HYDRODYNAMICS SECTION FIVE.

Concerning the motion of water from vessels being filled constantly.

§. 1. Vessels may be kept full, with just as much water pouring in continually, as the amount flowing out ; but the water pouring in can be either put in place in the same direction as the motion of the surface and with the same velocity at every instant, evidently as if the surface were being created anew continually, to which the velocity of the water now approximates, or from the side and without impetus, just as if the surface, which is devised to be created continually anew, shall be endowed with no motion and finally requiring to be moved by the water below. I shall pass by the remaining ways of introducing new water, which are infinite.

Meanwhile the rule has been received about this motion, especially in the latter case, the water to flow out with a velocity agreeing with the height of the surface above the opening: yet it is seen easily that cannot prevail, unless for a vessel with infinite width everywhere, moreover in the remaining cases to be such that the motion begins from rest slowly and may be increased slowly through some interval of time, and finally after an infinite time it may acquire the whole of the velocity. Nevertheless, if it may be said that the thing is, that these accelerations generally happen so very quickly, that only in the shortest possible times is the whole velocity not present: truly this matter is had otherwise in extended water ducts, in which the increases of the velocities do not escape notice and when they can be observed by separate measurements.

But of whatever nature it may be, since the mathematical accuracy shall displease no one, I have put in place to consider and pursue the motion of the water from first principles.

§. 2. All the properties of this motion allow themselves to be reduced to three particular equations

1st: Between the amount of water ejected and the corresponding velocity;
2nd: Between the time and the velocity,
& 3rd: Between the amount of water and the time.
If one of these equations may be found, then the rest flow on from it at once.

Therefore in the first place we will scrutinize only the first more closely : Truly here we are mindful of these matters, which were discussed in the preceding section, concerning the contraction of the jet through simple orifices, or flowing out through converging tubes, and the spreading out of the same when it is ejected through diverging tubes. Moreover we have indicated there in §. 3 Art.1 Sect. IV that as far as the jet is to be considered, the velocities of the particles (by abstracting the mind from the changes which gravity produces on the particles beyond the vessel) will not be changed further, and all that part of the jet or the motion is required to be judged within the vessel, evidently as if the surface of the jet becomes well established to that point. Therefore since hence the discussion will be about the vessel through which the water flows out, it being understood that vessel will be ideal, of which the efflux opening shall be a cross-section of the jet henceforth subject to no change, except what is due to the descent or the ascent of the jet.

Problem.

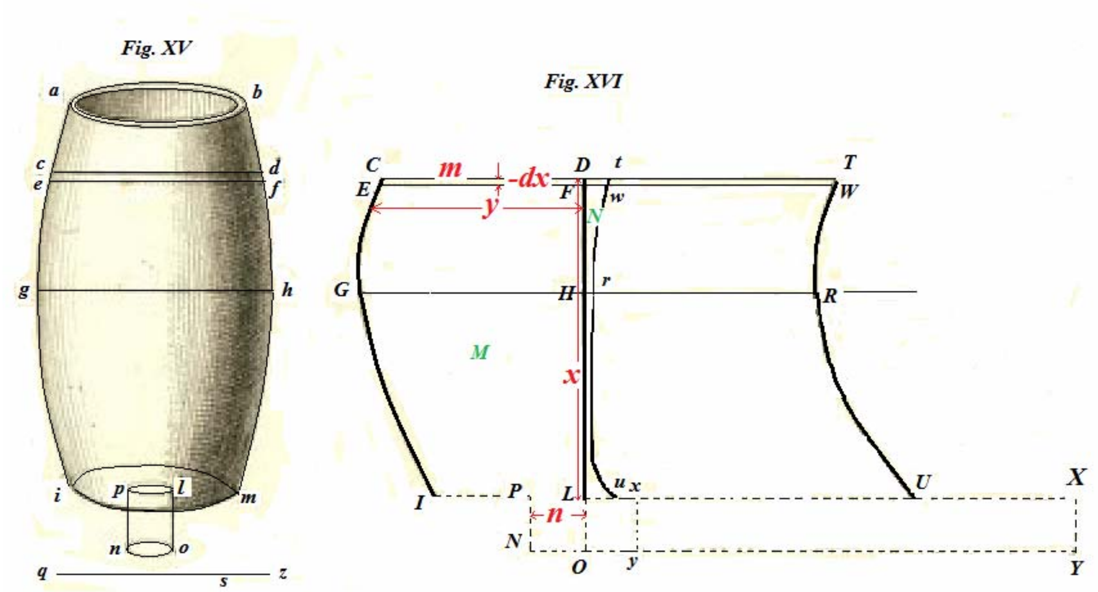
§. 3. To find the velocity of the water flowing out from a vessel constantly filled, after a given quantity of water has flowed out.

Solution.

Two ways of pouring the water are especially worthy of consideration, of which either postulates the solution of the other problem : indeed either the water is put to rain down vertically into the vessel and thus indeed, so that the same may flow on with exactly the same velocity that the water of the surface has, or the water may flow on sideways, and thus without impetus, so that its water shall follow the surface freely, and at last the motion is as required to be moved [by the surface].

Case I.

So that we may find the equation between the amount of water ejected and the corresponding velocity for the first case, with a single circumstance changed, it will be by insisting on treading the same path, which I have followed in the first paragraph of the third section.



Therefore the proposed vessel shall be *aimb* (Fig.15 & 16) as in § 6 Sect. III, because by the water flowing into the vessel, it is constantly kept full as far as *cd* ; moreover the water flows out by the opening *pl;* and that amount of water now going to flow out may be put present in the upright cylinder above the opening *pl* of height *x*, [*i.e.* along the vertical *x*-axis] but finally a droplet has flowed out with a velocity, by which it could rise to the height *qs* or *v;* thus now it will be required to find the equation between *x* & *v*.

The curve *CGI* shall be a measure of the [area of the] cross-section, truly such that with *HL* denoting the height above the opening, *HG* may express the cross-section of the vessel at that place. Then the third curve *tru* comes about, the applied line *Hr* of which everywhere shall be equal to the third continued proportional for *GH* & *PL* or applied line of which *Hr* shall be $= PL^2 : GH$ $\left[= n^2 : m = \frac{f_{out}^2}{f} \text{ in Sect.III} \right]$.

Calling the volume $DCIL = M$, the volume $DtuL = N$, and the *ascent potential* of the water contained in the vessel, after the preceding amount has now flowed out (by §. 2 Sect. III)$= \frac{N}{M} v$. [These are represented as areas in the diagram, but correspond to the actual volumes associated with the original surfaces of rotation.]. Again it is understood a small amount *plon* flows out, and the surface *cd* falls to *ef*, now the height corresponding to the velocity for the small amount *plon* $= v + dv$;and if now the parallelogram *LxyO* may be constructed, of which the side *LO* shall be *lo* and the other $Lx = PL$, the *ascent potential* of the same water in the place *efmlonpie* equals the fourth proportion to the volume *EFLONPIE* (which again is $= M$, because *PLON* expresses the magnitude of the droplet *plon*, while *CDFE* expresses the minimum amount equal to this droplet *cdfe*) of the volume *wuxyOLF* (which is $=$ to the volume $N - DtwF + LxyO$, from which if *PL* or *Lx* is put equal to $= n$, $CD = m,\ LO = lo = dx,$ there will be $Dt = \frac{nn}{m}, DF = \frac{n}{m} dx$, hence the small volume $DtwF = \frac{n^3}{mm} dx$ and the volume $LxyO = ndx$ and finally the volume $wuxyOLF = N - \frac{n^3}{mm} dx + ndx$) and the height $= v + dv$. Therefore the *ascent potential* in the manner said

$= (N - \frac{n^3}{mm} dx + ndx) \times (v + dv) : M =$ (with the differentials of the second order rejected)

$$\frac{N}{M} v + \frac{N}{M} dv - \frac{n^3}{mmM} vdx + \frac{n}{M} vdx,$$

thus so that the increment of the *ascent potential*, which is added to the water while the droplet *plon* flows out, shall be

$$= \frac{N}{M} dv - \frac{n^3}{mmM} vdx + \frac{n}{M} vdx,$$

where the volumes *N* & *M* are of constant magnitude on account of the continuous outflow of the water. We do not consider in this case the first *ascension potential* of the droplet *cdfe,* which has flowed in while the other equal one *plon* has flowed out, because this ascent is not generated by the internal force, nor indeed is the water below afterwards itself put to pull on the small amount *cdfe,* as rather we consider this rather to be poured in continuously by a continuous force, and that is performed with neither a greater or smaller velocity than that of the surface *ef.*

Therefore the whole increment [of the ascent potential] is required to be considered here as we have said :

$$\frac{N}{M}dv - \frac{n^3}{mmM}vdx + \frac{n}{M}vdx.$$

Truly this same increment must be equal to the actual descent of the centre of gravity. And this descent, on putting $DL = a$ [*i.e.* a constant height], is by paragraph 7 of Sect. III $= \frac{nadx}{M}$; therefore such an equation may be had :

$$\frac{N}{M}dv - \frac{n^3}{mmM}vdx + \frac{n}{M}vdx = \frac{nadx}{M},$$

or

$$dx = Ndv : \left(na - nv + \frac{n^3}{mm}v \right).$$

Truly this, if it may be integrated thus, so that *v* & *x* may vanish at the same time, gives :

$$x = \frac{mmN}{n^3 - nmm} \log \frac{mma - mmv + nnv}{mma},$$

which equation, on putting *c* for the number of which the logarithm is unity, is equivalent to this other form

$$v = \frac{mma}{mm - nn} \times \left(1 - c^{\frac{n^3 - nmm}{mmN}x} \right).$$

Truly this solution is squared [to find the speed] for the first case, where the water is poured onto the surface with a motion in common with the descent of the nearest surface.

Case II.

[According to Mikhailov, in the *Werke*, D'Alembert noted that the principle of living forces did not apply in this case, as pouring the water in sideways essentially introduced a new source of kinetic energy, in modern terms; Bernoulli of course had said that this on-pouring was to be done infinitely slowly.]

Because if now the small amount *cdfe* may be put to be continually poured in from the side, then on account of its inertia it offers resistance to the motion of the water below and hence the *ascent potential* comes into the calculation in another way. But then in the first place the *ascent potential* is required to be considered of the increased mass of water *cdmlpic* with drops being added soon; then the ascent potential of the same water in place at *cdmlonpic* is required to be found, the differential of which *nadx* is to be equated with the *actual descent* $\frac{nadx}{M}$. Truly the *ascent potential* of all the before mentioned water before the affusion [*i.e.* pouring on] of the particles and of the same after the affusion thus is found: certainly the *ascent potential* of the water *cdmlpic* is $=\frac{Nv}{M}$, and the *ascent potential* of the particles affused appears to be zero, because the laterally affused particles do not yet have the common motion of the mass below :Therefore the *ascent potential* of each amount of water (which evidently may be obtained respectively by multiplying each mass by its *ascent potential*, and by dividing the product of the sum by the sum of the masses) is

$$=\left(M\times\frac{Nv}{M}+ndx\times 0\right):(M+ndx)=\frac{Nv}{M+ndx}.$$

Truly after the small amount *ndx* now has been added above, it will acquire the common motion with the water nearest below, and thus the *ascent potential* of the same water in the situation *cdmlonpic* shall be made equal to the fourth proportional to the space [*i.e.* volume] *CDLONPIC* $(M+ndx)$, the space *DtuxyOLD* $(N+ndx)$ and the height $v+dv$, that is,

$$=\frac{(N+ndx)\times(v+dv)}{M+ndx},$$

of which the excess above the former *ascent potential* is

$$=\frac{Ndv+nvdx+ndxdv}{M+ndx}=$$

(with the differentials of the second order rejected)

$$\frac{Ndv+nvdx}{M}.$$

Therefore the equation of such may be obtained :

$$\frac{Ndv+nvdx}{M}=\frac{nadx}{M},$$

which treated as before and deduced at the end gives :

$$x=\frac{N}{n}\log\frac{a}{a-v},$$

or

$$v=a\times\left(1-c^{-\frac{nx}{N}}\right),$$

which solution prevails for the lateral affusion.

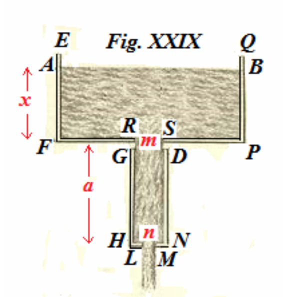
Scholium 1.

§. 4. Certainly these equations are different from each other ; but the difference is greater there when the vessel is of smaller cross-section ; and if indeed the cross-section of the vessel above at *cd* shall be as if infinite besides the cross-section of the opening, *n* vanishes besides *m* in the first case and becomes just as in the latter:

$$v = a \times \left(1 - c^{-\frac{nx}{N}}\right),$$

Therefore here in the hypothesis the motion is the same in both places as anyone could have foreseen without difficulty. But the motion is always quicker in the first affusion than the second, with all else being equal.

