, because the surface in the ascending tube was with bubbles always: yet it was seen, now the tube was required to be immersed to a depth of 4 inches with two or three lines, thus with the water remaining outside around four lines, in short as the theory indicates.

Experiment 4.

A cylindrical glass tube, which had a diameter of around three lines, I immersed to a depth of 20 inches and I made, so that the water in that would be balanced with the water first raised to a height of around one inch. It could not make more than four or five noteworthy complete swings to and fro, nor thus was I able with any rigor to measure the length of the simple isochronous pendulum ; yet that was seen by me to be 22 or 23 inches; from which I inferred the adhesion of the water to the walls of the tube not only diminished the displacements but also delayed the times of the oscillations a little : for following §. 19 it ought to have been made by the aforementioned length of only twenty inches. I have found the same in oscillations, which we have handled in the above section.

The rest blocked off, with the lower orifice almost reduced to half, I was unable to observe whether the displacements thence were to be diminished or the oscillations retarded, which agrees with these, which may be found in §.7 & §.18.

Experiment 5.

I immersed a conical tube with a length of 21 inches with the wider end opening to the water, thus so that a single inch might project from the water: moreover the other opening was a little greater than twice the former. I found the length of the pendulum isochronous with the vibrations of the water in the tube to be found to balance at fifteen inches ; but following §.23 it must be the same with the length a little less than fourteen inches. Finally likewise with the same tube used, but in the inverted position, the length of the isochronous pendulum was required to be taken a little more than twice that, which it was before, just as is indicated in the paragraph cited.

HYDRODYNAMICAE SECTIO SEPTIMA

De motu aquarum per vasa submersa, ubi exemplis ostenditur, quam insigniter utile sit principium conservationis virium vivarum, vel iis in casibus, quibus continue aliquid de illis perdi censendum est.

PARS PRIMA.

De descensu aquarum.

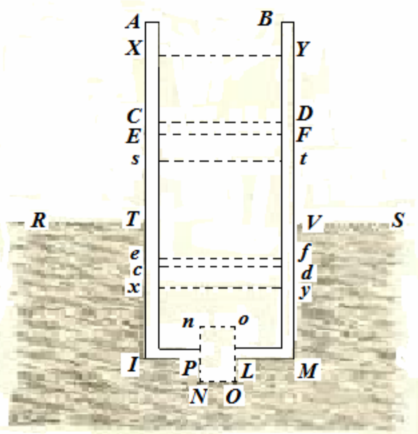
§. 1. Finge cylindrum aqua plenum, cujus fundum perforatum sit, illudque ad certam altitudinem aquae stagnanti veluti infinitae submersum, & facile intelliges superficiem aquae in cylindro contentae descensuram, & quidem infra superficiem aquae exterioris, dein rursus ascensuram & sic porro. Hae vero oscillationes admodum differunt ab oscillationibus in praecedente sectione consideratis, in quibus nempe motus reciproci semper sunt inverso ordine iidem cum motibus, qui praecesserunt. Quis autem hic praesumat refluxum aquarum seu ascensum eundem fore, qui fuerat descensus? Talia si quis statueret, is certe vehementer falleretur, etiamsi vel nihil motus diminuatur ab adhaesione aquarum ad latera vasis hujuscemodique aliis impedimentis, non secus atque regulae motuum a percussione pro corporibus elasticis valde diversae sunt ab iis, quae pro corporibus mollibus valent, utut in utroque casu corpora liberrime moveri censeantur. Utor hoc simili, quod argumentum nostrum egregie illustrat: Prouti enim regulae motuum in corporibus mollibus recte determinantur, si post collisionem ea *vis vivae* pars deperdita censeatur, quae in compressionem corporum impensa fuit (neque enim haec ut in corporibus elasticis restituitur motui progressivo); ita ascensus fluidi non minus recte definietur, si accurate examinetur, quantum *vis vivae* singulis momentis motui particularum aquearum intestino communicetur, nunquam rediturum ad motum progressivum, de quo sermo est.

§. 2. Cum itaque res eo deducta sit, ut exploretur, quantum *vis vivae* in motibus istis reciprocis continue perdatur, disquisitionem ab hoc incipiemus.

Primo autem patet omnem *vim vivam* quae particulis effluentibus inest transire ad aquam externam nec ullo modo promovere subsequentem ascensum seu influxum aquae externae in tubum: Nimis haec est clara hypothesis, quam ut majori explicatione opus habeat: respicit autem aquarum effluxum & in hoc unica est consideranda. Venit jam altera, quae pertinet ad aquarum influxum.

Secundo igitur non minus perspicuum mihi quidem est, quod irruente aqua per foramen majori velocitate, quam quae aquae internae ascendenti inest, excessus ille rursus motum quendam intestinum in eadem aqua interna cieat, parum aut nihil ad ascensum conferentem. Hoc si ita sit, ponaturque amplitudo foraminis $=1$, amplitudo cylindri $=n$, *ascensus potent.* guttulae irrumpentis $=nnv$, ejusque velocitas $=n\sqrt{v}$, retinebit haec particula motu suo, quem cum reliqua aqua interna communem habet, velocitatem $\sqrt{v}$, conservabitque proinde *ascensum potent.* v ; reliquum autem *ascensum potent.* nempe $nnv-v$ ad motum particularum intestinum transiisse censendum est. Hypothesis ista, quamvis Physica sit & proxime tantum vera, tamen magnam habet utilitatem ad motus fluidorum sine notabili errore determinandos, quoties in vase uniformis continuitas, quae hactenus assumta fuit, praerumpitur, vel uti cum aqua per plura foramina transire cogitur; imo crediderim unicam esse, cujus ope hujusmodi motus mira phaenomena recte explicari possint. Quapropter velim, ut recte animo perpendatur, antequam ad alia divertatur lector.

Fig. XXXVI



§. 3. Jam igitur quaestionem ipsam examinabimus, incipiendo ab aquarum descensu. Concipiatur cylindrus *AIMB* (Fig. 36), aqua plenus usque in *XY* & aquae infinitae *RTVS* submersus, ita ut longitudo ejus sit in situ verticali; habeat ejus fundum lumen *PL*, per quod aqua ex vase in aquam circumfluam effluere possit. Quaëritur velocitas aquae internae, postquam superficies ejus per datum spatium *XC* vel *YD* descendit, posita *MY* vel $IX=a$, $MV=b$, $MD=x$, amplitudine foraminis $=1$, & denique amplitudine cylindri $=n$.

