HYDRODYNAMICS SECTION THREE.

Concerning the velocities of fluids flowing from some kind of vessel through an opening of any kind.

§.1. Before we attempt to define the motion of water arising from its own weight, we will ruminate over what had been brought forwards by us from the principles required to be used for this, in the first section, paragraphs §§.18, 19, 20, 21 & 22.

Certainly we will be remembering the *potential ascent* [*i.e. the power or ability to rise to a certain height*] of the system, the individual parts of which are moving with some velocity, to indicate the vertical height, to which the centre of gravity of that system would reach, if the individual parts were understood to be able to rise by changing the velocity of their motion by going upwards, as far as they are able, and the *actual descent* to denote the vertical height, through which the centre of gravity would fall, after the individual particles were at rest [perhaps to return to the initial situation above, depending on the situation]. Then also we will remember that by necessity the *potential ascent* is equal to the *actual descent*, when all the motion stays together in the extended material, and nothing moves across from that into unaffected matter or some other matter not pertaining to the system, and then the motion of the fluids to be approximately such, that everywhere the velocity shall be inversely proportional to the corresponding size [i.e. cross-sectional area] of the vessel, concerning which we will add certain other matters in their place. Now it is agreed to examine the following proposition.

[There are a number of assumptions implicit in D. Bernoulli's presentation, some of which are indicated here:

We must stress initially the actuality of the first situation of the *potential ascent*: here the water is actually flowing horizontally in bulk at all times in a symmetrical manner along a pipe with a simple bulge; thus, there is no actual *ascent* of the centre of gravity of the water at any time in the first situation, which is purely hypothetical or *potential*, as Bernoulli points out. This would have to be the origin of the idea of potential energy. It is of course the speed of the water that changes between normal cross-sections of different sizes as the pressure changes, and so the ability or potential of the water to rise to different heights ; for a given volume that may be incremental or finite, Bernoulli evaluates the vertical location of the centre of gravity of the potential curve, that relates to the square of the speed of the water flowing across a given cross-section or amplitude, according to the *vis vivans* or living force principle, introduced earlier by Leibnitz and also used by Huygens and others, but ignored by Newton in his later editions of the Principia : and which later became known as the conservation of mechanical energy principle. The actual position of the centre of gravity of this hypothetical curves is effected by taking moments of the incremental masses about the *x*-axis in this first situation, and only the height corresponding to the position is required. The reason for introducing this principle is simply that all properties are reduced to lengths, which lead to integrals that can be evaluated. Thus the solution of the differential equation arising is time independent, and corresponds to the steady state solution of the d.e. developed from mass conservation principles. The evaluation of these integrals is set out below, and hence is a hypothetical mathematical procedure based on mean values theorems, rather than directly on physical happenings involving the movement of water up or down, and involving the actual change of the position of the centre of gravity of the fluid by masses of water moving.

This latter situation is considered later in § 7, and is called the *actual descent*, as it involves the actual rise or descent of water, and is the other contributing factor leading to changes in the living force, due to water falling or rising; in this case moments are taken about the *y*-axis, and the change in the vertical position of the real centre of gravity located along the *x*-axis is located for a small flow change. Finally, the equality of these two effects at the differential level leads to the conservation of energy principle applied to the flow in question.

Finally we note that Bernoulli was well aware of the limitations of his theory; deep water in vessels may introduce hydrostatic pressures in addition to the dynamical pressure he considers ; there is a propensity for individual particles of water to go astray if the outlet is too wide and the flow rate too great: the first acknowledgement of turbulent flow; and other causes. We need to remark also that K.F.* has pointed out some confusion with the ratio *m:n*, which occasionally refers to the diameters of the pipes and openings, rather than to the cross-sections of the same: a dimensional analysis can always be performed quickly to clarify this point.]

[*K. Flierl : Publications of the Research Institute of the German Museum for the History of Natural Science and Technology. Series C: Source Texts & Translations (1965).]

Problem.

§. 2. If water may flow through a channel of some form, the velocity of which shall be known at some place, to find the *potential ascent* of all the water contained in the channel.

Solution.

The channel shall be of any shape *ST* (Fig. 13 & 14) through which the water *bcfg* flows; it is assumed, if some point *n* may be taken on the axis *ae*, through which the plane *pm* may pass perpendicular to the axis, to be so that all the particles of water present in that plane flow with an equal velocity, and indeed of such [a magnitude], which shall be inversely proportional everywhere to the magnitude of the [cross-] section *pm.* But the velocity of such water at *gf* shall be that due to the vertical height *qs*, that is, the *potential*

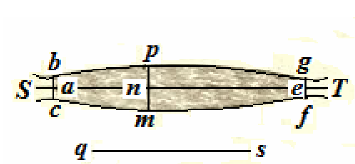


Fig. XIII

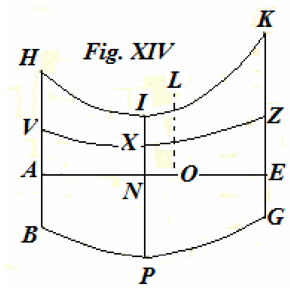
ascent of the layer of water at *gf* shall be equal to the line *qs*, and because heights of this kind are in the square ratio of the velocities, the *potential ascent* of the water at *pm* follows to be equal to the fourth proportion to the square of the magnitude *pm*, the square of the magnitude *gf* and the height *qs*, surely $=\frac{gf^2}{pm^2}\times qs$.

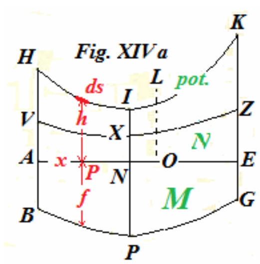
[$i.e.\quad potential\ ascent\times cross\text{-}section\ area^2 = const.$ It may be helpful to comment further on this result for future use, where to some extent we follow KF :
Let *Q* be the volume of water flowing through the pipe in unit time, and we let f_o be the outlet cross-section area and f_n the cross-section area of the pipe at some reference point *n* ; it follows that the corresponding speeds at these points are $v_n=\frac{Q}{f_n}$ and $v_{out}=\frac{Q}{f_{out}}$, and the appropriate height for the speed or rising potential is given, in modern terms, by $h_n=\frac{v_n^2}{2g}$ and $h_{out}=\frac{v_{out}^2}{2g}$. Note that according to the *vis vivans* or living force idea of the time, the potential height could be either proportional or equal to the square of the speed. (The experiments conducted by s'Gravesande, with heavy balls indenting the surface of smooth clay by differing degrees, on falling from differing heights, had already shown this to be the case experimentally). Combining which with

$Q=v_n f_n = v_{out} f_{out}$, we have $h_n f_n^2 = h_{out} f_{out}^2 = hf(x)^2$ in general, as required.]

Thus being forewarned by these, we may put *BPG* in Fig. XIV to be the scale of the width of the channel, thus so that with $AN = an$ put in place, *NP* shall indicate the magnitude [*i.e.* area] of the channel at *pm:* then the curve *HIK* shall be the scale of the *ascending potentials*, thus so that there shall be

$NI=\frac{EG^2}{NP^2}\times qs.$ Now we may consider the individual elements of the curve *HIK* to have a weight equal to the corresponding layer of water, and the centre of gravity of this curve to fall on the point *L*, and *LO* may be drawn perpendicular to the axis *AE;* thus *LO* will be the *ascent* of the whole potential sought. But from mechanics is agreed, if a third curve shall become *VXZ*, of which the applied line *NX* everywhere shall be equal to $\frac{EG^2}{NP}$, *LO* shall be equal to the fourth proportional for the spaces *AEGB*, *AEZV* and the line *qs* or *EK*. [$i.e.\ AEGB : AEZV = EK : LO$]
Therefore the answer to the question becomes apparent. Q. E. I.





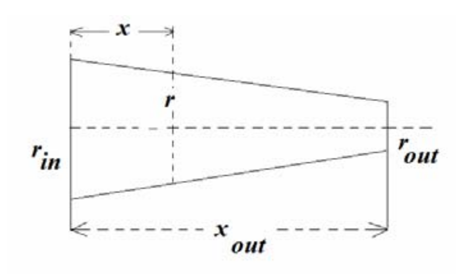
[Further explanation : Thus, in Fig. XIV*a*, the incremental mass *dm* is the amount of water flowing through the section of the vessel at *P* in time *dt*, and is associated with the individual element *ds* of the curve *HIK* with coordinates (*x*, *h*) on the *x*-axis *AE* : $dm=\rho Qdt=\rho f(x)vdt=\rho f(x)dx$, where ρ is the constant density of the water, *v*(*x*) the speed across the section at *P*, and *f*(*x*) the area of cross-section at the position *x*. The moment of this incremental mass about the *x*-axis is given by: $h \times \rho\, fdx = h_{out} \times \frac{f_{out}^2}{f^2} \times f \times \rho dx = h_{out} \times \frac{f_{out}^2}{f} \times \rho dx$; the sum of the moments of the *potential ascend* curve *HIK*, corresponding to the centre of gravity LO is given by:

$LO \times \int \rho\, fdx = \int hf \times \rho dx = h_{out} \int \frac{f_{out}^2}{f} \times \rho dx,$ etc., were the integrations are taken over the whole length *AE*, and from which it follows readily that
$AEGB : AEZV = (h_{out} =) EK : LO$, where
$AEGB = \int fdx$ [M in green represents the fluid volume] and $AEZV = \int \frac{f_{out}^2 dx}{f}$ [N in green].

§. 3. For example in a conical channel, in which the front *gf* and rear *be* [circular] surfaces may have diameters as *m* to *n*, the *potential ascend* of the water

$$= \frac{3m^3}{n(mm + mn + nn)} \times qs.$$

[Be warned : here and elsewhere *m*:*n* represents the ratio of the diameters rather than the ratio of the areas of cross section.
Following KF, pp.16-17 , in which case the curve *BPG* is a straight line, corresponding to the slope of the truncated cone.
We note that the left and right radii r_{in} and r_{out} are related by : $r_{out} = r_{in} \frac{m}{n}$, and the radius of any intermediate section at *x* is given by :



$$r = \left[1 - \frac{x}{x_{out}}\left(1 - \frac{m}{n}\right)\right] r_{in}$$

In addition, $f(x) = \pi r^2 = \pi r_{in}^2 \left[1 - \frac{x}{x_{out}}\left(1 - \frac{m}{n}\right)\right]^2$. From above,

$LO = \frac{N}{M} qs,$ where $N = \int_0^{x_{out}} \frac{f_{out}^2}{f} dx$ and $M = \int_0^{x_{out}} fdx.$ With these integrals evaluated, the stated result follows.]

Problem.

§. 4. With infinitely small variations given both in the ratio as well as in the velocity, which correspond to the forwards surface of the water, to find the variations in the *potential ascent* belonging to all the water.

Solution.

Let the volume $AEGB = M$, the volume $AEZV = N$, $qs = v$, the whole *potential ascent* will be equal to $= \frac{Nv}{M}$: because truly the amount of water in the channel is put to be constantly the same, the space $AEGB$ is invariable, and thus $dM = 0$, thus so that the differential of the *potential ascent* shall be more simply $= \frac{Ndv + vdN}{M}$, but dN is found from the variation of the situation of the water. Therefore the proposition is apparent. Q. E. I.

Scholium.

§. 5. These propositions are of use for the motion of fluid moving within the vessel, that is, not by being required to be defined from the efflux, as I will show in its place : but truly with the fluid flowing through an orifice, it will be more convenient to put in place another calculation, as follows.

Problem.

§. 6. To find the differential of the *potential ascent* after a little drop has flowed out through the orifice.

Solution.

We may devise water to flow from the vessel *aimb* (Fig. 15) formed in some manner, the base *im* shall be pierced by a hole *pl:* the amount of water remaining in the vessel, after a given quantity of this has flowed out, now shall be *cimd;*

Fig. XV

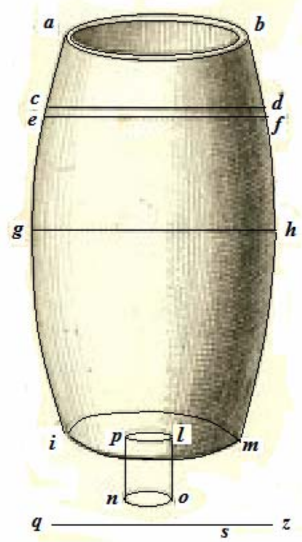
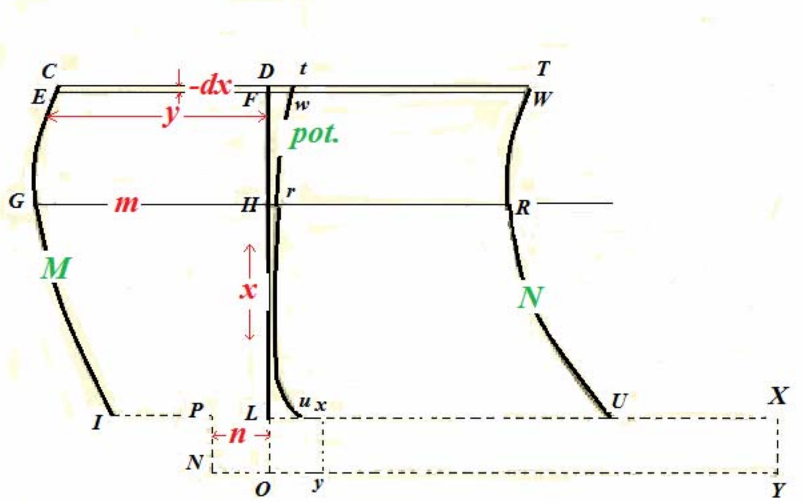


Fig. XVI



moreover in an infinitely small time the droplet *pnol* shall flow out, with the surface *cd* falling to the position *ef:* the section *gh* at the middle of the water may be considered parallel to the surface *cd* or even *ef* itself to be parallel with the base *im;* and thus the velocity of any one such particle at [the middle position] *gh* shall be so that it shall be able to ascend to the height *qs*, or *v*, since the droplet has not yet flowed, and to have risen to the height *qz* or $v+dv$, after that droplet itself has flowed out. Thus with all these in place, the increment of the *potential ascent* of the water is sought after the situation *cimd* has been changed into the situation *eipnolmf*, that is, after the droplet has emerged.