It is convenient to explain this matter physically also, so that we shall be able to understand that more clearly in all phenomena. There shall be for the sake of a shorter delineation, a vertical cylinder in place of any vessel and in whatever direction, with the opening at the base, namely *GHND* (Fig. 29) and then the vessel *EFPQ* shall have a hole at *RS*; the openings *RS* & *GD* are produced perfectly equal, and corresponding perfectly to the smallest distance itself, thus so that water flowing out from the upper vessel all flows into the cylinder placed below.



The water may begin to flow out from each vessel, but put to flow from that upper one constantly with the velocity that the surface of the water has in the cylinder placed below. Thus is appears to satisfy the first condition of affusion. Now truly we will investigate the phenomena of this motion, to see whether they will agree with the preceding.

Therefore we may consider the upper vessel to be as if infinite, so that the water flowing out through *RS* at individual times may have a velocity which agrees with the height *PB* or *FA:* thus this height *PB* will be required to be devised from the beginning very small, because then the water must flow out with an infinitely small velocity, then truly to increase slowly, and that continually to become more and more, then after an infinite time the motion may remain uniform, moreover it is sought whether a height of the water *PB* shall be reached at last after an infinite time or truly or truly a certain limit is not going to be passed over. Thus that will become known.

The height shall be *GH* or *RH* (for neither between those is agreed to differ) $= a$, $AF = x$, the cross-section of the opening $LM = n$, the cross-section of the opening $RS = m$; because truly, as it has been shown, each vessel hangs together and can be considered to be one, after an infinite time the velocity of the water (by §. 23 Sect. III) in $LM = \sqrt{a+x}$, and in $RS = \sqrt{x}$ (because it will be apparent later, if the vessels may again now be considered to be separated, since each can be considered [on its own]); but the velocities must be in the inverse ratio of the cross-sections of the openings: and thus $\sqrt{a+x} : \sqrt{x} :: m : n$, from which $a+x : x :: mm : nn$, or $a : x :: (mm-nn) : nn$, therefore $x = \frac{nna}{mm-nn}$ from which we see $a + x = \frac{mma}{mm-nn}$

therefore the height of the water due to the velocity at *LM* to be in this way $=\frac{mma}{mm-nn}$, evidently after an infinite amount of water now has flowed out : but above we have found the same height, or

$$v=\frac{mma}{mm-nn}\times\left(1-c^{\frac{n^3-nmm}{mmN}x}\right),$$

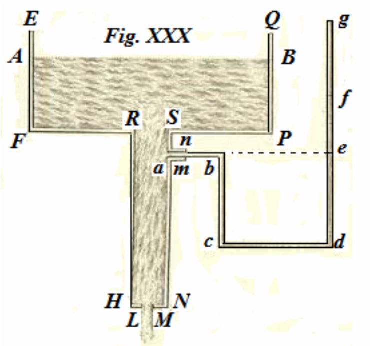
where if there is put $x=\infty$ (for in an infinite time an infinite amount of water will have flowed across) the term of the exponential vanishes, but only if *m* shall be greater than *n* and thus equally there becomes

$$v=\frac{mma}{mm-nn}.$$

This agreement is remarkable, because the paths we have followed are very different. For the remaining, if *m* shall not be greater than *n*, the motion shall never be permanent not even after an infinite time ; for then the velocity will increase indefinitely, since otherwise the height due to the velocity at no point may pass beyond the height $\frac{mma}{mm-nn}$. We shall say nothing about these cases.

Scholium 2.

§. 5. Now another question arises here worthy of note ; truly from which it shall be possible to measure the mechanics of the affusion, so that the upper vessel shall remain full according to the height due to the whole flow. This problem may be difficult on account of the inconstancy of the height sought, unless this particular trick may be used, that we now examine. Moreover it depends on that situation above, because the water in the smallest volume *RSDG* may experience neither a positive nor negative compression, because from the usual hypothesis since the water shall be moving with a velocity near the bottom, and thus it makes no attempt either to retain or expel any particles. Therefore each vessel becomes as I have said, and it shall be a strong tube with the upper vessel (for neither in any other way than for the sake of demonstration have we put these separate before); moreover the tube may have at the top *a* (Fig. 30) a small hole, to which the tubule *am* may correspond, in this tubule the curved glass tube may be inserted *abcdg,* with the opening *mn* protected by wax*:* and the horizontal line *ae* may be drawn and the point *e* is noted. Thus with these prepared, thus it will be required to be done, that in the course of the experiment the surface of the water shall remain constantly at the point *e;* and you will see towards this to be required, that from the beginning the surface of the water shall be approximately at the bottom *FP*, then, so that it is raised continually, and at last so that after a time even if infinite it will at no point pass beyond the height



$\frac{nna}{mm-nn}$; moreover thus it will be easy to moderate the affusion of water, so that the surface may not wander excessively from the point *e*, but only if the circumstances shall not be prepared thus, so that the water shall not be flooding in exceedingly swiftly.

Because if you should notice the surface in the tube were to be raised above *e*, restrain the affusion a little, because I will show it is required to be done in another way; if it were otherwise, then with a greater inundation of water.

This kind of experiment that I have done often presents nothing of difficulty, but lest an error may creep into the experiment, the effect of the glass capillary tube is required to be examined; this effect arises, if with the opening *IM* stopped up, prior to wetting the tube, the cylinder may be filled with water as far as the top, and thus you find the surface of the water in the tube to reach as far as to f, clearly a place higher than *e,* but you substitute this point f for that, about which we have talked just now, by taking into account the nature of capillary tubes.

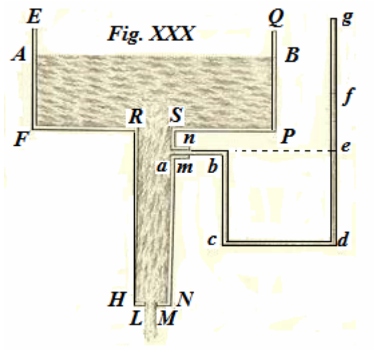
Therefore the affusion is effected correctly in this manner according to our standard hypothesis and thus thereafter experiments can be undertaken concerning this motion. Truly thus after we have explained this matter at length well enough, I think there is no need to remind that the upper vessel shall not be otherwise than attached to the lower cylindrical vessel, that we may consider only, so that that cylinder thus, when it must arise, must be kept full only and thus it is not understood by *m* to be the cross-section of the upper vessel but rather the cross-section of the opening *RS,* which properly for us is the surface of the water, since the water above *RS* must serve only to be affusing into the lower cylinder.

Scholium 3.

§. 6. Here I must not disregard, what thus may be considered which belongs to the hydrostatic case, knowledge about which I have made certain mention in Sect. I §.8; clearly we know now how great a velocity the water must flow past at *a* so that its pressure on the wall of the tube shall be precisely nothing. Truly while I may write about these, I had now discovered the general laws of *hydrostatics,* and I saw not without pleasure, that this case deduced in another way as a corollary from the theory, clearly acquires a similar solution from the general theory. Thus everywhere all these matters are mutually tied together, and they show a legitimate application of the principles.

Scholium 4.

§. 7. Certain affusions of water now follow by another way. The cylinder *RHNS* may be considered for some vessel, and this is required to be constantly full by affusion from the side : that will be able to happen by injecting a sufficient quantity of water through the tube *ma;* but since that cannot happen without motion, yet, because this is horizontal, soon all is taken away, and by itself can neither move the flow through the cylinder nor retard it; but there is another way in addition, which we understand for the same to be removed by a calculation performed correctly: evidently if we may consider the vessel *EFPQ* to be of infinite cross-section, and we understand its base to be covered over continually with water, but thus, so that the height of the water in the



upper vessel shall be required to be had infinitely small; the upper vessel will supply water by a tube connected to it, thence neither may other motion arise, than from the side affusion, but only if the opening *RS* may remain always covered up; moreover it will become easily as if a certain cataract may be formed, if the opening *LM* shall be large, and the tube *RSNH* long. Because this other way must exert the same effect in the motion of the water as the former, and anyone may be seen from that, how in each manner the water entering the tube shall be required to overcome the inertia from all the water below. But the same also can be demonstrated from first principles by inquiring into the motion, which thence must begin, following the equation of paragraph eight of Sect. III, which is this:

$$Ndv - \frac{mmvydx}{nn} + \frac{mmvdx}{y} = -yxdx \ ;$$

moreover it will be adapted to the present case, if for $m,\ x\ \&\ -dx$ you substitute respectively $n,\ a,\ \&\ \frac{ndx}{y}$ (the ratio of which will be apparent, if you compare these with those) and likewise you put *y* to be infinite; for then the third term of the equation vanishes, and altogether it shall be, as we have found from the present investigation above,

$$Ndv + nvdx = nadx \,.$$

Afterwards in these scholia the nature of each motion, as far as the simple physical consideration of the matter permits, and we have shown the difference of these, and likewise the manner these are required to be produced according to the hypothetical mechanical laws we have treated, it remains, that the remaining more remarkable phenomena also will be indicated now, which I shall now do.

Corollary 1.

§. 8. If in the vessel *RSNH* all the base shall be absent, then the opening $LM =$ to the opening *RS;* it is possible here even to out do that, if indeed the sides of the vessel may diverge. Moreover in these cases no term is had for the altitude *v* in the equation

$$v = \frac{mma}{mm - nn} \times \left(1 - c^{\frac{n^3 - nmm}{mmN}x} \right)$$

and it becomes infinite, if the quantity of water ejected indicated by *nx* is infinite.

That indeed is itself evident from the equation, when *n* is greater than *m;* but when the sizes of the openings are equal, it is required to return to the differential equation of paragraph three, from which that nearby equation was deduced, truly :

$$\frac{N}{M}dv - \frac{n^3}{mmM}vdx + \frac{n}{M}vdx = \frac{n}{M}adx\,,$$

which on putting $n = m$ gives $Ndv = nadx$, that is, $v = \frac{nax}{N}$,where *v* evidently shall be infinite if *x* is infinite.

Corollary 2.

§. 9. But if moreover there shall be a base for the proposed vessel, and in that a opening, the cross-section of which to be indicated by *n* shall be less than the cross-section of the opening *RS* expressed by *m, v* has a value that indeed it can never reach, but still it can almost approach, and to which it converges very quickly, unless for a given need the vessels may be used thought out contrary to this matter, so that after the shortest time interval the flow, which shall be perceived by the senses, shall not be perceptibly different from *mma*. But that term is such, $v = \frac{mma}{mm-nn}$: therefore in the case of the second Scholium of §. 5 the final term is $PB\ = v - a = \frac{nna}{mm-nn}$. I will illustrate the approach to its final term with an example of the quickest velocity, after I have put in place the equation between *v* and the time corresponding to the height *v*.

Corollary 3.

§.10. In the case of affusion, that we call from the side, the final height becomes $v = a$, whatever ratio may exist between each of the openings of the vessel.

Corollary 4.

§. 11. If the vessel is cylindrical and the length of that may be put $= b$, there becomes (see §. 3) $N = \frac{nnb}{m}$: but is must be observed that the values of the letters *a* and *b* are not to be confused, for the first expresses the height of the upper orifice above the lower, the other the length of the channel; and thus therefore in this case the values agree between themselves anyhow, when the axis of the vessel is a right line and vertical; but if the axis is twisting, or perhaps not vertical, in turn they differ from each other: Thus I have wished expressly to remind about this, lest anyone would himself feel, from the figures I have made of the vessels, that the axes of which be established everywhere from right lines and verticals.

Because if therefore for cylindrical vessels there may be put $N = \frac{nn}{m}b$, there shall be for the vertical affusion

$$v = \frac{mma}{mm-nn} \times \left(1 - c^{\frac{nn-mm}{mnb}x}\right)$$

and for the other from the side there becomes

$$v = a\left(1 - c^{\frac{-mx}{nb}}\right).$$

Problem.

§. 12. To find the velocity of the outflow of the water from a vessel filled constantly, after the flow has endured for a given time.

Solution.

