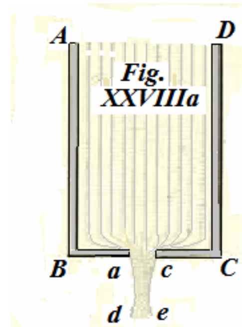
HYDRODYNAMICS SECTION FOUR.

Concerned with the various times, which are desired in the efflux of the water.

§.1. The matter would be considered entirely geometrical by many, which evidently may have no need for physical consideration, so that, with the water flowing out from a given vessel through a known opening with the velocity determined in every situation, the time may be defined in which a given quantity of water may flow out. Nevertheless a trial may teach the opposite ; for it follows that water may flow through holes in much smaller amounts, which are present in a thin plate, than may follow from simple consideration of the velocities, and (nor indeed may matters themselves be agreed on in different circumstances) that generally is in the ratio of 1 to $\sqrt{2}$; this motivated Newton, as he had affirmed in the first edition of the *Princ.Math.* that water flows from a vessel with that velocity, which is generated by half the height of the water above the opening, to which all the experiments taken concerning velocities immediately contradict [*i.e.* performed at a later date, the experiments referred to by Newton indicated a half]. Afterwards, investigating the origin of this contradiction, the great man had observed that to be put in place due to a contraction of the jet of the water, which contraction was accustomed to happen just before the opening. Another observation made by me was that the change was made at one time in this manner and at another in the opposite. Clearly when the water does not flow out through a simple opening, but truly through a tube, on the one hand the jet is contracted if the exterior tube converges together, but on the other hand expands if the same diverges. In the book *Libro de castellis*, Giovanni Poleni set out the most accurate experiments concerning the contraction of jets of water flowing out through converging tubes, p. 15 & seqq.; it was observed by the most celebrated man that the contraction of the jet was greater there, where the inner orifice of the conical tube was greater, and with the outer orifice and the length of the tube remaining the same, which is the reason why similar amounts of water, with all else being equal, had flowed out slower from that, because the greater the inner opening was, although the internal impediments from the adhesion of the water to the sides of the tube continued to have an effect: but there would be a diminution of these impediments, so that the water in that place would flow out with a greater velocity, where the jet had contracted especially, and nevertheless the water would be dispensed more sparingly : truly that is deduced from the observed times of efflux, and from the sizes of the jets, where they have contracted the most. Therefore since the whole matter hinges on these changes in the jets, from that the phenomena will be examined and explained in more detail.

§. 2. For example, we may assume a vertical cylinder, which may have an opening placed horizontally in the middle of the base, but the water may be considered to be divided internally into horizontal layers: Thus with these in position, we may agree that the motion of any horizontal layer to be the same and such indeed, so that the horizontal position shall be conserved in these, where yet as I have warned, it is not possible for this hypothesis to be extended to layers close to the opening, because truly no sensible error shall arise in the velocity of the water flowing out, there shall not be worth the effort, that an account of this matter may be had. Now truly, when I turn to other phenomena arising from the internal motion of the water, such as especially depend on being close to the opening in the aforementioned layers, we will illuminate this a little.

§. 3. But for me such internal motion of the water is to be considered, such as if the water may be carried by an infinite number of tubules placed next to each other, of which the intermediate fall towards the opening nearly perpendicular to the surface, with the remainder slowly curving themselves towards the opening, as Fig. 28a shows, from which it is apparent, the individual particles descend in this way so far with the motion not to be vertical, then they may extend towards the base, and these then slowly run themselves slowly bending towards the opening, thus so that the particles near the base to be carried by a horizontal motion, the others more vertically placed to the opening may flow out. It is possible often to see a motion of this kind, with particles of wax, that they call Spanish wax, swimming in the water. But thence it is understood the individual particles are not able to assist in watching over its whole direction to the opening, nor yet that thus curving, so that the motion they assume will be clearly parallel to the axis, but to be rather, so that the jet of water flowing out may be contracted as far as to *de*, where thus notably it will be more slender, then in arising around the opening *ac*. But this contraction of the jet flowing vertically is not to be confused with that other contraction, which shall be from the acceleration of the water. Then also it is apparent, that when the direction of the individual particles standing at the opening shall be different, by necessity from the impetus, that the same particles make mutually against each other, the jet shall be compressed, and thus becomes more slender. And from that compression it happens, that another contradiction will be involved, so that at one time or another the water even now is accelerated before escaping from the opening, and thus the *potential ascent* shall increase, even if we do not attend to the other common acceleration from all the bodies falling, or thus it may not be related, and henceforth we shall not make mention of this. But unless I am mistaken I am of the opinion, the matter is required to be treated again in this manner.



(I) The jet of water has been considered up to that point, then the velocities of the particles may not change more, because although at no time does everything become done with all rigor, nevertheless this can be agreed upon not far from the opening, as at *de*. But if this were thus the case, and water from the vessel *ABCD* were considered to flow from the opening *ac*, in place of the simple vessel *ABCD*, another composite vessel is required to be considered *ABadecCD.*

Therefore whatever was set out in the preceding section, for determining the velocities everywhere, that generally will have a place, if in place of the vessel a vessel may be considered placed below, which I have said to be constructed from a contracted tube. Nor yet with this correction, from the account set out of our method of determining the velocity of the water flowing out, can it produce a noticeable change on account of the shortness of the tube *adec,* but it is able very notably [to produce such a change] on account of the quantity, because the water not only flows from the opening *ac*, but also may be considered to flow out from *de*.

(II) Thus the velocities in different places of the jet will be inversely as the magnitudes of the corresponding areas of cross-section and since in the largest vessels the velocity at *de* shall be such as which may agree with the whole height of the water, and likewise it will be agreed from experiments, the areas of cross-section *ac* and *de* to be approximately as $\sqrt{2}$ to 1, Newton thus thought it possible to confirm his theory, which stated that water from an opening truly flowed out with a velocity which must be half the height of the water above the opening, however the velocity of the water in progress may increase: from which it was observed by me regarding the matter to be of the preconceived opinion of the exceeding stickiness : for neither is the ratio of the opening *ac* to *de* always the same, nor thus can the motion of the water from the vessel be explained, to which the [hypothetical] tubule may adhere : Truly in a word the attenuation of the jet is accidental, for the whole can be impeded, by appending to the opening a very small cylindrical tube or by increasing the thickness of the plate in which the hole is present, and then without any correction of the situation the theorems occur, both in the account of the velocities as well as of the quantities, which were shown in the preceding section.

(III) Moreover it is apparent from that explanation given above about the contraction of the jet, it is not possible for that not to change with different circumstances ; thus we learn from experiments, the same to be diminished with an increase in the thickness of the sides of the opening : but I know that the height of the water above the opening does not bring matters together well enough : I might believe almost for an increase in the height of the water inside to increase the contraction by a small amount, although I may have easily foreseen that to be small : it is also plausible, there with all else being equal the contraction of the jet to be small, especially vertical, so that a greater ratio may be had with the cross-section of the opening to the cross-section of the cylinder, because the motion of the internal water near the base shall become less at an angle there, thus so that if the opening took up the whole cross-section of the cylinder, certainly no attenuation of the water jet would be able to arise. I may wish to turn my mind to those who will consider, perhaps, that they are required to have an account of this contraction in that determination of the velocities. For when the cross-section hole is not much less in that of the vessel, no notable contraction can arise, and when the opening is small, on the other hand almost no difference arises concerning the velocities whether the opening may be increased or diminished a little.

§. 4. The ratio of water flowing out horizontally is almost the same, so that I shall remain silent about other directions : for from any direction the water flows out in a similar manner through the opening; indeed also it ascends from the lowest part as far as to the opening so that it shall be able to flow out, which itself I have been able to observe often. Therefore by a similar cause the same happens in the attenuation of jets flowing out, as that is easier to be observed by eye, because here the attenuation may not be had in any other way than arising from the acceleration of the water now passed out. And on account of this reason, if anyone should decide to make measurements about the contraction of the jet, this may be done better by my judgment, by using the jet horizontally than by flowing out in any other direction.

§. 5. But how great the contraction shall be, that is, what ratio shall lie between the cross-section of the orifice and the minimum cross-section of the jet flowing horizontally, can be found either by taking actual measurements of these diameters themselves with the corresponding cross-sections, or even from the mean quantity of water flowing out in a given time, with the given velocities, where the velocities still shall not be deduced from the height of the water above the opening, but rather from the cross-section of the jet ; since the impediments now greater and again smaller at no time allow all the velocity of the water, as by the strength of the theory it should acquire, where no account of these impediments may be had.

§. 6. I think from these premises now shown well enough, to be perfect agreement between the quantity of water flowing out and its velocity, but only if for the opening, which is in the vessel, as far as another diminished opening may be substituted there, then the maximum contraction of the cross-section of the jet shall not be exceeded : and likewise there will be, in whatever place of the jet, or in whatever depth of the water from the surface this opening shall be put in place, either at *ac* or at *de,* seeing that the velocities will always correspond approximately to the whole height of the water above that place, where the opening is devised: the cross-section of its opening held in mind I will call thenceforth the *cross-section of the contracted jet.*

§. 7. But if now that section, about which we have just spoken, should have a constant ratio to the orifice, the opening of the outflow may be diminished in the same known ratio, and afterwards a calculation may be put in place concerning the amount of water flowing out in a given time. Thus surely with that ratio put $=\frac{1}{\alpha}$ and with the cross-section of the orifice called n, the *cross-section of the uniform jet* shall be agreed to be called $=\frac{n}{\alpha}$.
[Thus, a jet of uniform cross-section will be used in calculations, corresponding to that of the waist or narrowest cross-section.]
But when the variable shall be under different circumstances, the rules in that matter are not allowed to be given from those used formerly: moreover, the cross-section may be changed especially by the thickness of the plate, in which the opening is present, either increased or decreased [according, perhaps, to Bernoulli's idea of slightly conical openings ; the part played by surface tension does not seem to have entered into the argument, regarding the contraction of the jet, as this phenomenon was not understood correctly at the time] : also otherwise, although small, it can bring together the magnitude of the opening, the cross-section of the vessel, and these both absolute as well as relative, as well perhaps the height of the water above the opening. Meanwhile with the plate assumed to be thin, with the widest vessel, and with an opening from 4 or even 6 lines in diameter arising [a line $=\frac{1}{12}^{\text{th}}$ inch]; the ratio between the opening and the cross-section of the contracted jet was not accustomed to depart much from that, as Newton stated, truly as $\sqrt{2}$ to 1. But often it was observed to be greater or less than the other values.

§. 8. Truly whatever it shall be, in any case we will indicate that as before, by $=\frac{1}{\alpha}$. And for this in place we will show now the calculation for the times; but for the sake of brevity we will consider only cylindrical vessels, and in these we will examine two kinds of times chiefly; in the first place how to define the point of maximum velocity, and secondly, what corresponds to the time of emptying. Truly in each case we will put the motion to begin from rest.

§. 9. Therefore there was put in place a vertical cylindrical vessel full of water, and the height of the water at the beginning of the flow shall be $=a$, the cross-section of the cylinder $=m$, the cross-section of the jet $=n$, the *cross-section of the uniform jet* $=\frac{n}{\alpha}$; now water has flowed out during a time *t*, then the height of the water left above the opening shall be $=x$, at the same point of the time the surface of the water shall have an internal velocity, which may correspond to the height *v*: this velocity itself will be $=\sqrt{v}$, moreover the element of the time *dt* will be proportional to the element of the space $-dx$ divided by the velocity $\sqrt{v}$, from which $dt=\frac{-dx}{\sqrt{v}}$.

Indeed the value of *v* was determined in Sect. III, where we have used the same denominations with which now we make use. But because it is required to dispense the correct measure of water, so that for the opening *n, the contracted cross-section of the jet* $\frac{n}{\alpha}$ may be substituted, it follows, that the same may become substituted into the value of *v*, and thus there may be put in place :

$$v=\frac{nna}{2nn-mm\alpha\alpha}\left(\left(\frac{a}{x}\right)^{1-\frac{mm\alpha\alpha}{nn}}-\frac{x}{a}\right).$$

Truly if this value may be substituted into the equation : $dt=\frac{-dx}{\sqrt{v}}$, there arises

$$dt=-dx:\sqrt{\frac{nna}{2nn-mm\alpha\alpha}\left(\left(\frac{a}{x}\right)^{1-\frac{mm\alpha\alpha}{nn}}-\frac{x}{a}\right)},$$

with the aid of this equation all the times wished for can be defined by approximations or series, but only if the value of α may be known for the individual points: But we will assume that to be of a constant value, since in the present case there shall be nothing, by which it shall be able to be changed except the different heights and speeds of the fluid, which will be able to confer little or nothing with any perceived magnitude to that calculation.

§. 10. Now, so that the equation desired shall be shown by a series, we will consider the quantity

$$1:\sqrt{\frac{nna}{2nn-mm\alpha\alpha}\left(\left(\frac{a}{x}\right)^{1-\frac{mm\alpha\alpha}{nn}}-\frac{x}{a}\right)}$$

according to this form :

$$\left(\frac{nnx}{mm\alpha\alpha-2nn}\right)^{-\frac{1}{2}}\times\left(1-\left(\frac{a}{x}\right)^{\frac{mm\alpha\alpha}{nn}-2}\right)^{-\frac{1}{2}}$$

and we may resolve the latter factor by the customary rules into this series :

$$1+\frac{1}{2}\left(\frac{a}{x}\right)^{\frac{mm\alpha\alpha}{nn}-2}+\frac{1\cdot 3}{1\cdot 2\cdot 4}\left(\frac{a}{x}\right)^{\frac{2mm\alpha\alpha}{nn}-4}+\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3\cdot 8}\left(\frac{a}{x}\right)^{\frac{3mm\alpha\alpha}{nn}-6}+\text{etc.},$$

from which now the form of the equation will be had, with a little changed :

$$dt=-\frac{dx\sqrt{mm\alpha\alpha-2nn}}{n\sqrt{a}}\times\left(\left(\frac{x}{a}\right)^{-\frac{1}{2}}+\frac{1}{2}\left(\frac{x}{a}\right)^{\frac{mm\alpha\alpha}{nn}-\frac{5}{2}}+\frac{1\cdot 3}{1\cdot 2\cdot 4}\left(\frac{x}{a}\right)^{\frac{2mm\alpha\alpha}{nn}-\frac{9}{2}}+\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3\cdot 8}\left(\frac{x}{a}\right)^{\frac{3mm\alpha\alpha}{nn}-\frac{13}{2}}+\text{etc.}\right).$$

This equation thus is to be integrated, so that on putting $x=a$ there becomes $t=0$; moreover thus there arises

$$t=\left(2+\frac{nn}{2mm\alpha\alpha-3nn}+\frac{3nn}{16mm\alpha\alpha-28nn}+\text{etc.}\right)\times\frac{\sqrt{(mm\alpha\alpha-2nn)\cdot a}}{n}$$
$$-\left(2\left(\frac{x}{a}\right)^{\frac{1}{2}}+\frac{nn}{2mm\alpha\alpha-3nn}\left(\frac{x}{a}\right)^{\frac{mm\alpha\alpha}{nn}-\frac{3}{2}}+\frac{3nn}{16mm\alpha\alpha-28nn}\left(\frac{x}{a}\right)^{\frac{2mm\alpha\alpha}{nn}-\frac{7}{2}}+\text{etc.}\right)$$
$$\times\frac{\sqrt{(mm\alpha\alpha-2nn)\cdot a}}{n},$$

where $2\sqrt{a}$ expresses the time that a body spends while falling freely from the height *a*.

[There is a very useful note, by the editor G.K.Mikhailov, for a modern researcher into the units used in the 18th Century, in the *General Introduction* to Volume 5 of the Collected Works of Daniel Bernoulli, (die Werke von Daniel Bernoulli, Bande 5) starting on p.10, regarding the change from the units then used, to modern units; essentially the acceleration of gravity is taken as $\frac{1}{2}$, to be regarded as a constant without dimensions, while time is not a fundamental quantity, though mass and weight or force are all given the same fundamental dimension, as is length. Thus, at this stage in the development , everything in dynamics is measured in terms of length and weight, such as feet and pounds, in either English, German, or French units of the same. Thus, speeds are equivalent to lengths, etc. Later, Euler and Bernoulli were adept at readjusting the resulting equations so that the acceleration of gravity was given the value considered correct at the time, around 32 ft/sec^2.]

Truly if there may be put into this equation :

$$x = a : \left(\frac{mm\alpha\alpha - nn}{nn}\right)^{nn:(mm\alpha\alpha - 2nn)}$$

which is the height of the water with the maximum velocity (by §. 16 Sect. III & §.8 Sect. IV*)*, then the time may be found that from the start of the flow as far as to pass the point of the maximum velocity ; and when there is put $x = 0$, the time arises, when the whole vessel is emptied, and finally if x may be put equal to any quantity *c*, *t* will express the time which the surface spends in falling through the height $a - c$*;* moreover we will see for these cases, what must happen, when the vessel is very large, and the number *m* thus contains the other number *n* many times.