Solutio eadem erit, quam pro simili quaestione, sed ea admodum generali, dedimus in sectione tertia: observetur tantum, quod sumta particula aquae infinite parva *CDFE* aequali guttulae *PLON* eo ipso tempore ejectae, *descensus actualis* sit nunc aestimandus ex altitudine *DV* vel *CT*, cum in altero casu definiendus erat ex tota altitudine *DM*.

Sit nempe velocitas superficiei aqueae *CD* ea, quae debetur altitudini *v*, & in situ infinite propinquo *EF* respondebit eadem velocitas altitudini $v-dv$; et cum *ascensus potentialis* aquae *CDMLPIC* sit *v*, obtinebitur *ascensus potent.* ejusdem aquae in situ proximo *EFMLONPIE*, si multiplicetur massa *EFMLPIE* $(nx-ndx)$ per suum *ascensum potent.* $(v-dv)$, ut etiam guttula *LONP* (ndx) per suum itidem *ascensum potentialem* *nnv*, aggregatumque productorum dividatur per summam massarum (*nx*): habetur itaque iste *ascensus potentialis*

$$=\frac{(nx-ndx)\times(v-dv)+ndx\times nnv}{nx}$$

seu

$$\frac{xv-vdx-xdv+nnvdx}{x}.$$

Est proinde incrementum *ascensus potent.*

$$=\frac{-vdx-xdv+nnvdx}{x}$$

(conf. §. 6 Sect. III). Istud vero incrementum aequale censendum est cum *descensu actuali* infinite parvo, qui (per§. 7 Sect. III & per annotationem modo datam) est $=\frac{(x-b)dx}{x}$. Habetur itaque talis aequatio

$$-vdx-xdv+nnvdx=\left(x-b\right)dx\,,$$

quae debito modo integrata mutatur in hanc

$$v=\frac{1}{nn-2}\times\left(x-\frac{x^{nn-1}}{a^{nn-2}}\right)-\frac{b}{nn-1}\times\left(1-\frac{x^{nn-1}}{a^{nn-1}}\right).$$

Ex ista vero aequatione talia sequuntur corollaria.

§. 4. Fuerit amplitudo cylindri veluti infinita ratione foraminis, & erit censendum $v=\frac{x-b}{nn}$; ipsaque altitudo pro velocitate aquae, dum effluit, est $=x-b$. Unde consequens est, aquam effluere velocitate, quam grave acquirit cadendo ex altitudine superficiei internae supra externam, & eo usque effluet, donec ambae superficies sint ad libellam positae, tuncque omnis motus cessabit: adeoque eadem lege aquae effluunt, qua si fundum situm *IM* mutaret cum *TV*.

Cum vero foramen non potest ceu infinite parvum considerari, descendit superficies aquae internae infra externam; atque ut innotescat ad quamnam profunditatem *xy* sit descensura superficies *CD*, facienda est $v=0$, seu

$$\left(nn-1\right)\left(a^{nn-1}x-x^{nn-1}a\right)=\left(nn-2\right)\times\left(a^{nn-1}b-x^{nn-1}b\right);$$

nunquam autem superficies interna tantum descendet infra superficiem externam, quantum super eandem elevata fuerat; provenit iste defectus ab *ascensu pot.* aquae durante descensu ejectae, cui debet esse proportionalis.

§. 5. Notabile est, quod cum eo profundius descendat aqua in cylindro, quo magis ab initio descensus fuerit elevata & quo majori lumine perforatum est fundum, nunquam tamen omnis aqua ex cylindro effluere possit quantumvis fuerit ante descensum elevata & pars cylindri submersa utlibet parva, ipsumque simul foramen vel totum fundum exhaurire ponatur.

§. 6. Velocitas superficiei aquae internae maxima est, cum sumitur

$$x=\left(\frac{a^{nn-1}}{nna-nnb-a+2b}\right)^{1:(nn-2)}.$$

Si proinde $n=1$, existente scilicet orificio cylindri toto aperto, fit $x=b$, & maxima est velocitas, cum ambae superficies sunt in eadem altitudine positae. Quia vero multa sunt, quae ex hisce aequationibus dignosci nequeunt in duobus casibus, nempe $nn=1$ & $nn=2$, hique multa habent particularia, eosdem seorsim jam attingam.

§. 7. Sit primo $nn=1$, & erit $-xdv=(x-b)dx$ (per§. 3) vel $-dv=dx-\frac{bdx}{x}$, quae sic integrata, ut sit simul $v=0$ & $x=a$,

dat

$$-v=x-a+b\log\frac{a}{x},$$

seu

$$v=a-x-b\log\frac{a}{x}.$$

Exinde talia deduci possunt.

I. Ut obtineatur maximus descensus, faciendum est $a-x-b\log\frac{a}{x}=0$; patet autem ex ista aequatione, nunquam negativum valorem obtinere litteram *x*, imo nequidem totam evanescere sine contradictione, nisi ponatur $\frac{a}{b}=\infty$, quod indicat fieri non posse, ut omnis effluat aqua durante descensu in isto casu & multo minus in reliquis, quod confirmat paragraphum quintum.

II. Velocitas maxima talis est, quae debetur altitudini $a-b-b\log\frac{a}{b}$, atque si differentia inter *a* & *b*, quam ponam $=c$, sit valde parva, existentibus nimirum excursionibus fluidi perexiguis ratione longitudinis, ad quam cylindrus est submersus, poterit $\log\frac{a}{b}$ censeri $=\frac{c}{b}-\frac{cc}{2b}$ ipsaque proinde altitudo maximae debita velocitati seu $a-b-b\log\frac{a}{b}=\frac{cc}{2b}$ quod motum admodum lentum fore arguit.

