HYDRODYNAMICS SECTION SIX.

Concerning fluids not flowing out, or moving within the walls of the vessels.

§. 1. Up to this stage we have examined water flowing out ; now however we will consider water flowing, which does not flow beyond the boundaries of the vessel. All these motions can be reduced to two kind, both to be dealt with in turn:

1. When the fluid is moving continually towards the same end of an infinitely long tube.
2. When it may be disturbed by reciprocal or oscillatory motions.

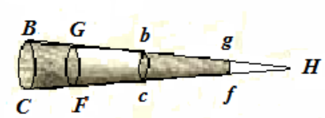
The motion of water along indefinitely long channels.

Case I.

§. 2. Initially the channel shall be put horizontally, but with a cross-section given changing according to some law : fluid may be thus placed into that tube, which usually happens in narrower tubes, so that both the end surfaces may be held perpendicular to the axis of the channel, and thus begins to move with a certain given velocity. if these shall be thus, and clearly no obstructions to the motion may be imagined to be present, for the motion of the water to be without end, just as a ball on a horizontal table may continue to progress freely without end. Yet a conspicuous difference intercedes between each motion: truly all the parts of the ball are progressing continually with the same velocity [only if it is sliding with zero friction!], but in water the parts are changing motion continually : Neither will it be difficult to define this motion, since we will be considered the motion to be such that the total ascent potential of the water will remain the same, which the motion was from the start : But we have determined the *ascent potential* of water moved in a channel with a certain velocity in the second section of paragraph three. Therefore nothing further is left for the solution of the question : nor again will more than one or two examples from this matter be brought forwards.

Example 1.

For example let *BgfC* (Fig. 31) be a channel which may have the figure of a truncated cone ; the part of this *BGFC* is understood to be filled with fluid in motion towards *gf;* and the particles of fluid at *GF* may have a velocity corresponding to the height *v;* and finally the fluid arrives at the position *bgfc.* With these in place the velocity of the fluid at *gf* is sought. But I will call the height owed to the velocity of the water at $gf = V$; the vertex of the cone shall be at *H;* the diameter at $BC = n$; the diameter at $GF = m$; with the length $BG = a$; $Gg = b$, the diameter $gf = \frac{ma + mb - nb}{a}$.

Fig. XXXI


[*i.e.* $n : m = BH : GH;\ n - m : m = a : GH;$

$\therefore GH = ma : n - m = \frac{ma}{n-m}$

$m : gf = GH : gH \quad \therefore gH = \frac{a \times gf}{n-m};$

$GH : gH = m : gf \quad \therefore b : gH = m - gf : gf \quad \therefore \frac{b(n-m)}{a} = m - gf$

$\therefore gf = \frac{ma + mb - bn}{a}$.]

Then because the volume *BGFC* is equal to the volume *bgfc*, there will be

$$BC^2 \times BH - GF^2 \times GH = bc^2 \times bH - gf^2 \times gH;$$

from which

$$bc^2 \times bH = BC^2 \times BH - GF^2 \times GH + gf^2 \times gH;$$

however,

$bH = \frac{BH}{BC} \times bc$; therefore

$$bc^3 = BC^3 - \frac{GF^2 \times GH \times BC}{BH} + \frac{gf^2 \times gH \times BC}{BH} = BC^3 - GF^3 + gf^3,$$

or

$$bc = \sqrt[3]{n^3 - m^3 + \left(\frac{ma + mb - nb}{a}\right)^3}.$$

Indeed by §.3 Sect. III, the *ascent potential* in the situation *BGFC*

$$= \frac{3m^3 v}{n(mm + mn + nn)};$$

and equally the *ascent potential* of the same water in the situation *bgfc* found

$$= \frac{3\alpha^3 V}{\beta(\alpha\alpha + \alpha\beta + \beta\beta)},$$

therefore for brevity on putting α & β for the values found for the diameters *gf* & *bc*. Therefore there will be

$$V=\frac{m^3\times(\alpha\alpha+\alpha\beta+\beta\beta)\times\beta\times v}{\alpha^3\times(mm+mn+nn)n}.$$

It is easily deduced from this formula, for the anterior particles to be moving forwards with a greater velocity, the latter ones with a smaller velocity, and thus, so that if the small opening *gf* may be considered to be infinitely small, the velocity of the water at *gf* becomes infinite & at *bc* infinitely small.

Example 2.

A channel was composed from two cylindrical tubes *BN* & *OP* (Fig. 32) with unequal cross-sections; in the greater branch the fluid *BGFC* was put to be moving towards *P* with a velocity which corresponded to the height *v*. Thus it is evident that no change in the motion to have been present before the surface *GF* arrived at *MN;* but from this instant of time the motion continually varied while the fluid all had entered into the narrower tube. Therefore with the fluid held in the position *bgfc*, it is asked what the velocity of the surface *fg* shall become; moreover we will designate the height corresponding to this velocity by *V.*

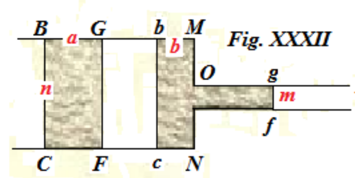


Fig. XXXII

The diameters *GF* & *gf* shall be as *n* & *m:* the length *BG* may be called $=a,\ bM=b$; there will be $Og=\frac{nn}{mm}\times(a-b)$; the *ascent potential* of the water $BGFC=v$;the *ascent potential* of the water *bgfc* therefore

$$=\frac{n^4a-n^4b+m^4b}{n^4a}\times V\ ;$$

therefore

$$V=\frac{n^4a}{n^4a-n^4b+m^4b}v\,.$$

[*i.e.* from §.3, Sect. III $v=\frac{N}{M}V$, where $N=\int_0^{x_{out}}\frac{f_{out}^2}{f}dx$ and $M=\int_0^{x_{out}}fdx$. In this case, $f=f_n=\pi n^2$ in the length *b*, and $f=f_m=f_{out}=\pi m^2$ in the narrower tube. Hence, as these are constant quantities, the integrals become :

$$N = f_{out}^2 / f_n \times b + f_{out}^2 / f_m \times Og = \pi(m^4 / n^2) \times b + \pi m^2 \times \frac{nn}{mm} \times (a-b) = \frac{\pi}{n^2}(m^4 b + n^4 a - n^4 b);$$

$$M = \pi n^2 \times b + \pi m^2 \times \frac{nn}{mm} \times (a-b) = \pi n^2 \times b + \pi n^2 \times (a-b)$$

Hence, $v = \frac{N}{M}V = \frac{\frac{\pi}{n^2}(m^4 b + n^4 a - n^4 b)}{\pi n^2 \times b + \pi n^2 \times (a-b)} = \frac{m^4 b + n^4 a - n^4 b}{n^4 a} \times V.$ This account has followed that of KF essentially.]

From these it is understood the velocity of the first drops bursting out into the narrower tube correspond to the height $\frac{n^4}{m^4}v$, however this velocity decreases quickly, thus so that after a very small amount of fluid had flowed through, it shall be possible now to be agreed, $V = \frac{a}{a-b}v$, and when all the fluid has flowed across, it assumes the former velocity. For example, if the diameter of the greater tube were ten times the other; and the first drops flowed out from the greater tube into the narrower one with a velocity corresponding to a height of 10 000*v*: however if now you put a tenth part of the fluid to have been transferred, you find the height, which may agree with the velocity of the fluid moving along in the narrower tube, to be approximately equal to 10*v*.