[Here we may note that the above Figure XIV has been turned on its side, *i.e.* become vertical, as Figure XVI ; however, the same kind of analysis is applied relating the pressure to a potential height, and the same kind of curves are constructed, and these do not include changes in pressure with depth which have been ignored until the next section, so that again this is an application of the above theorem involving pressure changes due to changes in the width of the channel, set out to show the differential changes, such as in ***M*** and ***N*** in Fig. XIV*a*; note that in the following sections, ***M*** and ***N*** may refer to the curve or the area under the curve, as the situation demands. There is one set of curves before the droplet emerges, and another set after it emerges.]

The curve *CG* (Fig. 16) becomes, as before, the scale of the magnitudes [of the cross-section areas] and thus where *CD* or *EF* shall represent the magnitude of the water surface before or after the outflow of the droplet, *GH* that assumed magnitude [*i.e*. to be known, as above], *IL* the magnitude of the base, *PL* the magnitude of the opening, while the attached minimum parallelogram *PNOL* corresponds to the cylindrical droplet *pnol:* then another curve *TRU* may be constructed, the applied lines of which shall be equal to the square of the line *GH,* divided by the corresponding applied line of the curve *CGI,* to which curve, with the same condition, is connected the small parallelogram *LOyx,* of which certainly the side *Lx* is equal to the square of the line *GH* divided by the line *PL.* [i.e. the same theorem applies to the cylindrical droplet as to the main curve.]

Therefore now it is apparent that the *potential ascent* of all the water before the outflow of the droplet to be = to the fourth proportional to the space *DCIPL*, the space *DTUL* and the height *qs*,

$$[i.e. = \frac{area\ DTUL}{area\ DCIPL} \times qs = \frac{N}{M} \times dv\ ]$$

and likewise after the outflow of the droplet, to be = to the fourth proportional to the space *FEIPNOL*, the space *FWUxyoL* and the altitude *qz*:

$$[i.e. = \frac{area\ FWUXYOL}{area\ FEIPNOL} \times qz = \frac{N+dN}{M} \times (v+dv)\ ]$$

but these analogous first terms are equal to each other (evidently the spaces *DCIPL* and *FEIPNOL*), therefore if either of these volumes may be indicated by *M*, the space *DTUL* by *N*, the space *FWUXYOL* by $N + dN$, with the height *qs* by *v* and *qz* by $v + dv$, the increment of the *potential ascent* arising from the efflux of the droplet

$$= \frac{Ndv + vdN}{M}.$$

$$[i.e.\ \frac{N+dN}{M} \times (v+dv) = \frac{Nv + vdN + Ndv + dNdv}{M} = \frac{Nv}{M} + \frac{vdN + Ndv}{M}$$
$$= LO + \frac{vdN + Ndv}{M},$$ from which the result follows.]

Because if now there may be put

$LD = x,\ FD = -dx, DC = y,\ HG = m,\ PL = n,$ there will be $DT = \frac{mm}{n}, LX = \frac{mm}{n}, LO = \frac{-ydx}{n}$

(because the volume $DFEC$ = volume $LONP$, [and *LO* is not equal to *OL* above!]), and hence

$$dN = LOYX - DFWT = -\frac{mmydx}{nn} + \frac{mmdx}{y},$$

from which now the increment sought of the ascending potential is

$$= (Ndv - \frac{mmvydx}{nn} + \frac{mmvdx}{y}) : M.$$

Q.E.I.

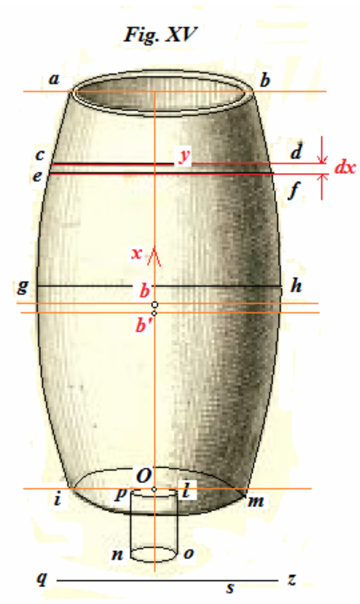
Problem.

§. 7. With the same positions retained, to find the infinitely small *actual descent* of the water, as a droplet flows out.

[*i.e*. here the change in the centre of gravity is considered due to a descent.]

Solution.

Since in Fig. 15 both the water situated at *cdmi* as well as in the situation *efmlonpi* changes, it is apparent in each case the centre of gravity of the part of water *efmi* to be in the same place, and therefore only a small part of the water *cdfe* needs to be considered (which is $=-ydx$, as the whole mass of water $=M$) to have descended to *lonp*. Now the height of the particle of water *cdfe* above the drop *lonp* is $=x$, the height of the centre of gravity of the water *efmi* from the bottom $=b$, therefore the height of the centre of gravity of all the water in the position *cdmi* above the base $=b-\frac{ydx}{M}\times(x-b)$ and in the situation *efmlonpi* the same height will be $=\left(\frac{M+ydx}{M}\right)\times b$; from which the difference of the heights, or of the *actual descent* sought



$$=-\frac{ydx}{M}\times x,$$

which equation shows, the droplet which flows out is required to be multiplied by the height of water above the opening, and the product to be divided by the amount of water, so that the actual descent [*descensus actualis*] may be found, which happens as the drop flows out. Q. E. I.

[We shall call δm the mass of water in the narrow cylinder *cdfe* ; the mass of the unmoved water is then $M-\delta m$; let x_{cg} be the height of the centre of gravity of the whole mass *M* initially, and *b* the position of the centre of gravity of the unmoved mass $M-\delta m$; initially : $M\times x_{cg}=(M-\delta m)\times b+x\delta m$, and finally,

$M\times x'_{cg}=(M-\delta m)\times b-\delta x\delta m$; hence

$M\times\left(x'_{cg}-x_{cg}\right)=(x-\delta x)\times\delta m$; or $bb'=x'_{cg}-x_{cg}=\frac{(x-dx)\times ydx}{M}=\frac{xydx}{M}$ downwards,

Where *y* is the cross-section at the position *x*. Physically, *xydx* represents the moment of the displaced volume element *ydx*, and its magnitude can be considered as the change in the descending potential.]

Problem.

§. 8. To determine the motion of a homogeneous fluid flowing out from a given vessel through a given opening.

Solution.

Because by hypothesis our *potential ascent* is equal to the *actual descent* for single instants, the increment of the former while a droplet flows out is equal to the increment of the latter which arises in a similar short interval of time. Therefore if again the surface of the water, after a given amount of which has flowed out, may be put $= y$, with some magnitude [*i.e.* normal diameter] of the vessel assumed $= m$ as desired, with n the magnitude of the hole, the height of the water above the hole $= x$; if besides the quantity N may be put in place by that law, which was indicated in §. 6, and by v the height may be understood owing to the velocity of the water at the assumed place, where the diameter of the vessel surely is $= m$, by §.6 the increment of the *potential ascent*

$$= (Ndv - \frac{mmvydx}{nn} + \frac{mmvdx}{y}) : M,$$

and the minimum of the *actual descent* $= -\frac{yxdx}{M}$ (per preced. §), from which there is had

$$(Ndv - \frac{mmvydx}{nn} + \frac{mmvdx}{y}) : M = -yxdx : M$$

or

$$Ndv - \frac{mmvydx}{nn} + \frac{mmvdx}{y} = -yxdx,$$

which equation can be integrated generally, whenever the letters N and y are given functions of x [of dimensions 3 and 2 respectively] and the letter v is only of a single dimension.

Corollary 1.

§. 9. Since the velocities shall be in the inverse ratio to the cross-section magnitudes, it is apparent the height, which shall correspond to the velocity of the water flowing out, to be $= \frac{mm}{nn} v$, from which hence if it may be called z, there will be

$$nnNdz - mmzydx + \frac{mmnnzdx}{y} = -mmyxdx.$$

Corollary 2.

§. 10. If the opening shall be very small compared with the diameter of the vessel, there becomes $n = 0$, and the whole equation will be changed into this :

$$-mmzydx = -mmyxdx \text{ or } z = x;$$

therefore then the water constantly flows out from that with a velocity, from which it shall be possible to rise as far as to the height of the uppermost surface, which up to this time was the only case the Geometers were able to understand correctly: and this proposition prevails for all vessels of whatever shape: but when the opening may not be considered to be infinitely small, by no means is the shape of the vessel to be disregarded. Yet it is possible to be observed, that unless the opening shall be very great, certainly without notable error, the same can be regarded as infinitely small.

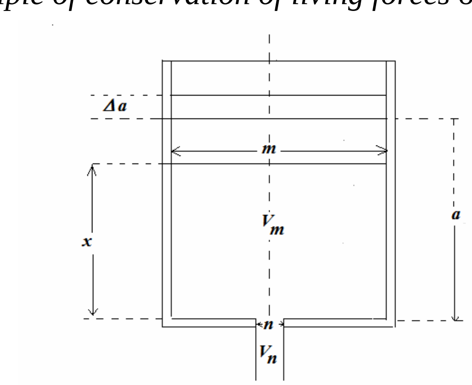
Corollary 3.

§. 11. When the fluid is not the same everywhere, the calculation is to be put in place in a similar manner, without doubt by examining both the increment of the *potential ascent* of the composite fluid, as well as the *actual descent*, and by requiring these to be equal to each other. Because if moreover the opening shall be very small, it is itself apparent, as the calculation shows also, so that the fluid would burst forth to such a height owed, as if the vessel were refilled with the same liquid to the same height as from which it escapes, the sides of the opening bearing the same pressure.

General Scholium.

§.12. Before we may deduce more specialized corollaries from our theory about the motion of fluids from cylindrical vessels, here it will be convenient to examine, until such times as the hypothesis assumed agree with the nature of things and what other causes shall be able to be introduced, of which we have no account in the calculation, diminishing the motion of fluid.

Because in the first place pertaining to the *Principle of conservation of living forces* or *the continual agreement between the potential ascent and the actual descent*, I can see nothing here, which shall be able to be particularly an impediment to that, but only if we remove it from consideration of friction, stickiness, resistance from the air, and from any other obstacles of this kind. Indeed it shall happen often, that that principle cannot be used without limitation, as in the following we will show, evidently since the individual water particles may be carried off with a diverse motion, so that it happens that something may be lost from the motion by



Extra Figure

individual movements, or if you wish, from the *potential ascent*. But in the present case nothing like this happens, whenever all the particles shall be moved almost similarly and especially, when the opening is very small, the motion of the particles internally is almost zero, and thus nothing hence detrimental can come about. But the other principle, where it is assumed the velocity of any particle to be that, which corresponds to the inverse ratio of the cross-section, indeed labors under a twofold inconvenience, *in the first place* namely, because the motion around the sides of the vessel shall be a little slower than in the middle, thus not all the particles corresponding to the same cross-section of the vessel with an equal speed, and *in the second place,* because the water not conveniently remote from the bottom shall be unable to have this motion that this principle demands [thus, wider cross-sections suffer from a pressure gradient due to the weight of water above, as we indicated initially]: But in both cases no sensible error later arises, when in this problem with the simple internal shape hardly anything pertains to the motion of the water. From the same reason it is understood the motion of the water flowing along some other direction is not able to be much different, because clearly the internal motion of the water in the deepest part of the vessel becomes so much different, and this diversity can be of almost no concern. Therefore it appears the hypotheses, on which the calculation of our problem is based, thus agrees with the nature of the question, so that thus no error shall be able to arise perceptible to the senses. But truly the impediments mentioned above, friction, the tenacity of the fluid and other similar causes are more efficacious, especially when the opening, through which the fluid springs out, is extremely small, either the height of the water above the opening is exceedingly great, or finally the tube is extremely thin, concerning which matters many experiments are extant in the work of Mariotte : *Tract. de mot. aquarum.* Indeed now I shall move on to examining the motion of water flowing from cylindrical vessels through openings of any magnitude. But we will consider the case of short and of more elegant solutions for vessels placed vertically.

Concerning these which relate to the efflux of water from vertically placed cylinders, through a hole of some kind, which is on the horizontal base.