Demonstrabo autem in sequentibus, totum motum caeteris paribus eundem manere, cum cylindri censentur infinite submersi, quocunque foramine fundum fuerit perforatum, ita ut motus aquae internae a diminuto foramine non retardetur; quod quamvis prima fronte admodum paradoxum videatur, non poterit tamen vera ejus ratio physica effugere animum ad haec attentiorem. In eo scilicet versatur, quod *vis viva*, quae in tubo generatur, veluti infinita sit prae vi viva aquae per foramen transeuntis nec adeoque hujus foraminis consideratio computum diversum faciat.

Demonstrabimus etiam similes esse motus reciprocos & oscillationes tam majores quam minores inter se esse Isochronas, atque pro hisce longitudinem penduli simplicis tautochroni determinabimus.

§. 8. Fuerit nunc $nn = 2$; ita vero habetur vi §. 3 $vdx - xdv = (x-b)dx$, vel

$$\frac{xdv - vdx}{xx} = \frac{(b-x)dx}{xx},$$

quae recte integrata abit in hanc

$$v = \frac{bx}{a} - b + x\log\frac{a}{x}.$$

Si fiat $\frac{bx}{a} - b + x\log\frac{a}{x} = 0$, dabit *x* locum maximi descensus; locus autem maximae velocitatis habebitur, faciendo $x = c^{\frac{b-a}{a}}a$, ubi per *c* intelligitur numerus, cujus logarithmus est unitas. Postquam sic varios perstrinximus casus pro diversis foraminum magnitudinibus, superest ut etiam consideremus, quid in diversis altitudinum *a* & *b* casibus succedere possit.

§. 9. Et primo quidem si *b* nulla statuatur prae *a*, quod fit cum cylindri fundum tantum radit superficiem aquae exterioris, tunc prodit

$$v = \frac{1}{nn-2}\left(x - \frac{x^{nn-1}}{a^{nn-2}}\right),$$

quae quidem aequatio non nisi forma differt ab illa, quae §.14 Sect. III data fuit pro eo casu, quo aquae ex cylindro in aërem ejici ponuntur. Et saepe etiam expertus sum cylindrum eodem tempore evacuari, sive aquae in aërem ejiciantur, sive fundum aquae stagnanti tantillum submergatur. Docet haec experientia parum aut nihil obstare aërem externum effluxui, cum resistentia plus quam octingenties major notabiliorem effectum non exerat. Quia adeoque iste casus nihil particulare habet, quod non loco citato monitum fuerit, huic non ulterius immorabimur: Inquiremus potius, quid fieri debeat, cum elevatio aquae internae super externam, quanta ab initio descensus est, sumitur valde parva & negligenda prae immersione cylindri; cui hypothesi satisfit, cum excessus altitudinis *a* super altitudinem *b* (quem excessum rursus vocabimus (ut§. 7) *c*) est admodum parvus.

§.10. Cum itaque ponitur $a - b = c$, ponendum etiam erit $a - x = z$, tumque utraque quantitas, nempe *c* & *z,* erunt negligendae prae quantitatibus *a* & *b,* sed si $a - x = z$, erit $x = a - z$ &

$$x^{nn-1} = (a-z)^{nn-1}$$
$$= a^{nn-1} - (nn-1)a^{nn-2}z + \left(\frac{\overline{nn-1}\cdot\overline{nn-2}}{2}\right)a^{nn-3}zz$$
$$-\left(\frac{\overline{nn-1}\cdot\overline{nn-2}\cdot\overline{nn-3}}{2\cdot 3}\right)a^{nn-4}z^3 + \text{etc.}$$

Haec series quantum ad institutum nostrum sufficit est continuanda; sufficiet autem ad tres usque terminos. Igitur in aequatione integrata quam dedimus §. 3 ponemus, $x = a - z$ *&*

$$x^{nn-1} = a^{nn-1}$$
$$= a^{nn-1} - (nn-1)a^{nn-2}z + \left(\frac{\overline{nn-1}\cdot\overline{nn-2}}{2}\right)a^{nn-3}zz$$
$$-\left(\frac{\overline{nn-1}\cdot\overline{nn-2}\cdot\overline{nn-3}}{2\cdot 3}\right)a^{nn-4}z^3 + \text{etc.}$$

sic erit

$$v = \frac{1}{nn-1}\left(a - z - a + (nn-1)z - \left(\frac{\overline{nn-1}\cdot\overline{nn-2}}{2}\right)\frac{zz}{a}\right)$$
$$-\frac{b}{nn-1}\left(1 - 1 + (nn-1)\frac{z}{a} - \left(\frac{\overline{nn-1}\cdot\overline{nn-2}}{2}\right)\frac{zz}{aa}\right).$$

In qua aequatione si termini se destruentes deleantur, atque ponatur $a-c$ pro *b*, rejiciaturque terminus qui affectatur quantitate $\frac{czz}{aa}$, prodit simpliciter

$$v = \frac{2cz - zz}{2a},$$

ex qua formula, cum littera *n* evanuerit, indicium habemus, nihil magnitudinem orificii pertinere ad motum aquae internae, cujus rei originem jam supra (§. 7) indicavi.

In sequentibus autem demonstrabimus, non differre hunc motum a subsequente motu refluo, hincque oscillationes fieri tautochronas. Priusquam vero ad alia pergam monendum duxi, in isto calculo quantitates $\frac{c}{a}$ & $\frac{z}{a}$ non solum prae unitate, sed & prae $\frac{1}{nn}$ ceu infinite parvas positas fuisse, ad quod animus probe est ad vertendus in instituendis experimentis; licet utique theoriam infinite parvorum ad experimenta sine notabili errore revocare diminuendo admodum quantitates, quae in theoria ceu infinite parvae consideratae fuerunt, sed faciendum est, ut in experimento omnia huic legi sint subjecta. Ita v. gr. si in cylindro omne fundum absit, posito $n=1$, idque submersum ponatur ad altitudinem triginta quinque pollicum, satis accurate sumetur experimentum, cum aqua ante oscillationes elevata tantum fuerit ad altitudinem unius pollicis supra superficiem aquae circumfluae; nec dum error notabilis erit, si vel orificium inferius ad dimidium obstruatur existente tunc $\frac{c}{a}$ ad $\frac{1}{nn}$ ut 1 ad 9, quae ratio in nostro experimento tuto adhuc negligi potest: at si jam diametrum tubi duplam ponas diametri orificii, occlusis tribus quartis aperturae integrae partibus, jam fiet $n=4$ & $\frac{c}{a}$ ad $\frac{1}{nn}$ ut 4 ad 9, quae ratio non satis parva amplius erit, ut experimentum conditionibus theoriae cum sufficienti praecisione satisfacere affirmari possit.