[*i.e.*
$$V = \frac{n^4 a}{n^4 a - n^4 b + m^4 b}v = \frac{n^4}{n^4 - n^4 b / a + m^4 b / a}v = \frac{10^4}{10^4 - 10^4 \times 9/10 + 9/10}v \simeq \frac{1}{1 - 9/10} = 10v.]$$

If you seek the time, so that the transfluxion of the fluid *Of* may come about, you will find that to equal

$$\frac{2(n^4 a - n^4 b + m^4 b)^{\frac{3}{2}} - 2m^6 a\sqrt{a}}{3mm(n^4 - m^4)\sqrt{av}}.$$

[The time is found as before by integrating the relation $dt = \frac{dx}{\sqrt{V}}$, where we are using 'natural units', where $b < x < a$ and the time progresses from zero when $x = a$ until the final value when $x = b$; thus, for the intermediate values of *x*, we have

$Og' = y = \frac{nn}{mm} \times (a-x)$. Hence, $dt = \frac{dx}{\sqrt{V}} = -\frac{nn}{mm\sqrt{V}}dy = -\frac{\sqrt{n^4 a - n^4 y + m^4 y}}{m^2\sqrt{av}}dy;$

Hence,

$$t=-\frac{1}{m^2\sqrt{av}}\int\sqrt{n^4a-n^4y+m^4y}dy=-\frac{1}{m^2\sqrt{av}}\int\sqrt{n^4a+\left(m^4-n^4\right)y}dy$$

$$=-\frac{2}{3\left(m^4-n^4\right)m^2\sqrt{av}}\left[\left(n^4a+\left(m^4-n^4\right)y\right)^{\frac{3}{2}}\right]_0^b$$

$$=\frac{2(n^4a-n^4b+m^4b)^{\frac{3}{2}}-2m^6a\sqrt{a}}{3mm(n^4-m^4)\sqrt{av}}.]$$

Therefore all the fluid has flowed across in the time

$$\frac{2n^6a\sqrt{a}-2m^6a\sqrt{a}}{3mm(n^4-m^4)\sqrt{av}}=\frac{2(n^4+mmnn+m^4)a}{3mm(nn+mm)\sqrt{v}},$$

where by $\frac{a}{\sqrt{v}}$ is understood the time, in which the fluid in the wider tube completes the distance *a*. These however, as I have said, thus will themselves be had only if there shall be no impediments to the motion, and likewise in the whole treatment of the channel taken together the velocities may be placed in inverse proportion to the cross-sections. Now meanwhile I have advised elsewhere that the water in the vicinity of the side *MN* is unable to observe this law. And thus when such a case occurs, there the real motion will agree more with the theory, so that the part *bM* were longer and so that the fewer obstacles would be present.

§. 3. Because if now the channel were not horizontal but placed obliquely to the horizontal, everything would itself appear to be kept similarly, except that the *ascent potential* of the water in any situation shall be equal to the initial increase of the actual descent, that is, to the required vertical descent of the centre of gravity. And if by no impulse the water by itself may begin by its own accord to move, the *actual descent* simply will put equal to the *ascent potential.*

Therefore the water will continue to progress, as long as the centre of gravity is put in a lower place, than it was from the start of the motion. But truly since the tube thus were formed and inflexible and filled with a quantity of fluid, so that the centre of gravity were able to reassume the initial height, then the motion of the fluid would be obtained backwards and it will oscillate without end. Soon we will be talking about the particular motion by making it a part of this section itself. Meanwhile it is permitted to observe, if it is possible, that all the water from the lower place may flow past higher by its own accord without previous suction, but only if all matters may themselves be had in the manner due.

Concerning the oscillations of fluids in curved tubes

Case II.

§. 4. My father gave certain theorems in vol. 2 of the *Comm. Acad. Scient. Petrop.*, [CP II, 1727 (1729)] which showed a significant use which the theory of living forces has in mechanical matters. Truly that found in the third place thus is as follows :

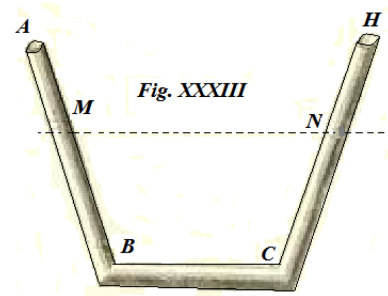


Fig. XXXIII

ABCH shall be a cylindrical tube (Fig. 33) *with an opening at each end and bent into two legs BA & CH to the horizontal part BC; the sine of the angle ABC* $= p$ *, and the sine of the angle* $HCB = q$ *, doubtless with the total sine being* $=1$; *again this tube shall be filled with water as far as the horizontal line MN; and the length of the part of the tube MBCN may be called L: The oscillations of the water disturbed in this tube both large and small, were all tautochrones and of the same duration as the smallest oscillations of a simple pendulum, of which the length* $= \frac{L}{p+q}$.

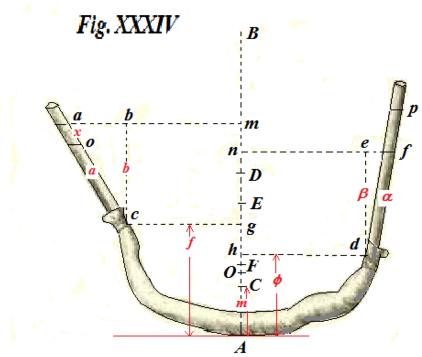
To this theorem such a corollary was appended by the same author:

If the angles ABC & HCB are right, which single case has been solved by Newton, it will be the length of the simple pendulum $= \frac{1}{2}L$*, which is isochronous to the oscillating water, as Newton found.* [See *Principia*, Book II, Prop. XLIV.]

§. 5. These are the matters which had been published at this time about the oscillations of fluids, and indeed in the first place by Newton, so that both the nature of waves as well as the fruitfulness of the principle of *living forces* had been shown by my father. Because truly it is our intention to give the fuller theory about the theory of motion of liquids, regarding this matter, it will be that kind of argument I will pursue to its full extent : Therefore I will investigate, in which ways the unequal oscillations of the fluid may become isochronous, and in which ways they may not be the same? Then for the former I will give the length of the simple tautochronous pendulum, for the other I will indicate the time of the continuation of the motion: moreover I will consider tubes bent in any manner and unequally wide.

Lemma.

§. 6. Let *cAd* (Fig. 34) be an animal hide or channel filled with water of any shape ending at each end in two cylindrical channels *ac* & *fd*, inclined to the horizontal in some manner and of any cross-section, one of which I put filled with water as far as to *a*, the other as far as *f;* it may be necessary to determine the height of the centre of gravity of all the water, from the given height of the centre of gravity of the water contained in the skin *cAd*, and from the others just as much suffices to be known beforehand.



Solution.

Let the centre of gravity of the water contained in the vessel *cAd* be at *C*, and it may be understood with the vertical line *AB* drawn through this point *C*, then the horizontal lines *am*, *cg*, *fn*, and *dh* may be drawn together with the vertical lines *cb* and *de*. Putting in place $ac = a$; $fd = \alpha$; $bc = b$; $ed = \beta$; the cross-section of the tube $ac = g$; the cross-section of the tube $fd = \gamma$; again the mass of the water or the capacity of the channel $cAd = M$, the lines $Ag = f$; $Ah = \phi$; $AC = m$. The lines *mg* and *nh* may be divided in two by the points *D* and *E* and thus the centres of gravity of the water contained in the cylindrical tubes will be at the heights of the points *D* and *E*.

With these in place there becomes $AD = f + \frac{1}{2}b$; $AE = \phi + \frac{1}{2}\beta$; the mass of the water in $ac = ga$, in $fd = \gamma\alpha$. Therefore if the centre of gravity sought for all the water *acAdf* may be understood to be put at the height *F*, *AF* will be found, so that it may agree with mechanics, by multiplying the mass of the water *ac* by *DA*, the mass of the water *fd* by *EA* and the mass of the water in *cAd* by *CA*, and the sum of these products on being divided by the sum of their masses. From which there is found :

$$AF = \frac{ga \times (f + \frac{1}{2}b) + \gamma\alpha \times (\phi + \frac{1}{2}\beta) + Mm}{ga + \gamma\alpha + M}.$$

Problem.

§. 7. To determine the velocities of the oscillating water everywhere, on putting the oscillations not to travel beyond the ends of the cylindrical tubes.

Solution.

The water shall begin the oscillation at the position *acAdf* and later it will arrive at the position *ocAdp*, with the denominations made in the preceding paragraph retained, putting $ao = x$; there will be $fp = \frac{gx}{\gamma}$: from which it will be (if surely the centre of gravity of all the water were considered to be falling from *F* to O) in the manner of the preceding paragraph :

$$AO = \frac{g \times (a - x) \times (f + \tfrac{1}{2} b - \frac{bx}{2a}) + \gamma(\alpha + \frac{gx}{\gamma}) \times (\phi + \tfrac{1}{2}\beta + \frac{\beta gx}{2\alpha\gamma}) + Mm}{ga + \gamma\alpha + M}.$$

Hence the fall of the centre of gravity is deduced, or the *actual descent*

$$FO = \frac{(b - \beta + f - \phi)gx - \left(\frac{bg}{2a} + \frac{\beta gg}{2\alpha\gamma}\right)xx}{ga + \gamma\alpha + M}.$$

Now the velocity of the water in the tube *ac* (truly when the surface is at *o*) shall be such as may correspond to the height *v*, and then the *ascent potential* of the water in the other tube $= -\frac{gg}{\gamma\gamma} v$; [*i.e.* the continuity condition again : $\text{velocity}^2 \times \text{cross-section}^2 = \text{constant}$] and equally the *ascent potential* of the water *cAd* will be proportional to the height *v*, and hence we may put that $= Nv$ (where *N* depends on the shape of the hide *cAd* an can be determined by §. 2 Sect.III). Now however if with the *ascent potentials* multiplied by their masses everywhere, and the products divided by the sum of their masses, the *ascent potential* of all the water *ocAdp* will be found

$$= \frac{\left(ga - gx + \frac{\alpha gg}{\gamma} + \frac{g^3 x}{\gamma\gamma} + MN\right)v}{ga + \gamma\alpha + M}.$$

And because here the *ascent potential* is equal to the *actual descent FO* found just above, there will be

$$v = \frac{(b - \beta + f - \phi)gx - \left(\frac{bg}{2a} + \frac{\beta gg}{2\alpha\gamma}\right)xx}{ga - gx + \frac{\alpha gg}{\gamma} + \frac{g^3 x}{\gamma\gamma} + MN}.$$

Q.E.I

Corollary1.