§.13. Geometers are accustomed to consider chiefly vertically placed cylinders, the discussion being about the sudden escape of water from the vessel : Therefore nothing at all will be deduced from that matter which pertain to this, that follows from our general theory. Let the cross-section area of the cylinder to the cross-section area of the hole be as *m* to *n;* the height of the water above the hole when the flow begins $= a$*;* the height of the water remaining $= x$, the height due to the internal speed of the water $= v$ [; the velocity here $= V_m$] ;in the canonical equation of paragraph eight there will be $y = m,\ N = [m^2 x / m] = mx$ (by §. 6), which therefore will change into this equation,

$$[c.f.\ Ndv - \frac{mmvydx}{nn} + \frac{mmvdx}{y} = -yxdx,]$$

$$mxdv - \frac{m^3}{nn} vdx + mvdx = -mxdx,$$

or
$$\left(1 - \frac{mm}{nn}\right) vdx + xdv = -xdx;$$

this latter equation is multiplied by $x^{-\frac{mm}{nn}}$, so there may be had

$$\left(1 - \frac{mm}{nn}\right) x^{-\frac{mm}{nn}} vdx + x^{1-\frac{mm}{nn}} dv = -x^{1-\frac{mm}{nn}} dx.$$

Now this equation can be integrated : but the addition of a constant into the integration is required to be observed, namely of such a kind, that from the initial flow, that is, when $x = a$, the velocity of the fluid shall be zero, and from that equally $v = 0$: thus truly there arises :

$$x^{1-\frac{mm}{nn}} v = \frac{nn}{2nn - mm} \left(a^{2-\frac{mm}{nn}} - x^{2-\frac{mm}{nn}} \right)$$

or

$$v = \frac{nna}{2nn - mm} \left(\left(\frac{a}{x}\right)^{1-\frac{mm}{nn}} - \frac{x}{a} \right).$$

§. 14. Therefore from this equation the height is known generating the internal speed of the water ; where it deserves to be noted, if the vessel shall be as large as possible, soon it can be considered to be $v = \frac{nn}{mm} x,$ clearly after only a little while the water falls, that is, once *x* is a little smaller than *a*. This rule fails notably only around the start of the motion and if the first element of that first motion be considered (where surely the height $a - x$ can be considered as infinitely small) the equation shall then be $v = a - x$. From which it follows, in every cylinder, whatever the hole were, the internal water can begin to accelerate from the start if the motion, in the manner of a freely falling body. Truly if the motion may be continued for a little while, there this rule fails less, where both the hole were greater, and the water higher in the tube ; if again that [*i.e.* potential] height may be desired, which corresponds to the velocity of the out flowing water, as in § 9. which we may put $= z,$ there will be $z = \frac{mm}{nn} v,$ or

$$z = \frac{mma}{2nn - mm} \left(\left(\frac{a}{x}\right)^{1-\frac{mm}{nn}} - \frac{x}{a} \right).$$

§.15. When n is $=m$, that is, when there is no base, it appears from the nature of the thing, the water falls freely and accelerates in the manner of heavy bodies ; but the equation itself may show that also; for it becomes in this position $z=a-x$. Truly if the hole is even infinitely small in relation to the size of the vessel, which we have now considered above, there is on putting $n=0$, and then there shall be $z=x$, which shows water flows from that constantly with a velocity, which can rise to the whole height of the water. Finally when $mm=2nn$, there arises $z=\frac{mm}{0}(x-x)$, since from which value nothing shall become known, it is required to descend to the differential equation §.13,

$$[c.f.\ \left(1-\frac{mm}{nn}\right)x^{-\frac{mm}{nn}}vdx+x^{1-\frac{mm}{nn}}dv=-x^{1-\frac{mm}{nn}}dx.$$
$$i.e.\ -x^{-2}vdx+x^{-1}dv=-x^{-1}dx;\ \text{finally, } vdx-xdv=xdx.]$$

which now is this :

$$-vdx+xdv=-xdx,$$

or

$$\frac{xdv-vdx}{xx}=-\frac{dx}{x},$$

which integrated with the addition of the constant due gives :

$$\frac{v}{x}=\log\frac{a}{x},$$

or
$$v=x\log\frac{a}{x},$$

or
$$z=2v=2x\log\frac{a}{x}.$$

§.16. The velocity of water flowing out from the start increases and afterwards decreases, and is a maximum somewhere, namely in that place, where the water has fallen to the height

$$a:\left(\frac{mm-nn}{nn}\right)^{nn:(mm-2nn)};$$

[This result follows at once from the differentiation of

$$z=\frac{mma}{2nn-mm}\left(\left(\frac{a}{x}\right)^{1-\frac{mm}{nn}}-\frac{x}{a}\right),\text{or of } z=\left(\left(\frac{a}{x}\right)^{1-\frac{mm}{nn}}-\frac{x}{a}\right);$$

For $\frac{dz}{dx}=\left(1-\frac{mm}{nn}\right)\cdot\left(\frac{a}{x}\right)^{-\frac{mm}{nn}}\cdot\frac{-a}{x^2}-\frac{1}{a}=0$ or $\left(\frac{mm}{nn}-1\right)\cdot\left(\frac{x}{a}\right)^{\frac{mm}{nn}}=\frac{x^2}{a^2}$;

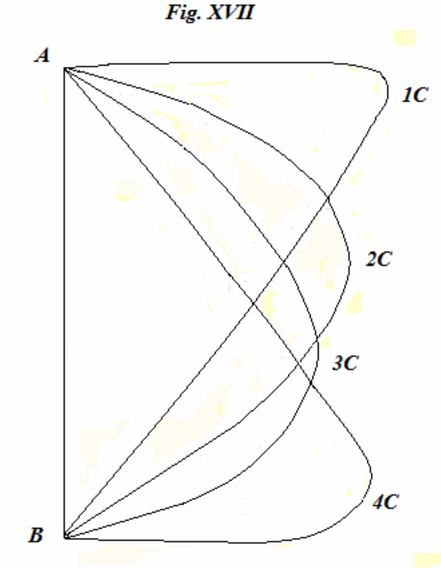
$$\left(\frac{mm-nn}{nn}\right)=\frac{x^{2-\frac{mm}{nn}}}{a^{2-\frac{mm}{nn}}}\therefore x=a\left(\frac{mm-nn}{nn}\right)^{\frac{nn}{2nn-mm}}.]$$

that likewise Mariotte has shown, in his well set out experiments in *Treatise on the motion of water, part.* 3, *disc.* 3, *exp.* 5, and that maximum velocity is such, which is owed to the [potential] height

$$\frac{mma}{mm-2nn}\times\left(\left(\frac{nn}{mm-nn}\right)^{nn:(mm-2nn)}-\left(\frac{nn}{mm-nn}\right)^{(mm-nn):(mm-2nn)}\right),$$

which quantity simplified $[z]$

$$=\frac{mma}{mm-nn}\times\left(\frac{nn}{mm-nn}\right)^{nn:(mm-2nn)}.$$



The time is understood from these formulas, in which the velocity is changed from zero to a maximum, clearly to be imperceptible, when the hole is not very small or the tube is not very long : but to become notable, when things are had otherwise, as we see in leaping fountains, to which water is drawn through long distances ; indeed these matters which are concerned with the time, will be explained more in the following section, and likewise it will be shown, how little water may be ejected from the widest vessels, before they flow out with the maximum speed. The natures of the velocities may be better understood from the opposite figure 17, in which if *AB* may represent the whole height of the fluid above the opening from the start of the flow, the curves *A1CB, A2CB, A3CB, A4CB* express the scales of the corresponding heights, to which the fluid flowing out with its velocity shall be able to ascend in holes of different sizes:

clearly the scale approaches to the figure *A1CB*, if the hole may have a small ratio to the size of the vessel, and to the figure *A2CB*, when it is assumed the bottom is to be perforated by a larger hole; and if now the ratio of the hole shall be to the size of the vessel as 1 to $\sqrt{2}$, that scale will be as *A3CB* (as in the case in which the velocity shall be less than in any other, and that is the one named to which the height $\frac{2a}{c}$ is due, on understanding by *c* the number of which the logarithm is one, that is, of a height a little less than $\frac{3}{4}a$) and finally there will be the scale *A4CB* when almost no base remains.

[In the first case, the potential height is given by $z=\frac{mma}{2nn-mm}\left(\left(\frac{a}{x}\right)^{1-\frac{mm}{nn}}-\frac{x}{a}\right)$; for the case in which $n\simeq 0$, where we can write :

$$z=\frac{mma}{2nn-mm}\left(\left(\frac{a}{x}\right)^{1-\frac{mm}{nn}}-\frac{x}{a}\right)=-a\left(\left(\frac{x}{a}\right)^{\frac{mm}{nn}-1}-\frac{x}{a}\right)\simeq -a\cdot\frac{x}{a}\left(\left(\frac{x}{a}\right)^{\frac{mm}{nn}}-1\right)\simeq x.$$

In the second case, where $m\simeq n$ we can write : $z=a\left(1-\frac{x}{a}\right)=a-x.$

In the third case, we are given $n^2:m^2=1:2$, and we can write $z=2v=2x\log\frac{a}{x}.$ The maximum value of the speed occurs when $\frac{dz}{dx}=0$; *i.e.*
$\frac{dz}{dx}=\log\frac{a}{x}-x\cdot\frac{x}{a}\cdot\frac{-a}{x^2}=\log\frac{a}{x}-1=0$; *i.e.* $\frac{a}{x}=e$ and $z_{\max}=2v=\frac{2a}{e}\log e=\frac{2a}{c}.$
The last case is similar to the second case.]

§. 17. Now truly we will illustrate with an example, what was indicated above, truly unless the opening shall be the largest, that to be possible without much error in the calculation to be considered as indefinitely small, and thus to assume, $z=x$ as was said in §§.10 and 15. That may be seen to prevail only in the works of some authors, as they had considered that no account was required to be had in the magnitude of the hole at any time, however great the size of the hole may be put in place, which certainly is a ridiculous thing : perhaps no-one up to the present that I know of has considered the size of the hole correctly for this matter. Therefore we may put in place a cylinder, the diameter of which shall be only four times the diameter of the hole, the biggest holes of which kind rarely are accustomed to occur in hydraulic devices, and we may devise the surface of the water to be falling only be a hundredth part of the total initial height (but I assume to be falling a little, because from the first no initial motion of the water can be present, still much less, so that the water flowing shall be able to rise to the whole height by its own motion); these in place make $m = 16n$ and $mm = 256nn$, and $x = \frac{99}{100}a$, from which there is produced :

$$z = \frac{128}{127}\left(\frac{99}{100} - \left(\frac{99}{100}\right)^{255}\right)a = \frac{92}{100}a,$$

[from $z = \frac{mma}{2nn - mm}\left(\left(\frac{a}{x}\right)^{1-\frac{mm}{nn}} - \frac{x}{a}\right) = \frac{256}{254}\left(\frac{99}{100} - \left(\frac{99}{100}\right)^{256-1}\right)$ etc.]

which indeed differs a little from the magnitude *x*, or from $\frac{99}{100}a$, but certainly still not by much, and the difference shall be much less, when the hole is less and the surface of the water falls a little more. Therefore this theory differs from the usual chiefly around the start of the flow, where the motion is less than was put in place ; on the other hand the water is ejected with a greater velocity than due to the second principle.

§.18. Up until now we have considered the motion of the water to have arisen on account of its own gravity; now we put in place the water to be ejected by a different force besides the force of gravity, and to be communicating so great a velocity to the water flowing out, by which it shall be able to rise to a much greater height, than if the motion of the water were produced by gravity only; then suddenly the other different force vanishes, and the water itself to be left ; but if that happens, experience teaches that the velocity of the water decreases rapidly and soon to be such, that remarkably it will not exceed that speed, which was arising from the gravity of the water alone. Thus we see happening a little in leaping fountains (of which I may say something about the true cause and size elsewhere) so that the water may leap up to three or four times or to even a greater height, than is usual; thus when that happens, the leap itself stops at once and the customary height is not exceeded, as great as that perceived able by the senses : but with tubes perforated with not very large holes ; for when the hole is a little larger, the jump of the water does not decease so quickly. And thus now we will examine, as far as the theory may agree with, and we may subjoin the accurate measures of these phenomena, such as thence follow. Truly so that we may pursue the matter generally, again we will put the cross section of the cylinder to the cross-section of the opening as *m* to *n*: the water to be driven out from that with a speed by which it is able to rise to a height α, and from that point of time the height of the water above the hole to be $= a$, of which gravity alone now expels the water ; then the surface of the water in the cylinder descends through the vertical height $a - x$, thus so that the final height left shall be $= x$ and then the speed of the water ejected to be such, that is due to the height *z*. Thus with these in place we shall use the general differential equation §. 9, which is this :

$$nnNdz - mmzydx + \frac{mmnnzdx}{y} = -mmyxdx$$

(where again, as §.13 had shown, there is $y = m$ and $N = mx$) and which in our particular case shall become of such a kind :

$$\left(1-\frac{mm}{nn}\right)zdx + xdz = -\frac{mm}{nn}xdx,$$

which multiplied by $x^{-\frac{mm}{nn}}$ and thus afterwards integrated [by parts], so that on putting $x = a$ there becomes $z = \alpha$, the desired final equation will be given:

$$z = \left(\frac{mm}{2nn-mm} + \frac{\alpha}{a}\right)x^{\frac{2nn-mm}{nn}} \times x^{\frac{mm-nn}{nn}} - \frac{mm}{2nn-mm}x$$

or $$z = \frac{mma}{2nn-mm}\left(\left(\frac{\alpha}{a}\right)^{1-\frac{mm}{nn}} - \frac{x}{a}\right) + \left(\frac{x}{a}\right)^{\frac{mm-nn}{nn}}\alpha$$

which height if it may be compared with that, which was indicated in paragraph 14, it is found the excess of one above the other

$$= \left(\frac{x}{a}\right)^{\frac{mm-nn}{nn}}\alpha;$$

so that now all those other phenomena may be confirmed, which were indicated just now ; for that excess, when the number *m* is much greater than *n*, becomes at once unmeasurable, even after the water has dropped a little, that is, after the shortest interval of time, yet at no time does everything vanish, as long as the flow lasts, and finally there remarkably it is to continue, when the greater ratio of the number *m* to *n* approaches equality. For example the diameter of the pipe were ten times greater than the diameter of the hole, and the water may be expelled with such a force, so that by its speed it shall be able to leap up to a height which shall be four times the height of *a*, or of the height of the water above the opening, it is sought to what height, by its velocity from the water flowing, it will be able to rise, after the surface of the water in the pipe has fallen by a thousandth part of *a*, if meanwhile the water may be acted on by its own weight only for the outflow, then what would the same height have become, if the water had no motion from the start: moreover there is : $m = 100n, mm = 10\,000nn,\ x = \frac{999}{1000}a,\ \alpha = 4a$, from which in the first case there shall be :

$$z = \left(\frac{10\,000}{9998}\left(\frac{999}{1000} - \left(\frac{999}{1000}\right)^{9999}\right) + 4\left(\frac{999}{1000}\right)^{9999}\right)a,$$

or

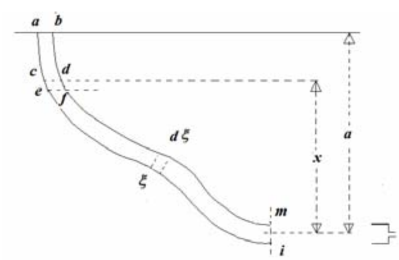
$$z = \frac{99915}{100\,000}a + \frac{18}{100\,000}a,$$

but in the latter case there becomes :

$$z = \frac{99915}{100\,000}a$$

from which example it is apparent, how small and clearly insensitive to measurement, the excess of the first height shall be over the second, and as to how quickly that water thrown may be diminished, whenever the whole change is made, while the surface of the water falls by a one thousandth part of the height *a*, as the time in hydraulic machinery is certainly not accustomed to be short. Then also it may be confirmed, as was said above in paragraph 17, clearly to be approximately $z = x$, when the hole is even moderately small, since in the present case, where the motion begins from rest, the difference between *z* and *x* shall be only fifteen hundredths of a thousandth part of its height *a;* because while the height *z* is a little greater than *x*, it is apparent for the water flowing to rise to a greater height, after the water has been flowing for some time, than the height of the water is above the opening.