Hic itaque jam porro inquirere conveniet, quid de his casibus statuendum sit, quibus $\frac{c}{a}$ ad $\frac{1}{nn}$ notabilem quidem inter se habent rationem, utraque vero quantitas sit admodum exigua, quod nimirum fit, cum cylindrus profundissime submergitur, simul autem fundum parvulo est pertusum foramine.

§. 11. Sed iste, quem modo finximus, casus melius ex aequatione differentiali paragraphi tertii, quam ex integrali, ut antea factum, deducitur: potest autem pro his circumstantiis rejici terminus $-vdx$ prae $nnvdx$, atque sic assumi,

$$-xdv+nnvdx=(x-b)dx,$$

in qua si rursus ponitur $a-b=c$ & $a-x=z$, prodit

$$adv-zdv+nnvdz=(c-z)dz,$$

cujus secundus terminus $z\,dv$ rursus prae primo negligi potest, ita vero habetur

$$adv-nnvdz=(c-z)dz.$$

Ponatur hic (sumto α pro numero, cujus logarithmus hyperbolicus est unitas) $v=\frac{1}{nn}\alpha^{\frac{-nnz}{a}}q$; hoc modo mutabitur postrema aequatio in hanc

$$\alpha^{\frac{-nnz}{a}}adq=nn(c-z)dz,$$

vel

$$adq=nn\alpha^{\frac{nnz}{a}}\times(c-z)dz.$$

Haec vero ita est integranda, ut z & v vel etiam z & q simul evanescant; habebitur igitur

$$q=\left(c+\frac{a}{nn}-z\right)\alpha^{\frac{nnz}{a}}-c-\frac{a}{nn},$$

vel denique

$$v=\frac{1}{nn}\left(c+\frac{a}{nn}-z\right)-\frac{1}{nn}\left(c+\frac{a}{nn}\right)\alpha^{\frac{-nnz}{a}}.$$

Ex ista vero aequatione deducitur:

I. Oriri rursus, ut paragrapho decimo alia methodo inventum fuit, $v=\frac{2cz-zz}{2a}$, si nempe rursus ponatur $\frac{nnz}{a}$ numerus valde parvus. Id vero ut pateat, resolvenda est quantitas exponentialis $\alpha^{\frac{-nnz}{a}}$ in seriem, quae est ipsi aequalis,

$$1-\frac{nnz}{a}+\frac{n^4z}{2aa}-\frac{n^6z^3}{2\cdot 3a^3}+\text{ etc.},$$

ex qua pro nostro scopo tres priores termini sufficiunt; eo autem substituto valore rejectoque termino rejiciendo, reperitur ut dixi

$$v=\frac{2cz-zz}{2a}.$$

II. At si vicissim $\frac{nn}{1}$ infinities major ponatur quam $\frac{a}{z}$ aut $\frac{a}{c}$, quia tunc $\alpha^{\frac{-nnz}{a}}=0$, ut & $\frac{a}{nn}=0$, fiere intelligitur $v=\frac{c-z}{nn}$, sive $v=\frac{x-b}{nn}$, ut § 4.

III. Neutram vero praemissarum formularum sine notabili errore locum habere patet, cum $\frac{nnc}{a}$ numerus est mediocris, nempe nec infinitus, nec infinite parvus, & tamen utraque quantitas $\frac{nn}{1}$ & $\frac{a}{c}$ infinita.

Fuerit v. gr. elevatio indicata per *c* unius pollicis, immersio cylindri *b* 80 *poll.,* ipsaque *a* 81 *poll.;* dein ponatur diameter tubi tripla diametri foraminis, id est, $nn=81$, erit $v=\frac{2-z-2\alpha^{-z}}{nn}$, atque st porro ponatur $z=c=1$, ut habeatur altitudo velocitatis, cum utraque superficies est ad libellam posita, erit $v=\frac{\alpha-2}{nn\alpha}$, id est, proxime $v=\frac{1}{307}$ *poll.*, cum secundum paragraphum decimum debuisset oriri $v = \frac{1}{162}\, poll.$ & secundum paragraphum quartum $v = 0$. In eodem exemplo fit spatium integrum, quod superficies percurrit, non omnino octo quintarum partium unius pollicis, locusque maximae velocitatis est praeterpropter sexaginta novem centesimarum partium ejusdem mensurae infra altitudinem initialem.

§. 12. Non difficilius esset ad omnes vasorum figuras extendere, quae hactenus dicta sunt, imo etiam ad spatia finita, quibus aqua externa determinetur: fiunt autem formulae plerumque adeo prolixae, ut consultius duxerim easdem silentio praeterire, & specimine saltem aliquo particularem ostendere modum, quo theoria ad quoslibet casus alios eruendos applicanda sit.

Attentionem particulariorem merentur, quae de motu aquarum in tubis inferius largiter apertis, & profundissime submersis indicavi, quia in his motus oscillatorius, ut in pendulis, constantis durationis est, & undarum in mari fluxus illustratur ab illis. Existimavi autem prius de refluxu aquarum in cylindris submersis generaliter tractandum esse, atque ostendendum in ista hypothesi refluxum non differre a praecedente fluxu, quam motus totus oscillatorius examinetur. Jam igitur de isto refluxu commentabimur, deinceps utrumque motum in diversis casibus combinaturi, ne aliquid in argumento desiderari possit.

PARS SECUNDA.

De ascensu aquarum.