With all the hypotheses and denominations retained, which we have used in §.3, and in addition on putting the time passed from the start $= t$, we will have equations requiring to be changed in the said paragraph given by others, which express the relation between *t* and *v*, with the quantities *x* and *dx* eliminated. Truly the first element of the infinitesimal time dt is in proportion to the smallest increment of distance *dx*, which is run through, divided by the velocity $\sqrt{v}$: therefore we may put $dt = \frac{\gamma dx}{\sqrt{v}}$, and thus the equation will be modified :

$$dx = Ndv : (na - nv + \frac{n^3}{mm}v)$$

which was given for the vertical affusion, with the velocity owed required to be put in place in this equation :

$$(\mathrm{I}) \qquad dt = N\gamma dv : (na\sqrt{v} - nv\sqrt{v} + \frac{n^3}{mm}v\sqrt{v}),$$

truly with the second affusion being looked after from the side, truly $dx = Ndv : (na - nv)$, it will be changed into this, after the same substitution:

$$(\mathrm{II}) \qquad dt = N\gamma dv : (na\sqrt{v} - nv\sqrt{v}).$$

Truly these two equations, in the due manner of the integration give, for the first :

$$(\alpha) \qquad t = \frac{mN\gamma}{n\sqrt{(mma - nna)}} \times \log\frac{m\sqrt{a} + \sqrt{(mmv - nnv)}}{m\sqrt{a} - \sqrt{(mmv - nnv)}}$$

[For :

$$dt = N\gamma dv : \ (na\sqrt{v} - nv\sqrt{v} + \frac{n^3}{mm}v\sqrt{v}) = \frac{N\gamma m^2}{n}\frac{dv}{m^2a\sqrt{v} - m^2v\sqrt{v} + n^2v\sqrt{v}};$$

putting $\sqrt{v} = y$ and $v = y^2$ giving $dv = 2ydy,$ then

$$dt = \frac{N\gamma m^2}{n}\frac{2ydy}{m^2ay - m^2y^3 + n^2y^3} = \frac{2N\gamma m^2}{n}\frac{dy}{m^2a - \left(m^2 - n^2\right)y^2}$$

$$= \frac{2N\gamma}{na}\frac{dy}{1 - \left(\frac{m^2 - n^2}{m^2a}\right)y^2}; \text{putting } \left(\frac{m^2 - n^2}{m^2a}\right)y^2 = z^2, \text{we have } \frac{z}{y} = \sqrt{\frac{m^2 - n^2}{m^2a}};$$

$$\left(\frac{m^2 - n^2}{m^2a}\right)ydy = zdz; dy = \left(\frac{m^2a}{m^2 - n^2}\right)\frac{z}{y}dz = \sqrt{\frac{m^2a}{m^2 - n^2}}dz;$$

$$dt = \frac{2mN\gamma}{n\sqrt{a\left(m^2 - n^2\right)}}\frac{dz}{1 - z^2}.$$

]

Hence, $t = \frac{2mN\gamma}{n\sqrt{a\left(m^2 - n^2\right)}}\int\frac{dz}{1 - z^2} = \frac{mN\gamma}{n\sqrt{a\left(m^2 - n^2\right)}}\log\frac{1 + z}{1 - z}$; But $z = y\sqrt{\frac{m^2 - n^2}{m^2a}}$,

hence $t = \frac{mN\gamma}{n\sqrt{a\left(m^2 - n^2\right)}}\log\frac{m\sqrt{a} + \sqrt{m^2v - n^2v}}{m\sqrt{a} - \sqrt{m^2v - n^2v}}$.]

and for the second, which was deduced previously, on putting $m = \infty$,

(β) $$t = \frac{N\gamma}{n\sqrt{a}} \times \log\frac{\sqrt{a} + \sqrt{v}}{\sqrt{a} - \sqrt{v}}.$$

Q.E.I.

Scholium.

§.13. If the vessel about which it has been spoken shall be cylinder twisted and inclined in some manner, the length of which may be put $= b$, with the height of the water remaining above the opening $= a$, again there will be, as in §. 11, $N = \frac{nn}{m} b$.

But because, since the constant $2\gamma\sqrt{A}$ expresses the time, which a body spends in falling freely from rest through a height *A*,
[For, $dt = \frac{\gamma dx}{\sqrt{v}}$ and in modern terms, setting $x = \frac{1}{2}gt^2$gives $dx = gtdt$ and $t = \frac{V}{g} = \sqrt{2H/g}$, giving $dx = gtdt = \sqrt{2gH}dt$; hence $\frac{\gamma}{\sqrt{A}} = \frac{1}{\sqrt{2gA}}$ or $\gamma = \frac{1}{\sqrt{2g}}$.
Hence in modern terms, $2\gamma\sqrt{A} = \sqrt{2A/g} = T$, the required time to fall.]
it is apparent that $\frac{2mN\gamma}{nn\sqrt{a}}(= 2\gamma\sqrt{\frac{bb}{a}})$, [since $N = \frac{nnb}{m}$] expresses the time in which a body beginning to move freely from rest falls through a height $\frac{bb}{a}$: we may accept this time for the general measure, and we may put that $= \theta$, and the equation(α) for the cylinder or channel will be changed into this :

$$t = \frac{n\theta}{2\sqrt{(mm - nn)}} \times \log \frac{m\sqrt{a} + \sqrt{(mmv - nnv)}}{m\sqrt{a} - \sqrt{(mmv - nnv)}},$$

truly for the second assigned (β), it will become such :

$$t = \frac{n\theta}{2m} \times \log \frac{\sqrt{a} + \sqrt{v}}{\sqrt{a} - \sqrt{v}},$$

and from each it is apparent, it is possible in the shortest time to acquire almost all the velocity, and with that the quicker as the wider is the tube, as it is shorter, and as it is more vertical: Nor are accelerations to be perceived, unless very long water ducts may be put in place and then also in a short time nearly all the levels of the accelerations will have run through, as each now I will show by example.

(I) The time is sought in which the velocity will be acquired which may be due to a height of $\frac{99}{100}a$, where the flow is from a vertical cylinder constantly filled, with a length of sixteen English feet and of which the diameter shall be five times that of the opening, and that in the hypothesis, to which the second equation pertains ; thus there is $\frac{n}{m} = \frac{1}{25}$, $v = \frac{99}{100}a$, $b = a$, from which the time spent in which a body falls from rest by falling through a distance $\frac{bb}{a}$ [or 16 ft.], or $\theta =$ one second ; hence the equation becomes $t = \frac{1}{50}\log 399$, that is, nearly the 9th part of a second, which short time is certainly imperceptible; when truly the time is assumed notable, the changes of the heights *v* become unobservable. Likewise if the time may be sought under the first hypothesis (when surely the velocity due is generated together with the ninety-nine parts of a hundred of the height, with the amount due after the time becomes infinite), evidently the time when it reaches $v = \frac{99}{100} \times \left(\frac{mma}{mm - nn} \right)$ is found a little greater than from the preceding, but by an unmeasurable excess: from which it is apparent in vessels of this kind, water is not able to be added nearly quickly enough into the upper vessel, so that it may satisfy the hypothesis, nor indeed on account of the same hypothesis can other experiments be prepared, so that it may be investigated, whether in fact the height *BP* shall be so great in Fig. 30, as to be due to the living force of paragraph five, so that the point placed at *e* or *f* may serve for the duration of the flow, which before the flow with the opening *LM* blocked, had no water present in the upper vessel.

(II) Now again the same time is sought for the second hypothesis, if the tube were the same size and constructed with the same opening, but placed at an angle and the length *b* would be had 184 poles or 1104 *Paris ft.* while the height of the surface of the water above the efflux of the opening shall be 16 *Paris ft.* thus there comes about $b = 1104$, & $\frac{bb}{a} = 76176$ and the time will be around $\theta = 72$ *sec.*, from which the time sought is between eight and nine seconds, which certainly is notable enough. Truly if the time may be desired, when the height *v* may be equal only to a fourth part of the height *a*, that may be found to be equal to $\frac{72}{50} \cdot \log 3 =$ one and a half seconds approximately.

I know not whether these may agree with those, which Mariotte referred to from his observations in Treatise on the motion of water [*Tract. de mot. aquar.*], Part. 5, Disc. 1, where he made a mention of some leaping fountain, which is at Chantilly , towards which water is carried downstream by a channel 184 poles long, but only if I may conjecture from the preceding, and the greatest height of the surface of the water above the efflux opening, indicated by *a*, was sixteen feet*:* the diameter of the water course was 5 inches, and moreover the opening had a diameter of one inch. It seems to me that Mariotte thus talked as if the accelerations were much slower than are shown by our formula, as I do not know whether it should be attributed to this opening or perhaps to another, besides the one about which mention has been made, or whether the flow began while the water channel was not full of water, because they were done much later, as I believe ; if neither were the case, I am confident the phenomena observed by Mariotte were of such a kind, and to be observed anew daily, were able clearly to be in agreement with our calculation. These are the remaining words of Mariotte: *That above,* he said, *happen for the same jet, which with the opening blocked by hand for a length of time of ten or twelve seconds and the same opened up after, the water does not burst out immediately, but rises little by little to a height of* 3 *inches, later to a height of a foot and finally to two feet successively for notable intervals. . . . But yet at last the water bursts forth with its whole impetus.*

Problem.

§. 14. To find the quantity of water flowing through a vessel in a given time, constantly being replenished.

Solution.

Again with the positions and denominations of the third and twelfth paragraphs used again, now the equation between x and t is required to be found : because truly, as we have seen in §.12, there is $dt = \frac{\gamma dx}{\sqrt{v}}$, and there will be $\sqrt{v} = \frac{\gamma dx}{dt}$ and this value will be substituted into the equations, which we have given with the integrals in §. 3 ; this was the first of these equations :

$$v = \frac{mma}{mm-nn} \times \left(1 - c^{\frac{n^3-nmm}{mmN}x}\right),$$

which before the following may be put in place may be changed into this (I)

$$\text{(I)} \qquad \frac{\gamma\gamma dx^2}{dt^2} = \frac{mma}{mm-nn} \times \left(1 - c^{\frac{n^3-nmm}{mmN}x}\right);$$

the second from §. 3 of the discussed equations was such

$$v = a \times \left(1 - c^{\frac{-n}{N}x}\right),$$

which thus in the present case provides the following :

$$\text{(II)} \qquad \frac{\gamma\gamma dx^2}{dt^2} = a \times \left(1 - c^{\frac{-n}{N}x}\right).$$

Now these equations (I) & (II) shall be integrated, which indeed is easy and because the first contains the second (for each is the same if $m = \infty$), we will handle this only, and we will consider that equation now in this form :

$$dt = \frac{\gamma\sqrt{(mm-nn)}}{m\sqrt{a}} dx : \sqrt{\left(1 - c^{\frac{n^3-nmm}{mmN}x}\right)}.$$

Moreover so that the manner of integration may become more clear there is put $c^{\frac{n^3-nmm}{mmN}x} = z$, and hence

$$dx = \frac{mmNdz}{(n^3 - nmm)z},$$

therefore for brevity the constant quantity

$$\frac{\gamma\sqrt{(mm-nn)}}{m\sqrt{a}} \times \frac{mmN}{n^3 - nmm}$$

or

$$\frac{-\gamma mN}{n\sqrt{(mm-nn)a}}$$

may be indicated by α , and there will be had

$$dt = \frac{\alpha dz}{z\sqrt{(1-z)}},$$

in which if in addition there becomes $1 - z = qq,$ or $z = 1 - qq,$ $dz = -2qdq$, there arises

$$dt = \frac{-2\alpha dq}{1-qq} = -\frac{\alpha dq}{1+q} - \frac{\alpha dq}{1-q},$$

of which the integral is

$$t = -\alpha \log(1+q) + \alpha \log(1-q) = \alpha \log \frac{1-q}{1+q}.$$

Nor is there a need for a constant, when indeed from the nature of the problem *t* and *x* must vanish at the same time ; for on putting $x = 0,$ there becomes $z = 1,$ and $q = 0$, therefore equally *t* and *q* likewise must begin from zero, to which condition the equation found $t = \alpha \log \frac{1-q}{1+q}$ gives satisfaction : It remains that we put reassume the first values by reversing the arrangement, thus there becomes :

$$t = \alpha \log \frac{1-\sqrt{(1-z)}}{1+\sqrt{(1-z)}}$$

or

$$t = \frac{\gamma mN}{n\sqrt{(mm-nn)a}} \times \log \frac{1+\sqrt{(1-z)}}{1-\sqrt{(1-z)}}$$

or finally

(I) $$t=\frac{\gamma mN}{n\sqrt{(mm-nn)a}}\times\left(\log\left(1+\sqrt{1-c^{\frac{n^3-nmm}{mmN}x}}\right)-\log\left(1-\sqrt{1-c^{\frac{n^3-nmm}{mmN}x}}\right)\right).$$

And this equation on putting $m=\infty$ gives the other equation sought :

(II) $$t=\frac{\gamma N}{n\sqrt{a}}\times\left(\log\left(1+\sqrt{1-c^{\frac{-n}{N}x}}\right)-\log\left(1-\sqrt{1-c^{\frac{-n}{N}x}}\right)\right).$$

Q.E.I.

Corollary 1.