§. 11. In the first place, if $\frac{m}{n}$ were an infinite number, the height of the water corresponding to the point of maximum velocity will be

$$a : \left(\frac{mm\alpha\alpha - nn}{nn}\right)^{nn:(mm\alpha\alpha - 2nn)} = a : \left(\frac{mm\alpha\alpha}{nn}\right)^{nn:mm\alpha\alpha};$$

but because $\frac{mm\alpha\alpha}{nn}$ is [still] an infinite number, it can be agreed :

$$\left(\frac{mm\alpha\alpha}{nn}\right)^{nn:mm\alpha\alpha} = 1 + \left(\log\frac{mm\alpha\alpha}{nn}\right) : \frac{mm\alpha\alpha}{nn};$$

the demonstration of which matter is such : an infinite quantity *A* shall be proposed and there may be had as in our example $A^{1:A}$, and anyone can see easily that the quantity is a little greater than unity, and indeed by an excessively small amount, that we will call *z;*

and thus there is had $A^{1:A} = 1 + z$, the logarithms of each side may be taken, and there will be $\frac{\log A}{A} = \log(1+z) = z$ (on account of the infinitely small value of z)*;* therefore $A^{1:A} = 1 + \frac{\log A}{A}$: and hence similarly there is, as we have said,

$$\left(\frac{mm\alpha\alpha}{nn}\right)^{nn:mm\alpha\alpha} = 1 + \left(\log\frac{mm\alpha\alpha}{nn}\right) : \frac{mm\alpha\alpha}{nn}.$$

Again because this quantity to be added to unity is infinitely small, [the height of the head of water for the max. velocity will be]

$$= a : \left(\frac{mm\alpha\alpha}{nn}\right)^{nn:mm\alpha\alpha}$$

or, [*i.e.* on setting $\frac{1}{1-x} \approx 1 + x$, for very small x]

$$a : \left(1 + \left(\log\frac{mm\alpha\alpha}{nn}\right) : \frac{mm\alpha\alpha}{nn}\right) = a - a\left(\log\frac{mm\alpha\alpha}{nn}\right) : \frac{mm\alpha\alpha}{nn},$$

therefore is the space through which the surface of the water has fallen, then from rest the maximum velocity arising ,

$$= a\left(\log\frac{mm\alpha\alpha}{nn}\right) : \frac{mm\alpha\alpha}{nn},$$

or

$$= \frac{2nna}{mm\alpha\alpha}\log\frac{m\alpha}{n}.$$

This equation shows that the descent of water in an infinitely wide vessel is infinitely slow, when the water now has reached the maximum step in the velocity : But this should not be doubted without hindrance, or not while a finite quantity of water may flow out, since the cylinder stands on an infinite base, of a height however infinitely small, it shall have an infinite magnitude : while it follows from our equation, this quantity also is infinitely small, and is called equal to

$$\frac{2nna}{m\alpha\alpha}\log\frac{m\alpha}{n}.$$

And this agrees exceptionally well with the progress of the phenomenon, which we find out in the efflux of water from a tank from a simple hole for a whole day. For when we block the opening with a finger, and soon with the finger removed we may allow the water to flow out horizontally, we observe no drops falling on the middle ground between the longest trajectory and the place which may correspond to the perpendicular of the opening.

§. 12. Just as in the nearby paragraph we have determined the quantities, however infinitely small, to be used in the descent of the water and the efflux of the water while it reached the maximum value of the velocity, thus now we will keep the same in the ratio of the time increments. Moreover I say the time to suffice in the equation expressed in §.10, so that the single first term of the series may be taken in each, which will be apparent when anyone has extended the calculation to two terms : therefore it is the increment of the time sought, or [on integrating with the constant of integration put in place] :

$$t=\left(2-2\sqrt{\frac{x}{a}}\right)\times\frac{\sqrt{(mm\alpha\alpha-2nn)\cdot a}}{n};$$

hence with the related value for *x* put in place here, which was defined in the preceding paragraph, there becomes :

$$t=\left(2-2\sqrt{1-\left(\log\frac{mm\alpha\alpha}{nn}\right):\frac{mm\alpha\alpha}{nn}}\right)\times\sqrt{\left(\frac{mm\alpha\alpha-2nn}{nn}\right)\cdot a}$$

or, on putting $1-\left(\log\frac{mm\alpha\alpha}{nn}\right):\frac{2mm\alpha\alpha}{nn}$ for the corresponding quantity with the sign of the root expanded,

$$[i.e.\ t=2\left(1-\left(1-\left(\log\frac{mm\alpha\alpha}{nn}\right):\frac{mm\alpha\alpha}{nn}\right)^{\frac{1}{2}}\right)\times\sqrt{\left(\frac{mm\alpha\alpha-2nn}{nn}\right)\cdot a}$$

]

$$=2\left(1-1+\left(\log\frac{mm\alpha\alpha}{nn}\right):\frac{2mm\alpha\alpha}{nn}\right)\times\sqrt{\left(\frac{mm\alpha\alpha-2nn}{nn}\right)\cdot a}\,]$$

it becomes

$$t=\left(\left(\log\frac{mm\alpha\alpha}{nn}\right):\frac{mm\alpha\alpha}{nn}\right)\times\sqrt{\left(\frac{mm\alpha\alpha-2nn}{nn}\right)\cdot a}$$

or finally with the quantity *2nn* discarded within the sign of the root, there arises :

$$t=\frac{2n\sqrt{a}}{m\alpha}\cdot\log\frac{m\alpha}{n}.$$

But this short time is infinitely small, because, as it has been noted, the logarithm of an infinitely small quantity is infinitely smaller than that quantity itself. But thus truly when from the beginning the water is expelled at once with its maximum velocity, perhaps to some it will seem strange at first sight, that a finite velocity can be generated in an instant: yet no-one will think it strange that an infinite mass of any kind such as the amount of water contained in the vessel, is able to produce a finite velocity in an infinitely small time, and that by the action of gravity alone.

§.13. If besides we may wish to express the time of depletion in that vessel of infinite size, which certainly will be infinite, it will be, as was indicated above, in the equation of paragraph ten, putting $x=0$, and likewise requiring the first term of the series to be used, and again on putting $m\alpha$ for $\sqrt{(mm\alpha\alpha-2nn)}$; and thus there becomes :

$$t=\frac{2m\alpha}{n}\sqrt{a}.$$

Then finally the time, which is expended in the fall of the surface through the height $a-c$, will be expressed by this equation in a similar hypothesis:

$$t=\frac{2m\alpha}{n}\left(\sqrt{a}-\sqrt{c}\right).$$

[See KL, p. 38, for a complete derivation of this result.]

§. 14. Indeed the equations not presented accurately, still may give approximate satisfaction when the vessel is not infinite, but yet may be of a great cross-section : indeed they are not very much deficient, when the number *m* even moderately exceeds the number *n*. Here it may be allowed to add a few words about the experiment that I have mentioned at the end of paragraph eleven, and this indulgence is given by our putting in place, which experiment chiefly is turned towards illuminating and the examination of known motions for phenomena. Moreover I have said in the paragraph mentioned when the water flows out horizontally, the first drop obtains at once the whole amplitude of the trajectory ; and likewise this indeed indicates the theory for the largest vessels ; but truly in the not so large vessels, certain droplets must flow out with smaller impetus, before the point of the maximum velocity may be present, and these droplets must be incident on some middle position between the maximum trajectory and the point which corresponds vertically to the opening ; and this also I have been observed to happen, from vessels with a cross-section just greater than ten times more than the opening. Truly when at some time I might begin an experiment with a vessel half a foot high, because besides the cross-section may be had around a hundred times that of the opening, lest indeed the smallest particle of water, as far as I was able to see, notably was absent from the full water of jet. And thus we may see in this case what amount of water must flow before the point of maximum velocity ; but it will be just as great, as the amount contained in a cylinder of the same cross-section with the height

$$a-a:\left(\frac{mm\alpha\alpha-nn}{nn}\right)^{nn:(mm\alpha\alpha-2nn)}$$

(see the end of §.10); nor does this minimum height hardly differ from this much shorter expression, namely $\frac{2nn}{mm\alpha\alpha}\log\frac{m\alpha}{n}$ (see §. 11), where now by $\frac{n}{m}$,$\frac{1}{100}$ is understood and by a half a foot, while for α there must be substituted $\sqrt{2}$ (for we do not wish to sum this more accurately) an by the log the hyperbolic logarithm is indicated; thus so that there arises,

$$\frac{2nna}{mm\alpha\alpha}\log\frac{m\alpha}{n}=\frac{1}{20\,000}\left(\log 100+\frac{1}{2}\log 2\right)=0,0002476\ ft.\text{ or }0,00297\ inch.$$

and because I had found the cross-section of the vessel equal to $6\frac{1}{5}$ sq.inches, I knew the amount of water sought, which surely had to flow before the maximum jet had arisen, to be approximately equal to the fifty-second part of a cubic inch, or , by putting an average drop to make six cubic lines [*i.e.* $\frac{1}{52}\div(6\times\frac{1}{12^3})\approx 5.5$ cubic inch.], more than five drops. But in the experiment I observed none, I suspect the reason for this occurance to be, because the first drops, although now ejected, yet even now are being propelled by the water following subsequently ; indeed the rest are following exceedingly faster, than so that the first meanwhile may be separated from these. Moreover it happens here, because the small interval of time from the initial flow as far as to the maximum [rate] (which certainly by §.12 is approximately $=\frac{2n\sqrt{a}}{m\alpha}\cdot\log\frac{m\alpha}{n}$, where here by $2\sqrt{a}$ the time is understood, in which a body is dropped freely through a height of half a foot, that is, around $\frac{2}{11}$ of a second), because I say that increment of time will not extend beyond a one hundred and fifty-eight part of a second.

Perhaps it has contributed a little, that it shall not be possible to remove the finger quickly enough from the opening. Truly this may be especially relevant, because the greater part of that water, which before it now erupts with the maximum velocity, thus must approach to the maximum trajectory, so that no difference shall be able to be observed and thus scarcely a single droplet were to be particularly distinguished by that deficiency, if it were itself free to be separated from the subsequent water.

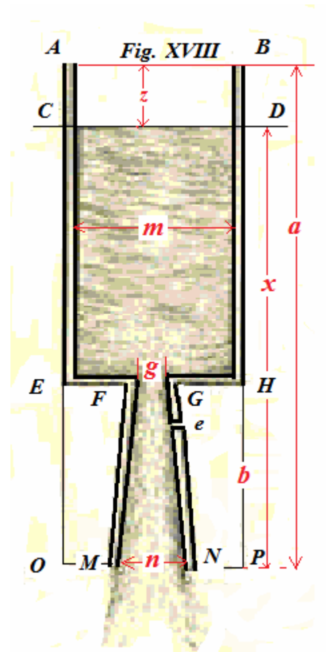
§. 15. Until now, we have been concerned with water flowing out through openings : now we may progress to the efflux of water from vessels either through converging or diverging cones. Because if water may flow out through a converging tube, the same account may be composed from the converging motion of the particles §.3 demanded for the exposition from simple openings, to be such that the water jet shall be contracted before the opening, and even now its particles shall be accelerated, and thus they shall indicate the quantity of water flowing out in a give time to be less than the measured efflux of the opening and of the velocities, with no account of the contracted jet. But this contraction is accustomed to be small in longer tubes. In diverging tubes everything happens in the opposite manner : for the jet shall be expanded in front of the opening ; the motion of the water is slowed down and a greater amount of water has flowed out in a given time, than may follow without this dilation, from the observed cross-section of the opening and from the velocity of the water flowing out through that. Finally with a water jet flowing out through a cylindrical tube it is neither contracted or expanded. Therefore it is required to attend properly either to these contractions or expansions in estimating the quantities of water flowing out in a given time, which question we will treat all the way to the end of the section.

But now it pleases to subject the changes to examination which follow in the efflux of the motion from the initial motion. Truly with these compended causes we shall not attend to changes in the jet ; for neither thus has the matter been prepared so that it shall agree accurately enough with experiments nor are the changes presented here of great concern ; but the matter itself is worth mentioning, because it may be inquired into anxiously so that its nature shall be able to be understood correctly.

Concerning vessels, which have tubes attached [or pipes ; in this translation I have preferred to use the word tube, as it refers to an attachment smaller than the original vessel] attached, we have already acted on in the above Section, in paragraphs §. 21, §. 22, §. 23, and indeed in paragraph 21 we have given the more general equations, whatever the ratio was between the cross-sections of the vessel and the tubes : but they are exceedingly intricate and they put in place excessively tiresome calculations : In the paragraph which follows that, I have investigated a hypothesis, the vessel of which everywhere is made in an infinite size to the ratio of the tube, in which hypothesis I have said, the water flows out with a velocity, which shall rise to the whole height of the water above the opening; but still at the end of the paragraph I have expressly warned, *from the beginning of the motion the water falls slower than thus it was defined, nor can that first rule be used, when the surface falls through some small space,* which matter by itself is apparent enough, since it is not possible to produce the maximum velocity in an instant from a state of rest in a tube, however it may be made simply by boring through an opening in the tube.

Thus bearing these in mind I have decided to investigate the initial changes, and to reduce these to certain measures. Moreover towards this end the aforementioned rules will suffice at least, where no account of these original changes may be had, although the remained is exactly true in an infinitely great vessel ; for all the changes which precede the state of the maximum velocities, happen while the surfaces descend through an infinitely small space; but still this descent, if only the vessel were infinite in the geometric sense, not only shall not happen in an infinitely small time, as in the case of a simple opening, but in an infinitely great time, and meanwhile also an infinite amount of water will have flowed out, since through the opening, with all else being equal, an infinitely small amount may flow out. But so that I may elicit this, I need to derive another equation from the general equation in §. 23 Sect. III, for the most simple this shall be $s = x$, with *s* put for the height, which may correspond to the velocity of the water flowing out, and *x* for the height of the water flowing out above the orifice ; but it will be understood for each, by putting in place our requirements, thus it has to be done so that an account may be had of the increments of the velocity, which before was not required.

§. 16. Therefore the cylinder *AEHB* (Fig. 18) may be had as in paragraph 22, Sect. III and this may be considered infinitely wide and full of water, and it may have the tube attached *FMNG* of finite cross-section of the form of a truncated cone, either with the cross-section increasing or decreasing towards the opening *MN*, through which the water flows out: it shall be so that the height of the initial water above the opening *MN*, truly $NG + HB = a$; with the height of the surface of the water situated at *CD* above *MN*, that is, $NG + HD = x$; with the length of the tube joined on or $NG = b$, with the cross-section of the opening $MN = n$, with the cross-section of the opening $FG = g$, with the cross-section of the cylinder, which is infinite, $= m$; and finally the velocity of the surface of the water at the position *CD* shall be such as which may agree with the height *v*, which height everywhere will be infinitely small. With these in place we have seen this equation to be obtained generally in the place mentioned :



$$m(x-b)dv + \frac{bmm}{\sqrt{gn}}dv - \frac{m^3}{nn}vdx + mvdx = -mxdx,$$

in which it is apparent, now to be able to ignore the first term $m(x-b)dv$ before the second $\frac{bmm}{\sqrt{gn}}dv$, and so that the fourth *mvdx* before $-\frac{m^3}{nn}vdx$, and thus to assume

$$\frac{bmm}{\sqrt{gn}}dv - \frac{m^3}{nn}vdx = -mxdx,$$

in which equation if on the contrary the first term may be ignored, because it can happen, unless the changes also may be desired, which endure from the start of the descent, even if they are infinitely small, the rule of the common ascent potential may arise of the water flowing out to the height of the whole water : now truly for our situation, in which we desire these first changes, this term will be required to be retained, and thus the final equation in its whole extension being treated. But for the indeterminate to be separated from each other in turn, there may by put $\frac{mm}{nn}v - x = s$, or $v = \frac{nn}{mm}(ds + x)$, and $dv = \frac{nn}{mm}(ds + dx)$ and thus there becomes

$$dx = \frac{-nnbds}{nnb - ms\sqrt{gn}}$$

which thus requiring to be integrated, so that by making $x = a$, there may be produced $v = 0$ and hence $s = -a$, thus truly there shall be

$$x - a = \frac{nnb}{m\sqrt{gn}} \log \frac{nnb - ms\sqrt{gn}}{nnb + ma\sqrt{gn}}$$

and with the assumed value for *s* put in place $\frac{mm}{nn}v - x$, there will be produced :

$$x - a = \frac{nnb}{m\sqrt{gn}} \log \frac{n^4 b - m^3 v\sqrt{gn} + mnnx\sqrt{gn}}{n^4 b + mnna\sqrt{gn}}.$$

Here again in the quantity involved within the sign of the logarithm, the term $n^4 b$ can be eliminated from the numerator, evidently infinitely smaller than the term $mnnx\sqrt{gn}$, and from the denominator the term $n^4 b$ equally to be infinitely smaller than the other $mnna\sqrt{gn}$. And thus there becomes

$$x - a = \frac{nnb}{m\sqrt{gn}} \log \frac{nnx - mmv}{na}.$$

From which it may be obtained, on putting *c* for the number whose logarithm is one :

$$v = \frac{nnx}{mm} - \frac{nna}{mm} \times c^{\frac{m\cdot(x-a)\sqrt{gn}}{nnb}}$$

or with $a - x = z$,thus so that *z* may denote the space, through which the surface of the water now has fallen, this more united form can be given to the equation :

$$v = \frac{nn(a-z)}{mm} - \frac{nna}{mm} : c^{\frac{mz}{nb}\sqrt{\frac{g}{n}}},$$

from which in turn it may be allowed so that when *z* will have had even a minimum ratio to *b*, the denominator of the other term becomes infinite, and

$$v = \frac{nn(a-z)}{mm} = \frac{nnx}{mm};$$

but truly otherwise the expression itself is found, while the descent z is infinitely small, which case we now consider.

§. 17. Now from these premises it is easy to define the descent of the fluid through a little space, while it acquires the maximum velocity, surely by requiring to make $dv = 0$, or

$$-\frac{nndz}{mm} + \frac{na}{mb}\sqrt{\frac{g}{n}}dz : c^{\frac{mz}{nb}\sqrt{\frac{g}{n}}} = 0,$$

that is, [on canceling common terms and reverting to logs:]

$$z = \frac{nb}{m}\sqrt{\frac{n}{g}} \times \log\left(\frac{ma}{nb}\sqrt{\frac{g}{n}}\right).$$

Moreover this height multiplied by the cross-section of the cylinder *m* gives the amount of water flowing out meantime, evidently

$$nb\sqrt{\frac{n}{g}} \times \log\left(\frac{ma}{nb}\sqrt{\frac{g}{n}}\right),$$

which quantity, as we have advised in §. 15, is infinite, although only logarithmically, an infinitude of this kind is less than the root of any dimension from the same infinitude ; evidently $\log\infty$ is less than $\infty^{\frac{1}{n}}$, however great the assignable number *n*. And this therefore I advise, so that it may be understood thus what happens, so that, if truly we may consider that infinitude to a very large magnitude, the amount of water that may escape shall be small enough. These are the remaining corollaries of this formula.