§. 19. Thus, after we have deduced from our general theory, what motion they observe of fluids in vertically placed cylinders, now also we will consider pipes placed obliquely, which are accustomed to be found extended in leaping fountains. In these indeed it is singular, that the acceleration of the motion thus does not happen suddenly, as when the cylinders are vertical, and thus it may allow agreement with the senses to be perceived with the real motion of water.



Extra Figure.

§. 20. We may devise a channel curved in some way, but still cylindrical, the cross-section of which again shall have the ratio *m* to *n* to the cross-section of the hole. The motion may start from rest, and the vertical height of the water above the opening from the start of the motion shall be $= a$; a certain amount of water will flow out, and the vertical height of the water left above the opening may be put $= x$, the length of the channel, which at that moment is full, $= \xi$, and then the internal velocity of the water may be had (here I assume the individual particles of its motion to be carried parallel to the axis of the channel), which shall correspond to the height *v;* thus with these in place, if we may use similar reasoning to the above, without doubt by seeking the increment of the *potential ascent* while the droplet flows out, we have made use of §.6, and also on putting = *actual descent*, now such an equation may be obtained :

$$\xi dv - \frac{mm}{nn}vd\xi + vd\xi = -xd\xi,$$

or

$$\left(1-\frac{mm}{nn}\right)vd\xi+\xi dv=-xd\xi,$$

of which the integral, because it is apparent with the terms multiplied by $\xi^{-\frac{mm}{nn}}$, this is

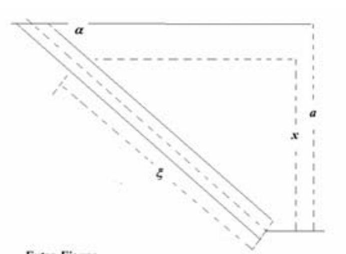
$$v=\xi^{\frac{mm}{nn}-1}\int -x\xi^{-\frac{mm}{nn}}d\xi.$$

[See KL p.25 for a detailed evaluation of this equation.]

For example, the channel was straight and inclined thus to the horizontal, so that the sine of the angle intercepted between each shall be to the total sine as 1 to *g*, there will be $\xi=gx$ [$=x/\sin\alpha$] *;*from which

$$v=\frac{nna}{2nn-mm}\left(\left(\frac{a}{x}\right)^{\frac{nn-mm}{nn}}-\frac{x}{a}\right),$$

which equation since it does not differ from the equation §. 13 for given vertical cylinders, it follows in each case the velocities of the water to be the same, after the vertical descents of the surface of the water are the same : Therefore similar accelerations in both homologous places are in the ratio of the vertical heights, and only with this difference interceding, that in the inclined channel they become slower, and that in the ratio as 1 to *g:* therefore we can easily see these accelerations in greatly inclined channels, which cannot be observed in vertical channels on account of the excessive speed. Moreover it is itself evident from that, because frictional forces are increased by the length of the tube, it is not possible for the speeds thus not to be diminished, towards which they draw attention, according to which experiments will be considered to be put in place.



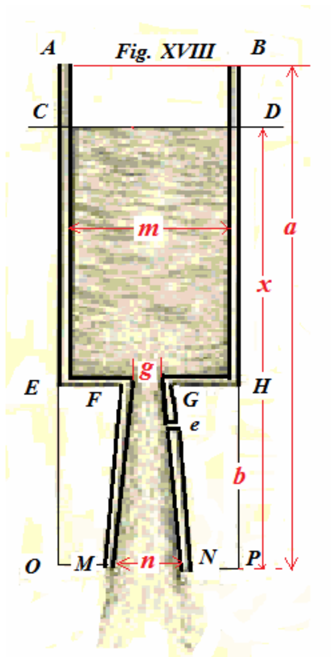
Extra Figure

Concerning the efflux of water from vertically placed cylinders, which end in other equally vertical but narrower tubes.

§. 21. It is agreed from several experiment, between two cylinders entirely equal and put in similar positions, of which the latter may correspond to a tube with a narrower hole, this to be emptied faster which has a tube attached, and indeed from that quicker, in which the tube increases in cross-section from the point of insertion towards the extremity, which have been set out by D. 'sGravesande in *Phys. Elem. Math., book.* 2, *ch.* 8. We will understand the whole matter from the following problem.

Problem.

§. 22. A cylindrical vessel *AEHB* (Fig. 18) perforated by a hole at *FG* was put in place, by which opening it could communicate with the conical tube *FMNG*, through which opening *MN* finally the water could flow out. The velocity of the surface of the water *CD* is sought, after it fell from rest through *AC* or *BD*.



Solution.

The initial height of the water above *MN* evidently shall be $NG + HB = a$, with the height of the surface of the water in the position *CD* above *MN*, that is, $NG + HD = x$; with the length of the tube joined on or $NG, = b$; the cross-sectional area of the orifice $MN = n$; the cross-sectional area of the orifice $FG = g$, with the cross-sectional area of the upper cylinder $= m$; the velocity of the surface of the water at *CD* such that may be due to the height *v*, in the general equation §. 8 there will be $y = m$ and $N = m(x-b) + \frac{bmm}{\sqrt{gn}}$, which substituting into the calculation in place conforms to be apparent with §. 6;

[$Ndv - \frac{mmvydx}{nn} + \frac{mmvdx}{y} = -yxdx$; *i.e.* the total change in the potentials up and down is zero; recall for the truncated cone, that the base has diameter $\sqrt{n}$, while the upper circle has diameter $\sqrt{g}$, while $N = \int_0^b \frac{mm}{r} dz$, where $\sqrt{r} = \sqrt{n} - \frac{z}{b}\left(\sqrt{n} - \sqrt{g}\right)$;

giving:

$$N = \int_o^b \frac{mm}{\left(\sqrt{n} - \frac{z}{b}\left(\sqrt{n} - \sqrt{g}\right)\right)^2} dz = \frac{mmb}{\left(\sqrt{gn}\right)},$$ on evaluating the integral.]

moreover the remaining positions are the same as before. Therefore the equation in paragraph 8 will change into this: [by the conservation of living force, or energy,]

$$m\left(x-b\right)dv + \frac{bmm}{\sqrt{gn}} dv - \frac{m^3vdx}{nn} + mvdx = -mxdx,$$ which divided again by the factor *m* and making $x - b + \frac{bm}{\sqrt{gn}} = d$, gives

$$\left(1-\frac{mm}{nn}\right)vdz+zdv=-zdz-bdz+\frac{mbdz}{\sqrt{gn}},$$

which multiplied by $z^{-\frac{mm}{nn}}$ makes

$$\left(1-\frac{mm}{nn}\right)z^{-\frac{mm}{nn}}vdz+z^{1-\frac{mm}{nn}}dv=-z^{1-\frac{mm}{nn}}dz-bz^{-\frac{mm}{nn}}dz+\frac{mbz^{-\frac{mm}{nn}}dz}{\sqrt{gn}},$$

from which, after integration with the constant *C* added, there arises :

$$z^{\frac{nn-mm}{nn}}v=C-\frac{nn}{2nn-mm}z^{\frac{2nn-mm}{nn}}-\frac{nnb}{nn-mm}z^{\frac{nn-mm}{nn}}+\frac{mnnb}{(nn-mm)\sqrt{gn}}z^{\frac{nn-mm}{nn}},$$

in which the value of the constant quantity *C* is defined from that, because from the initial flow (when truly $x=a$ or $z=a-b+\frac{mb}{\sqrt{gn}}$) there shall be $v=0$, because the motion cannon arise at an instant point of time ; hence therefore there shall be

$$C=\left(\left(a-b+\frac{mb}{\sqrt{gn}}\right)\times\frac{nn}{2nn-mm}+\frac{mb\sqrt{gn}-mnnb}{(nn-mm)\sqrt{gn}}\right)\times\left(a-b+\frac{mb}{\sqrt{gn}}\right)^{\frac{nn-mm}{nn}}.$$

Indeed everything can be defined from these equations; truly because the calculation shall be a little more long-drawn out, unless the cross-section of the upper vessel may be indicated by so great an *m*, so that it can be considered in an infinite ratio to the cross-sections *g* and *n*, hence we will only consider the case, and thus there no notable error thence will arise, even if the number $\frac{m}{n}$ or $\frac{m}{g}$ shall be the mean of the magnitudes.

§. 23. Because if hence we may put $m=\infty$, and likewise we may make use of the differential of the nearby paragraph for the first equation, and in this there may be put $v=\frac{nn}{mm}s$, thus so that the height may be found from the value of the letter *s* to which the water flowing through the orifice *MN* shall be able to rise according to its velocity, there will be in the first place

$$\frac{nn}{m}(x-b)ds+\frac{bnn}{\sqrt{gn}}ds-msdx+\frac{nn}{m}sdx=-mxdx$$

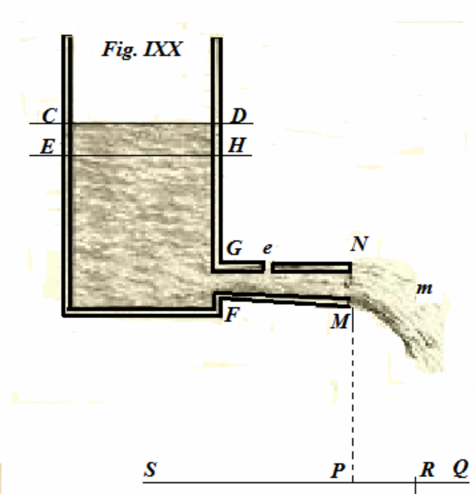
and because $m=\infty$ and it will be easily foreseen that the ratio between *s* and *x* will be finite, and between *ds* and *dx*, this same equation will be changed with the rejected terms again being rejected in this case $-msdx=-mxdx$ or $s=x$, which was shown equally in paragraph 10.

[Write the above in the form :

$$\frac{nn}{mm}(x-b)ds+\frac{bnn}{m\sqrt{gn}}ds-sdx+\frac{nn}{mm}sdx=-xdx,\text{etc.}]$$

Indeed I was led from that matter to demonstrate that here anew, because the present case can be seen to be different from that, about which mention was made in the previous paragraph. With these understood there is no need for more explanation about these phenomena, indicated by the author 'sGravesande in §. 21; for it is apparent, water cannot flow through the composite vessel *AEFMNGHB*, otherwise than through the simple vessel *AOMNPB*, evidently when the opening *MN* is very small ; hence the velocity of the surface of the water *CD* is to be greater, as if the water were to flow through the vessel *AEFGHB*, on putting the opening $MN=FG$ [*i.e.* the lower part becomes cylindrical], and much greater still if *MN* were greater than *FG*, which happens when the tube cross-section increases towards the lower cross-section : but yet it must be noted, from the beginning of the motion the water falls slower than were thus defined, nor does the first position have the same rule as the surface *CD* falls through a little space, because it happens still in a short time: we will examine the changes in the following section, which in this case are made from the start of the motion.

§. 24. The computation is to be put in place in the same manner, if to the vessel, which now we put to be of infinite cross-section always, a pipe shall be inserted not vertically but horizontally, as in fig. 19, or in some other direction; but always it will soon be found that the water flows from that through the opening *MN*, after the surface of the water in the main vessel has dropped a little, with nearly the velocity which shall correspond to the height of that surface above the opening ; thence it shall be clear that with both the height of the water above the small tube *GN* remaining constant, as well as with the opening *FG*, the quantity of water may be increased in a given time flowing from an increased cross-section of the opening *MN:* Therefore here we have given the demonstration thus, which was mentioned at the end of §.4 Section 1. Frontinus truly taught from experience that : *more water than due to be distributed from a water gauge both placed and measured legally, to which larger pipes have been attached directly.**



[Evidently the resistance to the flow has been decreased.] And indeed the quantities of water with all things being equal must be distributed through the openings *MN* themselves nearly in proportion, unless there were much resistance, which would greatly diminish this quantity, concerning which I may say something next : these impediments can be made, certainly so that an exceedingly small flow of water may be moved from the increased end of the orifice ; yet always a little amount will be moved.

[* Extra notes of a general nature from article by Philip Smith, B.A., of the University of London on pp108-115 in :

A Dictionary of Greek and Roman Antiquities, John Murray, London, 1875, by William Smith, D.C.L., LL.D. , in which some of the original specialized Latin terms are introduced :

The leaden cisterns, which each person had in his own house to receive the water laid on from the *castellum privatum*, were called *castella domestica*. All the water which entered the castellum was measured, at its ingress and egress, by the size of the tube through which it passed. The former was called *modulus acceptorius*, the latter *erogatorius*. To distribute the water was termed *erogare*; the distribution, *erogatio*; the size of the tube, *fistularum* or *modulorum capacitas*, or *lumen*. The smaller pipes which led from the main to the houses of private persons, were called *punctae*; those inserted by fraud into the duct itself, or into the main after it had left the castellum, *fistulae illicitae*.