§. 15. If there may be put $x=\infty$, as may be apparent from the nature of the question, now when an infinite amount of water has flowed through, and it may be assumed that *m* is greater than *n*, just as generally it is accustomed to be, and the size of the exponential is required to vanish in each term taken with the positive logarithm, and each will become log 2. But truly in the logarithm with the negative taken, it is required to put in place

$$\sqrt{1-c^{\frac{n^3-nmm}{mmN}x}}=1-\tfrac{1}{2}c^{\frac{n^3-nmm}{mmN}x}$$

and hence,

$$\log\left(1-\sqrt{1-c^{\frac{n^3-nmm}{mmN}x}}\right)=\log\left(\tfrac{1}{2}c^{\frac{n^3-nmm}{mmN}x}\right)=\frac{n^3-nmm}{mmN}x-\log 2.$$

These substitutions if made correctly, will be for the first method of affusion that we have devised :

(I) $$t=\frac{\gamma mN}{n\sqrt{(mm-nn)a}}\times\left(2\log 2+\frac{mmn-n^3}{mmN}x\right),$$

which again on putting $m=\infty$ gives for the second case

(II) $$t=\frac{\gamma N}{n\sqrt{a}}\times\left(2\log 2+\frac{n}{N}x\right).$$

From these formulas it follows, indeed for a smaller quantity of water to be transferred, than if suddenly from the start all the water were to flow out with the velocity that in each case was acquired after an infinite time : yet the difference at no time shall go past a certain term, and after an infinite time to be understood by finite terms.

Corollary 2.

§.16. When we invert these equations found, we obtain

(I) $$x = \frac{2mmN}{mmn - n^3}\left(\log\left(1 + c^{\frac{-t}{\alpha}}\right) - \log 2 + \frac{t}{2\alpha}\right),$$

& (II) $$x = \frac{2N}{n} \times \left(\log\left(1 + c^{\frac{-t}{\beta}}\right) - \log 2 + \frac{t}{2\beta}\right),$$

where α , as above, $= \frac{-\gamma mN}{n\sqrt{(mm - nn)a}}$ & $\beta = \frac{-\gamma N}{n\sqrt{a}}$.

If besides, as in the above corollary, there is put $t = \infty$, unity will vanish in comparison with the exponential quantities, which are infinite above all orders, and there becomes

$$\log\left(1 + c^{\frac{-t}{\alpha}}\right) = -\frac{t}{\alpha} \text{ and } \log\left(1 + c^{\frac{-t}{\beta}}\right) = -\frac{t}{\beta}:$$

from which then with the values of the letters α and β resumed,

(I) $$x = \frac{mt\sqrt{a}}{\gamma\sqrt{(mm - nn)}} - \frac{2mmN}{mmn - n^3}\log 2,$$

& (II) $$x = \frac{t\sqrt{a}}{\gamma} - \frac{2N}{n}\log 2.$$

Therefore if immediately from the beginning, the flow of all the waters in both places will flow out constantly with that velocity they can acquire, but will not exceed the size of those quantities after an infinite amount of time, for the same time corresponding to the theory except by a small amount, which in the first case is expressed by $\frac{2mmN}{mm - nn}\log 2$, and in the second case by *2N* log 2. And if in place of an infinite time you take a time only of some fractions of a second, the same theorem above will have a place; thus so that if for example after the first tenth of a second the amount *Q* will have flowed out, the efflux shall be almost just the same amount as by the following approximation $Q + \frac{2mmN}{mmN - n^3}\log 2$, or in the other case $Q + \frac{2N}{n}\log 2$.

Scholium.

§. 17. The motion of water through siphons also relates to the theory explained so far. Moreover the theory indicates, the axis of the siphon to be bent at any angle, nor thence is the motion of the water going to be disturbed, only the height of the surface of the water above the orifice of the efflux shall remain the same ; since besides aquaducts, siphons or diabetes [*i.e.* the original Greek $\delta\iota\alpha\beta\eta\tau\eta\varsigma$ for siphon] of this kind, other vessels shall usually be cylindrical, there will be as I reminded in §. 13, whenever that may happen, it is required to put $N = \frac{nn}{m}b$, on understanding by *b* the length of the channel or of the siphon: also in the formulas of paragraphs 14, 15, & 16, the quantities thus will be required to be interpreted, where it is sought from the times, that $2\gamma\sqrt{A}$ will represent the time which a body spends in descending through a vertical height A beginning from rest.

For the remainder, as I have said elsewhere, the theory of this section indicates nothing unusual that falls under the senses, except in exceedingly long water channels, inclined very obliquely to the horizontal and having openings which are not too narrow; indeed these three conditions concur towards requiring to be retarded and thus notably by having an effect on the accelerations, the measures of which chiefly commend the theory.

However, in these circumstances there is some middle value requiring to be observed, lest the impediments arising from the adhesion of the water shall be excessive.

What pertains to the affusion of water, I myself have been seen to observe, if it should be made vertically and with impetus, to depart so much, as then the motion is accelerated rather than retarded, unless the affusion of the water happens there equally on the whole surface, that I set out in §. 4 ; for if indeed it may be poured on otherwise, the motion of the water in the vessel shall be disturbed, and this confused motion retards the outflow.

§.18. Finally here in a certain manner the experiments set up by the most celebrated Giovanni Poleni are relevant, as he refers in *Book One, De motu aquae mixto* [*Libri duo, Patavii* 1717 : *Concerning the motion of mixed water in two books*, Padova], p. 21 ff., which I have considered here to be adduced [*i.e.* cited as examples], because they show especially the final speed everywhere in vessels filled constantly to be that, which may be appropriate for the whole height of the water, as long as the vessels shall not be submerged, or for the difference of the heights of the water inside and outside for submerged vessels, although in the remainder there shall be nothing in these which now at this point would be new, because no accelerations shall be considered there.

Consider a cylinder, whose axis may be had placed vertical, as if of infinite cross-section; the whole base shall be : but in the wall there shall be an opening parallel to the axis, the cleft forming a rectangular parallelogram, which shall be extended from the base as far as to the top of the cylinder. Consider again water to be poured equally into the cylinder, thus, so that in equal times equal amounts shall be injected, the water will flow out from the cylinder through the opening : nor yet from the start will the same amount flow out, which is being poured in above, but to be less: therefore the surface of the water will rise in the cylinder to a certain asymptotic height ; truly if this is now understood to be the end present, the height of the water will remain unchanged and the same amount of water will flow out constantly, which will be flooding in on top. It is apparent also, the height of the water in the cylinder to be greater there, when it is being poured on more plentifully : Therefore it is sought, with increased amounts of water flooding in a given time, in what ratio the heights must increase, to which the water in the cylinder will rise.

This is the solution. The height of the water shall be, when in a permanent state, $= \alpha$, and a part may be cut from the surface which shall be $= x$, together with the differential *dx;* let the width of the cleft $= n$. Just as we will have for the cross-section of the opening $= ndx$, through which the water flows out with the velocity $\sqrt{x}$: [In modern terms this velocity would be $\sqrt{2gx}$.] therefore the amount of water flowing out there in a given time is as $ndx\sqrt{x}$, the integral of which is $\frac{2}{3}nx\sqrt{x}$; which expresses the amount of water flowing out through a length of the cleft of abscissa *x* in a given time : and thus the amount of water flowing in through the cleft in the same time will be expressed by $\frac{2}{3}n\alpha\sqrt{\alpha}$. But just as much flows out as flows in ; hence if the amount of water affused in a given time may be called *q*, there will be $\frac{2}{3}n\alpha\sqrt{\alpha} = q$. That indicates the amount of water requiring to be affused in a given time to follow the three on two ratio of the power of the height, to which the water rises from the bottom of the cylinder : or in turn the heights follow the cube roots of the squares of the amounts, by which the water is poured on top in a given time.

§. 19. With the solution to this problem I come to another the most celebrated Poleni considered. There shall be the same cylinder, but submerged by stagnant water in a ditch as if in an infinite vessel; and the height of the submersion is said to be $= a$, now from the same in place there is found, as before, again an equation between the height α of the surface of the water inside above the external surface, and the amount of flooding required in a given time *q*.

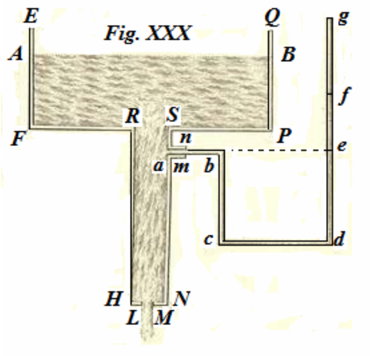
Because according to that part α of the cleft, which expels the water and emerges above the external water, we now see that in a given time the quantity $\frac{2}{3}n\alpha\sqrt{\alpha}$ to be set free: but the remaining part of the cleft submerged transfers the water everywhere with a common velocity, so that it will be apparent from what is said below, and indeed with a velocity $\sqrt{\alpha}$, thus, so that with this velocity multiplied by the magnitude of the submerged cleft *na*, the amount will be had, which is expelled in a given time, $= na\sqrt{\alpha}$. If each amount may be added into the sum, there will be had $(\frac{2}{3}\alpha + a)n\sqrt{\alpha} = q$.

With the aid of this equation, *q* is known from given heights *a* and α : or in turn the height *a* is known from the known amounts *a* & *q*.

Moreover this equation agrees exceedingly accurately with experiments, itself shown by the most celebrated author of these, whose solution does not differ from our one here. If follows from this equation, the heights α there to be greater for the same affusions of water, where the height of submersion *a* is less.

Experiments which pertain to Section V.

For §.5. With the vessel I have used in §.5 described with the glass tubule (Fig. 30). But at first I blocked the opening *LM* and filled the tube *RN* with water, then a little opening was scraped in its surface at *a:* then I observed the water advance in the tube to reach the extreme point *f*: after I releases the opening *LM*, and with water pouring in anew I flooded the above vessel *EFPQ* using the water with care so that the level of the water meanwhile at *f* neither rose nor fell. While these were happening, I was raising the surface *AB*, but at no time did it cross over a certain limit; certainly it was the case, as far as I was able to see, the maximum height *PB* or



$FA = \frac{nn}{mm - nn} a$, with $\frac{n}{m}$ denoting the ratio between the lower opening *LM* and the upper *RS*, and *a* the vertical height of the latter opening above the former.

Truly, this is the only experiment which I have set up myself, although there shall be many propositions contained in this section, which may deserve attention and these to be of an unexpected enough nature, yet I was unable to undertake experiments with these; thus as they were prepared in shorter vessels, so that anything unusual about them may escape the senses, that I have been unable to try out conveniently in longer conduits: when the opportunity may be given for this theory to be examined by others, they should attend to the following:

I. In leaping fountains the whole height of the jet should be observed; with the opening first stopped up, and with the same opened up again, soon an amount of water may be seen which flows out, while the water may arrive at half the height of the whole jet, or some other fraction, because indeed it will come about in the shortest time, the measure of this amount shall be the length of the cylinder above the orifice, through which the water leaps out, piled up to a length we have called *x*, however we have called the whole jet *a*, and the height of the jet observed which does not yet reach the whole height we have called *v*. Then at last with a calculation in place it may be investigated, whether these quantities may correspond correctly to the equations for each manner of pouring shown in paragraph three.

II. Everything may be done as before, perhaps with this difference, so that in place of the amount of the efflux, the time of the efflux may be noted, so that thus the formulas of paragraph ten shall be able to be examined, and at last the amount may be compared with the time of the flow, so that whether it may be apparent to correspond correctly to the formula of §.14.

III. Then that kind of experiment must be done especially, which I have indicated in paragraph sixteen, clearly by observing the amounts of water corresponding to half the times; moreover I hade said, however great a time may be taken, the difference of these amounts at no time to be equal to $\frac{2mmN}{mmn - n^3} \log 2$ in the former, which we have devised in the manner of pouring; or $\frac{2N}{n}\log 2$ in the latter. But with these differences, in as much as they never arise perfectly, yet to be in the shortest possible time arising.

What remains in this section are the corollaries and scholia that anyone will see easily, how they shall be able to be addressed to the experiments: But I may wish, before passing judgment, attention shall be paid to all the circumstances on account of all the impediments, of the narrowing of the jets, and of all the other matter that I am unwilling to repeat everywhere.

To §§. 18 & 19 The experiments for the confirmation of problem §.19 pertaining to vessels not submerged, see p. 26, *lib. cit.* of Poleni.

However since in a submerged vessel the height $a = 55$ *Paris lines.* (which height is said by him to be dead), five experiments have been put in place, in which the height, as he said, to be alive or α was the succession of the lines $8\frac{1}{2};\ 25;\ 42;\ 58\ \&\ 73\frac{1}{2}$: with these values substituted into the equation shown in §.18 it follows, the amounts of water affused in the given times to be as 100; 199; 299; 396 & 495: actually the affusions were in the ratio as 100, 200, 300, 400, & 500: the difference is so small, that is may be doubted, whether they would not be perfect agreement in the future, if all the measurements had been able to be taken with perfection.

Also the experiments remaining put in place by that most celebrated man are in complete agreement with the theory: the calculation of these is to be seen in the author's works. But I have been led to append the same with these matters here, because they pertain to the argument of this section, although I acknowledge freely that others need more experiments to be done by me, which depend on *momentary* changes, which I know no one has considered at this time, and which also support the *permanent* state.

HYDRODYNAMICAE SECTIO QUINTA.

De motu aquarum ex vasis constanter plenis.