(I) If the adjoined tube is cylindrical [*i.e.* $g = n$], there becomes $z = \frac{nb}{m}\log\frac{ma}{nb}$. Therefore from the remaining parts this quantity is found, so that the length of the tube connected, because generally it is true also : for from a changed value of *b* it is to be understood that the $\log\frac{ma}{nb}\sqrt{\frac{g}{n}}$ does not change, on account of the infinite number $\frac{m}{n}$.

(II) Accordingly, for the same opening *g* and with all else equal, it follows that the quantity *z* is in the three halves ratio to the extremity of the orifice : and if the same tube now with the opening narrower, again made wider be connected to the vessel, the quantity of water in the first case to the amount in the second shall be as the square of the wider orifice to the square of the lesser orifice.

(III) Finally it is required to observe the whole reasoning to prevail for all directions of the tube, as anyone may understand who will examine §. 22 of Sect. III correctly. Therefore the tube can be used both horizontal or in any other direction, and however curved, which especially will be called to mind in setting up experiments. But always the length of the tube will be understood by *b*, truly by *a* the vertical height of the water above the end of the orifice.

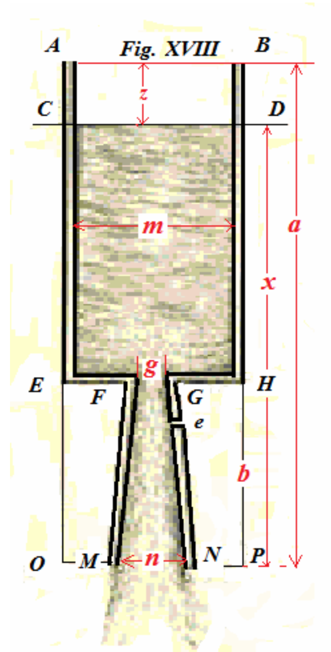
§. 18. Now I come to the time, so that these changes are made from rest to the maximum velocity: moreover I say it is possible in the calculation of times of this kind to put simply $v=\frac{nn}{mm}a$; for the remaining quantities vanish in the final equation in §.16, however small the height *z* may be taken, just as even the smallest assignable ratio may be had to that infinitely small height, which corresponds to the maximum of the velocity, evidently for $\frac{nb}{m}\sqrt{\frac{n}{g}}\times\log\left(\frac{ma}{nb}\sqrt{\frac{g}{n}}\right)$. It follows from thence the predicted time, which I shall call *t*,

$$=\frac{b\sqrt{n}}{\sqrt{ga}}\times\log\left(\frac{ma}{nb}\sqrt{\frac{g}{n}}\right)$$

and hence is infinite, although the same time shall be exceedingly brief, since the cross-section of the vessel is not infinite, but infinitum, great in some manner, which again can be deduced from the nature of the logarithmic infinitude.

[Bernoulli's formula for the time gives a comparison with the immediately foregoing equation for *z*, the height fallen to reach the maximum velocity, giving $t=z:\frac{n}{m}\sqrt{a}$, where $\frac{n}{m}\sqrt{a}$ is the final velocity, easily measured from the speed of the final jet ; the equation in §.16 then gives $z=\frac{nb}{m}\sqrt{\frac{n}{g}}\times\log\left(\frac{ma}{nb}\sqrt{\frac{g}{n}}\right)$.]

§. 19. Because the height of the velocity, as we have observed in the previous paragraph, can be considered at once $=\frac{nn}{mm}a$, that is, equal to the maximum, since the surface descends through the smallest assignable part of an infinitely small part, after which the velocity reaches the full maximum, it follows that most changes from rest as far as to the state of maximum velocity are not measurable, that is, infinitely small, indeed not only most, but besides all the infinitely small parts: thus the matter may itself be had clearly: the velocity from the first beginning plainly is zero, and after the water has fallen through an infinitely small distance, now so far it is not a maximum ; then while again it falls certainly through another small distance through infinitely small, yet infinitely greater than the first, it goes on moving with its velocity taking infinitely small increments, and then at last truly it reaches the maximum velocity: truly since these latter or infinitely small changes shall be unable to be perceived by the senses, otherwise we will handle that from the theorems we gave in §.17, by considering the place of the changes from rest as far as to the point of the maximum velocity, the same changes up to the given step of the velocity.



§. 20. And thus we will investigate, through how great a distance z the surface of the water descends from a state of rest, and how much water flows out, and finally how great a time must pass, so that the water inside may be moving with a velocity which is generated by the free fall from a given height, and this height we will call $\frac{nn}{mm}e$, thus so that e itself may denote a similar height for the velocity of the water flowing out. Towards this end, it is required that in the final equation of paragraph sixteen, for v there may be put $\frac{nne}{mm}$, thus moreover there shall be :

$$\frac{nne}{mm}=\frac{nn(a-z)}{mm}-\frac{nna}{mm}:c^{\frac{mz}{nb}\sqrt{\frac{g}{n}}}=\left[\frac{nn(a-z)}{mm}-\frac{nna}{mm}:\exp\left(\frac{mz}{nb}\sqrt{\frac{g}{n}}\right)\right]$$

and hence there is deduced :

$$\frac{mz}{nb}\sqrt{\frac{g}{n}}=\log\frac{a}{a-e-z};$$

here truly since e may be put to fall short notably from a, it is possible to reject the letter z involved within the sign of the logarithm, from which there may be found

$$z=[z_e]=\frac{nb}{m}\sqrt{\frac{n}{g}}\times\log\frac{a}{a-e}.$$

Truly this equation now may indicate a distance, which is infinitely small, and through which the surface of the water descends, while the velocity of the water flowing from rest shall be of such a size, that it may be due to the height e; and this increment itself may have [the ratio] to that indicated in paragraph seventeen,

$$[i.e.\ \ z_v=\frac{nb}{m}\sqrt{\frac{n}{g}}\times\log\left(\frac{ma}{nb}\sqrt{\frac{g}{n}}\right).\ ]$$

so that clearly the velocity to the maximum velocity arising, to be as

$$[\frac{z_e}{z_v}] = \log\frac{a}{a-e} \text{ to } \log\left(\frac{ma}{nb}\sqrt{\frac{g}{n}}\right),$$

thus so that the first shall be infinitely less than the other, even if that likewise shall be infinitely small.

If again the defined quantity *z* may be multiplied by *m*, the amount of water is produced flowing out while that velocity is due to the height *e*, which amount is hence equal to

$$nb\sqrt{\frac{n}{g}}\times\log\frac{a}{a-e}$$

[*i.e.* the element of volume] and thus shall be of a finite magnitude, and indeed of a greater amount, as the tube is taken longer, and where a greater jet may be anticipated.

Finally the time, in which the same happens, if the terms rejected shall be selected correctly, is found to equal

$$2\sqrt{\left(\frac{nbb}{ag}\log\frac{a}{a-e}\right)}$$

and thus finite but certainly small and in no example easily extended beyond a second. [The result quoted by Bernoulli is incorrect. The true time has been established by the editor G.K.Mikhailov, *Werke Bande V,* Section IV, note 21, to be rather $\sqrt{\frac{nb^2}{ag}}\cdot\log\frac{\sqrt{a}+\sqrt{e}}{\sqrt{a}-\sqrt{e}}$.]

§. 21. I wished to examine and pursue all these accurately, because of multiple phenomena, which are accustomed to be observed in the efflux of the water, as well also so that these changes, which are plainly sensible to the senses, we may think about correctly. There were many phenomena, which pass from the infinite to the finite or in turn, from the finite to the infinite in the flow of water, which do not follow on correctly and which are not able to be extracted themselves from many difficulties ; which others readily admit a solution, but if in place of the almost infinite vessel, of which kind there are none, a very large vessel may be taken, or even in many cases it will be sufficient, average size, the formulas will be approximately true, and the approach to the truth sometimes more and sometimes less according to the nature of the question: indeed I shall advise about these. Meanwhile thus it appears sufficient from the theory, because I had decided to explain chiefly, why water from the widest simple vessel generally may flow out with speed immediately, and why with the water ejected from a vessel by a tube it shall be otherwise: truly precise measures from these questions and from the equations themselves are to be deduced.

§. 22. Finally because it may be related to the time of emptying, it is apparent when the size of the vessel may be even moderately greater than the size of the attached tube, to be possible without sensible error to consider that $=\frac{m\alpha}{n}\theta$,on considering by θ the time, where a body by falling freely from rest has completed a height, which the water from the start of the flow had above the extreme opening of the tube, and by taking for $\frac{m\alpha}{n}$ the ratio which is between the cross-section of the tube and the *cross-section of the jet*, either *contracted* or *dilated.* Truly the impediments, which in these cases may be come upon by chance, certainly will increase that time. Truly if the time may be desired, in which the surface of the water may fall by a given height, that will require to be taken $=\frac{m\alpha}{n}(\theta-T)$, on taking for that the time T the body spends freely falling through the height, that the water has above the opening at the end of the flow.

Experiment which pertain to Section IV.

Since a great part of this section shall be put in place in the contraction of the water jet flowing through a hole in a thin plate, I have considered thinking about putting in place that contraction experiment accurately, indeed not by taking measures of the diameters, as I have proven that method cannot be made with sufficient accuracy, but by observing the actual velocities from the magnitude of the trajectory, and the quantities flowing from the given times ; in the experiments I have used an automaton [presumably modified from a pendulum clock, so that the beats or ticks could be heard easily], as in a time of one minute it beat 144 times in turn, and thus I have assumed to be the case in the following.

Concerning the theory of the contraction of water jets.

Experiment 1.

I used a cylindrical pipe, the diameter of which was 4 *in.* 3 *lin.* [*i.e.* $4\frac{1}{3}$ inches] in English measures made from a thin plate each had a hole in the side, that is, on the cylinder surface: the diameter of the hole was $=4\frac{52}{152}$ *lin.*; the water flowed out horizontally from the cylinder placed vertically, and from the beginning the height of the water above the centre of the opening was $=4$ *in.* 3 *lin.*, and likewise the height of the flow at the end was $=3$ *inch.*; moreover the whole flow lasted for an interval of eleven automatic pulses, which made a time of approximately 4 and a half seconds.

Again with the experiment repeated often and with observations made both from the height of the opening above the table placed horizontally, as well as with the cross-section of the jet, and this both at the start and the end of the flow, see from the *Lemma concerning the introduction of the experiments indicated in the previous section*, the velocity of the water flowing out at the place of the maximum contraction of the jet to have been such constantly, indeed however great it was able to be judged by the senses, which was due to the height of the water above the same place, which is at the same height as the opening.

Therefore if we put the contraction of the jet of water to have been the same everywhere and to this case we will apply the final equation of paragraph thirteen, namely

$$t=\frac{2m\alpha}{n}(\sqrt{a}-\sqrt{c}),$$

on putting, $t=4\frac{1}{2}\,sec.;$ there becomes $\frac{m}{n}=133;$ $2\sqrt{a}$ (= to the time which a body takes by falling freely from the initial height of water*) $=0{,}1483$ and $2\sqrt{c}$ is equal to the similar time for the final height of the water) $=0{,}1246:$ making $4\frac{1}{2}=3{,}15\alpha;$ so that $\alpha=1{,}43.$ Thus it follows, the cross-section of the opening to be to the cross-section of the contracted jet as 143 to 100; this ratio is a little greater than that which lies between $\sqrt{2}$ and 1 truly between 141 and 100; but if the velocities were to be observed with the greatest accuracy, there is no doubt, why these should not become a little smaller, than what the whole height of the water may demand ; and since the ratio of this quantity can be found, the value of α thus is to be taken a little diminished ; therefore it is possible to be deduced most safely from the whole experiment that the aforementioned ratio to have been as $\sqrt{2}$ to 1.

[*Recall that both speed *V* and the time *T* can be expressed from the vertical fall of a body from rest through a distance *h*, where the acceleration of gravity is taken as $\frac{1}{2}$ without dimensions , making use of the elementary formulas $V=\sqrt{h}$ and $T=2\sqrt{h}$.]

Experiment 2.

Then I wished to investigate by experiment, whether in all trajectories made in whatever direction, the contraction shall be the same, and hence in the end I have judged the matter to be approached thus, so that besides the change of its direction all the other circumstances are to be exactly the same. Truly thus I maintained arrangement this.

Clearly I used the same cylinder as before, but I have added that on to a vertically placed prismatic water reservoir, thus so that the axis of the cylinder should be horizontal, and thus I have turned around the added reservoir, so that at one time the centre of the hole of the water efflux was resolved to be on top, at another time in the middle, and again it occupied the down position: in the first case the water flowed out vertically, in the second horizontally, and in the third it was ejected vertically downwards ; truly in the individual cases I made sure that the heights of the water in the reservoir above the centre of the hole were exactly equal : this was the approach.

I have observed for equal times the surface of the water in the individual cases to descend in the reservoir through equal distances. Therefore in the jets projected upwards the upper water did not resist sensibly the water to be following below, which likewise I understand with the other manner, because evidently, if at a small distance from the opening such as of 3 lines I was removing the jet of water of any direction with some coin, thus so that the jet were to strike the coin perpendicularly, the efflux of the water would not be retarded. Again neither does the water descending vertically in jets draws the latter jets after the former ; and the contraction of the jets is the same everywhere, without the retardation or accretion of the water above or below the water being ejected, which cause the jet at some distance from the opening so that either it shall swell up or become thinner. For this I say is contracted by that manner, which arises from the oblique motion of the particles in the region of the opening.

Experiment 3.

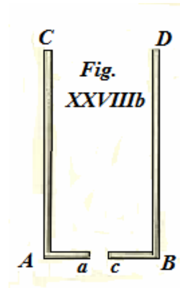
I have used the same equipment prepared in the previous manner for investigating whether the contraction of the jet, with all else being equal, will be changed by an increase in the height of the water above the opening. To this end, I have attached two needles to the inner walls of the water reservoir corresponding to the perpendicular of this, the first projected 13 *in.* and 10 *lines* [*i.e.* $13\frac{5}{6}$ inches] above the centre of the opening, the other 12 *in.*,$1\frac{3}{5}$ *lin.*[*i.e.* $12\frac{2}{15}$ inches] English measure*;* the cross-section of the reservoir was to the cross-section of the opening as 404 to 1; moreover I saw the surface of the water to have fallen from the upper needle to the lower one after an interval of 24 automatic beats, which made a time of 10 seconds.

But if indeed the same time were sought from the hypothesis, the jet itself not to be contracted, and likewise the water all to have flowed out with the same velocity, due to the theoretical force with no outside obstructions present, that time is found to be $=6\frac{7}{8}$ sec. Therefore it can be concluded thus, the cross-section of the opening to the cross-section of the contracted jet was as 10 to $6\frac{7}{8}$, that is, $\alpha = 1,45$, since in the first experiment for the same opening, with all circumstances considered carefully $\alpha = 1,41$.

Thus after I had demonstrated this, the rest was to investigate, whether all the water was flowing out with a perceived speed, concerning which I had more doubt, because with the increased speeds of the water, likewise the obstructions would increase, and these therefore would be noticeable in water from greater heights, such as are not present with lesser heights.

And thus I have attended with care to everything done (which chiefly is required for the precision of the experiment), so that the water flowed out in a perfectly horizontal direction, and with measurements taken both of the cross-section of the jet, as well as of the height of the opening above the horizontal table, see the calculation set out below, because when the height of the water was $=13$ *in.* with 10 *lin.* or 166 *lin.*, the water flowed out, or rather through the cross-section of the contracted jet, with a velocity, which agreed with a height of $158\frac{1}{2}$ *lin.;* therefore the velocity in the calculation was being diminished in the ratio of the square root of the height and the value of the letter α found was decreased in the same approximate ratio, which thus becomes a little less than 1,42 or again 1,41 and thus it is possible to deduce, the change in the height of the water alone does not change the contraction of the jet according to the senses.

Experiment 4.



I had a cylindrical tube of height 4 *in.*, a cross-section of which through the axis is shown by (Fig. 28b) *CABD;* the cross-section of the cylinder was to the cross-section of the opening *ac* as 110 to 1. This cylinder filled with water was emptied completely in a time of $21\frac{1}{2}$ seconds. But it must be noted, that the first outflow of the water is not to be conceded, as nothing of rotation is to be observed in its motion ; for otherwise the internal water soon is set into rotation, with the efflux enduring fast enough, and the efflux rather retarded, and with that the more, as the water within was set more quickly in rotation: because again in no circumstances with all the water flowed out, have I considered the time of the efflux, until it began to flow out drop by drop.

This experiment indicates here the contraction of the water was less than for the ratio $\sqrt{2}$ to 1; I had expected the time of emptying to be about 23 *sec.*, but the outcome was a little different as I have said, the reason for this I noticed afterward to be, because the lips of the opening were formed from the elongated tubule a little too short, as the Figure has shown, which prevented the contraction of the water jet : meanwhile the width of these lips did not reach two thirds of a line.

Experiment 5.