§. 13. Postquam aquae descenderunt in vase submerso, quantum id ipsis natura rei permittit, duo potissimum consideranda se offerunt; primo excessus altitudinis superficiei externae supra internam & secundo *vis viva* seu productum ex *ascensu potentiali* in massam illius aquae, quae ex cylindro in aquam circumstagnantem durante descensu ejecta fuit: haec enim *vis viva*, quae redire non potest ad aquam in cylindro, facit potissimum ut aquae multum absint, quo minus pristinam, ex qua ceciderant, in refluxu attingant altitudinem: nec tamen unica est haec ratio, etiamsi vel nihil obstent impedimenta tenacitatis, adhaesionis, hujusmodique alia: altera ratio indicata fuit §. 2. Istius vero rationis mensura ex ipso ascensu est deducenda, cum prior ad descensum pertineat & sola, abstrahendo animum ab impedimentis extrinsecis, in causa est, cur non aqua in ascensu tantum supra superficiem externam elevetur, quantum infra eandem depressa fuerat. Notandum enim est, futurum fuisse, aquis vel per minimum foramen influentibus, ut eadem velocitate ascenderent, tanquam si omne fundum deesset, plenoque orificio irrumperent, si modo post influxum impetum, quem in aquas internas faciunt, totum exererent ad earum ascensum promovendum: Verum quicunque hanc rem recte perpendit facile videt, plerumque impetum istum totum fere impendi in motum aliquem intestinum, qui nihil ascensum promoveat; dico autem notanter plerumque (quod bene notetur velim) quia cum foramen magnum admodum est, non difficulter praevidetur, impetum aquarum influentium ita apte fieri, ut motus internus haud parum inde promoveatur; at cum foramen minus est, liquet, rem secus se habere. Recte igitur adhibetur hypothesis nostra, cum vel fundum omne abest, aut fere totum est perforatum (sic enim excessus velocitatis aquae influentis supra velocitatem aquae internae nullus, aut valde exiguus est, & nullum illa in hanc impetum facit) vel etiam cum foramen minimum est, quia sic omnis impetus infringitur. Sed si foramen rationem habuerit ad amplitudinem tubi, veluti ut 1 ad $\sqrt{2}$, vel ut 1 ad 2, aut circiter, major paululum erit motus quam qui ex ista hypothesi sequitur, quia tunc notabilem impetum faciunt aquae irruentes, nec is omnis per rei naturam perditur.

Facile igitur est sine instituto calculo praevidere sequentes in aquarum, postquam ex certa altitudine delapsae fuerunt, refluxu affectiones.

I. Nullum nempe fore refluxum sensibilem, si foramen sit valde parvum.

II. Cum pars cylindri submersa non mutata maneat, nunquam aquas in refluxu certum terminum praetergressuras, si vel in infinitum elevatae fuerint aquae in praevio descensu: nunquam enim, ex quacunque altitudine incipiat descensus, omnes aquae ex cylindro effluunt, ut vidimus§§. 5 & 7.

III. Cum descensus incipere intelligatur ab altitudine *XY*, subsequensque ascensus fieri usque in *CD*, fore productum *descensus actualis* massae aquae *XYDC* usque ad *TV* in massam, mensuram rationis utriusque combinatae, quae, ut§. 2 dictum, ascensum a praecedente descensu differre faciunt, & cum ratio secundo loco recensita evanescat, si omne auferatur fundum *IM*, fore tunc istud productum aequale *vi vivae* omnis aquae, durante descensu ejectae, ita ut sine alio calculo, praeter hactenus jam positos, ascensus aquarum in cylindro toto aperto definiri possit.

IV. Ascensum fore aequalem descensui, cum cylindrus infinite submersus intelligitur evanescentibus tunc praefatis diminutionis causis.

V. Hinc igitur oscillationes sine fine fore, quia postremae oscillationes semper sint vel uti infinite parvae ratione submersionis altitudinum: faciunt autem impedimenta aliena, quorum nullam hucusque rationem habuimus, ut omnis motus cito admodum cesset.

§. 14. His generatim praemonitis, problema accuratiori calculo subjiciemus: duplicem autem dabo solutionem, alteram ad principia modo exposita accommodatam, alteram specie quodammodo diversam.

Igitur retentis tum figura, tum denominationibus §. 3 considerabimus aquam ex altitudine *XY* descendisse usque in *xy*, & ab hoc termino ascensum suum inchoare; dicatur *My* vel $Ix = \alpha$ & postquam jam ascendit usque ad *cd* vel *ef*, ponatur $Md = \xi,\ df = d\xi$. His ita ad calculum praeparatis, designataque rursus per *v* altitudine debita velocitati aquae in *cd* & per $v + dv$ simili altitudine in situ proximo *ef*, inquiremus in *incrementum ascensus potentialis* aquae accedens, dum cylindrum subit guttula *LONP*, superficiesque ex *cd* ascendit in *ef*; perspicuum autem est, cum ubique *ascensus potent.* aquae internae multiplicatus per suam massam exprimatur per $n\xi v$ (nec enim ulla attentio adhibenda est ad motum intestinum), fore ejusdem producti incrementum $n\xi dv + nvd\xi$. Si vero praeterea consideretur *ascensus potent.* $nnv - v$ (vid. §. 2), quem guttula influens $nd\xi$ perdit, quique pariter debetur *descensui actuali* particulae aqueae $nd\xi$ per altitudinem $b - \xi$, patet esse ponendum

$$n\xi dv + nvd\xi + (nnv - v)nd\xi = (b-\xi)nd\xi,$$

vel,

$$\xi dv + nnvd\xi = (b-\xi)d\xi.$$

Idem vero aliter sic invenitur. Consideretur scilicet guttulae *LONP* qua si nullam velocitatem fuisse, priusquam influere inciperet, eandem vero statim atque influere incipiat, acquirere *ascensum potentialem,* qui sit $= nnv$, quamvis mox post sui influxum (per annot. sec. §. 2) censenda sit motum continuare velocitate communi $\sqrt{v}$. Quo facto sic erit ratiocinandum. Ante influxum guttulae est *ascensus potent.* aquae *cdMLPlc* (cujus massa $= n\xi$) $= v$, & *ascens. potent.* guttulae *LONP* (cujus massa $= nd\xi$) $= 0$; ergo *ascensus potentialis* ommis aquae $cdMLONPIc = \frac{n\xi v}{n\xi + nd\xi} = \frac{\xi v}{\xi + d\xi}$.