§. 1. Vasa plena servantur, cum continue totidem affunduntur aquae, quot effluunt; affusio autem esse potest vel in eadem cum motus superficiei aqueae directione eademque singulis momentis velocitate, quasi scilicet nova continue crearetur superficies, cui velocitas aquae proximae jam insit, vel lateralis & sine impetu, veluti si superficies, quae continue nova creari fingitur, nullo motu praedita sit & demum ab aqua inferiore ad motum cienda. Reliquos affundendi novas aquas, qui infiniti sunt, modos praeteribo.

Regula interim circa hunc motum, praesertim posteriorem, recepta est, aquam effluere velocitate conveniente altitudini superficiei supra lumen: facile tamen est praevidere illam valere non posse, nisi pro vase ubique infinite amplo, in reliquis autem fore, ut motus a quiete incipiens sensim sensimque per aliqua temporis intervalla augeatur, & post infinitum demum tempus omnem velocitatem acquirat. Attamen, si dicendum, quod res est, fiunt istae accelerationes plerunque tam celeriter, ut minimo tempusculo tantum non tota velocitas adsit: Verum res secus se habet in praelongis aquae ductibus, in quibus velocitatum augmenta oculos non effugiunt & cum distinctis mensuris observari possunt.

Quicquid autem ejus rei sit, cum nullibi displicere possit accuratio mathematica, constitui motum aquarum a principio ad quemvis datum terminum considerare & prosequi.

§. 2. Omnes hujus motus proprietates ad tres praecipue aequationes se reduci patiuntur 1°. inter quantitatem aquae ejectae respondentisque velocitatis; 2°. inter tempus & velocitatem & 3°. inter quantitatem aquae & tempus. Harum aequationum si una habeatur reliquae inde sua sponte fluunt.

Primam igitur solam accuratius scrutabimur: Hic vero memores simus eorum, quae in praecedente sectione monita fuerunt circa contractionem venae per simplicia orificia, aut tubos convergentes effluentis, & dilatationem ejusdem, cum per tubos divergentes ejicitur. Indicavimus autem §. 3 art. 1 Sect. IV eo usque venam considerandam esse, donec particularum velocitates (abstrahendo animum a mutationibus quas gravitas in particulis extra vas producit) amplius non mutentur, & omnem illam venae partem ceu intra vas motam aestimandam esse, quasi scilicet superficies venae eousque indurescat. lgitur deinceps cum de vase per quod aquae effluunt sermo erit, subintelligendum erit vas illud ideale, cujus orificium effluxus sit sectio venae nulli deinceps mutationi subjectae, nisi quae descensui vel ascensui venae debetur.

Problema.

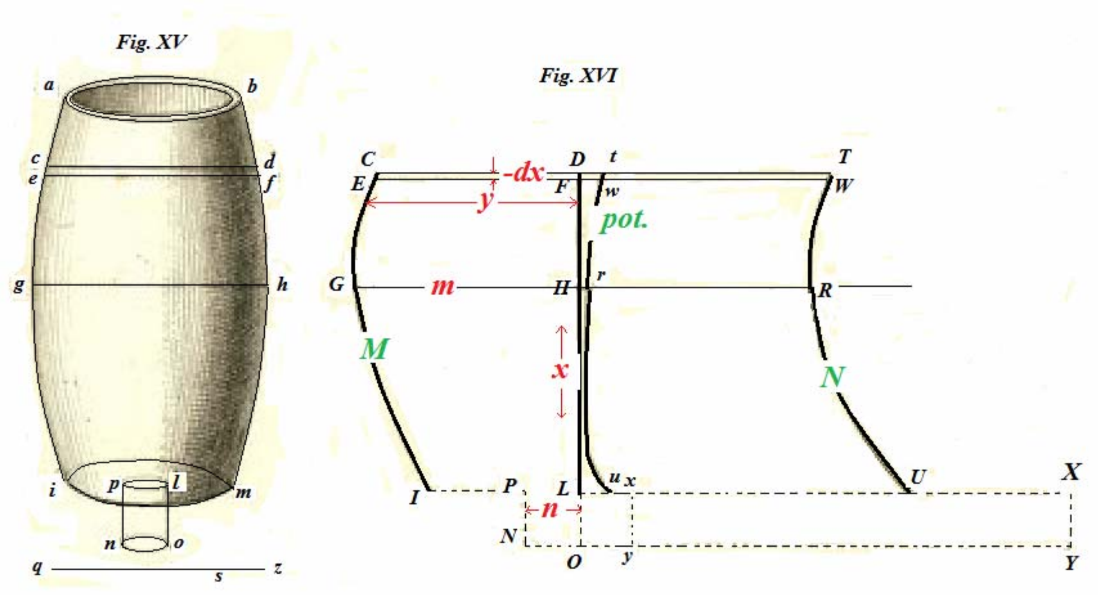
§. 3. lnvenire velocitatem aquae effluentis ex vase constanter pleno, postquam jam data aquae quantitas effluxit.

Solutio.

Duo sunt modi affundendae aquae praecipue consideratu digni, quorum quivis aliam postulat problematis solutionem: vel enim aqua verticaliter in vas depluere ponitur & ita quidem, ut eadem praecise affluat velocitate, quam habet aquae superficies, vellateraliter affluit aqua, sicque caret impetu, quo sua sponte aquae superficiem insequi possit, & in motum demum est cienda.

Casus I.

Ut pro primo casu aequationem inveniamus inter quantitatem aquae ejectae velocitatemque respondentem, iisdem unica mutata circumstantia vestigiis insistendum erit, quae in primis paragraphis sectionis tertiae secuti sumus.



Sit igitur ut in §. 6 Sect. III vas propositum *aimb* (Fig.15 & 16), quod affusione aquarum constanter plenum servatur usque in *cd;* effluant autem aquae per foramen *pl;* ponaturque eam aquae quantitatem jam effluxisse, quae contineri possit in cylindro super foramine *pl* erecto altitudinis *x,* ultimam autem guttulam effluxisse velocitate, qua ascendere possit ad altitudinem *qs* seu *v;* sic jam exhibenda erit aequatio inter *x* & *v*.

Sit curva *CGI* scala amplitudinum, talis nempe, ut, denotante *HL* altitudinem supra foramen, exprimat *HG* amplitudinem vasis in illo loco. Deinde fiat tertia curva *tru,* cujus applicata *Hr* sit ubique aequalis tertiae continue proportionali ad *GH* & *PL* seu cujus applicata *Hr* sit $=PL^2:GH$.

Dicatur spatium $DCIL=M$, spatium $DtuL=N$, & erit *ascensus potentialis* aquae in vase contentae, postquam praedicta quantitas jam effluxit (per §. 2 Sect. III) $=\frac{N}{M}v$. Effluere porro intelligatur particula *plon,* superficiesque *cd* descendere in *ef,* erit jam velocitatis altitudo pro particula *plon* $=v+dv$; atque si nunc construatur parallelogrammum *LxyO*, cujus latus *LO* sit *lo* & alterum $Lx=PL$, erit *ascensus potentialis* ejusdem aquae in situ *efmlonpie* aequalis quartae proportional ad spatium *EFLONPIE* (quod rursus est $=M$, quia *PLON* exprimit magnitudinem guttulae *plon,* dum *CDFE* exprimit quantitatem minimam *cdfe* isti guttulae aequalem) spatium *wuxyOLF* (quod est spatio $=N-DtwF+LxyO$, unde si *PL* seu *Lx* ponatur $=n$,

$CD=m,\ LO=lo=dx,$ erit $Dt=\frac{nn}{m}, DF=\frac{n}{m}dx$, hinc spatiolum $DtwF=\frac{n^3}{mm}dx$ &

spatium $LxyO = ndx$ & denique spatium $wuxyOLF = N - \frac{n^3}{mm}dx + ndx$) & altitudinem $v + dv$. Est igitur *ascensus mm potentialis* modo dictus

$= (N - \frac{n^3}{mm}dx + ndx) \times (v + dv) : M =$ (rejectis differentialibus secundi ordinis)

$$\frac{N}{M}v + \frac{N}{M}dv - \frac{n^3}{mmM}vdx + \frac{n}{M}vdx,$$

sic ut incrementum *ascensus potentialis*, quod aquae accessit dum guttula *plon* effluxit, sit

$$= \frac{N}{M}dv - \frac{n^3}{mmM}vdx + \frac{n}{M}vdx,$$

ubi spatia *N* & *M* sunt constantis magnitudinis ob aquae continuam affusionem. Non consideramus in hoc casu primo *ascensum potentialem* guttulae *cdfe*, quae affunditur dum altera aequalis *plon* effluit, quia iste ascensus non generatur vi interna, neque enim aqua inferior post se trahere ponitur particulam *cdfe*, quin potius hanc vi quadam extrinseca continue affundi consideramus, idque nec majore nec minore velocitate quam quae est superficiei *ef*.

Ergo omne incrementum hic considerandum est ut diximus

$$\frac{N}{M}dv - \frac{n^3}{mmM}vdx + \frac{n}{M}vdx.$$

Debet vero istud incrementum aequari *descensui actuali* centri gravitatis. Atqui iste descensus, posita $DL = a$, est per paragraphum septimum Sect. III $= \frac{nadx}{M}$; habetur igitur talis aequatio

$$\frac{N}{M}dv - \frac{n^3}{mmM}vdx + \frac{n}{M}vdx = \frac{nadx}{M},$$

seu

$$dx = Ndv : \left(na - nv + \frac{n^3}{mm}v \right).$$

Haec vero si ita integretur, ut *v* & *x* simul evanescant, dat

$$x = \frac{mmN}{n^3 - nmm}\log\frac{mma - mmv + nnv}{mma},$$

quae aequatio, posito *c* pro numero eujus Iogarithmus est unitas, aequivalet huic alteri

$$v=\frac{mma}{mm-nn}\times\left(1-c^{\frac{n^3-nmm}{mmN}x}\right).$$

Haec vero solutio quadrat pro casu prima, ubi aqua superne motu affunditur communi cum descensu superficiei proximae.

Casus II.

Quod si jam particula *cdfe* lateraliter continue affundi ponatur, tunc propter inertiam suam motui aquae inferioris resistit atque proinde *ascensus potentialis* ipsius aliter in computum venit. Tunc autem prius considerandus est *ascensus potentialis* massae aqueae *cdmlpic* auctae guttula mox affundenda; deinde indagandus *ascensus potent.* ejusdem aquae in situ *cdmlonpic*, postquam nempe guttula jam effluxit, eorumque *nadx* differentia est aequanda cum *descensu actuali* $\frac{nadx}{M}$. Verum *ascensus potentialis* omnis praedictae aquae ante affusionem particulae ejusdemque post affusionem ita invenitur: nempe *ascensus potentialis* aquae *cdmlpic* est $=\frac{Nv}{M}$, & *ascensus potent.* particulae affundi paratae nullus est, quia lateraliter affusa motum communem nondum habet cum massa inferiore: Igitur *ascensus potentialis* utriusque aquae (qui scilicet habetur multiplicando massam respective per suum *ascensum potentialem*, dividendoque productorum aggregatum per aggregatum massarum) est

$$=\left(M\times\frac{Nv}{M}+ndx\times 0\right):(M+ndx)=\frac{Nv}{M+ndx}.$$

Postquam vero particula $n\,dx$ superne jam affusa est, communem acquisivit motum cum aqua proxime inferiori, sicque fit *ascensus potentialis* ejusdem aquae in situ *cdmlonpic* aequalis quartae proportional ad spatium *CDLONPIC* $(M+ndx)$, spatium *DtuxyOLD* $(N+ndx)$ & altitudinem $v+dv$, id est,

$$=\frac{(N+ndx)\times(v+dv)}{M+ndx},$$

cujus excessus supra priorem *ascensum potentialem* est

$$=\frac{Ndv+nvdx+ndxdv}{M+ndx}=$$

(rejectis differentialibus secundi ordinis)

$$\frac{Ndv+nvdx}{M}.$$

Habetur igitur talis aequatio

$$\frac{Ndv+nvdx}{M}=\frac{nadx}{M},$$

quae ut prior pertractata & ad finem deducta dat

$$x = \frac{N}{n} \log \frac{a}{a-v},$$

vel

$$v = a \times \left(1 - c^{-\frac{nx}{N}}\right),$$

quae solutio valet pro affusione laterali.

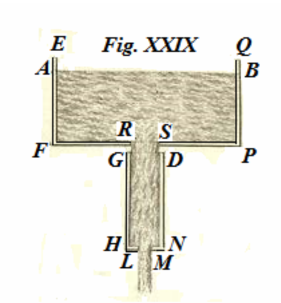
Scholion 1.

§. 4. Sunt hae aequationes inter se admodum diversae; diversitas autem eo major quo minoris est amplitudinis vas; & si quidem amplitudo vasis suprema in *cd* quasi infinita sit prae amplitudine foraminis, evanescit *n* prae *m* fitque in priori casu sicut in posteriori

$$v = a \times \left(1 - c^{-\frac{nx}{N}}\right),$$

Est igitur hac in hypothesi motus utrobique idem quod haud difficulter quisque praevidere potuerit. Celerior autem semper est caeteris paribus motus in priori affusione, quam in altera.