The erogatio was regulated by a tube called the *calix*, of the diameter required, and not less than a foot in length, attached to the extremity of each pipe, where it entered the castellum; it was probably of lead in the time of Vitruvius, such only being mentioned by him; but was made of bronze (aeneus) when Frontinus wrote, 'in order to check the roguery of the aquarii, who were able to increase or diminish the flow of water from the reservoir by compressing or extending the lead'. As a further security, the calix was stamped. Pipes which had no calix, were termed *solutae*. Frontinus also observes that the velocity of the water passing through the calix, and, consequently, the quantity given out, could be varied according to the angle which the calix made with the side of the reservoir: its proper position was, of course, horizontal.]

§. 25. From these premises it shall be apparent the velocity, by which the surface of the water *CD* in each case, from what we have said, with all else equal, to depend on the cross-section of the orifice *MN;* but this rests on that hypothesis, because the water may adhere to the sides of the tubes *GN* everywhere and flow out from the whole orifice *MN*, which hypothesis cannot be considered, if the orifice itself may be increased exceedingly. Then it is apparent also, when the water flows out through the vertical tube in Fig.18, the flow of which to be accelerated with the length of its tube increased: yet this also thus may be increased, so that finally the water may cease to continue in the tube, but rather to be divided into columns, which happens, if the tube may have a length of more or less thirty two feet [*i.e.* the pressure sustaining the water in the column has dropped to zero] and also, if likewise the cross-section may increase towards *MN;* thus if the opening *MN* shall be double the other opening *FG,* the length will not be able to be more than eight feet, without the subsequent danger of the water separating in the upper part of the tube, which matter I will show elsewhere : but there is another cause besides the above excessive length of the tube, which can produce separation of the water, namely because the height of the water *CEHD* shall be smaller than that so that it cannot burst into the tube quickly enough, so that it happens that air may flow in to the upper part together with the water, while the surface of the water assumes the form of a cataract or of a hollow funnel, thus so that the whole orifice *FG* may be covered over by water ; Indeed this occurance happens, so that the water may flow less abundantly, but not so that it shall have a smaller velocity, that a certain recent Italian author called Carolus Fontana considered, who considering this matter in his own language thus wrote *"But if here there were not,"* he says, *"as much water as would be sufficient to maintain the said pipe full, the water will attract air within itself in as great a quantity as water will be lacking to it, for intermixing within the water on all sides; but the velocity if the water will be lacking as much as will be the height, if all the air collected together with that will be in the pipe."* . [See below for the original; here we have used the translation offered by Carmody & Kobus.] The account of this, which I have said, cannot thence diminish the speed of the water, as anyone sees from that, because otherwise the descending potential cannot be equal to the ascending potential and the matter can be confirmed readily by experiment, with the end of the tube *MN* curved, so that the water may flow out horizontally, and from the magnitude the speed of the jet of water shall be able to be determined. But how can it happen for argument's sake, that with no changes from other circumstances air can mix with water around the top of the tube, as follows : truly make a small hole in the tube not far from the opening *FG* (Fig. 18 & 19) because if during the outflow you may block the small hole itself with a finger, pure water will flow out, and if you remove the finger, soon air through the little hole will interrupt and mix with that water itself flowing past . With these understood it will be easy to render an account of the phenomena, which are observed of the smoke in chimneys or in smoke ducts, for smoke desires to rise, because it is lighter than air, which agrees with experiments with smoke in a vacuum, where it was seen to be falling, therefore with the same assumed for smoke rising, as with water falling ; but this in fig.18 therefore flows out faster through the orifice *MN*, where it is wider, and where it is placed lower down: therefore also smoke therefore will pass through the chimney faster, and therefore rise higher with more fire in the hearth, so that the higher the chimney may draw, and so that the more the smoke may diverge to the upper regions, but only if it may not spread out a great deal ; which each may be confirmed by experience ; then in addition I have found, if the chimney may be perforated somewhere, only to be missing, that smoke may try to emerge through the hole itself, but rather air with a great impetus may rush in, and consequently rise through the chimney, mixing with the smoke, and likewise the air erupts into the tube *FGNM* (Fig.18 and 19) through the small hole *e* we have indicated in the tube. Thus truly the smoke certainly will be less copious, or perhaps may rise with more difficulty with the fire reduced.

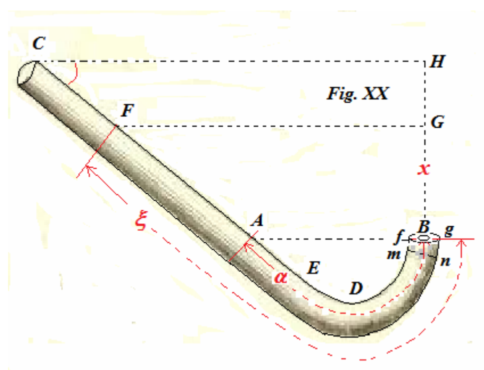
There are two other main reasons, the one quite distinct and the other appropriate to the nature of the matter in hand, which can retard the motion of the water greatly in Fig.18 & 19. The former is the adhesion of the water to the walls of the tube, and the other because when the tube increases in size the velocity of the water is nowhere to be constant in whatever part of the tube the cross-section may change; which change, if it arises, may be considered from the impulses of the infinitely small faster motions on water moving less quickly, it is apparent from the individual moments, from these impulses with these of soft bodies, [*i.e.* inelastic collisions] that a little is lost from the potential ascent, from which by necessity the efflux of the water is notably diminished.

§. 26. Now in the final place I may say something about recursive vessels, from which not all the water flows out: but for the sake of brevity I shall consider cylindrical channels, and of which a certain part, which the surface of the water does not pass over, shall be straight.

Problem.

Truly there shall be a cylindrical channel *CEDB* (Fig. 20) of which a sufficiently great part *CE* is straight, the remainder *EDB* curved in some manner ; should the whole channel be full of water flowing out through the opening *B*, and when the surface of the water will have arrived at *F* from *C*, the height may be sought corresponding to the velocity of the water at *F*.

Solution.



The vertical line *BH* and the horizontal lines *CH*, *FG*, *AB* are drawn, and the sine of the angle *HCE* shall be to the whole sine as 1 to *g:* Now truly if we may consider the whole business correctly, we will see the present problem to be contained in the other more general problem, which we have treated above in paragraph 20, where we had this equation :

$$v = \xi^{\frac{mm}{nn}-1}\int -x\xi^{-\frac{mm}{nn}}d\xi.$$

where for our present case by *v* is understood the height corresponding to the velocity of the surface of the water at the position *F*, by ξ the length *BDEF* and by *x* the height *BG*, and by $\frac{m}{n}$ an index of the ratio between the cross-sections of the tube and of the opening *B:* Because if truly the length $BDA = \alpha$, there will be $x = \frac{\xi - \alpha}{g}$, from which now there may be had

$$v = \xi^{\frac{mm}{nn}-1}\int -\left(\frac{\xi-\alpha}{g}\right)\xi^{-\frac{mm}{nn}}d\xi.$$

The whole length of the channel *BDEC* may be indicated by β, and there will be

$$\int -\left(\frac{\xi-\alpha}{g}\right)\xi^{-\frac{mm}{nn}}d\xi = \frac{nn\alpha}{g(nn-mm)}\left(\xi^{\frac{nn-mm}{nn}}-\beta^{\frac{nn-mm}{nn}}\right)$$
$$-\frac{nn}{g(2nn-mm)}\left(\xi^{\frac{2nn-mm}{nn}}-\beta^{\frac{2nn-mm}{nn}}\right),$$

and therefore :

$$v=\frac{nn\alpha}{g(nn-mm)}\left(1-\left(\frac{\beta}{\xi}\right)^{\frac{nn-mm}{nn}}\right)-\frac{nn\xi}{g(2nn-mm)}\left(1-\left(\frac{\beta}{\xi}\right)^{\frac{2nn-mm}{nn}}\right).$$

Q.E.I.

[See KL p.31-32 for a straight-forwards derivation of this result.]

§. 27. Because these equations are a little prolix, we shall not tarry on the general contemplation of the same, rather these particular cases are going to be considered, which shorten the calculation, nor can they be defined from that final equation.

If we may put the whole of the opening at *B* to be absent , there shall be $m=n$ and (which separately for this case that must be elicited, and equally for the other case to be discussed soon)

$$v=\frac{\beta-\xi+\alpha\log\xi-\alpha\log\beta}{g}$$

and then the maximum velocity is at *A*, and which corresponds to the so-called height

$$\frac{\beta-\alpha+\alpha\log\alpha-\alpha\log\beta}{g}.$$

And then the point *E* [where the pipe begins to curve] corresponding to the maximum descent may be obtained with the help of this equation [where *v* is taken as zero],

$$\xi-\alpha\log\xi=\beta-\alpha\log\beta.$$

[In this case $m=n$, and the second equation becomes:

$$v=\xi^0\int_\beta^\xi -\left(\frac{\xi-\alpha}{g}\right)\xi^{-1}d\xi=-\int_\beta^\xi\frac{d\xi}{g}+\int_\beta^\xi\frac{\alpha}{\xi g}d\xi$$

$$=\frac{\beta-\xi}{g}+\frac{\alpha}{g}\left[\ln\xi\right]_\beta^\xi=\frac{\beta-\xi+\alpha\ln\xi-\alpha\ln\beta}{g};$$

Again, the maximum velocity corresponds to

$$\frac{dv}{d\xi}=-1+\alpha\cdot\frac{1}{\xi}=0;\ \text{thus } \alpha=\xi \text{ and the max. velocity occurs at } A.$$

The height corresponding to the max. velocity is then
$v=\left(\beta-\alpha+\alpha\ln\alpha-\alpha\ln\beta\right)$ as required.]

The other separate case requiring to be deduced by calculation is, when $mm=2nn$, where there arises

$$v=\frac{\alpha\xi-\alpha\beta-\xi\beta\log\xi+\xi\beta\log\beta}{g\beta}$$

and if it may be taken, on putting *c* for the number, of which the logarithm is unity, $\xi=c^{\frac{\alpha-\beta}{\beta}}\beta$, [Thus, Euler had not yet put his stamp on the exponential function, as we would now replace this equation with $\xi=e^{\frac{\alpha-\beta}{\beta}}\beta$], thus the position of the maximum velocity will be determined, of which the generating height is $=\left(c^{\frac{\alpha-\beta}{\beta}}\beta-\alpha\right):g$ while the maximum descent, which is proportional to the total of the water flowing out, is defined by making

$$\alpha\xi-\alpha\beta-\xi\beta 1\text{og}\xi+\xi\beta\log\beta=0.$$

[In this case, $mm=2nn$, and the above equation for *v* becomes :

$$v=\xi\int_{\beta}^{\xi}-\left(\frac{\xi-\alpha}{g}\right)\xi^{-2}d\xi=-\frac{\xi}{g}\int_{\beta}^{\xi}\frac{d\xi}{\xi}+\xi\frac{\alpha}{g}\int_{\beta}^{\xi}\frac{d\xi}{\xi^{2}}$$
$$=\frac{-\xi\ln\xi+\xi\ln\beta}{g}-\frac{\alpha\xi}{g}\left(\frac{1}{\xi}-\frac{1}{\beta}\right);$$

Again, the maximum velocity corresponds to

$$\frac{dv}{d\xi}=\alpha-\beta\ln\xi-\beta\xi\cdot\frac{1}{\xi}+\beta\ln\beta=0;$$ and from that it follows that

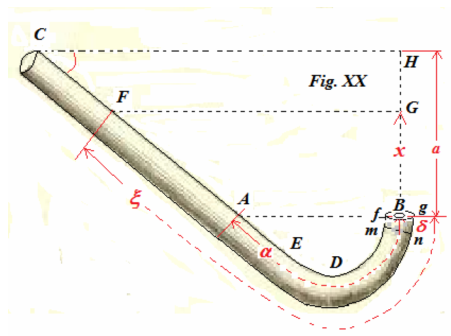
$$\beta\ln\frac{\xi}{\beta}=\alpha-\beta,\ \ \ln\frac{\xi}{\beta}=\frac{\alpha-\beta}{\beta},\text{ or }\xi=\beta e^{\frac{\alpha-\beta}{\beta}}.]$$

There is no reason to doubt why the experiments should not correspond to this precision, but only if the adhesion of the water to the walls of the pipe were not retard the motion ; yet I think, in the event of such experiments that it will be possible to understood, which of these impediments may need to be considered, and the experiments then shall show the truth of the propositions well enough.

§. 28. Finally I will communicate the true solution of a certain phenomenon, which at sight may seem very puzzling. For indeed from everything said up to now nothing could appear to be made clearer, than water may not flow out with a much greater velocity than it ought to flow from such a height above the opening (for they can be a little greater, especially if the openings are large, *compare that with what was said about the maximum velocities in* §. 16), it will be considered by many a wonder perhaps, *to be related now to leaping fountains, so that the water at certain instants of time may make a far higher leap, than would seem to be possible according to our rules.* Truly a jet of such a size disappears, as these may lose a little of their strength, but they may be confirmed in rather an outstanding manner. Moreover the solution to the puzzle is consistent with that, because until now we have considered continuous water, and separated by no empty air spaces : and Mr. De la Hire has correctly considered that no such irregular leaps can happen, unless air were introduced into the pipe together with the water near a bubbling spring, which I have indicated to happen often, see §. 25. Truly this air likewise is carried with the water as far as to the opening of the efflux, through which it soon erupts : as that happens, the mass of the water acquires an impetus, which is expended in expelling the water only, and with this agreed upon an immense jet is produced. Soon I will explain more clearly this cause of the phenomenon together with the due measurements, after presenting a little that has been published in the *Histor. Acad. Reg. Sc. Paris* for the year 1702. *One sees occasionally,* it says in the place cited, *water which leaves through an ajutage to jump three or four times higher than the height of the reservoir permits it to do, as well as returning itself very quickly to the height which the laws of hydrostatics prescribe for it. But how has it been able to depart for an instant ? Msr. De la Hire attributes that to the air enclosed in the conduit, which having been compressed and put under pressure by the water, which is descending always, it releases itself against that water which rises, and gives it this momentary speed.*

And thus Mr. De la Hire has noticed correctly that the leap must be due to the air, and there is no reason to doubt why that should not be the true account, by which the air would be able to produce that effect, which could be elicited, if the phenomenon, which was mentioned in passing, had been considered more carefully; certainly it would be observed easily that the air present between the intermediate volumes of water would sustain no pressure, unless due to the water resting above it (indeed this is not in the deepest water flowing, as I will show below in Sect. XII) and nor so far can the more compressed air expel the water preceding itself, than if its own position were water. Indeed I myself have foreseen (as I have shown often afterwards with the simplest experiment) that it is not to be the water in place before the air, but that which follows the air, [which is the cause of] the customary higher upsurge, which now I shall make more clear.