I arranged that the water from the vessel with the greatest cross-section flowed out horizontally from the tubule : moreover the tube was the shortest, truly the length not exceeding *3 lin.*, and it had a diameter of nearly *5 lin.*

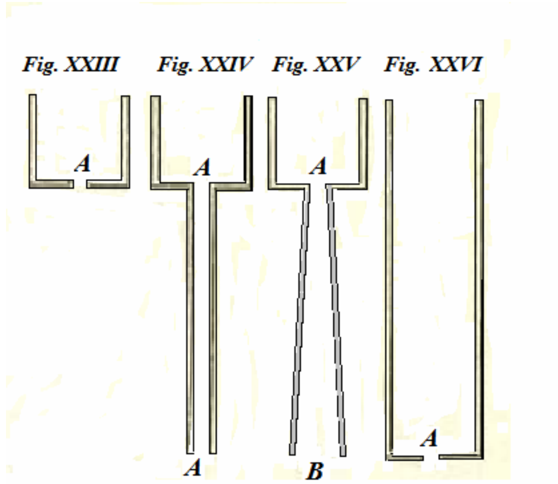
The given amount of water flowed out in a time of $11\frac{1}{4}$ *sec.* which ought to have flowed out in a time of $10\frac{2}{3}$ *sec.*, if neither the jet were contracted nor any impediments were present.

The actual velocities of the water I have considered need not be as I might measure, no one doubting such to exist, they must be of such a kind, that a given quantity of water may flow out through an observed orifice in an observed time, with no attention paid to the contraction of the jet.

I have used other tubes of a different diameter and length above and I have seen the amounts of water flowing out in a given time, and with given velocities, corresponding correctly to the efflux from the orifices: but the velocities there to be lacking more from the velocity due from the height of the water, where the tube was narrower and longer, and in order that the water was higher.

Concerning the theory of water flowing out of pipes.

Experiment 6.



Cylindrical vessels, of which Fig. 24 & 25 represent the cross-sections through the axis, the cylinders had the height 4 *Eng. inches* and a tube joined on of one foot ; the cross-sections of the cylinders were to the orifices *A*, as 110 to 1; but the orifice *B* was to the orifice *A* approximately as 25 to 16; the time of emptying the full first cylinder in Fig. 24 was $6\frac{1}{2}$ sec., in the others around $4\frac{1}{3}$ sec.

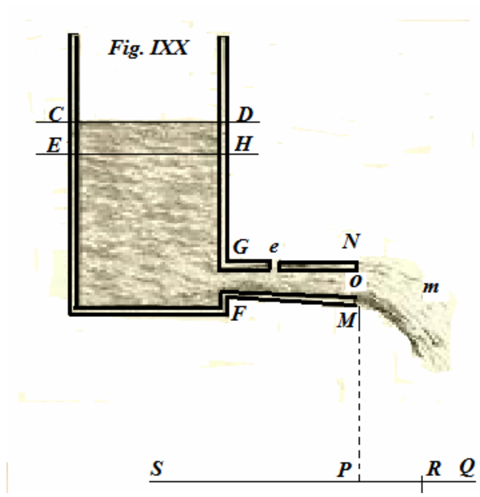
In these cases the vessels were to be enlarged enough on account of the connected tubes, so that the could be considered just as infinite; and by the rules indicated by us elsewhere the water ought to flow out through the distant opening corresponding to the whole height of the water, but only if you may remove the first instants of the flow, which here themselves are so short, that they are unable to be observed. And since besides, as we have advised elsewhere, the amount of water in a given time flowing out through the tube shall be estimated easier from the speeds and from the size of the orifices, to be found by the rule I have shown in §. 22, the time of emptying in the first case to be $4\frac{1}{3}$ sec., in the second case to be almost equal to 3 sec.

Because in the experiment these were observed a little greater, in Fig. 24 a maximum part is to be attributed to the adhesion of the water to the sides of the tube, but in Fig. 25 it is to be attributed to another additional reason indicated in paragraph 24, Sect. III.

Other phenomena in these vessels are to be noted: truly when the vessels are not yet emptied, indeed a sound is perceived from the air, which then mixes with the water above the orifice; truly I took this sound for the final moment of the flow : again it can happen easily, that the efflux of the water shall have gone just before it was reduced to perfect quiet (for the air and water were mixed up from filling and moved in the vortex of the water); but then the efflux certainly was slowed down and an internal sort of cataract was formed, and the air mixed with the water continued to flow out. Thus it can be the case that the efflux be retarded, if air and water are mixed up in the vortex before flowing out.

Experiment 7.

I used a prismatic vessel, to which a tubule was attached horizontally as in Fig. 19. It had an opening *GF* precisely five lines in diameter; the other *NM* $6\frac{1}{2}$ lines. Therefore the cross-sections of the openings *GF* and *NM* were as 100 to 169; but the cross-section of the vessel contained the cross-section of the opening *NM* two hundred and one times. The length of the tubule *GN* was 4 *in.*



Then the vessel was filled with water as far as to *CD*, the height of which above the axis of the tube was 13 *in.* 10 *lin.* With the hole *NM* opened the water flowed out and the surface fell as far as to *EH* in the time $8\frac{1}{3}$ *sec.;* truly the difference of the heights *CE* or *DH* was of two inches with eight lines. With the calculation reckoned according to the standard of paragraph 22 where no attention was paid either to impediments or to a change in the jet [diameter], we see that the predicted time of descent to be due approximately $=5\frac{1}{2}$ sec. Therefore it is required to be put in place in this manner, the whole mean velocity [measured] may be had to the whole velocity as the theory indices, as $5\frac{1}{2}$ to $8\frac{1}{3}$ or approximately as 2 to 3; and hence it can be concluded, the water to have flowed through the opening *MN* with a speed, which may be agreed to be with the $(\frac{2}{3})^2$ or four ninths parts of the height of the water above the opening *MN*, to have flowed across from the other opening *GF* with a speed of five parts [in nine] in addition on account of the same four parts due to the same height.

[Recall that $t=\frac{2m\alpha}{n}(\sqrt{a}-\sqrt{c})=\left[\frac{2m\alpha}{n\sqrt{2g}}(\sqrt{a}-\sqrt{c}) \text{ in modern terms}\right]$, is the time for a body to fall from the initial height *a* to a final height *c* expressed in feet, and the acceleration of gravity taken as 32 ft/sec.^{2} , and in this case we have the time to fall being equal to : $2\times 201\times\left(\frac{\sqrt{13\frac{10}{12}}}{12}-\frac{\sqrt{11\frac{2}{12}}}{12}\right)\times\frac{1}{\sqrt{64}}=5.48\,\text{sec}.$]

And thus it is apparent again the efflux of the water to be increased by the increase in the cross-section of the tube towards the outside, although neither the opening by which the tube is attached into the vessel, nor the position of the tube shall be changed.

Again on the table placed horizontally *PQ* I have observed the magnitude of the trajectory *PQ* for the height *oP*, which was 4 *in.* 8 *lin.* Moreover I found $PQ=9$ *in.* 6 *lin.*

It follows from this observation, that if with the consideration of the dilation of the jet set aside, the velocity of the water at *NM* ought to have a velocity, of such a magnitude as to be due to a height of 4 *in.*10 *lin.*,[by the Lemma in Section III, the velocity due is $PQ^2 : 4oP$, corresponding to 4 *in.*10 *lin.*], since again by the strength of projection of the experiment, evidently it would have a velocity due to a height of nearly 6 *in. 2 lin.* This observation confirms what I said in §.15 , namely in diverging tubes the jet of water to be expanded just as at *m*, and its motion to be retarded. Truly in the present case, so that both observations will agree, it should be said that the expansion shall be thus, so that it may have a cross-section reciprocally in the ratio of the opening *NM* as the aforementioned velocities or inversely as the roots of the heights due to the velocities themselves, evidently as $\sqrt{74}$ to $\sqrt{58}$ and hence the diameters of the dilated jets and of the orifices were as $\sqrt[4]{74}$ ad $\sqrt[4]{58}$ or as 1000 to 941.

Experiment 8.

I did another experiment that, although this may not yet be relevant, nevertheless I will review it: truly I made a hole *e* in the tube arising near the opening *GF* of nearly two lines, and again I observed the descent of the surface from *CD* to *EH* with the water flowing out through *NM*, and again I examined the size of the trajectory.

These two things I observed, which the first may seem to be almost a contradiction ; the descent from *CD* to *EH* was made slower than in the previous experiment, and now it lasted 10 *sec.* and also the trajectory *PQ* was still greater for the same height *oP;* for now it was $PQ = 10$ *in.*10 *lin.*

I can explain both phenomena thus : on account of the opening *e*, which was made near *GF* and because it allowed the free passage of air, it loosens the restriction that the water in the tube has otherwise, lest hence otherwise the water flows across where the little hole *e* is present, as if that itself in place should cut off the rest of the tube; but the flow will be slower, because as I have shown in different places, if the tube *GNMF* itself be diverging the time shall be made shorter. Because again although the quantity of water be smaller, yet the water shall not be able to flow through the opening *NM* with a greater impetus without involving a contradiction, the reason is through a mixture of air with water ; for the air continually intrudes into the tube through the tiny opening *e* and flows out together with the water through *NM.* Finally that phenomenon, as the water flowing through the opening *MN* actually speeds up, for with the hole *e* closed it does not seem possible by me to be explained otherwise, than because fewer extrinsic impediments than usual act on the water rarefied by the air.

Concerning the theory of water, which flows out from the largest vessel from the point of rest as far as to a given level of the velocity.

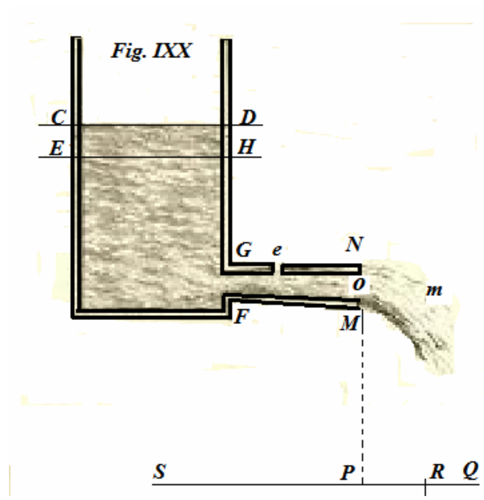
Experiment 9.

When water flows out through an opening made in a thin plate from the largest vessel, in the first place a droplet bursts out at once with the whole velocity, which is due to the height of the water above the hole.

This is in agreement with the theory indicated in §.11, if the vessel shall in fact be infinite, and although it were not to be infinite in the geometric sense, as long as it shall be very large, no drop equally can be observed, which does not flow out with the maximum velocity: I have explained this phenomenon in §.14, since evidently by the strength of the theory in some particular case examined in that place scarcely to be sensibly different by one or two droplets due to the maximum trajectory, I say it is not possible for such a small amount of water itself to be separated from the subsequent water on account of the mutual attraction of the water particles or their adhesion.

Experiment 10.

Truly when water was flowing out from the widest vessel through a tube inserted horizontally into the vessel, I observed before the jet flowing out had formed the maximum trajectory *omQ* (see. Fig.19), a noticeable enough amount of water had fallen down freely onto the horizontal table in the middle between *P* & *Q*, with that quantity to be greater both when the tube *GN* is longer and when it diverges more towards *N*, and finally that water to be spread out unequally, evidently to fall in greater abundance in the place which is further from the point *P*, than which is nearer the same point; but on account of the time, in which all these changes happen, I have observed that to be the shortest, and such that its measure cannot be grasped.



All these phenomena satisfy the accompanying propositions, which we have given in the eleventh paragraph as far as to the end of the section. But the measurements shown in that place cannot be confirmed correctly from the experiments, especially these, which have been indicated in §§. 15, 16 & 17, where evidently formulas are communicated, which express the amount of water flowing, while from rest the trajectory shall become a maximum : the reason is in the first place, because the first droplets which ought to fall close to the point *P* on the table may not be free to separate themselves from the following water; in the second place, because the amount of water near to the jet *oQ* (which indeed constitutes the maximum part to the strength of the theory itself) may not be able to be intercepted, and finally, because the motion of the water through the tube is accustomed to be excessively retarded from the external impediments alone, especially if the tubes diverge, and thus the actual motion shall be certainly different from the motion that the water shall be having from all the distant impediments. The remaining measurements indicated by us have been subjected to fewer and with these of smaller concern ; moreover in §.20 they may contain and express chiefly the amount of water, which flowed from the first point of the motion, while the water reached a given level of the velocity.

However on account of the reasons mentioned just now, especially in the case of diverging tubes, it shall be minimally possible to expect perfect agreement with the experiments, yet such I have found to be successful, so that I could easily understand the whole going to be in agreement if all the impediments together with the mutual adhesion arising from all the particles of water could be anticipated. Moreover I now explain the individual experiments taken both with the tube diverging as well as for the cylindrical case:

Experiment 11.

In Figure 19 a tube in the shape of a truncated cone was inserted into the vessel, the vessel itself was filled with water as far as to *CD*, thus, so that its height above the axis of the tube was equal to 433 equal small parts, which I used in the whole experiment. In the experiment I looked for that height which corresponds to the maximum trajectory at the point *Q*, and it was $PQ = 287$ *part.* while the height *oP* was $= 146$ *part.* Thus I saw the motion of the water to be very retarded both because of the adhesion of the water, as well as on account of the shape of the tube, which must happen in these cases as I have warned a number of times. But the distance ought to become, if nothing stands in the way of the motion $PQ = 503$ *part.*

Then I placed a dish on the horizontal table, the edges of which were at *S* and *R:* But first I soaked the dish, and I allowed all the water from that to flow away again : and with the measure *PR* taken, I found that to be 206 *part.*

And then the diameter *GF* was $= 13$ *part*. and *MN* $= 17$ *part.*, moreover the length of the tube was $= 125$ *part.* Thus from all these prepared, while I blocked the opening *MN* with a finger, with the finger removed the water was ejected immediately, and some part of this fell into the dish: this I gathered carefully in a glass cylinder, the diameter of which was $= 8\frac{1}{2}$ *part.*; the tube itself was filled to the height 210 *part.;* therefore the amount of water fallen into the dish $= 11922$ cubic particles.

Now truly this quantity by §.20 must be $= nb\sqrt{\frac{n}{g}} \times \log \frac{a}{a-e}$ where by *n* the cross-section of the opening *NM* or 227 *sq. parts* is understood, by *g* the cross-section of the opening $GF = 133$ *sq. parts.;* again *b* denotes the length of the tube, which was $= 125$ *part*.; by *a* is understood particularly the height of the surface *CD* above the axis of the tube, here truly it is required to be understood rather the height agreeing with the speed of the water incident at the point *Q*, or 141 *part.;* and similarly for *e* the height is required to be taken agreeing with the velocity of the particles incident at the point *R*, evidently $e = 73$ *part.* Finally the shortened word log signifies the hyperbolic logarithm. From these with numerical values substituted, there becomes

$$nb\sqrt{\frac{n}{g}} \times \log \frac{a}{a-e} = 227 \times 125 \times \frac{17}{13} \times \log \frac{141}{68} = 26830.$$

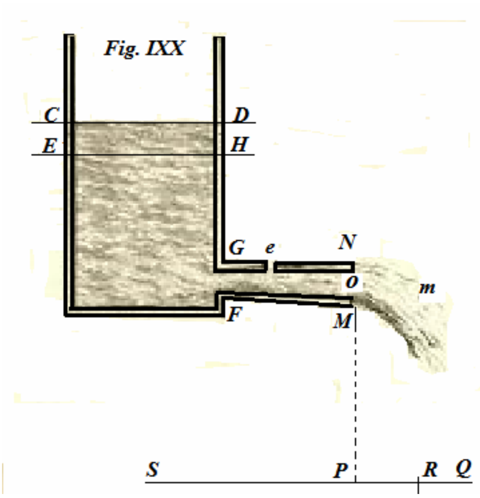
Therefore the amount of water found from the experiment was to the amount that the theory indicated with the consideration of all impediments set aside, as 11922 to $26\,830$; which numbers, although not a little different, still confirm the theory exceptionally well, which I shall now clearly bear that in mind.

In the formula $nb\sqrt{\frac{n}{g}}\times\log\frac{a}{a-e}$ we have put for *a* the height owed to the maximum velocity of the water flowing out, such as actually was the case in the experiment, it would not be with such obstacles about to be removed; evidently we can make $a=141$: in theory indeed there is $a=433$. Because if moreover we may assume this latter value, by retaining the value of the height $e=73$, [the amount of water flowing out] $nb\sqrt{\frac{n}{g}}\times\log\frac{a}{a-e}$ becomes approximately $=6700$, which number is now much less than the number elicited from the experiment, since that found previously was excessively greater. But such happens, when the height *e* serves to put the value [of the formula] in place: Truly just as the height *a* was increased from 141 as far as to 433, thus surely also the height *e* is required to be increased, and it shall come about with each height being increased in the same ratio, if the obstructions from the first droplets and the following ones are resisted equally: but the particles meet less resistance with all else being equal, the slower they are moving, and accordingly also the droplets which fall near the end *R* shall be retarded less, than those which have moved across from that side : Thence it is easy to deduce that the height *e* to be increased in a smaller ratio than the height *a*, truly we cannot name the ratio itself, unless afterwards, evidently with it being known, so that the theory may agree with the experiment; thus it is found to be by putting $e=120$, which number in mind clearly gives satisfaction to all the circumstances attended to well.

Thus it appears obvious to me, the success of an experiment to be such that clearly it should agreed with theory. Moreover, examples of this kind generally show that we have treated the true laws of motion in fluids, and amongst an infinitude of others I have selected that [rule], which has no connection nor relationship with the common rule, which decides that a fluid flows out [through an opening] with a velocity everywhere to be due to the whole height of the water above the opening, nor shall [such questions] be solved by the customary principles. Besides because in this experiment the motion of the water was retarded, I had wished to put another in place, so that all the impediments certainly would be diminished, so that thus it would appear in turn to approach more to the experimental and theoretical numbers, as there would be fewer impediments.

Experiment 12.

And thus now with the cylindrical tube I had used, through which the flow becomes easier, and with the same wider for the same reason: besides there was the water reservoir to which a much wider tube had been attached, and finally the height of the water contained in the region above the axis of the tube was much less, so that the water flowed out with a smaller velocity, and thus the obstacles were encountered with lesser effect: the rest was the same as before.