At vero postquam guttula *LONP* influxit situmque assumsit *LonP*, est ejus *ascens. potent.* $= nnv$, reliquae autem aquae *efMLonPIe* (cujus quidem massa rursus $= n\xi$) *ascensus potent.* est $= v + dv$*;* igitur *ascensus potent.* omnis aquae hic consideratae post influxum guttulae est

$$\frac{nd\xi \times nnv + n\xi \times (v+dv)}{n\xi + nd\xi} = \frac{\xi v + \xi dv + nnvd\xi}{\xi + d\xi},$$

cum ante eundem fluxum fuerit $\frac{\xi v}{\xi + d\xi}$: cepit igitur incrementum , $\frac{\xi dv + nnvd\xi}{\xi + d\xi}$ vel simplicius $\frac{\xi dv + nnvd\xi}{\xi}$. Istud vero incrementum aequandum est cum *descensu actuali* quem aqua facit mutando situm *cdMLONPIc* situ *efMLPIe*, qui descensus aequalis est quartae proportionali ad massam aquae internae $n\xi$, ad guttulam $nd\xi$ & altitudinem *Vf* vel $b - \xi$, sic ut praefatus descensus sit $= \frac{(b-\xi)d\xi}{\xi}$: unde iterum habetur talis aequatio

$$\xi dv + nnvd\xi = (b-\xi)d\xi;$$

hujus vero integralis post debitae constantis additionem talis fit

$$v = \frac{b}{nn}\left(1 - \left(\frac{\alpha}{\xi}\right)^{nn}\right) - \frac{1}{nn+1}\left(\xi - \left(\frac{\alpha}{\xi}\right)^{nn}\alpha\right),$$

quam nunc pro diversis ejus circumstantiis perpendemus.

§. 15. Et quidem cum fuerit amplitudo tubi infinities major quam amplitudo foraminis , patet fieri $v = \frac{b-\xi}{nn}$; & irruere proinde aquam velocitate quae debeatur altitudini superficiei externae super internam, neque tunc ultra superficiem aquae externae fiet ascensus.

Cum vero amplitudo foraminis rationem habet finitam ad amplitudinem tubi, ascensus fit ultra superficiem *RS* veluti usque in *st:* minor autem semper erit *Vt* quam *Vy,* nisi cum omne fundum abest, tunc enim erit $Vt = Vy$. Prouti monuimus §. *5* in descensu differentiam inter *VY* & *Vy* proportionalem esse & originem debere *ascensui potentiali* aquae durante descensu ejectae, ita nunc observari potest in ascensu differentiam inter *Vy* & *Vt* originem habere ab illisione guttularum *LonP* in massam aquae superjacentis, quae quidem illisio non promovet ascensum, sed in inutilem motum intestinum impenditur, prouti indicatum fuit §. 2. Ergo cum omne fundum *IM* abest, aqua tubum eadem velocitate ingreditur, qua jam gaudet aqua tubum antea ingressa & nulla fit collisio, quae causa est cur in isto casu tantum ascendat aqua ultra superficiem *RS*, quantum fuerat infra illam depressa, quod aequatio, uti mox videbimus, indicat.

§. 16. Determinabitur maximus ascensus *st,* faciendo $v = 0$. Igitur ut motus omnis recte definiatur, alternatim adhibendae erunt formulae §§. 3 & 14 erutae, quod nunc hoc unico illustrabo exemplo, quo $nn = 1$.

Si proinde $nn = 1$, fit

$$v = b\left(1 - \frac{\alpha}{\xi}\right) - \frac{1}{2}\left(\xi - \frac{\alpha\alpha}{\xi}\right),$$

eritque $v = 0$, cum sumitur $\xi = 2b - \alpha$, id est, cum sumitur $Vt = Vy$. Igitur si verbi gratia tubus *ABMI* aqua plenus omnique fundo destitutus fuerit ad medietatem usque immersus aquae exteriori, atque tota ipsius longitudo dicatur *a*, aqua sic agitabitur ut primo infra *TV* descendat spatio 0,297*a*, deinde simili spatio super eandem *TV* elevetur, rursusque infra eam deprimatur spatio 0,240*a*, eodemque lineam illam iterum transcendat, & sic porro.

§. 17. Patet etiam cum α est $= 0$, tubo scilicet ab omni aqua vacuo, fore generaliter

$$v = \frac{b}{nn} - \frac{\xi}{nn+1}$$

ascensumque integrum consequenter fore vel ascensum $\frac{nn+1}{nn}b$ supra superficiem exteriorem aquae $= \frac{b}{nn}$.

§. 18. Venio nunc ad tubos infinite submersos, in quibus descensum cum suis affectionibus determinavimus §. 10. Utemur autem eadem plane methodo ad hunc casum definiendum qua ibi usi sumus: erit nobis igitur depressio initialis $Vy\ (= b - \alpha) = c$, ascensus inde factus $yd\,(= \xi - \alpha) = z$. Sic est $\xi = \alpha + z$ & $b = \alpha + c$, ubi quantitates *z* & *c* sunt ceu infinite parvae considerandae ratione quantitatis *a*. Habetur hinc

$$\left(\frac{\alpha}{\xi}\right)^{nn} = \left(\frac{\alpha}{\alpha+z}\right)^{nn} = \left(1+\frac{z}{\alpha}\right)^{-nn} =$$

(adhibendo seriem notam & ex illa sumendo tres primos terminos)

$$1-\frac{nnz}{\alpha}+\frac{nn\cdot\overline{nn+1}zz}{2\alpha\alpha}$$

Substitutis istis valoribus pro b, ξ & $\left(\frac{\alpha}{\xi}\right)^{nn}$, mutatur aequatio ultima paragraphi decimi quarti in hanc,

$$v=\frac{\alpha+c}{nn}\times\left(\frac{nnz}{\alpha}-\frac{nn\times\overline{nn+1}zz}{2\alpha\alpha}\right)-\frac{1}{nn+1}\times\left(\alpha+z-\alpha+nnz-\frac{nn\times\overline{nn+1}zz}{2\alpha}\right)$$
$$=(\alpha+c)\times\left(\frac{z}{\alpha}-\frac{\overline{nn+1}zz}{2\alpha\alpha}\right)-\left(z-\frac{nnzz}{2\alpha}\right)$$
$$=\frac{cz}{\alpha}-\frac{zz}{2\alpha}-\frac{\overline{nn+1}czz}{2\alpha\alpha}:$$

Potest autem negligi iste ultimus terminus & sic fit simpliciter

$$v=\frac{2cz-zz}{2\alpha},$$

quam aequationem *n* non amplius ingreditur: Neque illa differt ab aequatione pro descensu §.10 data, nempe $v=\frac{2cz-zz}{2a}$, quandoquidem quantitates *a* & α non differunt nisi quantitate minima 2*c*.