In Fig. XX let *CADB* be a cylindrical water duct, as is customary, and that full of water, except the small part *mnB* is full of air. The lines *CH* and *HB* may be drawn to the horizontal and vertical : for brevity we may consider the weight of the air to be zero in comparison with the weight of the water, thus so that the passage of the air through the opening *B* offers no resistance to the flow of the water, although it would be easy to give an account of the inertia of the air in what follows, except that we may wish to avoid an extended calculation into the matter, where we may seek nothing precisely. The length of the channel shall be *CADf* or $CADm = \beta$ (for we may put the differential *mf* to be very small); *mf* or $ng = \delta$; $HB = a$; with the cross-section of the tube $= m$; and with the cross-section of the opening $B = n$; and then at last the water, when the surface is at *mn*, not to be moving, the height corresponding to the velocity is going to be sought, that the water with the surface *mn*, has when it arrives at the position *fg*; that height shall be $= v$, the *potential ascent* of all that water thus itself at that instant equally will be $= v$: but the *actual fall* is by §. 7 = to the fourth proportion to the whole mass of the water, the small amount of water *mngf* , and the height of the vertical *HB*, that is, $= \frac{\delta}{\beta} a$; therefore there is $v = \frac{\delta}{\beta} a$.



[Recall that the *actual fall* or change downwards in the centre of gravity of the whole mass $M = m\beta \cdot \rho$, is given by $\frac{-ydx}{M} \cdot x$, which is equal to the *potential rise* or *v*; in this case, where ρ is the water density, *m* is the area of cross-section, and the moment of the element changed is $-ydx \cdot x = -\rho m\delta \cdot a$; hence, $v = \frac{\rho m\delta}{\rho m\beta} \cdot a = \frac{\delta}{\beta} \cdot a$: this last result can be viewed in terms of the squares of speeds of the large and small masses of water, or the equality of their kinetic energies in modern terms, where it may be understood that the air bubble is trapped by a small mass of water δ above, which leaps up.]

Indeed this is diminished at once more quickly from the said height and water is forced to flow through the orifice *B*, which I have shown in §.18; but still in the first moment of time the water will maintain the motion that it has acquired, and thus a droplet will be ejected from the orifice with nearly the same velocity, which it will have due to the height $\frac{mm\delta}{nn\beta}a$. But this height not only is three or four times that of *a*, or even some greater amount : certainly for pleasure I have made water jumps of ten or twenty times the height of *a* with a glass tube; e.g., there was $\beta = 100$ feet, $\delta =$ one inch, but the diameter of the tube was ten times the diameter of the opening ; and there was $\frac{mm\delta}{nn\beta} = \frac{10\,000}{1200}$, thus so that in these circumstances the first drop, with the air resistance removed, must jump to a height of more than eight times the customary *a*. There are moreover many impediments and these of the greatest importance, which inhibit immense jumps ; namely a little of the motion is lost from the impulse of the surface of the water *mn* on the side *fg*, then also from the huge friction which the water suffers on being brought so very quickly through the little opening, because it must be very small: much also disappears, by how much less from all its speed, the water *CADm* may move on account of the adhesion of the water to the sides of the tube, which adhesion is very noticeable in such a long course.

Meanwhile there can be no doubt that this is the true solution of the phenomena, and for that solution to satisfy all the experiments I have made in every extent. Then the other momentary nature of the phenomenon also is solved correctly by this theory, because truly that jet shall be as if instantaneous, and after the shortest time interval the jet shall not to be more than usual: thus in the present circumstance, what we have just produced, we may investigate, if in the case that it must be changed a little by the rule §.18 (for there only mention was made about vessels placed vertically), how much water must flow so that the jet shall not be more than a thousandth part above the customary jet (which certainly will be the least to be observed in experiments of this kind), since from the start it was eight times greater than the same, we have found that quantity to be so small that the time, in which the whole was ejected, could not be measured in any way.

Experiments which relate to Section III.

Introduction.

Indeed there are many matters examined in this Section and those particular ones to be considered, which scarcely ever are able to be recalled *immediately* to experiments; and indeed with authors up to the present they have not considered any motion of fluids in the outflow, other than which are made through very small holes, and therefore since there shall be the new theory as we have deduced for the cross-sections of openings of any size, it is the confirmation this itself which would be a great help. But I do not see, how in vertical cylinders, concerning which chiefly we have acted, it is possible to observe the velocity of flowing water, especially when the opening is very large (indeed otherwise some judgment could be made from the time of depletion). Thus considering this carefully at last with our aim being served by paragraphs 16 & 20, in the first of which the maximum velocity was determined of water flowing from a vertically placed cylinder, moreover in the other it was shown, the motion to be the same with cylinders placed obliquely and vertically, if similar heights were assumed in both : therefore conveniently we used obliquely placed cylinders, so that from the maximum cross-section the jet of water could have the maximum velocity, or the height due to the same to be had from experiment: and here indeed by careful reasoning that maximum velocity, such as it actually is, can be examined, even if the openings shall be as large as you wish, which then if it shall be convenient may be observed by our rules, there will be no doubt remaining about our whole theory.

Truly before I shall approach the matter itself, the mechanical theorems which follow will be presented.

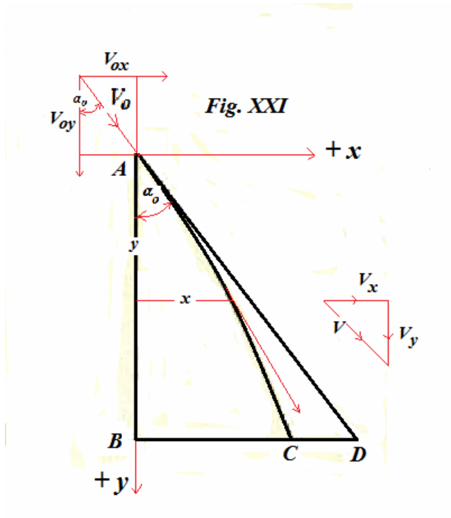
Lemma.

AB (Fig. 21) shall be a vertical line, *BD* horizontal; but the line *AD* may have any direction, under the direction of which the body at *A* may be understood to be projected, describing the parabolic arc AC, the tangent of which truly at *A* is the right line *AD*, the height corresponding to the velocity will be, by which the body at *A* were projected,

$$= \frac{BC^2 \times AD^2}{4AB{\cdot}BD{\cdot}CD}$$

and if *AD* were horizontal or the angle *BAD* right, that same height will become $= \frac{BC^2}{4AB}$.

Now truly I will explain what were observed by me.



[Adapted from KL, p.34 ; Bernoulli considers the parabolic trajectory *AC* in Fig. 21, which a body describes thrown from *A*, as given, and calculates the speed of the parabola backwards from that close to the origin, or the height due to the speed at *A*. As he gives only the final result, it is probably reasonable to find how the data was obtained : In the drawing the horizontal *X*-axis + *x* is positive to the right, and the vertical axis is drawn positive downwards +*Y*, so that at the starting point *A* the abscissa *x* and the ordinate *y* are equal to zero. At an arbitrary point (*x*, *y*) of the parabolic trajectory the speed of which is v, the casting speed V_0 is sought at *A* with the angle α_0 to the y-axis. The speed of the body in now resolved into components along the axis, v_x and v_y :

$$\text{(1) the vertical acceleration } \frac{dv_y}{dt} = g \text{ ; and}$$

$$\text{(2) the horizontal acceleration } \frac{dv_x}{dt} = 0.$$

From (1) it follows that $v_y = gt + C$; for when $t = 0$ $(\text{at } A)$ there is $v_y = v_{0y} = v_0 cos\alpha_0$,

from which $v_y = gt + v_0 cos\alpha_0$; since $v_y = \frac{dy}{dt}$, there is further $dy = gtdt + v_0 cos\alpha_0 dt$,

from which there becomes :

$$(3)\ \ AB = \tfrac{1}{2} gT^2 + v_0 cos\alpha_0 \cdot T.$$

From (2) it follows that $v_x = \text{const.} = v_{ox} = v_0 sin\alpha_0$, and, as above, $x = v_0 sin\alpha_0 t$.

Since $x = BC$ it follows that $BC = v_0 sin\alpha_0 \cdot T$, from which $T = \frac{BC}{v_0 sin\alpha_0}$. One puts T

into (3), then there becomes $AB = \frac{g \cdot BC}{v_0^2 sin^2\alpha_0} + v_0 \cos\alpha_0 \cdot \frac{BC}{v_0 \sin\alpha_0} = \frac{g \cdot BC^2}{2v_0 sin^2\alpha_0} + \frac{BC}{tg\alpha_0}$.

From which there is put in place :

$$ABv_0^2 = \frac{g \cdot BC^2}{2sin^2\alpha_0} + \frac{BCv_0^2}{tg\alpha_0} \text{ or } v_0^2\left(AB - \frac{BC}{tg\alpha_0}\right) = \frac{g.\, BC^2}{2sin^2\alpha_0}$$

and

$$v_0^2 = \frac{2g.\, BC^2 / 2sin^2\alpha_0}{2AB - 2BC / tg\alpha_0}$$ or the speed sought from the height

$$H = \frac{v_0^2}{2g} = BC^2 : \frac{4BD^2}{AD^2(AB - AB \cdot BC / BD)} = \frac{BC^2 \cdot AD^2}{4BD \cdot AB(BD - BC)} = \frac{BC^2 \cdot AD^2}{4BD \cdot BD \cdot CD},$$

as in the text.]

Concerning the maximum velocities of fluids flowing out through very large openings.
Referring to §§. 16 & 20.

First Experiment.

I have placed a cylindrical tube *FA* (Fig. 22) of length four inches obliquely to the horizontal, and I have put it firmly in place there ; but the size of the cross-section of the tube to the cross-section of the opening at *A* shall be as 2 to 1, and indeed the diameter of the tube will be besides equal to nearly seven twelfths of an inch ; then with the measures taken in the particular equations of the lines *FE, AB* and *BD* (of which the law from that figure itself is apparent) there are found : 81, 619, & 740.

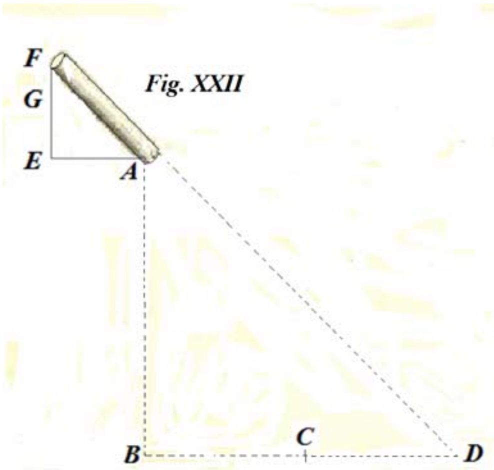


Fig. XXII

Thus with these prepared, I filled the tube with water, with the opening *A* stopped up with a finger meanwhile, and with that quickly removed all the water flowed out in the shortest time: yet I was able to observe, the first and last to fall nearer to the vertical *AB* than the middle; but the drops to be projected furthest to be falling on the place *C* and I have found BC after repeating the experiment more often to be 235 of the parts, of which I had made use before. Now truly if by the preceding lemma the height *EG* may be sought, to which the drops shall be able to ascend with the maximum velocity, there is found to be $EG = 56$ parts ; but it ought to be, by virtue of §§.16 and 20, $= 62$, and I did not expect a greater agreement, unless the friction of the water and its adhesion to the sides of the tube should bring some impediment to the motion.

II. With which put in place as at first, yet with the hole *A* diminished by half, thus so that the cross-section of the tube were four times the cross-section pertaining to the opening, I observed $BC = 252$; hence $EG = 68$ is deduced by experiment; but by theory is should be $= 70$; these numbers are less different than the preceding ones, because here there was much less friction impeding on account of the diminished velocity of the internal water.

Moreover each experiment confirms the theory with outstanding perfection.

Concerning the velocity of the water bursting out of the widest vessel.

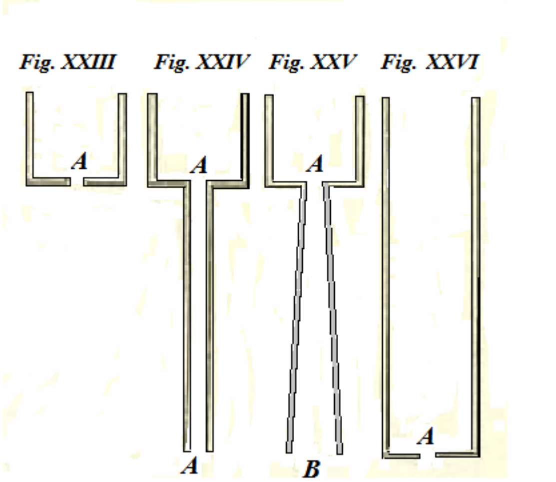
Referring to § .17. We will discuss in this paragraph, if the vessel shall be the widest, soon the water, after the internal surface has fallen some little amount, shall erupt with a velocity which shall correspond constantly to the height of the water above the opening. But in whatever direction you may be allowed to let the water flow out (nor indeed in the widest vessel is any direction of flow able to change the velocity), and you observe at some point of time, at how great a distance from the vertical the stream may strike the horizontal plane, and from that by the previous rule, to ask for the height corresponding there to the velocity of the water flowing out at that instant of time, thus always you will find that height equal to the height of water above the centre of the opening, but only if you remove the first small drops, which by virtue of §. 16 must flow out with a smaller velocity and by that action have gone : nor impediments, of which more often we have made mention, have made any noticeable delay to the flow, but only if the diameter of the opening may be equal to two or three twelfths of an inch, and the diameter of the vessel shall not be less than some inches, and finally the height of the water shall not exceed more than several feet.