Therefore the height of the water above the axis of the tube $=130$ *part*., $oP = 553$ *part*., $PQ = 453$ *part*., $PR = 297$; the diameter *GF* or $MN = 19$ *part.* and the length of the tube was 130 *part.* [Thus, the attached tube in Fig. 19 is now cylindrical.]

I saw the water fallen into the dish to have filled up the cylinder, which had a diameter $8\frac{1}{2}$ *part*., to a height of 281 *part*., and of which therefore the amount was 15 950 *cubic part.* In this case putting $a = \frac{453 \cdot 453}{4 \cdot 553} =$ 93 *part*., $e = 40$ *part*., $n = g = 284$ *square part.* and $b = 130$. Truly with these substitutions made, there becomes

$$nb\sqrt{\frac{n}{g}} \times \log \frac{a}{a-e} = 284 \cdot 130 \cdot \log \frac{93}{53} = 20760,$$

for which the number corresponding in the experiment, as we have seen, to be 15950. Truly this number takes up almost four fifths of the other, and thus can be agreed to be approximately the same, since in the preceding example on account of the similar ratios established a similar number from a similar one would be deficient by more than half.

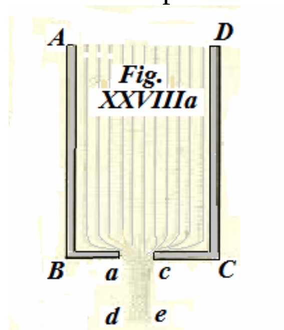
Now therefore it is abundantly clear, with only the external obstacles requiring to be assigned, because the experiments may not correspond to the precision of the formula ; yet meanwhile they are to be of such a kind, that they may not be able to show the strength of these formulas better.

HYDRODYNAMICAE SECTIO QUARTA.

De variis temporibus, quae in effluxu aquarum desiderari possunt.

§.1. Res videbitur multis omnino Geometrica, quae scilicet nulla consideratione physica opus habeat, ut, cum aquae ex dato vase per lumen cognitum velocitatibus in omni situ determinatis effluunt, tempus definiatur, quo data effluat aquae quantitas. Attamen experientia contrarium docet; nam multo minori quantitate aquae effluunt per foramina, quae sunt in lamina tenui, quam ex simplici velocitatum consideratione sequeretur, idque plerumque (nec enim res sibi constat in diversis circumstantiis) in ratione ut 1 ad $\sqrt{2}$; movit hoc Newtonum, ut affirmaret in prima *Princ. math.* editione aquam ex vase ea effluere velocitate, quae generetur altitudine dimidia aquae supra foramen, cui opinioni omnia experimenta de velocitatibus immediate sumta contradicunt. Explorans postmodum ipse magnus Vir hujus contradictionis originem, eam positam esse observavit in contractione venae aqueae, quae contractio mox prae foramine fieri solet. Alia quoque mihi observata fuit venae mutatio priori nunc similis nunc contraria. Nempe cum aquae non per simplex foramen, verum per tubulum effluunt, rursus contrahitur vena, si tubus exteriora versus convergit, sed dilatator si idem divergit. De contractione venae aqueae per tubas convergentes effluentis accuratissima sumsit experimenta Joh. Polenus in *Libro de castellis*, p. 15 & seqq.; contractio venae eo major a Viro Celeberrimo observata fuit, quo amplius erat orificium tubi conici internum manentibus orificio externo atque longitudine tubi, quae ratio est, quod similis aquae quantitas ceteris paribus eo tardius effluxerit, quo amplius fuerit orificium internum, quamvis impedimenta ab adhaesione aquae ad latera tubi minorem continue habuerint effectum: fecerunt autem istae impedimentorum diminutiones, ut aquae majori velocitate in loco, quo vena maxime erat contracta, fluerent, & nihilominus parcius erogarentur: verum id esse colligitur ex observatis effluxus temporibus & venarum, ubi maxime contrahuntur, amplitudinibus. Igitur cum in hisce venae mutationibus cardo rei vertatur, e re erit phaenomena uberius examinare & explicare.

§. 2. Assumamus v. gr. cylindrum verticalem, qui in medio fundi horizontaliter positi habeat foramen, aqua autem interna divisa concipiatur in strata horizontalia: His ita positis, censuimus motum cujusvis strati eundem esse & talem quidem, ut situs horizontalis in illis conservetur, ubi tamen monui, non posse hanc hypothesin extendi ad strata foramini proxima, quoniam vero inde nullus error sensibilis oriri possit ratione velocitatis aquarum effluentium, operae pretium non esse, ut ejus rei ratio habeatur. Nunc vero, quando alia phaenomena a motu aquae internae obliquo, qualis praesertim in praedictis stratis foramini proximis est, pendent, hunc paucis lustrabimus.



§. 3. Mihi autem videtur motum aquae internae talem esse concipiendum, qualis foret si aqua ferretur per tubulos infinitos juxta se positos, quorum intermedii proxime recta a superficie versus foramen descendunt, reliquis sensim se incurvantibus prope foramen, uti Fig. 28a ostendit, ex qua apparet, singulas particulas hoc modo descendere motu tantum non verticali, donec fundum prope attingant, easque tunc cursum suum sensim versus foramen inflectere, ita ut particulae fundo proximae motu fere horizontali, alterae magis verticaliter ad foramen effluant. Hujusmodi motus saepe oculis observare potui, cum particulae cerae, quam vocant Hispanicae, innatabant aquae. Exinde antem intelligitur non posse singulas particulas foramini adstantes directionem suam integram servare, neque tamen ita eam inflectere, ut motum axi plane parallelum assumant, sed fore potius, ut vena aquae effluentis contrahatur usque in *de*, ubi sic notabiliter gracilior erit, quam in ortu circa foramen *ac*. Haec antem contractio venae verticaliter fluentis non confundenda est cum alia contractione, quae fit ab acceleratione aquae. Dein patet quoque, quod cum singularum particularum foramini adstantium diversa sit directio, necessario ab impetu, quem in se mutuo faciunt eaedem particulae, vena comprimatur, atque sic gracilescat. Et ab ista compressione fit, quod alias contradictionem involveret, ut aqua jam jam egressa etiamnum prae foramine acceleretur, & sic *ascensus potentialis* crescat, etiamsi ad alteram accelerationem omnibus corporibus cadentibus communem non attendamus, ceu huc non pertinentem, & cujus deinceps mentionem non faciemus. Haec autem nisi me fallat opinio, res erit porro hunc in modum tractanda.

(I) Eousque vena aquae consideranda est, donec particularum velocitates amplius non mutentur, quod quamvis nunquam fiat omni rigore, attamen non procul a foramine fieri censendum est, veluti in *de*. Hoc autem si ita fuerit & aquae ex vase *ABCD* per foramen *ac* effluere ponantur, erit loco vasis simplicis *ABCD* concipiendum aliud compositum *ABadecCD*.

Quicquid igitur in praecedente sectione praemissum fuit, pro determinandis ubique velocitatibus, id omnino locum habebit, si loco vasis subjecti concipiatur vas, quod dixi tubulo contracto instructum. Nec tamen haec correctio, ratione praemissae nostrae methodi velocitatum aquae effluentis determinandarum, sensibilem mutationem producere potest ob brevitatem tubuli *adec*, potest autem valde notabilem ratione quantitatis, quia aquae non tam per orificium *ac*, quam per *de* effluere censendae sunt.

(II) Sic erunt velocitates in diversis locis ipsius venae reciproce ut amplitudines sectionum respondentium & cum in vasis amplissimis velocitas in *de* talis sit quae toti altitudini aquae conveniat, simulque experimentis constet, amplitudines *ac* & *de* proxime esse ut $\sqrt{2}$ ad 1, putavit Newtonus sic confirmari posse theoriam suam, qua statuit aquam ex foramine vero velocitate effluere quae debeatur dimidiae altitudini aquae supra foramen, quamvis in progressu velocitas aquae crescat: qua in re mihi videtur nimium adhaesisse praeconceptae opinioni: neque enim ratio orificii *ac* ad *de* semper eadem est, neque sic explicari potest motus aquarum ex vase, cui tubulus adhaeret: verbo! attenuatio venae prorsus accidentalis est, potest enim tota impediri, apponendo foramini parvulum tubulum cylindricum vel augendo tantum crassitiem laminae, cui foramen inest, & tunc sine ulla correctione locum habent tam ratione velocitatum quam quantitatum theoremata, quae in praecedente sectione exhibita fuerunt.

(III) Patet autem ex ipsa explicatione supra data de *contractione venae*, non posse non illam a diversis circumstantiis mutari; ita experimenta docent, diminui eandem ab aucta laterum foraminis crassitie: an altitudo aquae supra foramen aliquid conferat non satis scio: crediderim fere crescere aliquantulum contractionem ab aucta altitudine aquae internae, quamvis facile parum id fore praevideam: verisimile quoque est, eo minorem caeteris paribus fore contractionem venae, praesertim verticalis, quo majorem rationem habuerit amplitudo foraminis ad amplitudinem cylindri, quia motus aquae internae fundo proximae eo minus fit obliquus, ita ut si foramen totam amplitudinem cylindri occupet, nulla utique attenuatio venae aqueae oriri possit. Ad hoc animum advertant velim, qui hujus contractionis in ipsa velocitatum determinatione rationem habendam esse fortasse cogitabunt. Cum enim foramen non multo minus est amplitudine vasis, nulla oriri potest contractio notabilis & cum foramen est parvum, nulla rursus oritur fere differentia circa velocitates sive foramen aliquantum augeatur sive diminuatur.

§. 4. Eadem propemodum ratio est aquarum horizontaliter, ut de aliis directionibus taceam, effluentium: nam simili modo ab omni parte affluet aqua ad foramen; imo etiam ex inferiori parte ascendet usque ad foramen ut effluere possit, quod ipse saepe fieri observavi. Simili igitur causa similis fiet in vena effluente attenuatio, quam eo facilius est oculis perspicere, quod hic locum non habeat altera attenuatio ab acceleratione aquae jam egressae oriunda. Et ob hanc rationem, si quis observationes circa contractionem venae facere instituat, is meo judicio melius faciet, utendo venis horizontaliter, quam sub alia directione effluentibus.

§. 5. Quanta autem sit contractio, id est, quaenam ratio intercedat inter amplitudinem orificii sectionemque venae horizontaliter effluentis minimam, experiri licet vel sumendo actu mensuras diametrorum istis amplitudinibus respondentium, vel etiam mediante quantitate aquae dato tempore, datisque velocitatibus effluentis, ubi tamen velocitates non tam ex altitudine aquae supra foramen, quam ex amplitudine jactus deducendae erunt, quandoquidem impedimenta nunc majora nunc minora nunquam omnem aquae velocitatem permittant, quam vi theoriae, qua horum impedimentorum ratio nulla habetur, acquirere deberet.

§. 6. Ex praemissis nunc satis patere puto perfectum consensum fore inter quantitatem aquae effluentis ejusque velocitatem, si modo foramini, quod est in vase, substituatur aliud foramen eo usque diminutum, donec sectionem venae maxime contractae non superet: atque perinde erit, in quonam venae loco, aut in quanam profunditate a superficie aquae foramen hoc esse constituatur, sive in *ac* sive in *de*, quandoquidem velocitates semper proxime respondebunt toti altitudini aquae supra eum locum, quo foramen fingitur: amplitudinem hujus foraminis mente concipiendi vocabo deinceps *sectionem venae aqueae contractae.*

§. 7. Quod si jam *sectio* ista. de qua modo diximus, constantem haberet rationem ad orificium, in eadem ratione diminuendum cogitatione foret foramen effluxus, postmodumque calculus de quantitate aquae dato tempore effluentis instituendus. Ita nempe posita ista ratione $=\frac{1}{\alpha}$ nominataque amplitudine orificii *n*, censenda *n* esset *sectio venae solidae* $=\frac{n}{\alpha}$.

At variabilis cum sit sub diversis circumstantiis, regulas in hanc rem *a priori* dare non licet: mutatur autem maxime a crassitie laminae, in qua foramen est, aucta vel diminuta: aliquid etiam, quamvis id parum, conferre potest magnitudo foraminis, amplitudines vasis, haeque tam absolutae, quam relativae, ut & fortasse altitudo aquae supra foramen. Interim assumtis lamina tenui, vase amplissimo, foramine ad 4 vel 6 lineas in diametro assurgente; solet ratio inter foramen & *sectionem venae contractae* non multum recedere ab illa, quam Newtonus statuit, nempe ut $\sqrt{2}$ ad 1. Saepe autem ab aliis major observata fuit, atque ab aliis etiam minor.

§. 8. Quaecunque vero sit, in quolibet casu illam indicabimus, ut ante, per $=\frac{1}{\alpha}$. Huicque positioni nunc calculum pro temporibus superinstruemus; brevitatis autem gratia considerabimus tantum vasa cylindrica, atque in his duo potissimum examinabimus temporum genera; primum quod punctum maximae velocitatis definit, alterum, quod depletioni respondet. In utroque vero casu motum a quiete incipere ponemus.

§. 9. Fuerit igitur vas cylindricum verticaliter positum aqua plenum, sitque altitudo aquae ab initio fluxus $= a$, amplitudo cylindri $= m$, amplitudo foraminis $= n$, *sectio venae solidae* $= \frac{n}{\alpha}$; effluxerit jam aqua per tempus *t*, sitque tunc altitudo aquae residua supra foramen $= x$, eodemque temporis puncto habeat superficies aquae internae velocitatem, quae respondeat altitudini v: erit velocitas ipsa $= \sqrt{v}$, est autem elementum temporis *dt* proportionale elemento spatii $-dx$ diviso per velocitatem $\sqrt{v}$, unde $dt = \frac{-dx}{\sqrt{v}}$.

Determinatus equidem fuit valor ipsius *v* in Sect. III, ubi iisdem denominationibus usi sumus, quibus nunc utimur. At quoniam pro recta aquarum erogatarum mensura requiritur, ut foramini *n* substituatur *sectio venae contractae* $\frac{n}{\alpha}$, sequitur, ut in valore ipsius *v* eadem fiat substitutio atque sic statuatur

$$v = \frac{nna}{2nn - mm\alpha\alpha}\left(\left(\frac{a}{x}\right)^{1-\frac{mm\alpha\alpha}{nn}} - \frac{x}{a}\right).$$

Hic vero valor si substituatur in aequatione $dt = \frac{-dx}{\sqrt{v}}$, oritur

$$dt = -dx : \sqrt{\frac{nna}{2nn - mm\alpha\alpha}\left(\left(\frac{a}{x}\right)^{1-\frac{mm\alpha\alpha}{nn}} - \frac{x}{a}\right)},$$

ope cujus aequationis omnia tempora desiderata definiri possunt per approximationes seu series, si modo in singulis punctis valor ipsius α innotescat: Assumemus autem esse illum constantis valoris, quandoquidem in praesenti casu nihil sit, a quo mutari possit praeter diversas altitudines & velocitates fluidi, quae parum vel nihil quantum sensibus percipi potest ad id negotii conferunt.

§. 10. Jam ut aequatio desiderata per series exhiberi possit, considerabimus quantitatem

$$1 : \sqrt{\frac{nna}{2nn - mm\alpha\alpha}\left(\left(\frac{a}{x}\right)^{1-\frac{mm\alpha\alpha}{nn}} - \frac{x}{a}\right)}$$

sub hac forma

$$\left(\frac{nnx}{mm\alpha\alpha-2nn}\right)^{-\frac{1}{2}}\times\left(1-\left(\frac{a}{x}\right)^{\frac{mm\alpha\alpha}{nn}-2}\right)^{-\frac{1}{2}}$$

factoremque posteriorem per regulas solitas resolvemus in hanc seriem

$$1+\frac{1}{2}\left(\frac{a}{x}\right)^{\frac{mm\alpha\alpha}{nn}-2}+\frac{1\cdot 3}{1\cdot 2\cdot 4}\left(\frac{a}{x}\right)^{\frac{2mm\alpha\alpha}{nn}-4}+\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3\cdot 8}\left(\frac{a}{x}\right)^{\frac{3mm\alpha\alpha}{nn}-6}+\text{etc.},$$

unde nunc habetur mutata paullulum aequationis forma:

$$dt=-\frac{dx\sqrt{mm\alpha\alpha-2nn}}{n\sqrt{a}}\times\left(\left(\frac{x}{a}\right)^{-\frac{1}{2}}+\frac{1}{2}\left(\frac{x}{a}\right)^{\frac{mm\alpha\alpha}{nn}-\frac{5}{2}}+\frac{1\cdot 3}{1\cdot 2\cdot 4}\left(\frac{x}{a}\right)^{\frac{2mm\alpha\alpha}{nn}-\frac{9}{2}}+\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3\cdot 8}\left(\frac{x}{a}\right)^{\frac{3mm\alpha\alpha}{nn}-\frac{13}{2}}+\text{etc.}\right).$$

Haec aequatio ita est integranda, ut posita $x=a$ fiat $t=0$; sic autem oritur

$$t=\left(2+\frac{nn}{2mm\alpha\alpha-3nn}+\frac{3nn}{16mm\alpha\alpha-28nn}+\text{etc.}\right)\times\frac{\sqrt{(mm\alpha\alpha-2nn)\cdot a}}{n}$$
$$-\left(2\left(\frac{x}{a}\right)^{\frac{1}{2}}+\frac{nn}{2mm\alpha\alpha-3nn}\left(\frac{x}{a}\right)^{\frac{mm\alpha\alpha}{nn}-\frac{3}{2}}+\frac{3nn}{16mm\alpha\alpha-28nn}\left(\frac{x}{a}\right)^{\frac{2mm\alpha\alpha}{nn}-\frac{7}{2}}+\text{etc.}\right)$$
$$\times\frac{\sqrt{(mm\alpha\alpha-2nn)\cdot a}}{n},$$

ubi $2\sqrt{a}$ exprimit tempus quod corpus impendit dum libere delabitur per altitudinem *a*. Si vero in ista aequatione ponatur

$$x=a:\left(\frac{mm\alpha\alpha-nn}{nn}\right)^{nn:(mm\alpha\alpha-2nn)}$$

quae est altitudo aquae cum velocitas maxima est (per§. 16 Sect. III & §.8 Sect. *IV),* tum obtinetur tempus quod a fluxus principio ad punctum maximae velocitatis usque praeterit; & cum ponitur $x=0$, oritur tempus, quo vas totum depletur, ac denique si ponatur *x* cuicunque quantitati *c*, exprimet *t* tempus quod superficies insumit in descensum per altitudinem $a-c$; videbimus autem pro his casibus, quid fieri debeat, cum vas est valde amplum, numerusque *m* alterum *n* sic pluries continet.