§. 8. Because the line $mn = mg - nh + gh = b - \beta + f - \phi$, we may put $mn = c$, and likewise we will multiply the denominator and numerator by $2\gamma\gamma\alpha\alpha$ *:* Thus truly we will have

$$v = \frac{2g\gamma\gamma a\alpha c - \left(g\gamma\gamma\alpha b + gg\gamma\alpha\beta\right)xx}{2g\gamma\gamma aa\alpha - 2g\gamma\gamma a\alpha x + 2gg\gamma a\alpha\alpha + 2g^3 a\alpha x + 2\gamma\gamma a\alpha MN}.$$

Corollary 2.

§. 9. If there may arise $v = 0$, then it is apparent the value *x* denotes the whole displacement of the surface of the fluid in the tube *ac*, which thus is found to equal $\frac{2\gamma a\alpha c}{\gamma\alpha b + ga\beta}$, however in the other tube it becomes $= \frac{2ga\alpha c}{\gamma\alpha b + ga\beta}$.

Therefore the water in the narrower tube can be raised to a certain height, only if the ratio of the cross-sections *g* & γ may be taken large enough.

Corollary 3.

§. 10. That part of the vessel *cAd*, that we have put unable to be reached by either of the surfaces at any time, has no relevance to these excursions of the fluid, either to the increases or decreases: yet it can make the oscillations faster or slower, as we will show below.

Corollary 4.

§. 11. Each tube may be put with a common cross-section, evidently there will be, on putting $g = \gamma$,

$$v = \frac{2ga\alpha cx - (g\alpha b + ga\beta)xx}{2gaa\alpha + 2ga\alpha\alpha + 2a\alpha MN}.$$

In this case the maximum velocity of each surface is when they are placed in the middle of the whole displacement, and happens otherwise, when the tubes are of unequal cross-sections.

It is required to be observed also, the retardations and accelerations to be similar to each other at similar distances of the surfaces from the mid-points of their displacements, that is, from the places of maximum velocities.

Theorem.

§.12. When the cross-sections of the cylindrical tubes are equal in the manner mentioned, the oscillations among the isochrones themselves will be both greater and smaller, provided the surfaces at no time fall beyond the opening of the same tubes.

Demonstration.

It is agreed from mechanics, because if a rapid oscillation may complete the distance $= x$, and there may be had at the individual moments of time the element of time $dt = \frac{mdx}{\sqrt{nx - xx}}$, on understanding by *m* and *n* constant quantities, that can make their oscillations in the same time both larger or smaller.
[This equation can be rewritten in modern terms as the sum of the kinetic energy and potential energy of the pendulum bob to be constant about a certain origin.]
Because truly in our case there is

$$v = \frac{2ga\alpha cx - (g\alpha b + ga\beta)xx}{2gaa\alpha + 2ga\alpha\alpha + 2a\alpha MN},$$

and because that velocity is equal to $\sqrt{v}$, there will be

$$dt = dx\sqrt{\left(\frac{2gaa\alpha + 2ga\alpha\alpha + 2a\alpha MN}{g\alpha b + ga\beta}\right)} : \sqrt{\left(\frac{2ga\alpha cx}{g\alpha b + ga\beta} - xx\right)},$$

where equally all the letters have constant values except *x*, which denotes the distance traversed; it is apparent also these shall be isochronous oscillations of the fluid.
Q. E. D.

Problem.

§. 13. To find the length of the simple pendulum, which shall be tautochronous with the oscillations of the fluid mentioned before.

Solution.

In mechanics it may be shown, that, when $dt = \frac{mdx}{\sqrt{nx - xx}}$, the length of the simple tautochronous pendulum shall be $= \frac{1}{2}mm$ *:* Therefore in our case the length of the pendulum being discussed sought

$$= \frac{gaa\alpha + ga\alpha\alpha + a\alpha MN}{g\alpha b + ga\beta}.$$

Q.E.I.

$$[t=\int\frac{mdx}{\sqrt{nx-xx}}=\int\frac{2mdx}{n\sqrt{1-\frac{4}{n^2}(\frac{1}{2}n-x)^2}};\ \text{put } 1-\tfrac{2}{n}x=y;dx=-\tfrac{n}{2}dy;$$

$$\therefore\int\frac{2mdx}{n\sqrt{1-\frac{4}{n^2}(\frac{1}{2}n-x)^2}}=-m\int\frac{dy}{\sqrt{1-y^2}}=(\text{for a half period as 0 and } n \text{ give } v=0),$$

$$=-m\arcsin(1-\tfrac{2}{n}x)\Big|_0^n=m\pi.\ \text{Hence the period is } 2m\pi;\text{in modern units } T\equiv 2\pi\sqrt{l/g},$$

here $g=\frac{1}{2}$ and the period is $2\pi\sqrt{l/\frac{1}{2}}$,

and the length of the equivalent simple pendulum is $\frac{mm}{2}$.]

Corollary 1.

§.14. If the channel *cAd* may be put of the same cross-section with the tubes joined together, and its length may be called *l*, the mass of water contained in that, which we have called $M,=gl$; and the *ascent potential* of the water contained in that, which we have put $=Nv$, will be $=v$, thus so that there may be found $N=1$. Moreover with these values substituted for the letters *M* and *N*, the length of the tautochronous pendulum will be produced for this particular case

$$=\frac{aa\alpha+a\alpha\alpha+aal}{\alpha b+a\beta}=\frac{a\alpha}{\alpha b+a\beta}\times(a+\alpha+l)=\frac{(a+\alpha+l)}{\frac{b}{a}+\frac{\beta}{\alpha}}.$$

However $a+\alpha+l$ is the length of the whole vessel filled with water and $\frac{b}{a}$ signifies the ratio of the sine of the angle *bac* to the whole sine and equally $\frac{\beta}{\alpha}$ indicates the ratio of the sine of the angle *efd* to the whole sine, we see that our solution does not differ from that, which my father gave for that case, and which we have reconsidered above in §. 4.

Corollary 2.

§.15. If the channel *cAd* may be made of infinite cross-section everywhere, there will be $MN=0$ (by §. 2 Sect. III) and the length of the tautochronous pendulum

$$=\frac{a+b}{\frac{b}{a}+\frac{\beta}{\alpha}},$$

evidently as if the whole intermediate channel *cAd* were absent, and the cylindrical tubes were to be joined to each other at once.

Yet this will require a somewhat special consideration, which I will set out below.

Scholium.

§. 16. This theorem includes all the cases, which make tautochronous oscillations, where the tubes *ac* and *pd* are right: however when these tubes are curved, in which the surfaces of the fluid are moving, the others in addition give cases of tautochronism, which may be determined easily, if we were to linger with these for a long time. Moreover when these tubes are of unequal cross-sections, the times of oscillation also become of different magnitudes corresponding to the inequality, and just as such a time must be defined may be apparent to anyone from §. 8, where we have given the velocity of the fluid at some point.

But these are concerned with finite oscillations. If now we may consider the smallest oscillations, we will see all these become tautochrones amongst themselves, with the same quantity of fluid remaining, and with the same channel, whatever meanwhile shall be the figures and cross-sections of the channel. I shall explain this in the following paragraph.

Theorem.

§. 17. The smallest oscillations of a fluid oscillating in some channel, although they may be unequal amongst themselves, are all isochronous.

Demonstration.

When the oscillations are the smallest, those of a particular channel in which the surfaces of the fluid are disturbed, can be taken as cylinders, therefore with these denominations remaining the same, the value will remain, which we have assigned to the letter *v* in §.8, and from the same ratio it follows, the letters $a,\ b,\ \alpha,\ \beta\ \&\ x$, as it were of infinitely small value, can be ignored in comparison with $\frac{M}{g}$, thus so that in the present case there must be considered

$$v=\frac{2g\gamma a\alpha cx-(g\gamma\alpha b+gg\alpha\beta)xx}{2\gamma a\alpha M\cdot N}.$$

Therefore on account of paragraph twelve all the oscillations, as long as they are minimal, are isochronous amongst themselves. Q. E. D.