I have often tried all these, but the kind of experiment is too simple than it should merit to be described at length.

Concerning vessels constructed from vertical tubes.

Referring to §§. 22 & 23. Concerning these experiments the celebrated 'sGravesande made use of in *Phys. Elem. Math.* 32, which I have repeated; truly these have been made which correspond to the present situation mainly have been used.

Truly in the figures 23, 24, 25, 26 the individual openings equal to each other have been noted be the letter *A*, only with *B* present a little greater in the ratio as 16 ad 25, the cross-sections too, and so that the heights of the cylinders are equal, except the last, the length of which is four times as great: but the tubes connected by the two intermediate cylinders have three times the length of the cylinder. Therefore from these vessels filled with water an observation was made of its efflux.



I. The surface of the water does not fall from the beginning as quickly in Fig. 23 as in Fig. 24; truly after a little water has flowed from each, the motion becomes much quicker in the composite vessel than in the simple one ; and I have predicted each at the end of §. 23. But the situation can be better and more accurately understood from the differential equations which we have given in §§. 22 and 23, if we may use which for the first increments of the motions being found, both in the simple cylinder in Fig. 23 as in the composite one in Fig. 24, and in this at the end we may put the cross-sections of the cylinders to be as *m* ad *n*, the increment will be, that we have called dv in the simple vessel, to the increment in the composite vessel, as $1+\frac{3m}{n}$ to 4, and thus much longer in this case than in that. Hence if the first motion is considered to be perceived correctly, we shall at once be going to observe the greater speed of the other one, which happens in the simple cylinder ; since truly again it was shown in §. 16 and 23, the surface of the water, after it had fallen a little in each tube, to be nearly such, as which may correspond to the heights $\frac{nn}{mm}x$, on being understood by x the heights of the water above the opening, through which it has flowed, it soon follows that the water falls with a much greater velocity in Fig. 24 than in Fig. 23. Therefore the theory clearly agrees with the observations.

[From §. 22, for the first vessel we have :

$$z^{\frac{nn-mm}{nn}}v=C-\frac{nn}{2nn-mm}z^{\frac{2nn-mm}{nn}}-\frac{nnb}{nn-mm}z^{\frac{nn-mm}{nn}}+\frac{mnb}{(nn-mm)}z^{\frac{nn-mm}{nn}}.$$

If we now divide by $z^{\frac{2nn-mm}{nn}}$, we obtain :

$$v = Cz^{\frac{mm-nn}{nn}} - \frac{nn}{2nn-mm}z + K.$$

On differentiation, we find :

$$dv = C\frac{mm-nn}{nn}z^{\frac{mm-2nn}{nn}}dz - \frac{nn}{2nn-mm}dz.$$

For the simple vessel, in Fig. 23, in the designation for the height of the water z and the constant C by $x = a,\ b = 0$ and $g = n$, there becomes :

$$z_a = (x - b + \frac{nnb}{\sqrt{gn}})_a = a \text{ and } C = \left[\frac{annb}{2nn-mm}\right]a^{\frac{nn-mm}{nn}}.$$

Consequently,

$$\frac{dv}{dz} = a\frac{nn}{2nn-mm}\cdot\frac{mm-nn}{nn}a^{\frac{nn-mm}{nn}}a^{\frac{mm-2nn}{nn}} - \frac{nn}{2nn-mm}$$
$$= \frac{mm-nn}{2nn-mm} - \frac{nn}{2nn-mm} = -1.$$

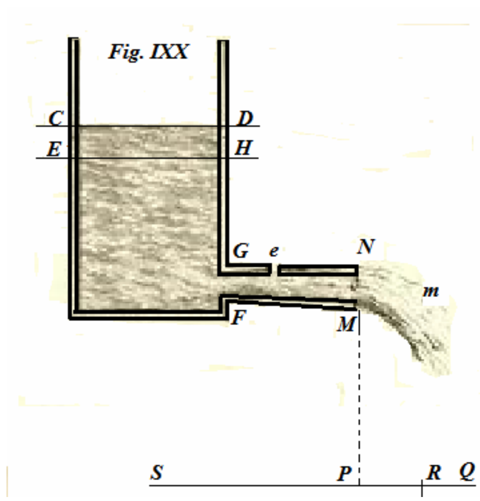
For the second composite vessel, Fig. 24, see KF, page 35, for a similar but lengthy calculation which agrees with the text.]

Il. The surface of the water fall a lot faster in Figure 26 than in 24, thus so that in the case of Fig. 26, it shall be as if in the middle between the cases of Figs. 23 and 26. Truly here it is again apparent, that the initial accelerations are to be much slower in the cylinder in Fig. 24 than in 26. Therefore in this respect, the theory indicates what would be observed; but truly there is not much difference, so that thence just as much shall arise as I have found by experiment, nor ought a greater difference be noticeable, after the surfaces has fallen a little in both places, by §. 23; but the left over difference must be attributed to the impediments which arise from the friction of the water in Fig. 24: for the water is carried off with a greater speed through the tube *AA*, and thus both on account of the increased speed, as well as because of the diminished cross-section of the vessel, the resistance to the motion of the water is most strongly proffered.

III. Finally, if you dismiss the first moment of time, the water surface descents the most quickly in the cylinder in Fig. 25 and that notably faster than in Fig. 26. This truly is in agreement with these principles which have been demonstrated in §. 23 ; moreover there must be soon after the common initial motion, surely on putting the heights of the water above the orifices with almost equal effluxes, the velocities in Figures 25 and 26 to be nearly as the cross-sections of the orifices *B* and *A*, that is, as 25 to 16; and because a small difference in the velocities may be observed, again the resistance due to friction is to be attributed besides the other cause I have indicated at the end of §. 25.

Concerning these vessels, in which horizontal tubes are inserted.

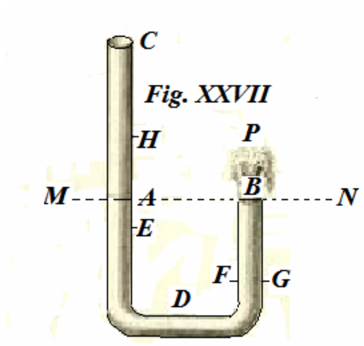
Referring to §.24. When water flows out from a very full vessel such as *CDG* (Fig. 19) through a horizontal tube *GM* wider at the end *NM* than at the beginning *GF*, that shall be carried with a greater speed through the opening *GF* (if again we except the first drops) than if either the tube is missing or it should be cylindrical. That also Frontinus well taught be experience, had emphasized long since to the doubtful, truly to be denied by all modern men.
Therefore as it was worth the effort I was led to explore the matter by experiment. Moreover there was the height of the vessel which I used , above the axis of the tube $=5\frac{1}{2}$ *English inches*, the length of the tube $GN = 2$ and 5/12 inches, the diameter of the opening *GF* was $= 3,36$ twelfths of an inch, the diameter of the opening $MN = 5,48$ twelfths of an inch*;* hence the cross-sections of the openings were nearly as 3 to 8, the cross-section of the vessel was so great, that it could be considered as infinite besides the cross-section of the tube. I wished to select all the measurements, so that it would be possible to repeat any experiment. Moreover with this vessel filled with water I observed the cross-section of the jet, and from that I knew all the required measurements afterwards to be able to calculate from that the height, due to the velocity of the water flowing first across *GF*, then across *NM*: this I found to be eleven twelfth of an inch for the first, and $6 \,\&\, 6\frac{2}{9}/12$ inches in fractions for the second, which I found also to arise from all the other experiments. But since the height $6 \,\&\, 6\frac{2}{9}/12$ inches owed for the speed is greater than $5\frac{1}{3}$ inches, our theory about the acceleration of the internal water from the cross-section of the tube towards the extremity shall be confirmed, however much there should be lacking, so that from the two chosen ratios I have mentioned introduced into §. 25, as it may not actually be accelerated so much by the force according to §. 24 with the obstacles removed, of which no account has been taken in the calculation.



Referring to §. 25. In this paragraph in passing I have warned, that it is possible in many ways, for air to be mixed with the water flowing through the tube. But then it is going to happen, that the water indeed may flow less copiously, but not with a smaller velocity, as I could test initially with the tubes *AA* and *AB* (Fig. 24 & 25), as I made a very small hole in each, not a long way from the same start ; the fact is, that water will flow through the tube with some noise and confusion, moreover the surface is accustomed to fall much more slowly ; then the tube of Fig. 19 equally may be perforated, not far from *G*, and again I have observed, the inner surface to fall a little more slowly, I made sure of this for myself because I counted the number of oscillations of a certain pendulum, during which the surface fell through a given space: but on account of the water flowing I saw a little water to flow from the full hole and then the water to be less than with the customary clarity, but the jet to be made of the ordinary clarity, or with a little more of the ordinary clarity ; but most often the water and the air were carried near each other, the one in the part of the tube placed below the side *FM*, that in the upper near *GN* and then the water to be clear and with the usual speed of ejection rather than being diminished but rather to be much greater, because I had foreseen that it was not possible to be obscured. I will add another experiment set up with greater precision concerning this matter in the next section.

But perhaps a place will be given elsewhere showing that water mixed with a sufficient amount of air nearby to flow out almost abundantly from there, as if it were flowing out there in that place with the tube cut off where it is perforated, to which matter also we may think about a corresponding experiment.

Concerning bending channels.



Referring to §. 27. With the horizontal line *MN* (Fig. 27) drawn on a wall, the whole cylindrical tube *CDB* is filled with water, and having both legs parallel to each other, thus put in place so that one end *B* may just scrape against *MN*, and likewise the legs shall be vertical, while meanwhile I was blocking the opening *C* with a finger, thus restraining the outflow of the water.

Then with the finger removed I observed the maximum height *BP*, to which the out flowing water was rising, and in turn on other occasions I attended to the place *E,* to which the water surface dropped; but I performed the experiment under two different circumstances ; for initially I had placed no cover over *B* ; then I used a cover perforated by a hole of such a size, that it had a cross-section in the ratio to the cross-section of the tube to be as 1 to $\sqrt{2}$. Meanwhile the measurements were such : $CA = 345;\ ADB = 530;\ BP = 33;\ \ \&\ AE = 88$ small parts, of which 375 were equal to the length of a London foot. Thus there were in the first case, but in the other with the rest remaining I saw that $BP = 64\ \ \&\ AE = 54$. Here I noted in passing, that wishing to explore the maximum descent *AE* in another way, after the end of the experiment I had inclined the tube, then water was seen once again to have flowed out through the nearest opening *B*, at which moment of time I measured the distance of the surface from the position *A* noted before ; that distant, the same that I was about to consider with the maximum descent *AE*, with the opinion it should be far less ; from which I was thoroughly taught that the part of the water, which in the experiment had flowed out through *B*, on the contrary was now flowing back in.

Thus from these observations, I sought the magnitudes *BP* and *AE* from the calculation according to the standard §. 27, by putting *m* initially $= n$,then by putting $mm = 2nn$ *;* but to have found in the first case $BP = 79$, which in the experiment did not exceed 33, and the maximum descent *AE* I found to be nearly $= 250$, that the experiment gave 88; then in the case for $mm = 2nn$, *BP* besides arose near the double of that which was observed, and $AE = 186$, which was to have observed 54 of the small lengths.

These immense maximum differences I attribute in part to the adhesion of the water to the walls of the tube, which adhesion certainly in cases of this kind can exert an incredible effect ; for the tube I used having scarcely $\frac{1}{6}$ of an inch in diameter, certainly greater agreement was going to be found with a tube of greater cross-section. Meanwhile it is plausible that the curvature of the tube in the lower part also takes away a little of the motion.

Referring to §. 28. With the same curved tube, that I have used in the manner described: but I have put the cover at *B* to be bored through with the smallest hole : I had arranged it so that the whole should be filled with water except the small part *FGB*, in which situation I have kept the water with the aid of a finger applied to the opening *C*. With the finger removed the water falls, and when it should arrive at the position *HDB*, some drops passed through the small hole at *B* as if they had exploded, so that they rose to a height greater than ten feet, although the height *HA* scarcely exceeded a height of half a foot. Meanwhile on account of the extreme smallness of the opening, the water encounters so much resistance while it passes through the opening, so that not a single fraction ascends to the height AH from the impetus of the water (upon which still with all the impediments removed it ought to continue a little), but scarcely one drop or two was expressed with a notable delay in the time, thus so that I could persuade myself, if without the impetus, but only by the natural pressure of the water, so great a leap were required to be produced, that would not be able to happen unless from a minimum height of one hundred feet.

Then also I had observed the throw of the water to be diminished more or less there before the trial had left the space *GB;* with which all the theories are in agreement. Taking measurements would be superfluous, because on account of the excessive impediments certainly so many prevent the jet of water, so many of these would need to be removed. Nevertheless so that I may confirm these to agree with the formulas of the experiment, I have taken the tube *CDB* wider, so that I may have removed the main part of the impediments, the part *DFB* was unimportant, and also the small part *GB,* as in the experiment I left empty without water and finally a cover was bored through by a not exceedingly small hole. And then I saw a leap not exceedingly short of the height $\frac{mm\delta}{nn\beta}a$, as I gave in §. 28 for this affair, indeed I remember to have predicted correctly the height of the jump to a friend present, after careful consideration about how much should be given in the calculation, besides the nearby impediments.