§. 11. Fuerit primo $\frac{m}{n}$ numerus infinitus, erit altitudo aquae puncto maximae velocitatis respondens seu

$$a:\left(\frac{mm\alpha\alpha-nn}{nn}\right)^{nn:(mm\alpha\alpha-2nn)}=a:\left(\frac{mm\alpha\alpha}{nn}\right)^{nn:mm\alpha\alpha};$$

quoniam autem $\frac{mm\alpha\alpha}{nn}$ est numerus infinitus, poterit censeri :

$$\left(\frac{mm\alpha\alpha}{nn}\right)^{nn:mm\alpha\alpha}=1+\left(\log\frac{mm\alpha\alpha}{nn}\right):\frac{mm\alpha\alpha}{nn};$$

cujus rei demonstratio talis est: proposita sit quantitas infinita A habeaturque ut in nostro exemplo $A^{1:A}$, facile quisque videt esse hanc quantitatem paullo majorem, quam est unitas, & quidem excessu infinite parvo, quem vocabimus *z;* habetur itaque $A^{1:A}=1+z$, sumantur utrobique logarithmi & erit $\frac{\log A}{A}=\log(1+z)=($ ob infinite parvum valorem ipsius z) *z;* igitur est $A^{1:A}=1+\frac{\log A}{A}$: proindeque similiter est, ut diximus,

$$\left(\frac{mm\alpha\alpha}{nn}\right)^{nn:mm\alpha\alpha}=1+\left(\log\frac{mm\alpha\alpha}{nn}\right):\frac{mm\alpha\alpha}{nn}.$$

Porro quia quantitas haec unitati addita est infinite parva, erit

$$=a:\left(\frac{mm\alpha\alpha}{nn}\right)^{nn:mm\alpha\alpha}$$

seu

$$a:\left(1+\left(\log\frac{mm\alpha\alpha}{nn}\right):\frac{mm\alpha\alpha}{nn}\right)=a-a\left(\log\frac{mm\alpha\alpha}{nn}\right):\frac{mm\alpha\alpha}{nn};$$

est igitur spatium per quod superficies aquae descendit, dum a quiete maxima oritur velocitas,

$$=a\left(\log\frac{mm\alpha\alpha}{nn}\right):\frac{mm\alpha\alpha}{nn},$$

seu

$$= \frac{2nna}{mm\alpha\alpha} \log \frac{m\alpha}{n}.$$

lndicat haec aequatio descensum aquae in vase infinite ampio infinite parvum esse, cum aqua jam maximum velocitatis gradum attigerit: Potuisset autem hoc non obstante dubitari, an non interea quantitas aquae finita effluat, quandoquidern cylindrus super basi infinita erectus, utut altitudinis infinite parvae, magnitudinem possit habere infinitam: at sequitur ex nostra aequatione, hanc quoque quantitatem infinite parvam esse, & nomimatim aequalem $\frac{2nna}{mm\alpha\alpha} \log \frac{m\alpha}{n}$.

Atque convenit hoc egregie profecto cum phaenomenis, quae in effluxu aquarum ex castellis per simplex foramen toto die experimur. Cum enim foramen digito obturamus, moxque remoto digito aquas horizontaliter effluere sinimus, nullam guttulam in terram delapsam observamus mediam inter jactum longissimum & locum, qui foramini ad perpendiculum respondeat.

§. 12. Prouti in proximo paragrapho determinavimus quantitates utut infinite parvas, descensus aquae internae uti & effluentis aquae dum maximum velocitatis gradum aqua attingit, ita nunc idem praestabimus ratione tempusculi. Dico autem sufficere in aequatione §. 10 tempus exprimente, ut in utraque serie unicus accipiatur terminus primus, quod apparebit cum quis calculum ad duos extenderit terminos: est igitur tempusculum quaesitum sive

$$t = \left(2 - 2\sqrt{\frac{x}{a}}\right) \times \frac{\sqrt{(mm\alpha\alpha - 2nn)\cdot a}}{n};$$

hinc posito pro *x* valore huc pertinente, qui in praecedente paragrapho fuit definitus, fit

$$t = \left(2 - 2\sqrt{1 - \left(\log \frac{mm\alpha\alpha}{nn}\right) : \frac{mm\alpha\alpha}{nn}}\right) \times \sqrt{\left(\frac{mm\alpha\alpha - 2nn}{nn}\right)\cdot a}$$

vel posito $1 - \left(\log \frac{mm\alpha\alpha}{nn}\right) : \frac{2mm\alpha\alpha}{nn}$ pro responente quantitate signo radicali involuta prodit

$$t = \left(\left(\log \frac{mm\alpha\alpha}{nn}\right) : \frac{mm\alpha\alpha}{nn}\right) \times \sqrt{\left(\frac{mm\alpha\alpha - 2nn}{nn}\right)\cdot a}$$

aut denique rejecta quantitate *2nn* in signo radicali, oritur

$$t=\frac{2n\sqrt{a}}{m\alpha}\cdot\log\frac{m\alpha}{n}.$$

Est autem hoc tempusculum infinite parvum, quia, ut notum est, logarithmus quantitatis infinitae infinities minor est ipsa quantitate. At vero cum sic statim ab initio fluxus aqua maxima sua velocitate expellitur, mirum prima fronte videbitur fortasse aliquibus, motum in instanti generari finitum: nemo tamen absurdum putabit, massam infinitam, cujusmodi est quantitas aquae in vase infinito contentae, posse tempusculo infinite parvo motum producere finitum, idque sola gravitatis actione.

§.13. Si praeterea in ista vasis infinite ampli positione tempus depletionis, quod utique infinitum erit, exprimere velimus, erit, ut supra indicatum fuit, in aequatione paragraphi decimi ponendum $x=0$, simulque solus primus seriei terminus adhibendus rursusque ponendum $m\alpha$ pro $\sqrt{(mm\alpha\alpha-2nn)}$; atque sic fit

$$t=\frac{2m\alpha}{n}\sqrt{a}.$$

Tum denique tempus, quod impenditur in descensum superficiei per altitudinem $a-c$, exprimetur in simili hypothesi hac aequatione

$$t=\frac{2m\alpha}{n}\left(\sqrt{a}-\sqrt{c}\right).$$

§. 14. Praemissae aequationes non accurate quidem, proxime tamen satisfacient, cum vas non infinitae, permagnae tamen amplitudinis est: imo non multum admodum deficient, cum numerus *m* vel mediocriter superat numerum *n.* Liceat quaedam hic verba adjicere circa experimentum quod in fine paragraphi undecimi indicavi, deturque haec venia instituto nostro, quod in phaenomenis motuum experientia cognitis potissimum versatur illustrandis examinandisque. Dixi autem in citato paragrapho cum aqua horizontaliter effluit, primam guttulam totam statim obtinere amplitudinem jactus; atque idem hoc quidem indicat theoria pro vasis amplissimis; at vero in vasis mediocriter amplis, quaedam guttulae minori impetu effluere deberent, priusquam punctum maximae velocitatis adsit, haeque guttulae incidere deberent in locum aliquem medium inter maximum jactum & punctum, quod foramini verticaliter respondet; atque hoc etiam ita fieri observavi, ex vasis amplitudinis veluti decies foramine majoris. Verum cum experimentum aliquando sumerem de vase pedem dimidium alto, quod amplitudinem praeter propter centuplam haberet foraminis, ne minima quidem particula aquae, quantum videre potui, notabiliter a jactu aquae pleno defecit. Videamus itaque quaenam aquae quantitas in hoc casu effluere deberet ante punctum maximae velocitatis; erit autem tanta, quantam continet cylindrus ejusdem amplitudinis in altitudine

$$a-a:\left(\frac{mm\alpha\alpha-nn}{nn}\right)^{nn:(mm\alpha\alpha-2nn)}$$

(vid. §.10 sub fin.); nec differt fere haec minima altitudo ab hac multo compendiosiore, nempe $\frac{2nn}{mm\alpha\alpha}\log\frac{m\alpha}{n}$ (vid. §. 11), ubi nunc per $\frac{n}{m}$ intelligitur $\frac{1}{100}$ & per *a* pes dimidius, dum pro α substitui potest $\sqrt{2}$ (non desideramus enim hic summam accurationem) & per log indicatur logarithmus hyperbolicus; ita vero fit,

$$\frac{2nna}{mm\alpha\alpha}\log\frac{m\alpha}{n}=\frac{1}{20\,000}\left(\log 100+\frac{1}{2}\log 2\right)=0,0002476\ ped.\ \text{seu}\ 0,00297\ poll.$$

& quoniam amplitudinem vasis aequalem inveneram $6\frac{1}{5}$ *poll.* quadratis, intellexi quantitatem aquae quaesitam, quae nempe effluere debuisset priusquam jactus maximus oriretur, exaequare circiter partem quinquagesimam secundam unius pollicis cubici, seu, posito guttam mediocrem sex lineas cubicas efficere, plusquam quinque guttas. In experimento autem nullam observavi, cujus rei rationem esse suspicor, quod primae guttulae, quamvis jam ejectae, ab aqua subsequente tamen etiamnum propellantur; nimis enim celeriter alterae subsequuntur, quam ut primae ab illis interea divelli possint. Huc autem facit, quod tempusculum a fluxus initio ad maximam expulsionem usque (quod nempe per §.12 est proxime $=\frac{2n\sqrt{a}}{m\alpha}\cdot\log\frac{m\alpha}{n}$, ubi per $2\sqrt{a}$ hic intelligitur tempus, quo corpus per altitudinem dimidii pedis labitur, id est, circiter $\frac{2}{11}$ unius minuti secundi), quod inquam tempusculum illud non ultra partem centesimam quinquagesimam octavam unius minuti secundi excurrat.

Fortasse aliquid contribuit, quod non possit digitus sat celeriter a foramine removeri. Praesertim vero huc pertinet, quod maxima pars illius aquae, quae ante praesentem maximam velocitatem erumpit, ita ad maxim[u]m jactum accedat, ut nulla differentia observari possit & sic vix unica guttula notabili discrimine ab illo defectura fuisset, si se libere ab aqua subsequente separare potuisset.

§. 15. Hactenus de aquis per foramina effluentibus: progrediamur nunc ad effluxum aquarum ex vasis per conos seu convergentes seu divergentes. Quod si autem aquae effluant per tubum convergentem, dictat eadem ratio a motu particularum convergente petita §.3 pro foraminibus simplicibus exposita, fore ut aquae vena prae foramine contrahatur etiamnum ejusque particulae accelerentur & sic quantitas aquae dato tempore effluentis minor sit quam mensurae orificii effluxus & velocitatum, nulla habita ratione ad contractionem venae, indicant. Parva autem solet esse ista contractio in tubis longioribus. In tubis divergentibus omnia fiunt modo contrario: dilatatur enim vena prae foramine; aquae motus retardatur & major aquae quantitas dato tempore effluit, quam sine ista dilatatione sequeretur ex observatis amplitudine orificii & velocitatibus aquae per illud effluentis. Ex tubis denique cylindricis effluens vena aquea nec contrahitur nec dilatatur. Probe est itaque attendendum ad has sive contractiones sive dilatationes in aestimandis quantitatibus aquae dato tempore effluentis, quam quaestionem obiter tractabimus in fine sectionis.

Nunc autem libet examini subjicere mutationes quae in effluxu aquarum succedunt ab initio motus. In his vero compendii causa non attendemus ad mutationes venae; neque enim res ita est comparata ut possit experimentis satis accurate confirmari neque magni momenti hic sunt praefatae mutationes; res autem ipsa digna est, quae sollicite perquiratur ut ejus natura animo recte intelligi possit.

De vasis, quae tubos habent annexos, jamjam egimus in superiori Section §. 21, §. 22, §. 23, & quidem paragrapho 21 aequationes dedimus generaliores, quaecunque fuerit ratio inter amplitudines vasis & tubi: sed nimis sunt perplexae calculumque postulant admodum operosum: In paragrapho, qui hunc sequitur, hypothesin pertractavi, quae vas ubique amplitudinis infinitae ratione tubi facit, in qua hypothesi dixi, aquam effluere velocitate, qua ad integram altitudinem aquae supra orificium effluxus ascendere possit; sed tamen in fine paragraphi expresse monui, *ab initio motus aquam tardius descendere, quam sic definitum fuit, nec regulam istam prius locum habere, quam superficies per spatiolum aliquod descenderit,* quae res per se satis patet, quandoquidem non possit in instanti velocitas maxima produci a statu quietis in tubo, quamvis fiat in vase foramine simplici perforato.

Haec ita perpendens animo concepi mutationes initiales explorare, easque ad certas mensuras reducere. Ad hoc autem minime sufficit praememorata regula, qua istarum mutationum initialium nulla ratio habetur, quamvis caeterum exacte vera in vase infinite amplo; omnes enim mutationes quae statum maximae velocitatis praecedunt, fiunt dum superficies per spatiolum infinite parvum descendunt; attamen descensus iste, si modo vas fuerit sensu Geometrico infinitum, non solum non fit tempore infinite parvo, proud in casu foraminis simplicis, sed tempore infinite magno, intereaque etiam quantitas aquae infinita effluit, cum per foramen quantitas caeteris paribus infinite parva effluat. Haec autem ut eruerem, opus habui aliam elicere aequationem ex aequatione generali §. 23 Sect. III quam simplicissimam hanc $s = x$, posita s pro altitudine, quae velocitati aquae effluentis respondeat, & x pro altitudine aquae supra orificium effluxus; intelliget autem quisque rem pro instituto nostro ita esse efficiendam, ut habeatur ratio incrementorum velocitatis, quod antea non requirebatur.

§. 16. Fuerit igitur ut in paragrapho 22 Sect. III cylindrus *AEHB* (Fig. 18) isque censeatur infinite amplus & aqua plenus, habeatque tubum annexum *FMNG* finitae amplitudinis formae coni truncati, sive crescentis amplitudine sive decrescentis versus orificium *MN*, per quod aquae effluunt: sit ut ibi altitudo initialis aquae supra foramen *MN*, nempe $NG + HB = a$; altitudo superficiei aqueae in situ *CD* supra *MN*, id est, $NG + HD = x$; longitudo tubi annexi seu $NG = b$, amplituda orificii $MN = n$, amplituda orificii $FG = g$, amplituda cylindri, quae est infinita, $= m$; sitque tandem velocitas superficiei aquae in situ *CD* talis quae conveniat altitudini v, quae altitudo utique infinite parva erit. His positis vidimus loco citato obtinere generaliter hanc aequationem:

$$m(x-b)dv + \frac{bmm}{\sqrt{gn}}dv - \frac{m^3}{nn}vdx + mvdx = -mxdx,$$

in qua patet, posse nunc negligi terminum primum $m(x-b)dv$ prae secunda $\frac{bmm}{\sqrt{gn}}dv$, ut & quartum $mvdx$ prae tertia $-\frac{m^3}{nn}vdx$, atque sic assumi

$$\frac{bmm}{\sqrt{gn}}dv-\frac{m^3}{nn}vdx=-mxdx,$$

in qua aequatione si rursus negligatur primos terminus, quod fieri potest, nisi mutationes etiam desiderentur, quae durante primo descensu, etsi infinite parvo fiunt, orietur regula vulgaris *ascensus potentialis* aquae effluentis ad altitudinem integram aquae: nunc vero pro nostro negotio, quo mutationes illas primas desideramus, terminus iste retinendus erit, atque sic aequatio ultima in tota sua extensione pertractanda. Ponatur autem pro separandis ab invicem indeterminatis $\frac{mm}{nn}v-x=s$, sive $v=\frac{nn}{mm}(ds+x)$, atque $dv=\frac{nn}{mm}(ds+dx)$ sicque fiet

$$dx=\frac{-nnbds}{nnb-ms\sqrt{gn}}$$

quae ita est integranda, ut facta $x=a$, prodeat $v=0$ hincque $s=-a$, ita vero fit

$$x-a=\frac{nnb}{m\sqrt{gn}}\log\frac{nnb-ms\sqrt{gn}}{nnb+ma\sqrt{gn}}$$

& posito pro s valore ejus assumto $\frac{mm}{nn}v-x$, prodit

$$x-a=\frac{nnb}{m\sqrt{gn}}\log\frac{n^4b-m^3v\sqrt{gn}+mnnx\sqrt{gn}}{n^4b+mnna\sqrt{gn}}.$$

Hic rursus in quantitate signo logarithmicali involuta potest ex numeratore eliminari terminus n^4b, infinities nempe minor termino $mnnx\sqrt{gn}$, nec non ex denominatore terminus n^4b infinities pariter minor altero $mnna\sqrt{gn}$. Et sic fit

$$x-a=\frac{nnb}{m\sqrt{gn}}\log\frac{nnx-mmv}{na}.$$

Inde habetur, posito c pro numero cujus logarithmus est unitas:

$$v=\frac{nnx}{mm}-\frac{nna}{mm}\times c^{\frac{m\cdot(x-a)\sqrt{gn}}{nnb}}$$

aut posita $a-x=z$, sic ut z denotet spatium, per quod superficies aquae jam descendit, poterit aequationi haec concillari forma:

$$v=\frac{nn(a-z)}{mm}-\frac{nna}{mm}:c^{\frac{mz}{nb}\sqrt{\frac{g}{n}}},$$

de qua iterum liquet quod cum z vel minimam habuerit rationem ad b, fiat denominator alterius termini infinitus &

$$v=\frac{nn(a-z)}{mm}=\frac{nnx}{mm};$$

at vero aliter se res habet, quamdiu descensus z infinite parvus est, quem casum nunc consideramus.