Problem.

§. 18. To determine the length of the simple pendulum to be tautochronous with the minimal oscillations of the fluid disturbed in some channel.

Solution.

Because in the whole of the motion the element of the time $dt = \frac{dx}{\sqrt{v}}$, there will be now

$$dt = dx\sqrt{\left(\frac{2\gamma a\alpha M \cdot N}{g\gamma\alpha b + gga\beta}\right)} : \sqrt{\left(\frac{2\gamma a\alpha cx}{\gamma\alpha b + ga\beta} - xx\right)}.$$

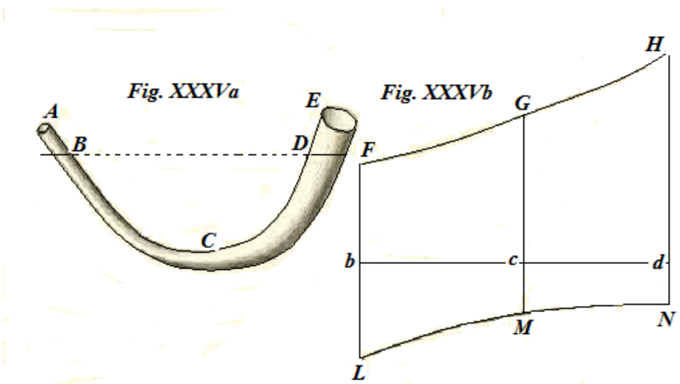
Therefore on account of paragraph thirteen the length of the pendulum sought to be tautochronous with the above oscillations

$$= \frac{\gamma a\alpha M \cdot N}{g\gamma\alpha b + gga\beta}.$$

Q.E.I.

Scholium.

§. 19. Now whenever I have pointed out in passing, how it shall be required to be understood by the quantities *M* & *N*, yet here I will put in place the whole construction, so that the nature of the matter may be more apparent to anyone.



Let there be a channel *ABCDE* of some kind, full of water as far as to *B* and *D* (Fig. *35a* & *b*); the whole sine is put $=1$, the sine of the angle $DBC = \frac{b}{a} = m$, the sine of the angle $BDC = \frac{\beta}{\alpha} = n$, the length of the tautochronous pendulum will be $\frac{\gamma M \cdot N}{mg\gamma + ngg}$, where *g* denotes the cross-section of the channel at *B* and γ its cross-section at *D*.

Now we may consider the length of the channel *BCD* filled with fluid extended straight in *bcd*, above which as it were with the axis made to the curve *FGH*, which shall be scaled to the cross-section at the equivalent places, thus, so that by putting $bc = BC$ there shall be *cG* to *bF*, as the cross-section at *C* to the cross-section at *B*. Therefore if *bF* represents the cross-section at *B*, then the interval *bdHF* will represent the magnitude *M*. Then upon the same axis *bd* another curve *LMN* can be constructed, the applied line of which *cM* shall be everywhere $\frac{bF^2}{cG}$ and there will be (per §.2, Sect. III) $N =$ volume *bdNL* divided by the volume *bdHF*, thus so that there shall be $M \times N =$ volume *bdNL*, which multiplied by $\frac{\gamma}{mg\gamma + ngg}$, will give the length of the tautochronous pendulum.

Corollary 1.

§. 20. If the tube *BCD* shall be of the same cross-section everywhere, and its length may be called *l*, the straight line *FH* will be parallel to *bd* itself, and equally *LN:* hence the volume $bdNL = gl$ and the length of the simple tautochronous pendulum will be $= \frac{l}{m+n}$.

Corollary 2.

§. 21. *BCD* shall be a cone of length *l; cG* (by putting $bc = x$) will be

$$= \left(\frac{x}{l} (\sqrt{\gamma} - \sqrt{g}) + \sqrt{g} \right)^2 ;$$

from which

$$cM = gg : \left(\frac{x}{l} (\sqrt{\gamma} - \sqrt{g}) + \sqrt{g} \right)^2 ;$$

therefore the volume *bcML*

$$= \frac{ggl}{\sqrt{g\gamma} - g} - \frac{ggl}{\sqrt{\gamma} - g} : \left(\frac{x}{l} (\sqrt{\gamma} - \sqrt{g}) + \sqrt{g} \right),$$

and therefore the whole volume *bdNL*

$$= \frac{ggl}{\sqrt{g\gamma} - g} + \frac{ggl}{\sqrt{g\gamma} - \gamma} = \frac{ggl}{\sqrt{g\gamma}}.$$

Therefore the length of the pendulum tautochronous with the water oscillations is

$$= \frac{l\sqrt{g\gamma}}{m\gamma + ng}.$$

Hence it is understood with everything else equal the water to oscillate the slowest when the cross-sections at *B* and *D* are in the inverse ratios of the sines of the corresponding angles *DBC* & *BDC:* then where the longer shall be the part filled with water and where just as the said angles shall be smaller, there equally the oscillations become slower.

Again with the comparison between cylindrical tubes and cones, and with the equal angles *BDC* and *DBC* [so that $m = n$], it is evident, the water to oscillate quicker in the cone than in the cylinder, because evidently $\frac{l\sqrt{g\gamma}}{\gamma + g}$ is less always than $\frac{1}{2}l$, [i.e. as the geometric mean of the two different numbers is less than their arithmetic mean] whatever the ratio of the inequality to be placed between g & γ . If again the aforementioned angles may be put unequal, it can happen that the water may oscillate either slower or faster in the one kind of tubes with respect to the other, as which I may confirm by example, I may put the angle *DBC* to be right, that is, $m = 1$, and the sine of the other angle *BDC* or $n, = \frac{1}{4}$, thus the length of the pendulum for the cylindrical tube $= \frac{4}{5}l$: However if under the same circumstances you may substitute a conical tube for the cylindrical one, which shall have a cross-section at *B* four times greater than the cross-section at *D*, you will have, on putting $\gamma = \frac{1}{4}g$, the length of the pendulum $= l$: $[i.e. = \frac{l\sqrt{g\gamma}}{m\gamma + ng} = \frac{l\sqrt{gg/4}}{1\cdot g/4 + g\cdot 1/4} = l.]$, and thus the tautochronous pendulum is longer for a conical tube than for a cylindrical tube with all else being equal, and the oscillations made in the former are slower than in the latter, as here: but if now, with everything again remaining, we may put the narrower tube at *B* rather than at *D*, the opposite will arise: if there were for example $\gamma = 4g$, the length of the pendulum would be $= \frac{8}{17}l$, and hence to be smaller than if the tube were cylindrical ; and again it will be smaller, if nevertheless we may put the cross-section at *B* greater than it is at *D*: thus if there were $\gamma = \frac{1}{64}g$, the length of the pendulum $= \frac{8}{17}l$, as before. It is to be noted, as we have seen also in the previous example, that with the cross-section remaining at *B*, and with the length of the channel *BCD* the same, there shall always be two different cross-sections to be defined at *D* for the same length of the tautochronous pendulum, except when the angle *DBC* & *BDC* are equal. The example is a particular example of this result, because either the cross-section at *D* shall be equal to the cross-section at *B*, or the ratio may be had for the same: the square of the sine of the angle *BDC* and the sine of the angle *DBC* , the oscillations of the fluid are completed in the same time in each tube.

[Thus there are two cases to consider, either
$m : n = g : \gamma$ or $m : n = \gamma : g$; hence either $n = m\gamma / g$, or $n = mg / \gamma$.

In the first case, $l_1 = \frac{l\sqrt{g\gamma}}{\gamma + g} = l\sqrt{g\gamma} : 2m\gamma.$

In the second case, $l_2 = \frac{l\sqrt{g\gamma}}{m\gamma + ng} = \frac{l\sqrt{g\gamma}}{\left(m\gamma + mg^2/\gamma\right)} = \frac{l\sqrt{g\gamma}}{m\gamma\left(1 + g^2/\gamma^2\right)}.$

Thus, if $g < \gamma$, $l_2 > l_1$, while if $g > \gamma$, $l_2 < l_1$; for $g = \gamma$, $l_2 = l_1$.]

General Scholium.