Caeterum hic omnia etiam sunt subintelligenda, quae eodem §. 10 de tubo non nimis obstruendo dicta sunt.

§. 19. Sunt igitur descensus & ascensus sibi aequales; nam ex aequationibus nostris patet, liquorem aequaliter librari ultra superficiem aquae externae. Deinde vero potissimum sequitur ex istis formulis, esse vel oscillationes inaequales inter se isochronas, modo omnes possint infinite parvae censeri ratione submersionis: Pendulum autem simplex tautochronum esse ejusdem longitudinis cum parte tubi submersa.

Differt istud theorema ab illo, quod§. 4 Sect. VI de oscillationibus in tubo cylindrico ex duobus cruribus verticalibus composito citatum fuit, in eo, quod ibi oscillationes omnes non exclusis oscillationibus finitae magnitudinis sint tautochronae, cum in praesenti casu oscillationes finitae sint inaequalis durationis; deinde quod ibi longitudo penduli sit aequalis dimidiae longitudini tubi, cum hic sit aequalis integrae, quamvis si recte res perpendatur, hic potius sit consensus quam dissensus dicendus ob tubi, quae in priori casu est, duplicationem.

§. 20. Utroque oscillationum genere illustratur natura undarum vento agitatarum: neque enim aliter moventur, quam quod aquae in illis continue ascendant rursusque descendant. Ita patet quod dicit Newtonus, tempora undulationum esse in ratione dimidiata latitudinum undarum, quia ponit undarum formam sibi constanter esse similem & proinde earum latitudinem proportionalem profunditati, ad quam aquae agitantur. Verisimile autem est profunditatem eam esse, quae pendulo simplici cum undis tautochrono, nempe v. gr. $60\frac{1}{3}$ *ped. Paris.* si singulis binis minutis secundis fiat undarum ascensus descensusve.

§. 21. Quamvis noluerim, ad prolixitatem calculi evitandam, hoc argumentum in omni sua extensione prosequi, propterque ea de cylindricis vasis tantum egerim, attamen quia in casu submersionis infinitae enunciationes & theoremata parum de sua concinnitate perdunt, superaddam theorema generale pro oscillationibus aquae in tubo utcunque inaequali, omissa tamen demonstratione, quae ex alibi dictis unicuique obvia erit, praesertim vero ex iis quae in Sect. VI§§. 6, 7 & seqq. usque ad 20 exposita fuerunt. Faciendum autem est, ut cylindricae sit structurae pars illa vasis superior, in qua excursiones fiunt.

§. 22. Fuerit igitur *bd* longitudo vasis submersi (Fig. 35b). Repraesentet *bF* ejus amplitudinem in loco superficiei, ponaturque vas ita formatum, ut sit curva *FGH* scala amplitudinum: sumatur linea *bc* fiatque curva *LMN*, cujus applicata *cM* sit ubique $=\frac{bF^2}{cG}$, & erit longitudo penduli isochroni cum oscillationibus aqueae superficiei = spatio *bdNL* diviso per *bL*.

Corollarium.

§. 23. Ex praecedente paragrapho sequitur, si tubus submersus conicus fuerit, habeatque amplitudinem in regione aquae superficiei, quae sit ad orificium submersum ut *m* ad *n,* fore longitudinem penduli Isochroni cum vibrante aqua ad longitudinem submersi tubi, ut $\sqrt{m}$ ad $\sqrt{n}$, id est, ut radices praedictarum amplitudinum, atque si tubus idem situ modo recto modo inverso submergatur tantum non totus, fore longitudines pendulorum isochronorum in ratione contraria orificiorum submersorum.

Scholium Generale.

§. 24. Quae in hac sectione continentur, quia novis hypothesibus innituntur pleraque, eo magis operae pretium erit experimentis tentare. Ego quidem diversa institui, non vacavit autem singula quae mente conceperam exequi: quae feci inferius recensebo. Interim ut tutius judicium ferri possit de consensu experimentorum cum theoria, dispiciendum prius erit pro rerum circumstantiis, an & quantum fere contractio venae effluentis (cujus naturam exposui in Sect. IV) calculum turbare possit: quod incommodum maxima parte tolli poterit, si fiat ut orificii inferioris latera parvulum aliquem cylindrum efforment, vix dimidiae lineae altitudinis, qua de re animo revolvatur experimentum quartum ad sectionem quartam pertinens. Deinde etiam animus advertendus ad resistentias ab adhaesione aquae oriundas, quae quidem parum retardant motus, si tempora oscillationum respicias, multum autem excursionibus detrahunt, praesertim si tubi strictiores & longiores sumantur. Igitur magis fidendum erit experimentis, quae circa oscillationum tempora facta fuerint, quia haec tempora a diminutione excursionum non multum admodum alterantur. Ratione primi experimentorum generis, quo excursiones fluidorum in tubis, tam descensus quam ascensus inquirendi observandique veniunt, hac usus fui circumspectione, ut filum tubo circumvolverem eo in loco, ad quem aquas descensuras vel ascensuras esse expectabam, idemque filum post saepe repetitum experimentum ita tandem locavi, ut superficies fluidi oscillantis nec ultra nec citra excurreret. Reliqua etiam loca, quae in tubo observanda erant, pariter filo circumvoluto notavi. Quod deinde ad tempora oscillationum pertinet, quia hae citissime decrescunt fiuntque imperceptibiles & plane nullae, non potui illa aliter inquirere, quam explorando post saepissime iteratum experimentum longitudinem penduli simplicis isochroni, quod dum oscillabat digitum orificio tubi superimposui eumque eo praecise temporis puncto removi, ut & pendulum & fluidum oscillationem simul inciperent.