Conveniet hic rem etiam physice explicare, ut eam distinctius in omnibus phaenomenis percipere possimus. Sit loco vasis cujuscunque & quamcunque directionem habentis brevioris delineationis gratia cylindrus verticalis cum foramine in fundo, nempe *GHND* (Fig. 29) sitque dein vas *EFPQ* perforatum in *RS;* fingantur orificia *RS* & *GD* perfecte aequalia, & ad minimam distantiam sibi perfecte respondentia, ita ut aquae ex superiori vase effluentes omnes in cylindrum subjectum influant.



Incipiant aquae ex utroque vase effluere, ex superiori autem constanter ea effluere velocitate ponantur, quam habet superficies aquae in cylindro supposito. Ita patet satisfieri primae affusionis conditioni. Jam vero hujus motus phaenomena investigabimus, visuri num cum praecedentibus conveniant.

Consideremus igitur vas superius esse veluti infinitum, ut aquae per *RS* effluentes singulis momentis habeant velocitatem quae conveniat altitudini *PB* seu *FA:* sic fingendum erit esse hanc altitudinem *PB* ab initio infinite parvam, quia tunc aquae velocitate infinite parva effluere debent, deinde vero sensim crescere, idque continue magis magisque, donec post tempus infinitum motus uniformis maneat, quaeritur autem an altitudo aquae *PB* tandem infinita futura sit an vero certum terminum non transgressura. Id sic cognoscetur.

Sit altitudo *GH* vel *RH* (neque enim illas inter se differre censendum est) $= a$, $AF = x$, amplitudo orificii $LM = n$, amplitudo orificii $RS = m$; quia vero, ut manifestum est, utrumque vas cohaerere & unum efficere putari potest, erit post tempus infinitum (per §. 23 Sect. III) velocitas aquae in $LM = \sqrt{a+x}$, & in *RS* $= \sqrt{x}$ (quod posterius patet, si nunc iterum separata vasa censentur, nam utrumque sine errore fingi potest); debent autem velocitates esse in inversa ratione amplitudinum orificiorum: est itaque $\sqrt{a+x}.\sqrt{x} :: m.n$, unde $a+x.x :: mm.nn$, vel $a.\,x :: (mm-nn).\,nn$, ergo $x = \frac{nna}{mm-nn}$ unde $a+x = \frac{mma}{mm-nn}$ videmus igitur altitudinem velocitati aquae in *LM* debitam esse hoc modo $= \frac{mma}{mm-nn}$, postquam scilicet infinita aquae quantitas jam effluxit: superius autem habuimus eandem altitudinem, seu

$$v = \frac{mma}{mm-nn} \times \left(1 - c^{\frac{n^3 - nmm}{mmN}x}\right),$$

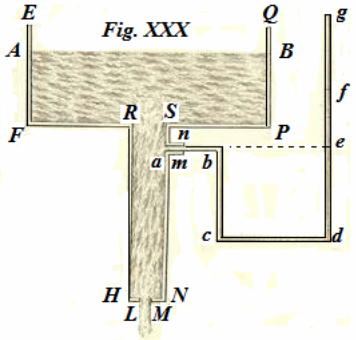
ubi si ponitur $x = \infty$ (infinito enim tempore infinita quantitas transfluit) evanescit terminus exponentialis, si modo *m* major sit quam *n* & sic fit partier

$$v = \frac{mma}{mm-nn}.$$

Mirabilis est iste consensus, quia valde diversae sunt viae, quas secuti sumus. Caeterum si *m* non sit major quam *n,* motus nunquam fit permanens nequidem post tempus infinitum; crescit enim tunc velocitas in infinitum cum secus altitudo velocitatis nunquam transgrediatur altitudinem $\frac{mma}{mm-nn}$. De his casibus nihil est quod dicamus.

Scholion 2.

§. 5. Quaestio hic nunc alia occurrit notatu digna; nempe quis esse possit modus affusionis mechanicus, ut vas superius ad debitam durante toto fluxu altitudinem plenum servetur. Difficile foret istud Problema ob inconstantiam altitudinis quaesitae, nisi peculiare hic artificium occurreret, quod nunc tradam. Nititur autem super eo, quod aqua in spatio minimo *RSDG* nullam patiatur compressionem neque affirmativam neque negativam, quia ex hypothesi communi velocitate movetur cum aqua proxime substrata, atque sic nulla particula nullam nec propellere nec retinere tentet. Fiat igitur vas quod dixi utrumque, sitque tubus cum vase superiore firmatus (neque enim aliter quam demonstrationis gratia illa posuimus antea separata); habeat autem tubus in summitate *a* (Fig. 30) foraminulum, cui respondeat tubulus *am*, in hunc tubulum immittatur tubus vitreus recurvus *abcdg*, obtectis cera oris *mn:* ducatur horizontalis *ae* noteturque punctum *e.* His sic praeparatis, sic erit faciendum, ut durante toto experimento summitas aquae constanter permaneat in puncto *e;* & ad hoc requiri videbis, ut ab initio superficies aquae sit fundo *FP* proxima, deinde, ut



continue elevetur, & denique ut post tempus etsi infinitum nunquam tamen transcendat altitudinem $\frac{nna}{mm-nn}$; facile autem erit aquarum affusionem ita moderari, ut superficies a puncto *e* non admodum divagetur, si modo circumstantiae non sint ita comparatae, ut aquae ab initio nimis celeriter sint affundendae.

Quod si autem superficiem in tubulo supra *e* elevatam animadvertis, inhibe paullo affusionem, quod faciendum esse alibi demonstrabo; si secus fuerit, largius aquas affunde.

Nihil habet difficuItatis istud experimenti genus cujusmodi saepe feci, sed ne error in experimentum irrepat, examinandus est tubi vitrei effectus capillaris; hunc effectum invenies, si obturato orificio *IM*, priusque madefacto tubo, cylindrus aqua impleatur usque ad summitatem, atque sic invenies superficiem aquae in tubo pertingere usque in *f*, locum nempe altiorem quam *e*, hoc autem punctum *f* illi, de quo modo diximus, abstrahendo animum a natura tubulorum capillarium, substitues.

Hoc igitur modo recte efficietur affusio ad normam hypotheseos nostrae & sic deinceps de hoc motu experimenta sumi poterunt. Postquam vero sic prolixe satis rem explicuimus, non opus puto monere vas superius non aliter pertinere ad vas cylindricum inferius, quod solum consideramus, quam ut cylindrus eo, quo fieri debet, modo plenus servetur atque sic per *m* non intelligendam esse amplitudinem vasis superioris sed amplitudinem orificii *RS*, quae proprie nobis est superficies aquae, cum aquae supra *RS* tantum debitae affusioni in cylindrum inferiorem inserviant.

Scholion 3.

§. 6. Non debeo hic praeterire, quod sic casus habeatur qui pertinet ad *hydraulicostaticam*, de qua scientia quaedam monui in Sect. I §. 8; cognoscimus nempe nunc quanta velocitate aqua in *a* praeterfluere debeat ut pressio ejus in latera tubi praecise nulla sit. Haec vero dum scriberem, jam detexeram leges *hydraulico-staticae* generales, & non sine voluptate vidi, quod iste casus ceu corollarium ex theoria plane alia deductus similem acquirat solutionem ex theoria generali. Sic omnia ubique mutuo cohaerent nexu, legitimamque principiorum applicationem demonstrant.

Scholion 4.

§. 7. Sequuntur nunc quaedam de alio aquae affundendi modo. Ponatur cylindrus *RHNS* pro vase quocunque, sitque is constanter plenus conservandus affusione laterali: poterit id fieri injiciendo sufficientem aquae quantitatem per tubulum *ma;* quamvis autem id non fiat sine motu, attamen, quia hic horizontalis est, mox omnis tollitur, & per se neque promovet fluxum per cylindrum neque eundem retardat; sed est alius insuper modus, quem subducto recte calculo eodem recidere intelligimus: nempe si vas *EFPQ* infinite amplum censemus, & ejus fundum aqua continue obtectum intelligimus, sed ita, ut aquae altitudo in vase superiori sit pro infinite parva habenda; subministrabit vas superius aquam tubo sibi annexo, neque alius inde motus orietur, quam ab affusione laterali, si modo orificium *RS* semper obtectum maneat; facile autem fit ut ibi cataracta quaedam formetur, si orificium *LM* amplum, tubusque *RSNH* longus sit. Quod hic alter modus eundem cum priori effectum in motum aquarum exerere debeat, quisque videt ex eo, quod in utroque modo omnis aquae tubum ingredientis inertia sit ab aqua inferiore superanda. Sed idem etiam *a priori* demonstrari poterit inquirendo in motum, qui inde oriri debeat, secundum aequationem paragraphi octavi Sect. III, quae haec est:

$$Ndv - \frac{mmvydx}{nn} + \frac{mmvdx}{y} = -yxdx ;$$

accommodabitur autem ad praesentem casum, si pro $m,\ x$ & $-dx$ substituas respective $n,\ a,$ & $\frac{ndx}{y}$ (cujus rei ratio patebit, si haec cum illis contuleris) simulque y infinitum ponas; tunc enim evanescit tertius aequationis terminus, fitque omnino, ut pro praesenti negotio supra invenimus,

$$Ndv + nvdx = nadx .$$

Postquam in his scholiis motus utriusque indolem, quantum simplex rei consideratio physica permittit, eorumque differentiam ostendimus, simulque modum illos producendi ad legem hypotheseos mechanicum tradidimus, superest, ut reliqua phaenomena notabiliora etiam indicentur, quod nunc faciam.

Corollarium 1.

§. 8. Si in vase *RSNH* omne fundum absit, erit orificium $LM =$ orificio *RS;* potest etiam hoc ab illo superari, si nempe vasis divergant latera. In his autem casibus nullum habet terminum altitudo *v* in aequatione

$$v = \frac{mma}{mm-nn} \times \left(1 - c^{\frac{n^3 - nmm}{mmN}x} \right)$$

& fit infinita, si quantitas aquae ejectae indicata per *nx* est infinita.

Id quidem per se patet ex aequatione, cum *n* est major quam *m;* at cum amplitudines orificiorum sunt aequales, recurrendum est ad aequationem differentialem paragraphi tertii, ex qua ista aequatio proxima deducta fuit, nempe

$$\frac{N}{M}dv - \frac{n^3}{mmM}vdx + \frac{n}{M}vdx = \frac{n}{M}adx ,$$

quae posito $n = m$ dat $Ndv = nadx,$ id est, $v = \frac{nax}{N}$, ubi *v* fit manifeste infinita si *x* est infinita.

Corollarium 2.

§. 9. Sin autem vasi proposito fundum sit, atque in eo foramen, cujus amplitudo indicata per *n* minor sit amplitudine orificii *RS* expressa per *m,* habet *v* valorem quem nunquam attingit quidem, sed tamen proxime assequitur, & ad quem tam cito convergit, nisi data opera vasa huic rei contraria excogitata adhibeantur, ut post minimum fluxus tempusculum, quod sensibus percipi possit, notabiliter ab eo non *mma* deficiat. Est autem terminus ille talis, $v = \frac{mma}{mm - nn}$: igitur in casu Scholii secundi §. 5 ultimus terminus *PB* est $= v - a = \frac{nna}{mm - nn}$. Exemplo citissimam velocitatis ad ultimum suum terminum accessionem illustrabo, postquam aequationem inter *v* & tempus altitudini *v* respondens apposuero.

Corollarium 3.

§.10. In casu affusionis, quam vocamus, lateralis, fit ultima altitude $v = a$, quaecunque inter utrumque vasis orificium ratio intercesserit.

Corollarium 4.

§. 11. Si vas est cylindricum ejusque longitudo ponatur $= b$, fit(vid. §. 3) $N = \frac{nnb}{m}$: notetur autem non confundendos esse valores litterarum *a* & *b*, primus enim exprimit altitudinem supremi orificii supra inferius, alter longitudinem canalis; sic itaque conveniunt inter se valores in hoc saltem casu, cum axis vasis linea est recta & verticalis; at si axis tortuosus est, vel saltem non verticalis, differunt a se invicem: Haec ideo expresse monere volui, ne quis sibi a figuris vasorum, quorum axes ubique rectos & verticales feci, imponi patiatur.

Quod si igitur pro vasis cylindricis ponatur $N = \frac{nnb}{m}$, fit pro affusione verticali

$$v = \frac{mma}{mm - nn} \times \left(1 - c^{\frac{nn - mm}{mnb} x} \right)$$

& pro altera laterali fit

$$v = a \left(1 - c^{\frac{-mx}{nb}} \right).$$

Problema.

§. 12. Invenire velocitatem aquae ex vase constanter pleno effluentis, postquam fluxus per datum tempus duravit.

Solutio.