§. 22. Thus I have taken up experiments concerned with the oscillations of fluids, so that repeatedly I could find by trial the length of the simple isochronous pendulum, and by means of such I have been able besides to observe this length in different cases, as the theory indicates in this section ; yet sometimes I have found that length must be a little greater ; the reason for this I have been able to see without difficulty, because frictions not only diminish the displacements of a fluid, but also produces retardations, and so that, when tubes in place there where they are bent, are accustomed to be narrower : That latter if it may be avoided with all care, and if these bends are not made by a single angle but slowly, and if finally the purest mercury may be used for the liquid oscillating, no doubt remains for me, to be so that the experiments confirm the theory presented precisely, thus, so that I will not need to inquire anxiously about others.

Yet it may be added to that account of the experiments put in place by me, that I had investigated accurately the cross-sections of the tubes before the experiment in different locations of these with the help of a column of mercury, which while it ran through the whole length of the tube, with its different lengths, the measurements of which I took with great care, which showed the variations of the cross-section everywhere : And indeed these cross-sections thus were found in the tube, now after it was curved, for the cross-sections certainly decreased by the curvature. This was the reason, because initially in the experiment undertaken by me, my success was less than expected : however a glass tube, of the kind accustomed to be used for making barometers, wide enough and the same to be nearly a perfect cylinder, I made to be curves, so that it showed more or less the twenty seventh figure, and that then I filled the greater part with mercury, the oscillations of its length I found to be slower than I had expected, because by not attending to the curvature at *D*, especially where the angles were formed. Therefore so that I might take this matter into account, henceforth I made use of tubes with a slow curvature, such as Fig 35*a* shows, and in these I have examined carefully the cross-sections after the curvature.

HYDRODYNAMICAE SECTIO SEXTA.

De fluidis non effluentibus seu intra latera vasorum motis.

§. 1. Hactenus consideravimus aquas effluentes; nunc vero contemplabimur motus aquarum, quae vasorum limites non praeterfluunt. Omnes hos motus ad duo reducam genera, ambo seorsim pertractanda:
1. Cum fluidum in tubo infinite longo continue movetur versus eandem plagam.
2. Cum motibus reciprocis seu oscillatoriis agitatur.

De motu aquarum per canales indefinite longos.

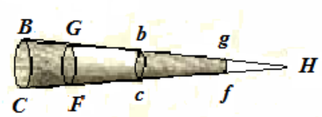
Casus I.

§. 2. Sit primo canalis horizontaliter positus, sed amplitudinibus data quacunque varians lege: ponatur fluidum in illo ita positum, quod fieri solet in tubis strictioribus, ut ambae superficies extremae situm obtineant ad axem canalis perpendicularem & sic data quadam velocitate moveri incipere. Haec si ita sint, nullaque plane motus impedimenta adesse fingantur, perspicuum est, motui aquarum nullum finem fore, quemadmodum globus super tabula horizontali liberrime progrediens motum sine fine continuat. Attamen insignis inter utrumque motum intercedit differentia: globi nempe partes omnes uniformi continue progrediuntur velocitate, in aqua perpetuo motum mutant: Neque difficile erit motum istum definire, cum considerabimus, motum talem esse debere, ut *ascensus potentialis* totius aquae idem conservetur, qui ab initio motus fuit: Determinavimus autem *ascensum potent.* aquae certa velocitate in canali quocunque motae in sectionis tertiae paragrapho secundo. Igitur nihil ad Solutionem quaestionis amplius residuum est: Neque tamen abs re erit unum alterumve ejus rei exemplum attulisse.

Exemplum 1.

Sit v. gr. canalis *BgfC* (Fig. 31) qui figuram habeat coni truncati; intelligatur pars ejus *BGFC* fluido plena moto versus *gf;* habeantque particulae fluidi in *GF* velocitatem debitam altitudini *v;* ac denique pervenerit fluidum in situm *bgfc.* His positis quaeritur velocitas fluidi in *gf.* Vocabo autem altitudinem velocitati aquae in *gf* debitam $=V$; sit vertex coni in *H;* diameter in $BC=n$; diameter in $GF=m$; longitudo $BG=a$; $Gg=b$, erit

Fig. XXXI



diameter $gf=\dfrac{ma+mb-nb}{a}$. Deinde quia solidum *BGFC* est aequale solido *bgfc*, erit

$$BC^2\times BH-GF^2\times GH=bc^2\times bH-gf^2\times gH ;$$

unde

$$bc^2 \times bH = BC^2 \times BH - GF^2 \times GH + gf^2 \times gH ;$$

est vero

$bH = \frac{BH}{BC} \times bc ;$ igitur

$$bc^3 = BC^3 - \frac{GF^2 \times GH \times BC}{BH} + \frac{gf^2 \times gH \times BC}{BH} = BC^3 - GF^3 + gf^3,$$

seu

$$bc = \sqrt[3]{n^3 - m^3 + \left(\frac{ma + mb - nb}{a}\right)^3}.$$

Est vero per §. 3 Sect. III *ascensus potent.* aquae in situ *BGFC*

$$= \frac{3m^3 v}{n(mm + mn + nn)};$$

pariterque *ascensus potent.* ejusdem aquae in situ *bgfc* reperitu

$$= \frac{3\alpha^3 V}{\beta(\alpha\alpha + \alpha\beta + \beta\beta)},$$

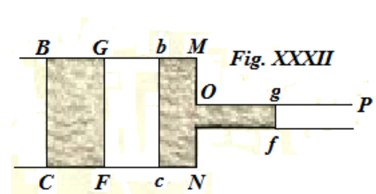
posito brevitatis ergo α & β pro inventis valoribus diametrorum *gf* & *bc*. Erit igitur

$$V = \frac{m^3 \times (\alpha\alpha + \alpha\beta + \beta\beta) \times \beta \times v}{\alpha^3 \times (mm + mn + nn) n}.$$

Ex hac formula facile colligitur, majori continue velocitate moveri particulas anteriores, minori posteriores, & sic, ut si foraminulum *gf* censeatur infinite parvum, fiat velocitas aquae in *gf* infinita & in *bc* infinite parva.

Exemplum 2.

Fuerit canalis compositus ex duobus tubis cylindricis *BN* & *OP* (Fig. 32) inaequalis amplitudinis; in ramo ampliore moveri ponatur fluidum *BGFC* versus *P* velocitate quae respondeat altitudini *v*. Ita perspicuum est nullam motus mutationem adfore, priusquam superficies *GF* pervenerit in *MN;* ab hoc autem temporis puncto



motum continue variari donec fluidum omne subingressum fuerit tubum strictiorem. Quaeritur itaque cum fluidum situm tenet *bgfc*, quaenam futura sit velocitas superficiei *fg*; altitudinem autem hujus velocitatis designabimus per *V*.

Sint diametri *GF* & *gf* ut *n* & *m:* longitudo *BG* vocetur $=a$, $bM=b$; erit $Og=\frac{nn}{mm}\times(a-b)$; *ascensus potent.* aquae $BGFC=v$; *ascensus potent.* aquae *bgfc* ergo

$$=\frac{n^4a-n^4b+m^4b}{n^4a}\times V\ ;$$

ergo

$$V=\frac{n^4a}{n^4a-n^4b+m^4b}v\ .$$

Ex his intelligitur velocitatem primae guttulae in tubum strictiorem irrumpentis respondere altitudini $\frac{n^4}{m^4}v$, hanc vero velocitatem citissime decrescere, ita ut postquam parvula fluidi pars transfluxit, jam possit censeri $V=\frac{a}{a-b}v$, & cum omne fluidum transfluxerit, pristinam assumat velocitatem. Fuerit v. gr. diameter tubi amplioris decupla alterius; & effluet prima guttula ex tubo ampliore in strictiorem velocitate debita altitudini 10 000*v* : si vero decimam fluidi partem jam transfluxisse ponas, invenies altitudinem, quae conveniat velocitati fluidi in tubo strictiori progredientis, proxime aequalem 10*v*.

Si tempus quaeras, quo fiat transfluxus fluidi *Of*, invenies illud aequale

$$\frac{2(n^4a-n^4b+m^4b)^{\frac{3}{2}}-2m^6a\sqrt{a}}{3mm(n^4-m^4)\sqrt{av}}.$$

Igitur omne fluidum transfluit tempore

$$\frac{2n^6a\sqrt{a}-2m^6a\sqrt{a}}{3mm(n^4-m^4)\sqrt{av}}=\frac{2(n^4+mmnn+m^4)a}{3mm(nn+mm)\sqrt{v}},$$

ubi per $\frac{a}{\sqrt{v}}$ intelligitur tempus, quo fluidum in tubo ampliori libere motum absolvit spatium *a*. Haec vero, ut dixi, se ita habebunt si nulla sint motus impedimenta, simulque in toto tractu canalis compositi velocitates amplitudinibus reciproce proportionales ponantur. Interim jam alibi monui non posse aquas lateri *MN* proximas hanc legem servare. Cum itaque talis casus occurrit, eo magis conveniet motus realis cum theoria, quo longior fuerit pars *bM* & quo pauciora adfuerint obstacula.