Experimenta ad Sectionem septimam referenda.

Experimentum 1.

Tubum adhibui vitreum cylindricum diametri fere quatuor linearum, inferius totum apertum. Cum aquae, in vase pellucida amplissimo stagnanti, submersi ad altitudinem *44lin.* digitumque orificio admovi superno, ne extrahendo tubi partem descenderet in illo aqua: extraxi deinceps tubum ad alt. *22lin.* ita ut tam pars tubi submersa, quam altitudo aquae internae supra externam esset *22lin.* moxque remoto digito observavi descensum superficiei in tubo infra superficiem aquae stagnantis eumque vidi fuisse $9\frac{1}{2}$ *lin.*

Debuisset autem vi §. 7 descendere tredecim lineis; defectus trium linearum cum dimidia unice fere adhaesioni aquae ad latera tubi tribuendus videtur.

Observato descensu totum experimentum repetii, ut ascensum quoque proximum experirer: Visus autem mihi fuit *8lin.*, qui vi paragraphi decimi sexti, habito respectu ad praevium descensum, esse debuerat $9\frac{1}{2}$ *lin.*, nempe tantus, quantus fuit praecedens descensus. Hic vero experimentum unica tantum linea cum dimidia defecit, cum in prima experimenti parte ad tres usque lineas cum dimidia defectus adfuit, quia nimirum major ibi facta fuit excursio eaque velocitate majori, ita ut impedimenta, quae una cum velocitatibus crescunt, admodum majora offenderit.

Experimentum 2.

Eodem tubo usus sum, sed eo lamina munito, quae foramine erat pertusa amplitudine $\sqrt{\frac{1}{2}}$ ratione amplitudinis tubi; cum superficies tubi esset octodecim lineis elevata supra aquam stagnantem, totidemque lineis fundum submersum, vidi superficiem tubi in descensu quinque fere lineis infra aquam stagnantem descendisse. Paragraphus octavus autem descensum arguit $7\frac{1}{2}$ *lin.;* defectum, qui plusquam $2\frac{1}{2}$ *lin.* fuit, rursus adhaesioni aquae ad latera tubi adscribo.

Deinde tubum hunc eadem lamina instructum admoto superius digito aquae immisi ad profunditatem 18 *lin*. totum ab aqua vacuum: remoto digito emersit superficies tubi supra aquam stagnantem integris octo lineis, cum §.17 earum novem indicat pro isto casu.

Quod hic defectus minor admodum fuerit, quam in descensu, rationi adscripsi, quam prolixe paragrapho decimo tertio indicavi, cum dicerem motum paullo majorem oriturum, cum foramen amplitudinem respectu tubi notabilem veluti in ratione $\sqrt{\frac{1}{2}}$ ad 1, aut circiter habuerit, quam qui ex hypothesi sequitur: atque ut ea de re certus plane fierem, tubum adhibui breviorem & ampliorem, ut omnis fere impedimentis alienis effectus praeriperetur, & experimentum cepi, quod sequitur.

Experimentum 3.

Tubum adhibui cujus diameter erat plus quam septem linearum, quem ex ferro confieri curavi, quia vitreus bene cylindricus non fuit ad manus: longitudo ejus fuit quatuor pollicum cum sex lineis & semisse: amplitudo ejus ratione foraminis indicata per *n* fuit $=1,860$ & $nn=3,458$. De isto tubo experimentum ita sumsi:

Obturato scilicet orificio superiori identidem tentavi, ad quam profunditatem submergendus esset aquae in area amplissima stagnanti, ut remoto protinus digito, qui orificium obtegebat, aqua ad limbum ejusdem orificii praecise ascenderet, nihilque praeterflueret. Istam vero profunditatem expertus sum 3 *poll.* cum tribus *lineis;* fuit igitur ascensus supra aquam externam unius pollicis & trium linearum cum dimidia, cum vel omnibus remotis impedimentis parum ultra undecim lineas ascensus fieri debuerit vi paragraphi 17. Recte igitur praemonitum fuit §. 13, non posse non ascensus fieri paullo majores in istiusmodi casibus, quam hypothesis postulat. Mox eidem tubo aliud applicui fundum; erat jam $n=3,68,$ & $nn=13,54$: difficile fuit experimenti successum recte dignoscere, quia superficies in tubo ascendens semper fuit bullata: visum tamen fuit, tubum nunc immergendum fuisse ad altitudinem 4 *poll.* cum duabus tribusve lineis, manentibus sic extra aquam praeterpropter quatuor lineis, prorsus ut theoria indicat.

Experimentum 4.

Tubum cylindricum vitreum, qui tres praeterpropter lineas habebat in diametro, immersi ad altitudinem 20 *poll.* fecique, ut aqua in illo libraretur, elevata prius aqua ad altitudinem unius fere pollicis. Ultra quatuor vel quinque itus reditusque bene notabiles non fecit, nec adeoque omni rigore longitudinem penduli simplicis isochroni examinare potui; mihi tamen illa visa fuit 22 aut 23 pollicum; ex quo intuli adhaesionem aquae ad latera tubi non solum diminuere excursiones, sed & morari paulisper tempora oscillationum: debuisset enim secundum §. 19 esse praefata longitudo viginti tantummodo pollicum. Idem expertus sum in oscillationibus, quas in superiori sectione pertractavimus.

Caeterum obturato vel ad dimidium fere orificio inferiori, observare non potui, excursiones inde fuisse diminutas aut oscillationes retardatas, quod conforme est cum iis, quae § §. 7 & 18 habentur.

Experimentum 5.

Tubum conicum longitudine 21 *poll.* immersi aquae orificio ampliore, ita ut unicus pollex extra aquam emineret: fuit autem alterum orificium alterius paululum plusquam duplum. Longitudinem penduli isochroni cum vibrationibus aquae in tubo libratae inveni quindecim *poll.;* debuisset autem secundum §. 23 esse eadem longitudo paullo minor quatuordecim pollicibus. Denique similiter eodem tubo usus, sed situ inverso, deprehendi longitudinem penduli isochroni tantillo plusquam duplam ejus, quae antea fuerat, prouti citato paragrapho indicatur.