§. 17. Hisce praemissis facile nunc est definire per quantulum spatium descendat fluidum, dum maximam velocitatem acquirit, faciendo nempe $dv=0$, sive

$$-\frac{nndz}{mm}+\frac{na}{mb}\sqrt{\frac{g}{n}}dz:c^{\frac{mz}{nb}\sqrt{\frac{g}{n}}}=0,$$

id est

$$z=\frac{nb}{m}\sqrt{\frac{n}{g}}\times\log\left(\frac{ma}{nb}\sqrt{\frac{g}{n}}\right).$$

Haec autem altitudo multiplicata per amplitudinem cylindri m dat quantitatem aquae interea effluentis, nempe

$$nb\sqrt{\frac{n}{g}}\times\log\left(\frac{ma}{nb}\sqrt{\frac{g}{n}}\right),$$

quae quantitas, ut supra §. 15 praemonui, est infinita, quamvis tantum logarithmicaliter, cujusmodi infinitum minus est, quam radix cujuscunque dimensionis datae ex eodem infinito; est scilicet $\log\infty$ minor quam $\infty^{\frac{1}{n}}$, quantuscunque fuerit numerus n assignabilis. Atque hoc ideo moneo, ut sic intelligatur, qui fiat, ut, si a vero infinito ratiocinamur ad quantitates valde magnas, quantitas ista aquae sat parva evadat. Caeterum corollaria formulae haec sunt.

(I) Si tubus annexus est cylindricus, fit $z = \frac{nb}{m}\log\frac{ma}{nb}$. Igitur caeteris paribus haec quantitas se habet, ut longitudo tubi annexi, quod generaliter etiam verum est: nam a mutato valore ipsius *b* censenda est non mutari quantitas $\log\frac{ma}{nb}\sqrt{\frac{g}{n}}$ ob valorem infinitum numeri $\frac{m}{n}$.

(II) Pro eodem orificio *g* caeterisque etiam paribus, sequitur quantitas *z* sesquiplicatam rationem orificii extremi: atque si idem tubus modo orificio strictiori modo ampliori vasi applicetur, erit quantitas aquae in casu priori ad similem quantitatem in posteriori, ut quadratum orificii amplioris, ad quadratum orificii minoris.

(III) Denique observandum est valere totum ratiocinium pro omnibus directionibus tubi, quod quivis perspiciet qui §. 22 Sect. III recte examinabit. Poterit igitur tubus adhiberi etiam horizontalis aut sub quacunque alia directione & utcunque incurvus, ad quod praesertim in instituendis experimentis animus erit advertendus. Semper autem intelligetur per *b* longitudo tubi, per *a* vero altitudo aquae verticalis supra orificium extremum.

§. 18. Venio nunc ad tempus, quo istae mutationes a quiete ad maximam velocitatem fiunt: Dico autem posse in calculo hujusmodi temporum simpliciter poni $v = \frac{nn}{mm}a$; reliquae enim quantitates in aequatione ultima §.16 evanescunt, quantumlibet parva sumatur altitudo *z*, modo habeat rationem vel minimam assignabilem ad altitudinem illam infinite parvam, quae respondet maximae velocitati, nempe ad $\frac{nb}{m}\sqrt{\frac{n}{g}}\times\log\left(\frac{ma}{nb}\sqrt{\frac{g}{n}}\right)$.

Sequitur exinde esse praedictum tempus, quod vocabo *t*,

$$= \frac{b\sqrt{n}}{\sqrt{ga}}\times\log\left(\frac{ma}{nb}\sqrt{\frac{g}{n}}\right)$$

& proinde infinitum, quamvis idem tempus admodum exiguum sit, quum amplitudo vasis non est infinita, sed utcunque magna, quod rursus ex natura infiniti logarithmicalis est deducendum.

§. 19. Quia altitudo velocitatis, ut vidimus in proximo paragrapho, potest statim censeri $= \frac{nn}{mm}a$, id est, aequalis maximae, cum superficies per minimam partem assignabilem descensus infinite parvi, post quem velocitas maxima plena adest, descendit, sequitur mutationes plerasque a quiete usque ad statum maximae velocitatis esse insensibiles, id est, infinite parvas, imo non solum plerasque, sed & omnes praeter particulam infinite parvam: res scilicet sic se habet: velocitas a primo initio plane nulla est, & postquam aqua per spatiolum infinite parvum descendit, jam est tantum non maxima; dein dum per aliud spatiolum rursus quidem infinite parvum, priori tamen infinite majus, descendit, pergit velocitate sua moveri, incrementa sumens infinite parva, & tunc demum vere maximam velocitatem attingit: Cum vero posteriores illae mutationes ceu infinite parvae non possint sensibus percipi, aliter pertractabimus ea quae a§.17 dedimus theoremata, considerando loco mutationum a quiete usque ad punctum maximae velocitatis, easdem mutationes usque ad datum gradum velocitatis.

§. 20. lndagabimus itaque, per quantum spatiolum z superficies aquae a statu quietis descendere, quantaque aqua effluere, ac denique quantum tempus praeterire debeat, ut aqua interna velocitate moveatur, quae generetur lapsu libero per datam altitudinem, quam vocabimus $\frac{nn}{mm}e$, ita ut ipsa e denotet similem altitudinem pro velocitate aquae effluentis. Ad hoc requiritur, ut in aequatione ultima paragraphi decimi sexti ponatur $\frac{nne}{mm}$ pro v, sic autem erit

$$\frac{nne}{mm}=\frac{nn(a-z)}{mm}-\frac{nna}{mm}:c^{\frac{mz}{nb}\sqrt{\frac{g}{n}}}$$

hincque deducitur

$$\frac{mz}{nb}\sqrt{\frac{g}{n}}=\log\frac{a}{a-e-z};$$

hic vero cum e ponatur deficere notabiliter ab a, potest rejici littera z signo logarithmicali involuta, unde obtinetur

$$z=\frac{nb}{m}\sqrt{\frac{n}{g}}\times\log\frac{a}{a-e}.$$

Haec vero aequatio jam indicat spatiolum, quod est infinite parvum, & per quod descendit superficies aquae, dum a quiete velocitas aquae effluentis tanta sit, quae debeatur altitudini e; seque habet hoc spatiolum ad illud paragrapho decimo septimo indicatum, quo nempe velocitas maxima oritur, ut $\log\frac{a}{a-e}$ ad $\log\left(\frac{ma}{nb}\sqrt{\frac{g}{n}}\right)$ ut primum sit infinities minus altero, etsi pariter infinite parvo.

Si porro definita quantitas z multiplicetur per m, obtinetur quantitas aquae effluentis dum illa velocitas altitudini e debita producitur, quae proin quantitas est aequalis

$$nb\sqrt{\frac{n}{g}}\times\log\frac{a}{a-e}$$

atque sic finitae magnitudinis, & quidem eo majoris, quo longior sumitur tubus, & quo major jactus expectatur.

Denique tempus, quo idem fit, si recte seligantur termini rejiciendi, reperitur aequale

$$2\sqrt{\left(\frac{nbb}{ag}\log\frac{a}{a-e}\right)}$$

atque sic finitum sed admodum parvum & in nullo exemplo ultra minutum secundum facile extendendum.

§. 21. Haec omnia accurate examinare ac prosequi volui, tum quod multorum phaenomenorum, quae in effluxu aquarum observari solent, solutio inde pendeat, tum etiam ut illas mutationes, quae sensibus plane sunt imperceptibiles, animo recte assequeremur. Multi fuerunt, qui transitus ab infinito ad finitum aut vicissim a finito ad infinitum in aquis fluentibus non recte assecuti a plurimis difficultatibus se extricare non potuerunt, quae alias facile admittunt solutionem, si autem loco vasis fere infiniti, cujusmodi nulla sunt, sumatur vas valde amplum, aut etiam quod in multis casibus sufficit, mediocriter amplum, erunt formulae proxime verae, & modo magis modo minus ad verum accedent pro indole quaestionis: de his quaedam monebo. Interim sic satis jam apparet ex theoria, quod potissimum explicare constitueram, cur aqua ex vase amplissimo simplici omni statim velocitate effluat, & cur secus sit de aquis ex vase per tubum ejectis: Mensurae vero praecisae de his quaestionibus ex aequationibus ipsis erunt deducendae.

§. 22. Tandem quod pertinet ad tempus depletionis, patet cum amplitudo vasis vel mediocriter superat amplitudinem tubi annexi, posse sine sensibili errore censeri illud $=\frac{m\alpha}{n}\theta$, intelligendo per θ tempus, quo corpus a quiete libere cadendo absolvit altitudinem, quam aqua ab initio fluxus habuit supra orificium tubi extremum, atque sumendo pro $\frac{m\alpha}{n}$ rationem quae est inter amplitudinem vasis & *sectionem vennae*, sive *contractam* sive *dilatatam*. Impedimenta vero, quae in his casibus fortuito superveniunt, tempus istud admodum augent. Si vero tempus desideretur, quo superficies aquae per datam descendat altitudinem, erit illud sumendum $=\frac{m\alpha}{n}(\theta-T)$ sumto pro T tempore quod corpus insumit libere cadendo per altitudinem, quam aqua in fine fluxus supra foramen habet.

Experimenta quae ad Sectionem IV pertinent.

Quum magna pars hujus Sectionis posita sit in contractione venae aqueae per foramen in lamina tenui factum fluentis, animo concepi de ista contractione experimenta instituere accurata, non quidem mensuras accipiendo diametrorum, quam methodum non sufficienti accuratione fieri posse expertus sum, sed observando velocitates *actuales* ex amplitudine jactus, & quantitates datis temporibus effluentes; in experimentis automato usus sum, quod tempore unius minuti primi 144 vicibus pulsabat, atque sic sequentia sumsi.

Ad theoriam contractionis venarum aquearum.

Experimentum 1.

Tubum cylindricum adhibui, cujus diameter erat 4 *poll.* 3 *lin. mens. Angl.* e lamina tenui factum quique foramen habebat in latere, id est, in superficie cylindrica: erat diameter foraminis $=4\frac{52}{152}$ *lin.*; aquae effluebant horizontaliter ex cylindro verticaliter posito, & fuit ab initio fluxus altitudo aquae supra centrum foraminis $=4$ *poll*. 3 *lin.* similisque altitudo in fine fluxus $=3$ *poll*.; duravit autem omnis fluxus intervallo undecim automati pulsuum, quae proxime efficiunt tempus 4 minutorum secundorum cum dimidio.

Porro repetito saepius experimento observatisque tum altitudine foraminis supra tabulam horizontaliter positam, tum amplitudine jactus, hacque tam in principio quam in fine fluxus, vidi ex *Lemm. in principio Experimentorum praecedentis Sect. indicato* velocitatem aquae effluentis in loco venae maxime contractae constanter talem fuisse, quantum quidem sensibus dijudicari potuit, quae deberetur altitudini aquae supra eundem locum, qui in eadem altitudine est qua foramen.

Igitur si contractionem venae aqueae ubique eandem fuisse ponamus & huic casui applicemus aequationem ultimam paragraphi decimi tertii, nempe $t=\frac{2m\alpha}{n}(\sqrt{a}-\sqrt{c})$, erit ponendum $t=4\frac{1}{2}$ *min. sec.*; $\frac{m}{n}=133; 2\sqrt{a}$ (= tempori quod corpus insumit libere cadendo per altitudinem aquae initialem) $=0,1483$ & $2\sqrt{c}$ tempori simili pro altitudine aquae ultima) $=0,1246$: fit $4\frac{1}{2}=3,15\alpha$; unde $\alpha=1,43$. Exinde consequens est, amplitudinem foraminis fuisse ad sectionem venae contractae ut 143 ad 100; haec ratio tantillo major est quam quae intercedit inter $\sqrt{2}$ & 1 nempe inter 141 & 100; sed si accuratissime velocitates observari potuissent, dubium non est, quin illae paullo minores futurae fuissent, quam quae toti altitudini aquae debeantur; & cum hujus rei ratio habetur, deprehenditur valorem ipsius *a* sic pauxillum diminuendum esse; potest igitur ex toto experimento colligi tutissime rationem praememoratam fuisse ut $\sqrt{2}$ ad 1.

Experimentum 2.

Deinde experimento explorare volui, an in omnibus jactibus sub quacunque directione contractio eadem sit, & hunc in finem existimavi rem sic esse aggrediendam, ut praeter directionis istius mutationem circumstantiae caeterae omnes essent prorsus similes. Id vero sic obtinui.

Eodem scilicet, quo antea, cylindro usus sum, eum autem arcae prismaticae verticaliter positae implantavi, ita, ut axis cylindri esset horizontalis, sicque implantatum circumverti, ut centrum foraminis aquarum effluxui destinati modo locum summum, modo medium, modo imum occuparet: in primo casu aquae verticaliter sursum effluebant, in secundo horizontaliter, in tertio verticaliter deorsum ejiciebantur; in singulis vero feci ut altitudines aquae in arca supra centrum foraminis essent perfecte aequales: successus hic fuit.

Observavi aequalibus temporibus superficiem aquae in singulis casibus per spatia aequalia in area descendere. Igitur in venis sursum projectis aqua superior non resistit sensibiliter aquae inferiori subsequenti, quod idem alio intellexi modo, quod scilicet, si ad parvam a foramine distantiam veluti 3 linearum nummo aliquo venam aqueam cujuscunque directionis excipiebam, ita ut vena in nummum perpendiculariter incideret, effluxus aquarum non fuerit retardatus. Porro nec aqua in venis verticaliter descendentibus anterior posteriorem post se trahit; ipsaque venae contractio similis ubique est, non considerata retardatione accelerationeque aquarum sursum vel deorsum ejectarum, quae faciunt ut vena in aliqua a foramine distantia vel intumescat, vel gracilescat. Hic enim sermo est de illa modo contractione, quae oritur a motu particularum obliquo in regione foraminis.

Experimentum 3.

Eadem machina praedicto modo praeparata usus sum ad explorandum, num contractio venae caeteris paribus mutaretur ab aucta altitudine aquae supra foramen. Hunc in finem duas acus infixi lateribus internis arcae ad perpendiculum sibi respondentes, prior eminebat supra centrum foraminis 13 *poll.* cum 10 *lineis,* altera 12 $poll. 1\frac{3}{5} lin$. *mens. Angl.;* amplitudo arcae erat ad amplitudinem foraminis ut 404 ad 1; vidi autem superficiem aquae a superiore acu ad inferiorem descendisse post intervalla 24 automati pulsuum, quae faciunt tempus 10 minutorum secundorum.

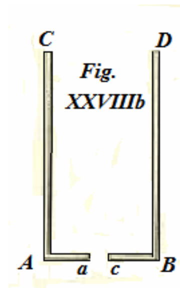
Quod si vero tempus idem quaeratur ad Hypothesin, venam se nihil contraxisse, simulque aquas omni velocitate, quam vi theoriae nullo praesente impedimento alieno habere debuissent, effluxisse, reperitur illud $=6\frac{7}{8}$ *min. sec.* Sic igitur concludi potest, fuisse amplitudinem foraminis ad *sectionem venae contractae* ut 10 ad $6\frac{7}{8}$, id est, $\alpha=1,45$, cum in primo experimento fuerit pro eodem foramine perpensis omnibus circumstantiis $\alpha=1,41$.

Postquam haec ita expertus fuissem, residuum erat explorare, an aquae omni velocitate ad sensus effluxerint, qua de re eo magis dubitavi, quod crescentibus velocitatibus aquae, crescant simul impedimenta, haecque proin notabilia esse possint in majoribus aquae altitudinibus, qualia in minoribus non sunt.

Feci itaque omni adhibita cura (quod potissimum ad praecisionem experimenti requiritur) ut aquae sub directione perfecte horizontali effluerent, & acceptis mensuris tum amplitudinis jactus, tum altitudinis foraminis supra tabulam horizontalem, vidi subducto calculo, quod cum altitudo aquae erat $=13\,poll.$ cum 10 *lin.* seu 166 *lin.,* aquae effluxerint, seu potius per *sectionem venae contractam* transfluxerint, velocitate, quae convenit altitudini $158\frac{1}{2}$ *lin.;* igitur velocitas in calculo diminuenda est in ratione subduplicata harum altitudinum atque in eadem ratione proxime decrescit valor inventus litterae α , qui ita fit paullo minor quam 1,42 seu rursus 1 ,41 & sic colligere licet, solam altitudinem aquae mutatam ad sensus non mutare contractionem venae.

Experimentum 4.

Tubum habui cylindricum altitudinis 4 *poll.*, cujus sectio per axem representatur per (Fig. 28b) *CABD;* amplitudo cylindri erat ad amplitudinem foraminis *ac* ut 110 ad 1. Cylindrus iste aqua plenus onmis evacuatus fuit tempore 21 minutorum secundorum cum dimidio. Notari autem debet, non prius aquis effluxum concedendum esse, quam nullus in illis motus turbinatorius observetur; secus enim aqua mox in turbinem vertitur, durante effluxu sat celerem, effluxusque valde retardatur, eoque magis, quo celerius aqua interna in gyrum agitur: quia porro nunquam omnis aqua effluit, effluxus tempus consideravi, usquedum stillatim effluere inciperet.