§. 3. Quod si nunc canalis fuerit non horizontaliter sed oblique ad horizontem positus, apparet omnia similiter se habere, nisi quod *ascensus potent.* aquae in omni situ aequandus sit *ascensui potent.* initiali aucto *descensu actuali*, id est, descensui verticali centri gravitatis. Atque si nullo impulsu aqua sua sponte se movere incipiat, erit simpliciter *descensus actualis* aequalis *ascensui potent.*

Igitur aqua continue progredi perget, quamdiu centrum gravitatis loco humiliori positum est, ac fuit ab initio motus. At vero cum tubus ita fuerit formatus & inflexus eaque fluidi quantitate repletus, ut centrum gravitatis pristinam altitudinem reassumere possit, tunc fluidum motum obtinebit retrogradum & sine fine oscillabitur. De isto motu praecipuam hujus sectionis partem faciente mox dicemus. Interea observare licet, fieri posse, ut aqua omnis ex loco humiliore per altiorem sua sponte sine praevia suctione praeterfluat, si modo omnia debito modo se habeant.

De oscillationibus fluidorum in tubis recurvis

Casus II.

§. 4. Dedit Pater meus in *Comm. Acad. Scient. Petrop. tom.* 2 theoremata quaedam, quae insignem manifestant usum quem theoria virium vivarum habet in rebus mechanicis. lllud vero quod tertio loco positum est ita se habet:

Sit tubus cylindricus ABCH (Fig. 33) *utrobique apertus atque inflexus in duo crura BA & CH ad partem horizontalem BC; sit sinus anguli* $ABC = p$, *& sinus anguli* $HCB = q$, *existente nimirum sinu toto* $=1$; *sit porro ille tubus aqua plenus usque ad horizontalem MN; voceturque L longitudo partis tubi MBCN aqua plenae: Erunt agitati liquoris in hoc tubo oscillationes tam majores, quam minores omnes tautochronae atque ejusdem durationis cum oscillationibus minimis penduli alicujus simplicis, cujus longitudo* $=\dfrac{L}{p+q}$.

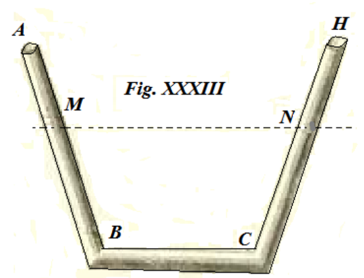


Fig. XXXIII

Huic theoremati eodem auctore subnectitur tale corollarium:

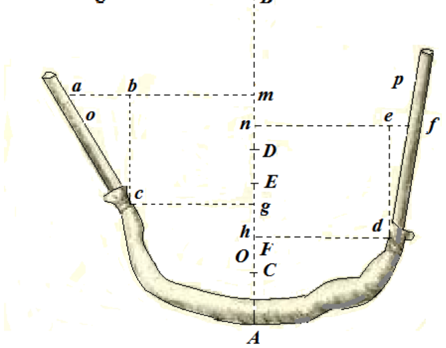
Si anguli ABC & HCB sunt recti, qui unicus casus est a Newtono solutus, erit longitudo penduli simplicis, quod oscillanti aquae isochronum est, $=\frac{1}{2}L$, *ut invenit Newtonus.*

§. 5. Haec sunt quae adhuc cum publico communicata fuerunt circa oscillationes fluidorum, & quidem primo a Newtono, ut undarum naturam, a Patre meo, ut fertilitatem principii *virium vivarum* ostenderet. Quia vero nostrum institutum est pleniorem dare de motibus aquarum theoriam, e re erit istud argumenti genus in tota sua extensione prosequi: Igitur disquiram, quibus modis oscillationes fluidi inaequales fiant isochronae, & quibus non item? Dein pro prioribus dabo longitudinem penduli simplicis tautochroni, pro alteris tempus durationis indicabo: tubos autem utcunque inflexos & inaequaliter amplos considerabo

Lemma.

§. 6. Sit *cAd* (Fig. 34) uter seu canalis aqua plenus formae cujuscunque datae desinens utrobique in duos canales cylindricos *ac* & *fd*, utcunque ad horizontem inclinatos & cujuscunque amplitudinis, quorum alterum plenum aqua ponam usque in *a*, alterum usque in *f;* oporteat determinare altitudinem centri gravitatis omnis aquae, ex data altitudine centri gravitatis aquae in utre *cAd* contentae, caeterisque quantum sufficit praecognitis.

Fig. XXXIV



Solutio.

Fuerit centrum gravitatis aquae in vase *cAd* contentae in *C*, ductaque intelligatur per istud punctum *C* verticalis *AB*, deinde ducantur horizontales *am*, *cg*, *fn*, & *dh* una cum verticalibus *cb* & *de*. Ponatur $ac = a$; $fd = \alpha$; $bc = b$; $ed = \beta$; amplitudo tubi $ac = g$; amplitudo tubi $fd = \gamma$; sit porro massa aquea seu capacitas canalis $cAd = M$, linea $Ag = f$; $Ah = \phi$; $AC = m$. Dividantur lineae *mg* & *nh* bifariam punctis *D* & *E* & sic erunt centra gravitatis aquarum in tubis cylindricis contentarum in altitudinibus punctorum *D* & *E*.

His positis fit $AD = f + \frac{1}{2}b$; $AE = \phi + \frac{1}{2}\beta$; massa aquae in $ac = ga$, in $fd = \gamma\alpha$. Igitur si centrum gravitatis quaesitum pro omni aqua *acAdf* intelligatur in altitudine *F* positum, habebitur, ut constat in mechanicis, *AF* multiplicando massam aquae in *ac* per *DA*, massam aquae *fd* per *EA* & massam aquae in *cAd* per *CA*, aggregatumque horum productorum dividendo per summam harum massarum. Unde invenitur

$$AF = \frac{ga \times (f + \frac{1}{2}b) + \gamma\alpha \times (\phi + \frac{1}{2}\beta) + Mm}{ga + \gamma\alpha + M}.$$

Problema.

§. 7. Determinare ubique velocitates aquae oscillantis, posito oscillationes ultra terminos tuborum cylindricorum non divagari.

Solutio.

Sit aqua oscillationem inchoans in situ *acAdf* perveneritque postmodum in situm *ocAdp*, retentisque denominationibus in praecedente paragrapho factis, ponatur $ao = x$;

erit $fp = \frac{gx}{\gamma}$: unde (si nempe centrum gravitatis ommis aquae descendisse putetur ex *F* in O) erit vi praecedentis paragraphi

$$AO = \frac{g\times(a-x)\times(f+\tfrac{1}{2}b-\frac{bx}{2a})+\gamma(\alpha+\frac{gx}{\gamma})\times(\phi+\tfrac{1}{2}\beta+\frac{\beta gx}{2\alpha\gamma})+Mm}{ga+\gamma\alpha+M}.$$

lnde deducitur descensus centri gravitatis seu *descensus actualis*

$$FO = \frac{(b-\beta+f-\phi)gx-\left(\frac{bg}{2a}+\frac{\beta gg}{2\alpha\gamma}\right)xx}{ga+\gamma\alpha+M}.$$

Sit nunc velocitas aquae in tubo *ac* (cum nempe superficies est in *o*) talis quae respondeat altitudini *v*, & erit tunc *ascensus potent.* aquae in altero tubo $= -\frac{gg}{\gamma\gamma}v$; pariterque *ascensus potent.* aquae *cAd* erit proportionalis altitudini *v*, eamque proinde ponemus $= Nv$ (ubi *N* pendet a figura utris *cAd* & determinari potest per §. 2 Sect.III). Jam vero si multiplicatis ubique *ascensibus potentialibus* per suas massas producta dividantur per summam massarum, habebitur *ascensus potent.* omnis aquae *ocAdp*

$$= \frac{\left(ga-gx+\frac{\alpha gg}{\gamma}+\frac{g^3x}{\gamma\gamma}+MN\right)v}{ga+\gamma\alpha+M}.$$

Et quia hic *ascensus potentialis* est aequalis *descensui actuali FO* paullo ante invento, erit

$$v = \frac{(b-\beta+f-\phi)gx-\left(\frac{bg}{2a}+\frac{\beta gg}{2\alpha\gamma}\right)xx}{ga-gx+\frac{\alpha gg}{\gamma}+\frac{g^3x}{\gamma\gamma}+MN}.$$

Q.E.I

Corollarium 1.