Retentis hypothesibus & denominationibus omnibus, quas in§. 3 adhibuimus, positoque insuper tempore a fluxus initio praeterito $= t$, mutandas habebimus aequationes in dicto paragrapho datas in alias, quae relationem exprimant inter *t* & *v*, eliminatis quantitatibus *x* & *dx*. Est vero elementum tempusculi *dt* proportionale minimo spatiolo *dx*, quod percurritur, diviso per velocitatem $\sqrt{v}$: ponemus igitur $dt = \frac{\gamma dx}{\sqrt{v}}$, & sic mutabitur aequatio

$$dx = Ndv : (na - nv + \frac{n^3}{mm} v)$$

quae data fuit pro affusione verticali debita velocitate instituenda in hanc

$$(\text{I}) \qquad dt = N\gamma dv :\ (na\sqrt{v} - nv\sqrt{v} + \frac{n^3}{mm} v\sqrt{v}),$$

altera vero affusioni inserviens laterali, nempe $dx = Ndv : (na - nv)$, abit in hanc post eandem substitutionem

$$(\text{II}) \qquad dt = N\gamma dv : (na\sqrt{v} - nv\sqrt{v}).$$

Hae vero aequationes debito modo integratae dant pro prima

$$(\alpha) \qquad t = \frac{mN\gamma}{n\sqrt{(mma - nna)}} \times \log \frac{m\sqrt{a} + \sqrt{(mmv - nnv)}}{m\sqrt{a} - \sqrt{(mmv - nnv)}}$$

& pro altera, quae ex priori deducitur, posito $m = \infty$,

$$(\beta) \qquad t = \frac{N\gamma}{n\sqrt{a}} \times \log \frac{\sqrt{a} + \sqrt{v}}{\sqrt{a} - \sqrt{v}}.$$

Q.E.I.

Scholium.

§.13. Si vas de quo sermo est sit cylindricum utcunque intortum & inclinatum, cujus longitudo ponatur = *b*, manente altitudine superficiei aqueae supra foramen $= a$, erit rursus, ut §. 11, $N = \frac{nn}{m} b$.

Quoniam autem, ut constat, $2\gamma\sqrt{A}$ exprimit tempus, quod corpus insumit cadendo libere & a quiete per altitudinem *A*, patet quantitatem $\frac{2mN\gamma}{nn\sqrt{a}} (= 2\gamma\sqrt{\frac{bb}{a}})$ exprimere tempus quo corpus moveri incipiens a quiete libere descendit per altitudi $\frac{bb}{a}$: accipiemus istud tempus pro communi mensura idemque ponemus $= \theta$, & mutabitur pro vasis seu canalibus cylindricis aequatio (α) in hanc

$$t = \frac{n\theta}{2\sqrt{(mm - nn)}} \times \log \frac{m\sqrt{a} + \sqrt{(mmv - nnv)}}{m\sqrt{a} - \sqrt{(mmv - nnv)}},$$

altera vera signata (β) talis fit

$$t=\frac{n\theta}{2m}\times\log\frac{\sqrt{a}+\sqrt{v}}{\sqrt{a}-\sqrt{v}},$$

ex quarum utraque apparet, non posse non brevissimo tempore aquas omnem fere velocitatem acquirere, idque eo citius quo amplior est tubus, quo brevior, & quo magis verticalis: Neque accelerationes ullo modo esse perceptibiles, nisi praelongi statuantur aquae ductus & tunc quoque brevi tempore omnes fere accelerationum gradus percurri, quod utrumque nunc exemplo illustrabo.

(I) Quaeritur tempus quo fluidum ex cylindro constanter pleno verticali, sedecim pedes anglicos longo & cujus diameter quintupla sit diametri foraminis, velocitatem acquirit quae debeatur altitudini $\frac{99}{100}a$, idque in hypothesi, ad quam aequatio secunda pertinet; sic est $\frac{n}{m}=\frac{1}{25},\ v=\frac{99}{100}a,\ b=a$, unde tempus quo corpus insumit cadendo libere per spatium $\frac{bb}{a}$, seu $\theta=$ uni minuto secundo; hinc fit $t=\frac{1}{50}\log 399$, id est, proxime nonae parti unius minuti secundi, quod tempusculum utique imperceptibile est; cum vero tempus notabile assumitur, fiunt mutationes altitudinum *v* insensibiles. Si tempus simile (quo nempe velocitas pariter nonaginta novem centesimis partibus altitudinis, quanta post tempus infinitum fit, debita generetur) in prima hypothesi quaeratur, nempe tempus quo obtinetur $v=\frac{99}{100}\times\left(\frac{mma}{mm-nn}\right)$ reperitur illud praecedente paullulum majus, sed excessu insensibili: unde patet in hujusmodi vasis non posse fere aquas sat celeriter affundi in vas superius, ut hypothesi satisfiat, nec adeoque ratione ejusdem hypotheseos experimenta alia sumi posse, quam ut exploretur, num revera tanta sit altitudo *BP* in figura trigesima, quanta vi paragraphi quinti esse debet, ut punctum *e* aut *f* durante fluxu situm servet, quem ante fluxum obturato orificio *LM*, nullaque existente aqua in vase superiore habuit.

(II) Quaeritur nunc idem tempus pro secunda rursus hypothesi, si tubus ejusdem fuerit amplitudinis eodemque foramine instructus, sed oblique situs longitudinemque *b* habuerit 184 *perticarum* seu 1104 *pedum Paris.* dum altitudo superficiei aqueae supra orificium effluxus sit 16 *ped. Paris.* Ita fiet $b=1104$, & $\frac{bb}{a}=76176$ atque praeterpropter $\theta=72$ *sec. min.*, unde tempus quaesitum medium est inter octo novemque minuta secunda, quod certe satis notabile est. Si vero tempus desideretur, quo altitudo *v* exaequet tantum quartam partem altitudinis *a*, reperietur illud aequale $\frac{72}{50}\log 3=$ proxime uni minuto secundo cum dimidio.

Nescio an haec conveniant cum iis, quas Mariottus a se observata refert in *Tract. de mot. aquar., part.* 5, *disc.* 1, ubi mentionem facit alicujus fontis salientis, qui est a *Chantilly* , ad quem aquae devehuntur per canalem 184 *perticas* longum, si modo recte ex antecedentibus conjeci, eratque summa superficiei aqueae altitudo supra orificium effluxus indicata per *a* sedecim *pedum:* diameter aquaeductus erat 5 *poll.,* orificium autem habebat diametrum unius pollicis. Videtur mihi Mariottus ita loqui ac si accelerationes multo fuissent tardiores, quam ab formula nostra indicantur, quod nescio an tribuendum sit huic quod fortasse alium, praeter orificium de quo hic sermo est, exitum habuerint aquae, an, quod aquae ductus dum fluxus inciperet non fuerit aqua plenus, quod posterius multa faciunt, ut credam; si neutrum fuerit, confido phaenomena qualia a Mariotto observata fuerunt & quotidie de novo observari poterunt plane convenisse cum calculo nostro. Caeterum verba Mariotti haec sunt : *Illud insuper*, ait, *singulari eidem jactui accidit, quod obturato manu orificio per decem aut duodecim scrupulorum secundorum temporis spatium eodemque postea reserato, aqua non protinus erumpat, sed paullatim assurgens jactus ascendat ad* 3 *poll., postea ad pedis altitudinem & denique ad duos pedes successive notabilibus intervallis. . . . Sed tandem tamen toto impetu suo aquae exiliebant.*

Problema.

§. 14. Invenire quantitatem aquae per datum vas, constanter plenum conservandum, dato tempore transfluentem.

Solutio.

Adhibitis rursus positionibus & denominationibus paragraphi tertii & duodecimi, invenienda nunc erit aequatio inter *x* & *t:* quia vero, ut vidimus §.12, est $dt = \frac{\gamma dx}{\sqrt{v}}$, erit $\sqrt{v} = \frac{\gamma dx}{dt}$ hicque valor substituendus erit in aequationibus, quas dedimus §. 3 integratis; prior harum aequationum haec fuit:

$$v = \frac{mma}{mm-nn} \times \left(1 - c^{\frac{n^3-nmm}{mmN}x}\right),$$

quae pro praesecuti instituto mutatur in hanc (I)

$$\text{(I)} \qquad \frac{\gamma\gamma dx^2}{dt^2} = \frac{mma}{mm-nn} \times \left(1 - c^{\frac{n^3-nmm}{mmN}x}\right);$$

altera ex §. 3 allegatarum aequationum talis fuit

$$v = a \times \left(1 - c^{\frac{-n}{N}x}\right),$$

quae adeoque subministrat in praesenti casu sequentem

$$\text{(II)} \qquad \frac{\gamma\gamma dx^2}{dt^2} = a \times \left(1 - c^{\frac{-n}{N}x}\right).$$

Erunt nunc aequationes (I) & (II) integrandae, quod quidem facile est & quia prior alteram continet (utraque enim eadem est si $m=\infty$), hanc solam pertractabimus, eamque nunc sub hac forma considerabimus.

$$dt=\frac{\gamma\sqrt{(mm-nn)}}{m\sqrt{a}}dx:\sqrt{\left(1-c^{\frac{n^3-nmm}{mmN}x}\right)}.$$

Ponatur autem ut integrationis modus eo magis patescat $c^{\frac{n^3-nmm}{mmN}x}=z$, atque proin

$$dx=\frac{mmNdz}{(n^3-nmm)z},$$

dein brevitatis ergo indicetur quantitas constans

$$\frac{\gamma\sqrt{(mm-nn)}}{m\sqrt{a}}\times\frac{mmN}{n^3-nmm}$$

seu

$$\frac{-\gamma mN}{n\sqrt{(mm-nn)a}}$$

per α, & habebitur

$$dt=\frac{\alpha dz}{z\sqrt{(1-z)}},$$

in qua si praeterea fiat $1-z=qq,$ seu $z=1-qq,$ $dz=-2qdq$, oritur

$$dt=\frac{-2\alpha dq}{1-qq}=-\frac{\alpha dq}{1+q}-\frac{\alpha dq}{1-q},$$

cujus integralis est

$$t=-\alpha\log\left(1+q\right)+\alpha\log\left(1-q\right)=\alpha\log\frac{1-q}{1+q}.$$

Nec opus est constante, quandoquidem ex natura rei *t* & *x* simul evanescere debent; posito autem $x=0,$ fit $z=1,$ & $q=0$, igitur pariter *t* & *q* simul a nihilo incipere debent, cui conditioni satisfacit aequatio inventa $t=\alpha\log\frac{1-q}{1+q}$: Superest ut

$$t=\alpha\log\frac{1-q}{1+q}$$

retrogrado ordine valores pristinos reassumamus, ita vero fit

$$t=\alpha\log\frac{1-\sqrt{(1-z)}}{1+\sqrt{(1-z)}}$$

vel

$$t=\frac{\gamma mN}{n\sqrt{(mm-nn)a}}\times\log\frac{1+\sqrt{(1-z)}}{1-\sqrt{(1-z)}}$$

vel denique

$$\text{(I)}\qquad t=\frac{\gamma mN}{n\sqrt{(mm-nn)a}}\times\left(\log\left(1+\sqrt{1-c^{\frac{n^3-nmm}{mmN}x}}\right)-\log\left(1-\sqrt{1-c^{\frac{n^3-nmm}{mmN}x}}\right)\right).$$

Istaque aequatio posito $m=\infty$ dat alteram aequationem quaesitam

$$\text{(II)}\qquad t=\frac{\gamma N}{n\sqrt{a}}\times\left(\log\left(1+\sqrt{1-c^{\frac{-n}{N}x}}\right)-\log\left(1-\sqrt{1-c^{\frac{-n}{N}x}}\right)\right).$$

Q.E.I.

Corollarium 1.

§. 15. Si ponatur $x=\infty$, ut appareat natura rei, cum infinita jam transfluxit aquae quantitas assumaturque *m* major quam *n*, prouti plerumque esse solet, evanescere censenda est, in utroque logarithmo affirmative sumto, quantitas exponentialis & habebitur utrobique log 2. At vero in logarithmo negative sumto statuenda est

$$\sqrt{1-c^{\frac{n^3-nmm}{mmN}x}}=1-\tfrac{1}{2}c^{\frac{n^3-nmm}{mmN}x}$$

& proinde,

$$\log\left(1-\sqrt{1-c^{\frac{n^3-nmm}{mmN}x}}\right)=\log\left(\tfrac{1}{2}c^{\frac{n^3-nmm}{mmN}x}\right)=\frac{n^3-nmm}{mmN}x-\log 2.$$

Hae Substitutiones si recte fiant, erit pro primo quem finximus affusionis modo

$$\text{(I)}\qquad t=\frac{\gamma mN}{n\sqrt{(mm-nn)a}}\times\left(2\log 2+\frac{mmn-n^3}{mmN}x\right),$$

quae posito rursus $m=\infty$ dat pro altero casu (ll)

$$\text{(II)}\qquad t=\frac{\gamma N}{n\sqrt{a}}\times\left(2\log 2+\frac{n}{N}x\right).$$

Sequitur ex istis formulis, minori quidem quantitate transfluere aquas, ac si statim ab initio omni velocitate, quam in utroque casu post tempus infinitum acquirunt, effluerent: differentiam tamen nunquam certum transgredi terminum & post tempus infinitum finitis comprehendi terminis.