Indicat hoc experimentum minorem hic aquae fuisse contractionem quam pro ratione $\sqrt{2}$ ad 1; expectaveram tempus evacuationis fore admodum 23 *min. sec.*, sed eventus paullo alius fuit ut dixi, cujus rei rationem esse postmodum animadverti, quod labia foraminis elongata tubulum fere quamvis brevissimum formarent, ut Figura ostendit, qui venae aqueae contractionem impediebat: interim latitudo istorum labiorum duas tertias lineae non attingebat.

Experimentum 5.

Feci ut aquae ex vase amplissimo per tubulum effluerent horizontaliter: erat autem tubus brevissimus, longitudinem nempe *3 lin.* non excedens, habebatque in diametro fere *5 lin.*

Effluxit data aquae quantitas tempore $11\frac{1}{4}$ *min. sec.* quae effluere debuisset tempore $10\frac{2}{3}$ *min. sec.*, si neque contractam fuisse venam, neque ulla adfuisse impedimenta statuatur.

Velocitates reales aquae non censui opus esse ut experirer, nullus dubitans tales fuisse, quales esse debeant, ut observato tempore per observatum orificium data quantitas aquae, nulla facta ad contractionem venae attentione, efflueret.

Alios insuper alius diametri longitudinisque adhibui tubulos & vidi quantitates aquae dato tempore datisque velocitatibus effluentis recte respondere orificiis effluxus: velocitates autem eo magis defecisse a velocitate integrae altitudini aquae debita, quo strictior & quo longior erat tubus, ut & quo altior erat aqua.

Ad theoriam aquarum per tubas effluentium.

Experimentum 6.

Vasa, quorum sectiones per axem representant Fig. 24 & 25, cylindrica altitudinem habebant 4 *poll. Angl.* tubosque annexos longitudinis unius pedis; amplitudines cylindrorum erant ad amplitudines orificiorum *A*, ut 110 ad 1; orificium autem *B* erat ad orificium *A* proxime ut 25 ad 16; tempus evacuationis repletis antea cylindris fuit in Fig. 24 sex *min. sec.* cum dimidio, in altera praeterpropter 4 hujusmodi minutorum cum triente.

In his casibus vasa satis ampla fuere ratione tuborum annexorum, ut veluti infinita censeri possent; debuissetque proin per Regulas passim a nobis indicatas aqua effluere per orificia extrema velocitatibus respondentibus toti altitudini aquae, si modo excipias prima fluxus momenta, quae ipsa tam brevia hic sunt, ut observari non possint. Et cum praeterea, ut passim monui, quantitas aquae dato tempore per tubos effluentis simpliciter aestimanda sit ex celeritatibus & magnitudine orificiorum, inveni per regulam §. 22 exhibitam, tempus evacuationis in primo casu $4\frac{1}{3}$ *min. sec.*, in posteriori = fere 3 *min. sec.*

Quod in experimento majora paullo fuerint observata, in Fig. 24 maximam partem adhaesioni aquae ad latera tubi, in Fig. autem 25 alii insuper rationi in paragrapho 24 *Sect.* III indicatae est tribuendum.

Phaenomena alia in his vasis sunt notanda: nempe cum vasa sunt tantum non evacuata, percipitur sonus quidem ab aëre, qui tunc aquae in orificio superiori se miscet; hunc vero sonum pro ultimo fluxus momento accepi: facile fit porro, ut aquae effluxus concedatur priusquam ad perfectam quietem fuerit reducta (nam ab impletione agitantur & in turbinem moventur aquae); tunc autem effluxus admodum retardatur & cataractae species interne formatur, continueque aër aquae effluenti se permiscet. Ita potest pro lubitu retardari effluxus, si in vorticem aquae agantur antequam effluant.

Experimentum 7.

Vase usus sum prismatico, cui tubulus infixus erat horizontaliter ut in Fig. 19. Habebat orificium *GF* in diametro praecise quinque lineas; alterum *NM* $6\frac{1}{2}$ *lin.* Erant proin ipsae amplitudines orificiorum *GF* & *NM* ut 100 ad 169; amplitudo vero vasis continebat amplitudinem orificii *NM* ducentis & una vicibus. Longitudo tubuli *GN* erat 4 *poll.*

Deinde vas aqua implevi usque in *CD*, cujus altitudo supra axem tubi erat 13 *poll.* 10 *lin.* Aperto orificio *NM* effluxerunt aquae descenditque superficies usque in *EH* tempore $8\frac{1}{3}$ *min. sec.;* erat vero altitudinum differentia *CE* vel *DH* duorum pollicum cum octo lineis. Subducto calculo ad normam paragraphi 22 ubi neque ad impedimenta, neque ad mutationem venae attenditur, videmus praedictum tempus descensus esse debuisse proxime $= 5$ *min. sec.* cum fere dimidio. Igitur statuendum est hoc modo, velocitatem mediam totalem se habuisse ad velocitatem integram, quam theoria indicat, ut $5\frac{1}{2}$ ad $8\frac{1}{3}$ seu proxime ut 2 ad 3; hincque concludi potest, aquam per orificium *MN* effluxisse velocitate, quae conveniat $(\frac{2}{3})^2$, seu quatuor nonis partibus altitudinis aquae supra foramen *MN*, per alterum vero orificium *GF* transfluxisse velocitate quinque praeter propter quartis ejusdem altitudinis partibus debita.

Apparet itaque rursus effluxum aquarum promoveri ab aucta amplitudine orificii tubi versus exteriora, quamvis nec orificium quo tubus in vas est implantatus, nec situs tubi sit mutatus.

Porro in tabula horizontaliter posita *PQ* observavi amplitudinem jactus *PQ* pro altitudine *oP*, quae erat 4 *poll.* 8 *lin.* Inveni autem $PQ = 9\ poll.\ 6\ lin.$

Sequitur ex ista observatione, quod si dilatationis venae consideratione seposita aquae in NM velocitatem debuerint habere, qualis debetur altitudini 4 *poll.* 10 *lin.*, cum tamen vi praemissi experimenti certe habuerit velocitatem debitam altitudini fere 6 *poll. 2 lin.* Confirmat haec observatio id quod §. 15 dixi, nempe in tubis divergentibus venam aqueam dilatari veluti in *m*, ipsiusque motum retardari. In praesenti vero casu, ut ambae observationes concilientur, dicendum erit venam ita dilatatam fuisse, ut amplitudinem haberet ratione orificii *NM* reciproce ut praedictae velocitates seu reciproce ut radices altitudinum istis velocitatibus debitarum, nempe ut, $\sqrt{74}$ ad $\sqrt{58}$ proindeque diametros venae dilatatae & orificii fuisse ut $\sqrt[4]{74}$ ad $\sqrt[4]{58}$ seu ut 1000 ad 941.

Experimentum 8.

Aliud feci experimentum quod, quamvis huc nondum pertineat, nihilominus recensebo: nempe in ortu prope orificium *GF* tubum perforavi foramine *e* duarum fere linearum, rursusque descensum superficiei ex *CD* in *EH* observavi effluente aqua per *NM*, simulque amplitudinem jactus examinavi.

Duo haec vidi, quae prima fronte sibi contradicere fere videntur; descensus ex *CD* in *EH* tardior factus est quam in praecedenti experimento fuerat, & nunc duravit 10 *min. sec.* & tamen amplior fuit jactus *PQ* pro eadem altitudine *oP;* jam enim erat $PQ = 10\, poll.$ 10 *lin.*

Ambo Phaenomena ita explico: ob foramen *e*, quod fuit factum prope *GF* quodque aëri liberum transitum concedit, solvitur nexus, quem alias inter se habent aquae in tubo, nec proin aliter transfluunt aquae ubi est foraminulum *e*, quam si eo ipso in loco esset rescissus tubus; fluerent autem tardius, quod passim demonstravi, si tubus *GNMF* ceu divergens brevior fieret. Quod porro aquae quamvis minori quantitate, tamen majori impetu per orificium *NM* non mutatum fluere possint sine implicita contradictione, ratio est permixtio aëris cum aqua; nam aër perpetuo irruit in tubum per foraminulum *e* & una cum aqua effluit per *NM*. Denique phaenomenon illud, quod aquae actu celerius fluant per *MN* aperto, quam clauso foramine *e*, aliter explicari non posse mihi videtur, quam quod impedimenta extrinseca minus agant in aquam aëre rarefactam quam naturalem.

Ad theoriam aquarum, quae ex vasis amplissimis a puncto quietis usque ad datum velocitatis gradum effluunt.

Experimentum 9.

Quum aquae per foramen in lamina tenui factum ex vase amplissimo effluunt, prima statim guttula omni velocitate, quae altitudini aquae supra foramen debetur, erumpit.

Conforme hoc est cum theoria §. 11 indicata, si vas sit revera infinitum, & quamvis etiam non fuerit sensu Geometrico infinitum, dummodo sit valde amplum, nulla pariter guttula ab initio fluxus observari potest, quae non maxima velocitate effluxerit: Phaenomenon hoc explicui §. 14, cum nempe vi theoriae in casu particulari aliquo ibidem recensito vix una aut duae guttulae sensibiliter a jactu maximo deficere debuissent, dixi non posse tantillam aquae quantitatem se ab aqua subsequente separare ob mutuam aquearum particularum attractionem seu adhaesionem.

Experimentum 10.

Quum vero aquae ex vase amplissimo per tubum vasi horizontaliter insertum effluebant, observavi priusquam vena effluens jactum formaret maximum *omQ* (vid. Fig.19), sat notabilem aquae quantitatem in tabulam horizontalem subjectam delabi mediam inter *P* & *Q*, eo majorem esse hanc quantitatem quo longior est tubus *GN* & quo magis versus *N* divergit, ac denique inaequaliter aquam illam distribui, multo copiosius scilicet decidere in locum, qui est remotior a puncto *P*, quam qui eidem est propior; ratione autem temporis, quo omnes istae mutationes fiunt, vidi illud brevissimum esse, & tale ut ejus mensura percipi non posset.

Omnia ista phaenomena ex asse satisfaciunt propositionibus, quas dedimus a paragrapho undecimo usque ad finem Sectionis. Mensurae autem ibidem exhibitae experimentis recte confirmari non possunt, praesertim illae, quae §§. 15, 16 & 17 indicatae sunt, ubi scilicet formulae communicantur, quae exprimant quantitatem aquae effluentis, dum a quiete maximus fit jactus: ratio est primo, quod primae guttulae quae prope punctum *P* in tabulam decidere deberent ab aqua subsequente non libere se separent; secundo, quod aquae quantitas venae *oQ* proxima (quae quidem maximam vi ipsius theoriae partem constituit) intercipi non queat, & denique, quod motus aquarum per tubos admodum retardari solet ab impedimentis extrinsecis, imprimis si tubi divergant, atque sic motus realis sit admodum diversus a motu quem aquae habiturae essent remotis omnibus impedimentis. Reliquae mensurae a nobis indicatae paucioribus iisque minoris momenti difficultatibus sunt subjectae; continentur autem §. 20 & exprimunt potissimum aquae quantitatem, quae a primo motus puncto effluit, dum aqua datum velocitatis gradum attingit.

Quamvis ob rationes modo dictas, praesertim in casu tuborum divergentium, perfectus consensus theoriae cum experimentis minime expectari possit, talem tamen expertus fui successum, ut facile intellexerim integrum futurum fuisse consensum si impedimenta omnia una cum aquearum particularum mutua adhaesione praeveniri potuissent. Experimenta autem sumsi tum de tubo divergente, tum de cylindrico: singula nunc exponam:

Experimentum 11.

In Figura 19 tubus forma coni truncati horizontaliter vasi erat insertus, vas ipsum aqua implevi usque in *CD*, ita, ut altitudo ejus supra axem tubi esset aequalis 433 particulis aequalibus, quibus in toto experimento usus sum. Pro illa altitudine experimento inquisivi in punctum *Q* maximo jactui respondens, & fuit $PQ = 287$ *part.* dum altitudo *oP* erat $= 146$ *part.* Sic vidi motum aquae tum propter aquae adhaesionem, tum propter figuram tubi fuisse valde retardatum, quod in his casibus fieri debere aliquoties monui. Debuisset autem, si nihil obstitisset motui, esse $PQ = 503$ *part.*

Deinde Patinam posui in tabulam horizontalem, cujus ora erant in *S* & *R:* Patinam autem prius madefeci, omnemque aquam ex illa depluere rursus sivi: sumtaque mensura *PR*, illam inveni 206 *part.*

Denique diameter *GF* erat $=13$ *part.*& *MN* $=17$ *part.*, longitudo tubi autem erat $=125$ *part.* His omnibus ita praeparatis, dum orificium *MN* digito obturarem, remoto confestim digito aquae ejiciebantur, earumque pars aliqua in patinam decidebat: hanc sollicite in tubum vitreum collegi cylindricum, cujus diameter erat $=8\frac{1}{2}$ *part.;* tubus iste impletus fuit ad altitudinem 210 *part.;* fuit igitur quantitas aquae in patinam delapsae $=11922$ particulis cubicis.

Jam vero deberet ista quantitas per §. 20 esse $=nb\sqrt{\frac{n}{g}}\times\log\frac{a}{a-e}$ ubi per n intelligitur amplitudo orificii *NM* seu 227 *part. quadratae*, per *g* amplitudo orificii $GF=133$ *part. quadr.;* denotat porro *b* longitudinem tubi, quae fuit $=125$ *part.;* per *a* proprie intelligitur altitudo superficiei *CD* supra axem tubi, hic vero intelligenda potius est altitudo conveniens velocitati aquae in punctum *Q* incidentis, seu 141 *part.;* similiterque pro *e* sumenda est altitudo conveniens velocitati particulae in punctum *R* incidentis, nempe73 *part.* Denique vox abbreviata log significat logarithmum hyperbolicum. Factis istis substitutionibus numericis, fit

$$nb\sqrt{\frac{n}{g}}\times\log\frac{a}{a-e}=227\times125\times\frac{17}{13}\times\log\frac{141}{68}=26830.$$

Fuit igitur quantitas aquae experimento inventa ad quantitatem, quam theoria seposita impedimentorum consideratione indicat, ut 11922 ad 26 830 ; qui numeri, quamvis non parum differant, tamen egregie theoriam confirmant, quod ipsum nunc clare ob oculos ponam.

In formula $nb\sqrt{\frac{n}{g}}\times\log\frac{a}{a-e}$ posuimus pro *a* altitudinem velocitati maximae aquae effluentis debitam, qualis revera fuit in experimento, non qualis remotis obstaculis futura fuisset; fecimus nempe $a=141$: in theoria vero est $a=433$. Quod si autem valor iste posterior assumatur, retinendo valorem altitudinis $e=73$, fit $nb\sqrt{\frac{n}{g}}\times\log\frac{a}{a-e}$ proxime $=6700$, qui numerus nunc multo minor est numero per experimentum eruto, cum antea fuerit admodum major. Talis autem fit cum altitudo *e* servare valorem ponitur: Verum prouti altitudo *a* aucta fuit ab 141 usque ad 433, ita certe etiam altitudo *e* est augenda, foretque utraque altitudo in eadem ratione augenda, si impedimenta primis guttulis aequaliter resisterent & sequentibus: sed minorem resistentiam offendunt caeteris paribus particulae, quo tardius moventur, atque proin etiam guttulae quae cadunt cis terminum *R* minus retardantur, quam quae terminum istum transgrediuntur: Facile est exinde colligere in minori ratione augendam esse altitudinem *e* quam alteram *a,* ipsam vero rationem dicere non possumus, nisi a posteriori, faciendo scilicet, ut theoria conveniat cum experimento; ita reperitur ponendum esse $e = 120$, qui numerus animo ad omnes circumstantias bene attento plane satisfacit.

Sic igitur manifestum mihi videtur, experimenti successum talem fuisse, ut plane cum theoria conveniat. Hujusmodi autem exempla omnino demonstrant, veras motuum leges in fluidis nos tradidisse, eaque inter infinita alia selegi, quod nullam habent nexum neque affinitatem cum regula communi, quae fluida ubique velocitate effluere statuit, toti altitudini aquae supra foramen debita, neque possint principiis consuetis solvi. Caeterum quoniam in hoc experimento motus aquae retardatus fuit, aliud instituere volui, quo omnia impedimenta admodum diminuerentur, ut sic appareret eo magis ad se invicem accedere numeros experimenti & regulae, quo minora essent impedimenta.

Experimentum 12.

Jam itaque usus fui tubo cylindrico per quem facilior fit transfluxus eoque ob eandem rationem ampliore: erat praeterea area cui tubus insertus fuit multo amplior, & denique altitudo aquae in area contentae supra axem tubi multo minor fuit, ut minori velocitate aquae transfluerent, sicque obstacula minoris momenti offenderent: caetera fuerunt, ut ante.

Fuit igitur altitudo aquae supra axem tubi $= 130$ *part.*, $oP = 553$ *part.*, $PQ = 453$ *part.*, $PR = 297$; diameter *GF* vel $MN = 19$ *part.* tubique longitudo 130 *part.*

Vidi aquam in patinam delapsam cylindrum explevisse, qui $8\frac{1}{2}$ *part.* in diametro continebat ad altitudinem 281 *part.* & cujus proinde capacitas erat 15 950 *part. cub.* In hoc casu ponendum est $a = \frac{453 \cdot 453}{4 \cdot 553} = 93$ *part.*, $e = 40$ *part.*, $n = g = 284$ particulis quadratis & $b = 130$. His vero factis substitutionibus fit

$$nb\sqrt{\frac{n}{g}} \times \log \frac{a}{a-e} = 284 \cdot 130 \cdot \log \frac{93}{53} = 20760,$$

cui numerus in experimento respondet, ut vidimus, 15950. Hic vero numeros fere quatuor quintas alterius explet, sicque eidem proxime accedit, cum in praecedenti exemplo ob rationes allatas similis numerus a simili plus quam dimidio defecerit.

Jam igitur abunde patet, solis obstaculis extrinsecis attribuendum esse, quod experimenta non ad amussim respondeant formulis; interim tamen talia esse, ut non possint melius harum formularum robur demonstrare.