§. 8. Quia linea $mn = mg - nh + gh = b - \beta + f - \phi$, ponemus $mn = c$, simulque multiplicabimus denominatorem & numeratorem per $2\gamma\gamma\alpha\alpha$ *:* Ita vero habebimus

$$v = \frac{2g\gamma\gamma a\alpha c - \left(g\gamma\gamma\alpha b + gg\gamma\alpha\beta\right)xx}{2g\gamma\gamma a a\alpha - 2g\gamma\gamma a\alpha x + 2gg\gamma a\alpha\alpha + 2g^3 a\alpha x + 2\gamma\gamma a\alpha MN}.$$

Corollarium 2.

§. 9. Si fiat $v = 0$, patet tunc valorem x denotare totam fluidi superficiei excursionem in tubo *ac*, quae sic invenitur aequalis $\frac{2\gamma a\alpha c}{\gamma\alpha b + ga\beta}$, in altero vero tubo fit $= \frac{2ga\alpha c}{\gamma\alpha b + ga\beta}$.

Igitur poterit aqua in tubo strictiori ad quamcunque elevari altitudinem, si modo ratio amplitudinum *g* & *y* sat magna sumatur.

Corollarium 3.

§. 10. Pars illa vasis *cAd*, quam neutra superficierum unquam attingi ponimus, nihil pertinet ad istas fluidi excursiones sive augendas sive diminuendas: facere tamen potest, ut inferius ostendetur, ad accelerandas retardandasque oscillationes.

Corollarium 4.

§. 11. Ponatur uterque tubus communis amplitudinis, erit, posito nempe $g = y$,

$$v = \frac{2ga\alpha cx - (g\alpha b + ga\beta)xx}{2gaa\alpha + 2ga\alpha\alpha + 2a\alpha MN}.$$

In hoc casu maxima superficiei utriusque velocitas est, cum in medio totius excursionis positae sunt, secus ac fit, cum tubi sunt inaequalis amplitudinis.

Notandum quoque est, similes esse inter se retardationes & accelerationes in distantiis similibus superficierum a punctis mediarum excursionum, id est, a locis maximarum velocitatum.

Theorema.

§.12. Cum amplitudines tuborum cylindricorum praedicto modo sunt aequales, erunt oscillationes tam majores quam minores inter se Isochronae, modo superficies nunquam descendant infra orificia eorundem tuborum.

Demonstratio.

Ex mechanicis constat, quod si mobile oscillans spatium perfecerit $= x$, habeatque in singulis locis elementum temporis $dt = \frac{mdx}{\sqrt{nx - xx}}$, intelligendo per m & n . quantitates constantes, id faciat oscillationes suas tam majores quam minores eodem tempore.

Quia vero in nostro casu est

$$v = \frac{2ga\alpha cx - (g\alpha b + ga\beta)xx}{2gaa\alpha + 2ga\alpha\alpha + 2a\alpha MN},$$

& quia velocitas ipsa est aequalis $\sqrt{v}$, erit

$$dt = dx\sqrt{\left(\frac{2gaa\alpha + 2ga\alpha\alpha + 2a\alpha MN}{g\alpha b + ga\beta}\right)} : \sqrt{\left(\frac{2ga\alpha cx}{g\alpha b + ga\beta} - xx\right)},$$

ubi pariter omnes litterae constantem habent valorem praeter *x*, quae spatium percursum denotat; patet has quoque fluidi oscillationes isochronas fore.
Q. E. D.

Problema.

§. 13. Invenire longitudinem penduli simplicis, quod sit tautochronum cum oscillationibus fluidi praefatis.

Solutio.

In mechanicis demonstratur, quod, cum $dt = \frac{mdx}{\sqrt{nx - xx}}$, sit longitudo penduli simplicis tautochroni $= \frac{1}{2}mm$ *:* Erit igitur in nostro casu de quo sermo est longitude penduli quaesita

$$= \frac{gaa\alpha + ga\alpha\alpha + a\alpha MN}{g\alpha b + ga\beta}.$$

Q.E.I.

Corollarium 1.

§.14. Si ponatur canalis *cAd* ejusdem amplitudinis cum tubis conjunctis, ejusque longitudo vocetur *l*, erit massa aquae in eo contentae, quam vocavimus $M, = gl$ *;* & *ascensus potent.* aquae in illo contentae, quem posuimus $= Nv,$ erit $= v$, ita ut habeatur $N = 1$. Substitutis autem istis valoribus pro litteris *M* & *N*, prodit longitudo penduli tautochroni pro isto casu particulari

$$= \frac{aa\alpha + a\alpha\alpha + aal}{\alpha b + a\beta} = \frac{a\alpha}{\alpha b + a\beta} \times (a + \alpha + l) = \frac{(a + \alpha + l)}{\frac{b}{a} + \frac{\beta}{\alpha}}.$$

Quia vero $a+\alpha+l$ est longitudo totius tractus aqua pleni & $\frac{b}{a}$ significat rationem *a* sinus anguli *bac* ad sinum totum pariter atque $\frac{\beta}{\alpha}$ denotat rationem sinus anguli *efd* ad sinum totum, videmus non differre nostram solutionem ab illa, quam Pater meus pro isto casu dedit, quamque supra recensui §. 4.

Corollarium 2.

§.15. Si ponatur canalis *cAd* infinitae ubique amplitudinis, erit $MN=0$ (per §. 2 Sect. III) & longitudo penduli tautochdroni

$$=\frac{a+b}{\frac{b}{a}+\frac{\beta}{\alpha}},$$

quasi nempe totus canalis intermedius *cAd* abesset, tubique cylindrici inter se immediate essent conjuncti.

Est tamen hic speciale aliquid considerandum, quod infra monebo.

Scholion.

§. 16. Complectitur hoc theorema omnes casus, qui oscillationes tautochronas faciunt, ubi tubi *ac* & *pd* sunt recti: cum vero hi tubi, in quibus fluidi superficies excurrunt, incurvati sunt, dantur alii insuper tautochronismi casus, quos facile foret determinare, si hisce diutius immorari vellemus. Caeterum cum tubi hi inaequalis amplitudinis sunt, fiunt quoque tempora oscillationibus diversarum magnitudinum respondentia inaequalia, & quomodo tempus tale definiri debeat unicuique apparet ex §. 8, ubi velocitatem fluidi in quolibet puncto dedimus.

Haec autem de oscillationibus finitis. Si nunc oscillationes minimas esse censeamus, videbimus illas fieri omnes inter se tautochronas, manente eadem fluidi quantitate, eodemque canali, quaecunque interea sint canalis figura & amplitudines. Id exponam in sequenti paragrapho.

Theorema.

§. 17. Oscillationes minimae fluidi in quocunque canali oscillantis, quamvis inaequales inter se, sunt omnes Isochronae.

Demonstratio.

Cum oscillationes sunt minimae, possunt illae canalis particulae, in quibus superficies fluidi agitantur, pro cylindricis haberi, igitur manentibus denominationibus iisdem, manebit valor, quem assignavimus litterae *v* in§. 8, & ex eadem ratione sequitur, litteras

$a,\ b,\ \alpha,\ \beta\ \&\ x$ ceu infinite parvi valoris negligi posse prae $\frac{M}{g}$, sic ut in praesenti casu censeri debeat

$$v = \frac{2g\gamma a\alpha cx - (g\gamma\alpha b + gg\alpha\beta)xx}{2\gamma a\alpha MN}.$$

Sunt igitur vi paragraphi duodecimi oscillationes omnes, quoad minimae sunt, inter se Isochronae. Q. E. D.

Problema.

§. 18. Determinare longitudinem penduli simplicis tautochroni cum oscillationibus minimis fluidi in canali quocunque agitati.

Solutio.