Corollarium 2.

§.16. Quum convertimus aequationes inventas, obtinemus

(I) $$x=\frac{2mmN}{mmn-n^3}\left(\log\left(1+c^{\frac{-t}{\alpha}}\right)-\log 2+\frac{t}{2\alpha}\right),$$

& (II) $$x=\frac{2N}{n}\times\left(\log\left(1+c^{\frac{-t}{\beta}}\right)-\log 2+\frac{t}{2\beta}\right),$$

ubi α, ut supra, $=\frac{-\gamma mN}{n\sqrt{(mm-nn)a}}$ & $\beta=\frac{-\gamma N}{n\sqrt{a}}$.

Si praeterea, ut in proximo Corollario, ponatur $t=\infty$, evanescit unitas prae quantitatibus exponentialibus, quae supra omnem ordinem infinitae sunt, & fit

$$\log\left(1+c^{\frac{-t}{\alpha}}\right)=-\frac{t}{\alpha} \text{ atque } \log\left(1+c^{\frac{-t}{\beta}}\right)=-\frac{t}{\beta}:$$

unde tunc erit resumtis valoribus litterarum α & β

(I) $$x=\frac{mt\sqrt{a}}{\gamma\sqrt{(mm-nn)}}-\frac{2mmN}{mmn-n^3}\log 2,$$

& (II) $$x=\frac{t\sqrt{a}}{\gamma}-\frac{2N}{n}\log 2.$$

Igitur si statim a fluxus initio utrobique aquae omni, quam acquirere possunt, velocitate constanter effluerent, non excederet earum quantitas post tempus infinitum quantitatem pro eodem tempore theoriae respondentem nisi parvula quantitate, quae in primo casu exprimitur per $\frac{2mmN}{mm-nn}\log 2$, & in secundo per *2N* log 2. Atque si loco temporis infiniti sumas tempus tantum aliquot scrupulorum secundorum, idem *theorema* proxime locum habebit; ita ut si v. gr. post decem prima minuta secunda effluxerit quantitas *Q*, effluxura fere sit totidem minutis secundis proxime sequentibus $Q+\frac{2mmN}{mmN-n^3}\log 2$, vel in altero casu $Q+\frac{2N}{n}\log 2$.

Scholium.

§. 17. Ad theoriam hactenus expositam pertinet etiam motus aquarum per siphones. Indicat autem theoria, posse siphonis axem utcunque inflecti, neque inde motum aquarum deturbatum iri, modo altitudo superficiei aqueae supra orificium effluxus eadem maneat; cum praeterea aquaeductus, siphones aut diabetae hujuscemodique vasa alia soleant esse cylindrica, erit ut monui §. 13, quoties id contingit, ponendum $N = \frac{nn}{m}b$, intelligendo per *b* longitudinem canalis aut siphonis: in formulis quoque paragraphorum 14, 15, & 16, erunt quantitates sic interpretandae, ubi de temporibus quaestio est, ut $2\gamma\sqrt{A}$ repraesentet tempus quod corpus impendit in descensum per altitudinem verticalem *A* a quiete coeptum.

Caeterum, ut dixi passim, nihil indicat singulare theoria hujus sectionis, quod sub sensus cadat, nisi in aquae ductibus admodum longis, ad horizontalem valde obliquis & orificium non admodum strictum habentibus; haec tria enim concurrunt ad retardandas sicque notabiles efficiendas accelerationes, quarum mensurae potissimum theoriam commendant.

Est tamen & in his circumstantiis medium aliquod observandum, ne impedimenta ab adhaesione aquae oriunda nimia sint.

Quod attinet ad affusionem aquarum, mihi visus sum animadvertere, si verticaliter fiat & cum impetu, tantum abesse, ut inde motus acceleretur, quin potius retardetur, nisi aquarum affusio fiat in totam superficiem aequabiliter eo, quem §. 4 exposui, modo; si enim aliter affundantur, motus aquarum in vase perturbatur, isque motus confusus effluxum retardat.

§.18. Denique huc quodammodo pertinent experimenta ab Clar. Joanne Poleno instituta, ut refert in *Libro primo de motu aquae mixto, p.* 21 & *seqq.*, quae ideo hic alleganda esse censui, quod egregie demonstrant, ubique celeritatem ultimam in vasis constanter plenis eam esse, quae integrae aquae altitudini conveniat, si vasa non sint submersa, aut differentiae altitudinum aquae internae & externae in vasis submersis, quamvis de caetero nihil in illis sit, quod nunc novum adhuc sit, quia nullae illic considerantur accelerationes.

Finge cylindrum, cujus axis habeat situm verticalem, amplitudinis veluti infinitae; fundum integrum sit: in pariete autem fissura sit axi parallela, foramen habens parallelogrammi rectanguli, quae a fundo ad cylindri usque summitatem extendatur. Puta porro aquam in cylindrum affundi aequabiliter, ita, ut aequalibus temporibus quantitates injiciantur aequales, effluent aquae ex cylindro per fissuram: nec tamen ab initio eadem effluent quantitate, qua superne affunduntur, sed minori: igitur assurget superficies aquae in cylindro ad certam usque altitudinem asymptoton; si vero is jam intelligatur adesse terminus, immutata manebit altitudo aquae & eadem quantitate effluent constanter aquae, qua affunduntur. Apparet quoque, altitudinem aquae in cylindro eo majorem fore, quo largius affundantur: Quaeritur itaque auctis quantitatibus aquarum dato tempore affundendis, in quanam ratione crescere debeant altitudines, ad quas aquae in cylindro assurgent.

Solutio haec est. Sit altitudo aquae, cum est in statu permanente, $= \alpha$, & abscindatur a superficie pars quae sit $= x$, una cum differentiali *dx;* sit latitudo rimae $= n$. Habebimus veluti foramen amplitudinis $= ndx$, per quod aquae effluunt velocitate $\sqrt{x}$: igitur quantitas aquae dato tempore ibi effluentis est ut $ndx\sqrt{x}$, cujus integralis est $\frac{2}{3}nx\sqrt{x}$; quae exprimit quantitatem aquae dato tempore per rimae longitudinem abscissam *x* effluentem: & sic quantitas aquae eodem tempore per rimam integram effluens exprimetur per $\frac{2}{3}n\alpha\sqrt{\alpha}$. Tantum autem effluit, quantum affunditur; hinc si quantitas aquae dato illo tempore affusae dicatur *q*, erit $\frac{2}{3}n\alpha\sqrt{\alpha} = q$. Id indicat quantitates aquarum dato tempore affundendarum sequi rationem sesquiplicatam altitudinum, ad quas aquae a fundo cylindri ascendunt: aut vicissim altitudines sequi rationem subtriplicatam quadratorum quantitatum, quibus aquae dato tempore affunduntur.

§. 19. Soluto hoc problemate venio ad alterum Cl. Poleno consideratum. Sit idem cylindrus, sed aquis in fossa veluti vase infinito stagnantibus submersus; dicaturque altitudo submersionis $= a$, quaeritur nunc iisdem positis, ut antea, rursus aequatio inter altitudinem α superficiei aqueae internae supra extemam, & quantitatem *q* dato tempore affundendam.

Quod ad illam rimae partem α , quae aquas ejicit & supra aquam extemam eminet, illam jam vidimus dato tempore erogare quantitatem $\frac{2}{3}n\alpha dx\sqrt{\alpha}$: residua autem rimae pars submersa aquas ubique communi velocitate transmittit, ut ex infra dicendis patebit, & quidem velocitate $\sqrt{\alpha}$, ita, ut multiplicata hac velocitate per magnitudinem rimae submersae *na*, habeatur quantitas, quam dato tempore ejicit, $= na\sqrt{\alpha}$. Si utraque quantitas in summam conjiciatur, habebitur $(\frac{2}{3}\alpha + a)n\sqrt{\alpha} = q$.

Ope hujus aequationis cognoscitur *q* ex datis altitudinibus *a* & α : aut vicissim altitudo *a* ex cognitis quantitatibus *a* & *q*.

Convenire autem hanc aequationem admodum accurate cum experimentis, ipse ostendit celeberrimus eorum auctor, cujus solutio ab hac nostra non differt. Sequitur ex ista aequatione, elevationes α eo majores esse pro iisdem aquarum affusionibus, quo minor est altitudo submersionis *a*.

Experimenta quae ad Sectionem V pertinent.

Ad§.5. Vase usus sum §. 5 descripto cum tubulo vitreo (Fig. 30). Primo autem obturavi orificium *LM* tubumque *RN* aqua implevi, donec superficies ejus raderet foraminulum in *a:* aquam tunc tubo ingressam observavi extremitate attigisse punctum *f*: postea reserato orificio *LM*, & aquis effluentibus novas affundebam in vas superius *EFPQ* adhibita diligentia, ut extremitas aquae in *f* interea nec ascenderet nec descenderet. Haec dum fierent elevabatur superficies *AB*, nunquam autem certum terminum transgrediebatur; fuit nempe, quantum videre potui, maxima altitudo *PB* seu $FA = \frac{nn}{mm-nn}a$, denotante $\frac{n}{m}$ rationem inter orificium inferius *LM* & superius *RS*, & *a* altitudinem verticalem orificii posterioris supra alterum.

Id vero solum est, quod ipsemet institui experimentum, quamvis multae sint propositiones in hac sectione contentae, quae mereantur attentionem eaeque satis inexpectatae, non potui tamen de illis experimenta sumere; sunt enim ita comparatae in vasis brevioribus, ut quod singulare habent, id sensus effugiat, rem autem experiri in longis aquaeductibus commode non potui: cum aliis haec dabitur occasio, theoriam hanc examinaturis, animum advertent ad sequentia:

I. In fontibus salientibus observetur altitudo jactus integra; postmodum obturato prius orificio eodemque mox reserato videatur aquae quantitas, quae effluat, dum aqua ad dimidiam altitudinem jactus integri, aut aliam partem quamcunque perveniat, quod quidem brevissimo eveniet tempore, illius quantitatis mensura sit longitudo cylindri super foramine, per quod aquae exiliunt, exstructi, quam longitudinem vocavimus x, altitudinem vero jactus integram nominavimus a, altitudinemque jactus qui nondum totam attigerit altitudinem, observatam designavimus per v. Tum denique instituto calculo exploretur, num hae quantitates recte respondeant aequationibus pro utroque affundendi modo exhibitis in paragrapho tertio.

II. Fiant omnia, ut ante, hoc saltem discrimine, quod loco quantitatis effluentis tempus effluxus notetur, ut sic examinari possint formulae paragraphi decimi tertii, & denique comparetur quantitas cum tempore fluxus, ut appareat num recte respondeat formulae §.14.

III. Tum praecipue fiat id experimenti genus, quod indicavi paragrapho decimo sexto, observando scilicet quantitates aquarum dimidiis temporibus respondentes; dixi autem, quantumvis magnum sumatur tempus, differentiam harum quantitatum nunquam exaequare $\frac{2mmN}{mmn-n^3}\log 2$ in priore, quem finxtmus, affundendi modo; aut $\frac{2N}{n}\log 2$ posteriori. Istas autem differentias, utut nunquam perfecte orituras, minimo tamen tempore proxime adfuturas esse.

Quae reliqua sunt in hac sectione Corollaria & Scholia quisque facile videbit, quo modo ad experimenta vocari possint: Velim autem, priusquam judicium ferat, attentus sit ad onmes circumstantias ratione impedimentorum, contractionis venae, aliorumque, quas nolo ubique repetere. Ad §§. 18 & 19 Experimenta pro confirmatione problematis §.19 ad vasa non submersa pertinentis, vide p. *26 lib. cit.* Ill. Poleni.

Cum vero in vase submerso esset altitudo $a = 55$ *lin. Paris.* (quae altitudo ei dicitur mortua), quinque instituit experimenta, in quibus altitudo, quam dicit, viva seu α erat successive linearum $8\frac{1}{2};\ 25;\ 42;\ 58\ \&\ 73\frac{1}{2}$: his substitutis valoribus in aequatione §.18 exhibita sequitur, quantitates aquarum dato tempore affusarum fuisse ut 100; 199; 299; 396 & 495: actu affusae fuerunt in ratione ut 100, 200, 300, 400, & 500: differentia tantilla est, ut dubitari possit, an non perfectus consensus futurus fuisset, si onmes mensurae rectissime haberi potuissent.

Reliqua etiam experimenta a Viro Cl. instituta cum theoria perfecte consentiunt: calculum eorum videre est apud ipsum Auctorem. E re autem duxi eadem hic apponere, quia ad argumentum hujusce sectionis pertinent, quamvis caeterum libenter fatear, me magis desiderare illa experimenta, quae a calculo mutationum *momentanearum*, nemini quod sciam adhuc consideratarum, pendent, quam quae statum *permanentem* supponunt.