Quia in omni motu est elementum temporis $dt = \frac{dx}{\sqrt{v}}$, erit nunc

$$dt = dx\sqrt{\left(\frac{2\gamma a\alpha MN}{g\gamma\alpha b + gga\beta}\right)} : \sqrt{\left(\frac{2\gamma a\alpha cx}{\gamma\alpha b + ga\beta} - xx\right)}.$$

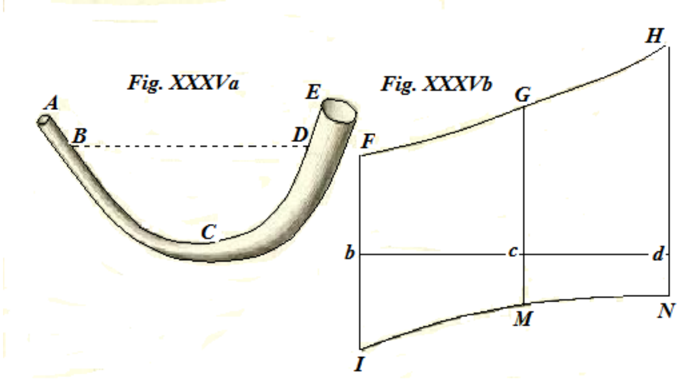
Igitur vi Paragraphi decimi tertii erit longitudo quaesita penduli cum praedictis oscillationibus tautochroni

$$= \frac{\gamma a\alpha MN}{g\gamma\alpha b + gga\beta}.$$

Q.E.I.

Scholium.

§. 19. Quamvis jam passim monuerim, quid intelligendum sit per quantitates *M* & *N*, tamen hic apponam totam constructionem, ut natura rei eo magis unicuique pateat.



Fuerit canalis qualiscunque *ABCDE* (Fig. *35a* & b) aqua plenus usque in *B* & *D;* ponatur sinus totus $=1$, sinus anguli $DBC = \frac{b}{a} = m$, sinus anguli $BDC = \frac{\beta}{\alpha} = n$, erit longitudo pendulumi tautochroni $\frac{\gamma MN}{mg\gamma + ngg}$, ubi *g* denotat amplitudinem canalis in *B* & γ amplitudinem ejus in *D.*

Concipiatur nunc longitudo canalis *BCD* fluido plena in rectam extensa *bcd*, super qua ceu axe fiat curva *FGH*, quae sit scala amplitudinum in locis homologis, ita, ut posita $bc = BC$ sit *cG* ad *bF*, ut amplitudo in *C* ad amplitudinem in *B.* Igitur si *bF* repraesentet amplitudinem in *B*, tunc spatium *bdHF* repraesentabit magnitudinem *M.* Deinde super eodem axe *bd* construatur alia curva *LMN*, cujus applicata *cM* sit ubique $\frac{bF^2}{cG}$ & erit (per §. 2 Sect. III) $N =$ spatio *bdNL* diviso per spatium *bdHF*, ita ut sit $M \times N =$ spatio *bdNL*, quod multiplicatum per $\frac{\gamma}{mg\gamma + ngg}$ dabit longitudinem penduli tautochroni.

Corollarium 1.

§. 20. Si tubus *BCD* sit ubique ejusdem amplitudinis, ejusque longitudo dicatur *l*, erit *FH* linea recta ipsi *bd* parallela, pariter atque *LN:* hinc spatium $bdNL = gl$ & longitudo penduli tautochroni $= \frac{l}{m+n}$.

Corollarium 2.

§. 21. Sit *BCD* canalis conicus longitudinis *l;* erit *cG* (posita $bc = x$)

$$= \left(\frac{x}{l} (\sqrt{\gamma} - \sqrt{g}) + \sqrt{g} \right)^2 ;$$

unde

$$cM = gg : \left(\frac{x}{l}(\sqrt{\gamma} - \sqrt{g}) + \sqrt{g}\right)^2;$$

ergo spatium *bcML*

$$= \frac{ggl}{\sqrt{g\gamma} - g} - \frac{ggl}{\sqrt{\gamma} - g} : \left(\frac{x}{l}(\sqrt{\gamma} - \sqrt{g}) + \sqrt{g}\right),$$

& proinde totum spatium *bdNL*

$$= \frac{ggl}{\sqrt{g\gamma} - g} + \frac{ggl}{\sqrt{g\gamma} - \gamma} = \frac{ggl}{\sqrt{g\gamma}}.$$

Est igitur longitudo penduli tautochroni cum oscillante aqua $= \dfrac{l\sqrt{g\gamma}}{m\gamma + ng}$.

Hinc intelligitur caeteris paribus oscillari aquam tardissime cum amplitudines in *B* & *D* sunt in ratione reciproca sinuum angulorum respondentium *DBC* & *BDC:* dein quo longior sit pars aqua plena & quo minores anguli modo dicti, eo pariter tardiores fieri oscillationes.

Porro comparatis inter se tubis cylindricis & conicis, positisque angulis *BDC* & *DBC* aequalibus, perspicuum est, citius oscillari aquam caeteris paribus in conicis quam cylindricis, quia nempe $\dfrac{l\sqrt{g\gamma}}{\gamma + g}$ semper minor est quam $\frac{1}{2}l$, quaecunque ratio inaequalis intercedat inter g & γ. Si porro praedicti anguli inaequales ponantur, fieri potest tam ut tardius quam ut citius oscilletur aqua in uno tuborum genere respectu alterius, quod ut exemplo confirmem, ponam angulum *DBC* rectum, id est, $m = 1$, & sinum alterius anguli *BDC* seu $n = \frac{1}{4}$, ita erit longitudo penduli pro tubis cylindricis $= \frac{4}{5}l$: Si vero sub iisdem reliquis circumstantiis tubo cylindrico substituas conicum, qui amplitudinem in *B* habeat quadruplo majorem, quam est amplitudo in *D*, habebis, posito $\gamma = \frac{1}{4}g$, longitudinem penduli $= l$: longius est itaque caeteris paribus pendulum tautochronum pro tubo conico quam pro cylindrico, & tardius fiunt oscillationes in illo, quam in hoc: sed si nunc, manentibus rursus reliquis, tubum conicum strictiorem ponamus in *B* quam in *D*, contrarium erit: fuerit v. gr. $\gamma = 4g$, erit longitudo penduli $= \frac{8}{17}l$, & proinde minor, quam si tubus cylindricus foret; rursusque minor erit, si amplitudinem in *B* admodum majorem ponas, quam est in D: ita si fuerit $\gamma = \frac{1}{64}g$, erit longitudo penduli $= \frac{8}{17}l$, ut ante. Notabile est, ut in praecedente etiam vidimus exemplo, quod, manentibus amplitudine in *B*, situ canalis *BCD* ejusdemque longitudine, duae semper diversae definiri possint amplitudines in *D* pro eadem penduli tautochroni longitudine, nisi cum anguli *DBC* & *BDC* sunt aequales. Hujus rei exemplum est particulare, quod, sive amplitudo in *D* aequalis sit amplitudini in *B*, sive rationem ad eandem habeat quadratam sinus ang. *BDC* & sin. ang. *DBC*, eodem tempore oscillationes fluidi absolvantur in tubo utroque.

Scholion Generale.

§. 22. Experimenta de oscillantibus fluidis ita sumpsi, ut crebra tentatione longitudinem penduli simplicis isochroni invenirem, hancque longitudinem in diversis casibus talem praeter propter esse observare potui, quam theoria in hac sectione indicat; aliquando tamen longitudinem illam debita paullo majorem inveni; cujus rei rationem haud difficulter hanc esse vidi, quod frictiones fluidi excursiones non solum diminuant, sed & retardent, ut &, quod tubi eo in loco, quo inflectuntur, strictiores esse soleant: Id posterius si omni cura evitetur, sique ipsae inflexiones non uno angulo sed lente fiant, & si denique pro liquore oscillante mercurius purissimus adhibeatur, dubium mihi nullum superest, fore ut experimenta praemissam theoriam ad amussim confirment, ita, ut operae pretium non duxerim anxie de illis inquirere.

Id tamen ratione experimentorum a me institutorum superaddam, quod amplitudines tuborum ante experimentum in diversis eorum locis accurate exploraverim ope columellae mercurii, quae dum gradatim totam longitudinem tubi percurreret, longitudinibus suis diversis, quarum mensuras assidue accipiebam, amplitudinum variationes ubique manifestabat: Et hae quidem amplitudines ita in tubo erunt explorandae, postquam jam fuerit incurvatus, nam ab incurvatione amplitudines admodum decrescunt. Haec ratio fuit, quod in primo hanc in rem a me sumto experimento, successus expectationem meam fefellerit: Tubum nempe vitreum, cujusmodi pro barometris conficiendis adhibere solent, satis amplum eundemque fere perfecte cylindricum, incurvare feci, ut ostendit propemodum Figura vigesima septima, eoque deinde mercurio maximam partem repleto, oscillationes ejus longe tardius fieri vidi, quam expectaveram, quia non attendi, tubum ab incurvatione in *D* insigniter fuisse constrictum, praesertim ubi anguli formantur. Hujus igitur rei ut rationem haberem, tubis deinceps lente incurvatis usus fui, quales ostendit Fig 35*a*, in iisque amplitudines post incurvationem diligenter exploravi.