You will obtain readily a similar momentary water explosion and that from a similar cause arising with fountains, which are ejected through the opening of a delivery pipe full of water. For if you suddenly put a finger on the opening of the pipe, so that a part of the opening may remain, you will see at once water to be expelled with great impetus, and soon the narrow jet is reduced to the limit of its initial velocity. Also you will notice the water to be projected from that with a greater velocity and further when you leave a smaller opening with the finger, and for that same opening remaining, the unusual jet from that is more protracted (though always very short) and to become more readily observed, so that the longer the pipe shall be, thus as in leaping fountains, to which the water is carried from a reservoir through a long conduit, if the conduits should not be exactly full and the water flows out from an orifice, there is no doubt why over a notable space of time thus a strong jet of water can be projected, being reduced to its customary velocity gradually : All these are in agreement with those matters, which § §. 28 & 18 had foretold.

This experiment was conducted initially some time ago by me and in turn I remember in the presence of the most venerable and celebrated masters De Maupertuis & Clairaut, with whom before I had fallen by chance into a discussion about these matters with water. But although here there shall be no air which can be accused, actually still this phenomenon does not differ from that, which D. de la Hire had noted, and each may come about from this, because the motion of the water contained in the conduit, or at any rate a part of its motion, cannot cease without some effect thence arising, and that itself consists of an enormous jet of water.

HYDRODYNAMICAE SECTIO TERTIA.

De velocitatibus fluidorum ex vase utcunque formato per lumen qualecunque effluentium.

§.1. Priusquam motum aquarum a gravitate propria ortum definire tentemus, ruminabimur quod in Sectione prima §§.18, 19, 20, 21 & 22 a nobis allatum fuit de principiis ad hoc adhibendis.

Recordabimur nempe *ascensum potentialem* Systematis, cujus singulae partes velocitate qualicunque moventur, significare altitudinem verticalem, ad quam centrum gravitatis illius Systematis pervenit, si singulae particulae motu sursum converso sua velocitate, quantum possunt, ascendere intelligantur, & *descensum actualem* denotare altitudinem verticalem, per quam centrum gravitatis descendit, postquam singulae particulae in quiete fuerant. Tum etiam memores erimus necessaria *ascensum potentialem* aequalem esse *descensui actuali,* quando omnis motus in materia substrata haeret, nihilque de eo in materiam insensibilem aut aliam ad systema non pertinentem transit, & denique motum fluidorum talem proxime esse, ut ubique velocitas reciproce sit proportionalis amplitudini vasis respondenti, qua de re suo loco alia quaedam interjiciemus. Nunc convenit examinare sequentem propositionem.

Problema.

§. 2. Si aqua per canalem utcunque formatum fluat, ejusque velocitas cognita sit aliquo in loco, invenire *ascensum potentialem* omnis aquae in canali contentae.

Solutio.

Sit canalis utcunque formatus *ST* (Fig. 13 & 14) per quem aqua fluit *bcfg;* assumitur, si in axe *ae* accipiatur punctum quodcunque *n,* per quod planum ad axem perpendiculare *pm*

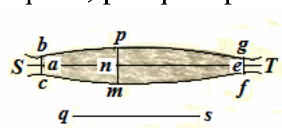
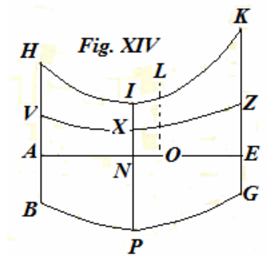


Fig. XIII

transeat, fore, ut omnes particulae aqueae in illo piano existentes aequali velocitate fluant, & quidem tali, quae sit ubique reciproce proportionalis magnitudini sectionis *pm.* Sit



autem velocitas aquae in *gf* talis, quae debetur altitudini verticali *qs,* id est, sit *ascensus potentialis* strati aquei in *gf* aequalis lineae *qs*, & quoniam hujusmodi altitudines sunt in ratione quadrata velocitatum, sequitur esse *ascensum potentialem* aquae in *pm* aequalem quartae proportionali ad quadratum amplitudinis *pm*, quadratum amplitudinis *gf* & altitudinem *qs,* nempe $=\frac{gf^2}{pm^2}\times qs.$ His ita praemonitis ponemus in Figura decima quarta esse curvam *BPG* scalam amplitudinum canalis, ita ut posita $AN = an$, denotet *NP* amplitudinem in *pm:* dein curvam *HIK* esse scalam *ascensuum potentialium,* ita ut sit $NI=\frac{EG^2}{NP^2}\times qs.$ Fingatur nunc elementa singula curvae *HIK* habere pondus aequale ponderi strati aquei respondentis, & cadere centrum gravitatis istius curvae in punctum *L,* & ducatur *LO* perpendicularis ad axem *AE;* sic erit *LO ascensus potentialis* totius aquae quaesitus. Ex mechanicis autem constat, si fiat tertia curva *VXZ,* cujus applicata *NX* sit ubique aequalis $\frac{EG^2}{NP}$, fore *LO* aequalem quartae proportionali ad spatium *AEGB* & *AEZV* atque lineam *qs* vel *EK.* Patet igitur quaesitum. Q. E. I.

§. 3. Fuerit v. gr. canalis conicus, in quo superficies anterior *gf* & posterior *be* diametros habeant ut *m* ad *n,* erit *ascensus potentialis* aquae

$$=\frac{3m^3}{n(mm+mn+nn)}\times qs.$$

Problema.

§. 4. Datis variationibus infinite parvis tam ratione situs quam velocitatis, quae superficiei aquae anteriori respondent, invenire variationes ad *ascensus potentiales* totius aquae pertinentes.

Solutio.

Sit spatium $AEGB = M$, spatium $AEZV = N$, $qs = v$, erit *ascensus poten.* $=\frac{Nv}{M}$: quia vero quantitas aquae in canali constanter eadem ponitur, erit spatium *AEGB* invariabile, adeoque $dM = 0$, ita ut differentiale *ascensus potent.* sit simpliciter $= \frac{Ndv + vdN}{M}$ habetur autem *dN* ex variatione situs aquae. Patet igitur propositum. Q. E. I.

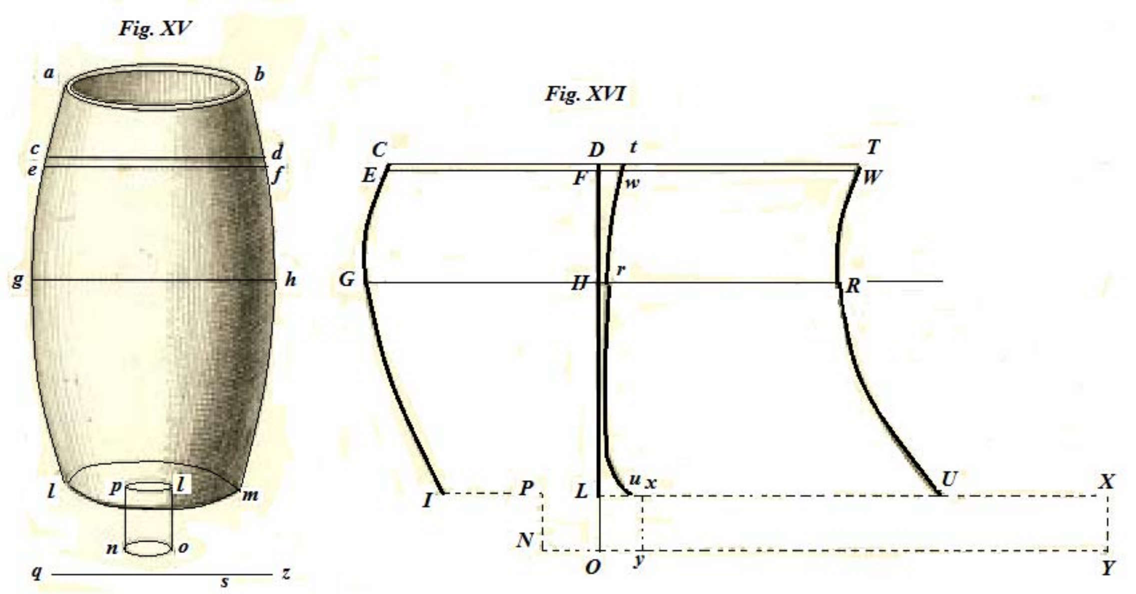
Scholion.

§. 5. Poterunt hae propositiones inservire pro motu fluidi intra vasa moti, id est, non effluentis definiendo, uti suo loco ostendam: at vero cum fluidum per foramen effluit, aptius instituetur aliter calculus, nempe ut sequitur.

Problema.

§. 6. Invenire differentiam *ascensus potentialis* postquam guttula per foramen effluxit.
Solutio.

Fingamus aquam effluere ex vase *aimb* (Fig. 15) utcunque formato, fundum sit *im* perforatum foramine *pl:* quantitas aquae, postquam jam data ejus quantitas effluxit,



residua in vase sit *cimd;* effluat autem tempusculo infinite parvo guttula *pnol,* superficie *cd* descendente in situm *ef:* concipiatur in medio aquae sectio *gh* parallela superficiebus *cd* vel *ef* ipsique fundo *im;* sitque velocitas unius cujusvis particulae in *gh* talis, ut possit ascendere ad altitudinem *qs* seu *v*, cum nondum effluxit guttula, & ad altitudinem *qz* sive $v + dv$, postquam ea ipsa guttula effluxit. Omnibus his ita positis, quaeritur incrementum *ascensus potentialis* aquae postquam situm *cimd* commutavit cum situ *eipnolmf,* id est, postquam guttula emanavit.

Fiat, ut antea, curva *CG* (Fig. 16) ceu scala amplitudinum, ubi adeoque *CD* vel *EF* repraesentabunt magnitudinem superficiei aqueae ante vel post effluxum guttulae, *GH* amplitudinem illam assumtam, *IL* magnitudinem fundi, *PL* magnitudinem foraminis, dum adhaerens parallelogrammum minimum *PNOL* respondet guttulae cylindricae *pnol:* dein construatur alia curva *TRU*, cujus applicatae sint rursus aequales quadrato lineae *GH*, diviso per applicatam respondentem curvae *CGI*, cui curvae eadem conditione annexum est parallelogrammulum *LOYX*, cujus nempe latus *LX* est aequale quadrato lineae *GH* diviso per lineam *PL*. Jam igitur apparet *ascensum potent* aquae ante effluxum guttulae esse $=$ quartae proportionali ad spatium *DCIPL*, spatium *DTUL* & altitudinem *qs*, eundemque post effluxum guttulae esse $=$ quartae proportionali ad spatium *FEIPNOL*, spatium *FWUXYOL* & altitudinem *qz:* sunt autem in utraque analogia termini primi (nempe spatium *DCIPL* & spatium *FEIPNOL*) inter se aequales, igitur si quodvis horum spatiorum indicetur per *M*, spatium *DTUL* per *N*, spatium *FWUXYOL* per $N+dN$, altitudo *qs* per *v* & *qz* per $v+dv$, erit incrementum *ascensus potentialis* durante guttulae effluxu

$$=\frac{Ndv+vdN}{M}.$$

Quod si nunc ponatur

$$LD = x,\ FD = -dx,\ DC = y,\ HG = m,\ PL = n, \text{erit } DT = \frac{mm}{n}, LX = \frac{mm}{n}, LO = \frac{-ydx}{n}$$

(quia spatium $DFEC =$ spatio $LONP$), hincque

$$dN = LOYX - DFWT = -\frac{mmydx}{nn} + \frac{mmdx}{y},$$

unde nunc incrementum quaesitum *ascensus potentialis* est

$$=(Ndv - \frac{mmvydx}{nn} + \frac{mmvdx}{y}) : M.$$

Q.E.l.

Problema.

§. 7. Retentis iisdem positionibus invenire *descensum actualem* infinite parvum aquae, dum guttula effluit.

Solutio.

Cum in Figura decima quinta aqua situm *cdmi* mutat cum situ *efmlonpi*, patet in utroque situ centrum gravitatis partis aquae *efmi* in eodem loco esse, posseque proin concipi solam particulam *cdfe* (quae est $= ydx$ dum tota aquae massa $= M$) descendisse in *lonp.* Sit jam altitudo particulae aqueae *cdfe* supra guttulam *lonp* $= x$, altitudo centri gravitatis aquae *efmi* a fundo $= b$, erit altitudo centri gravitatis omnis aquae in situ *cdmi* supra fundum $= b - \frac{ydx}{M} \times (x-b)$ & in situ *efmlonpi* erit eadem altitudo $= \left(\frac{M+ydx}{M}\right) \times b$*;* unde differentia altitudinum seu *descensus actualis* quaesitus

$$= -\frac{ydx}{M} \times x,$$

quae aequatio indicat, guttulam quae effluxerit multiplicandam esse per altitudinem aquae supra foramen, productumque dividendum per quantitatem aquae, ut habeatur *descensus actualis*, qui fit dum guttula effluit. Q. E. I.

Problema.

§. 8. Determinare motum fluidi homogenei ex vase dato per foramen datum effluentis.

Solutio.

Quoniam per hypothesin nostram *ascensus potentialis* singulis momentis aequalis est *descensui actuali*, erit incrementum prioris dum guttula effluit aequale incremento posterioris, quod simili tempusculo oritur. Igitur si rursus superficies aquae, postquam data ejus quantitas effluxit, ponatur $= y$, amplitudo vasis quocunque in loco ad libitum assumta $= m$, amplitudo foraminis n, altitudo aquae supra foramen $= x$; si praeterea quantitas N ea lege construatur, quae§. 6 indicata fuit, atque per v intelligatur altitudo debita velocitati aquae in loco assumto, ubi nempe amplitudo vasis est $= m$, erit per §. 6 incrementum *ascensus potentialis*

$$= (Ndv - \frac{mmvydx}{nn} + \frac{mmvdx}{y}) : M,$$

minimusque *descensus actua is* $= -\frac{yxdx}{M}$ (per praeced. §), unde habetur

$$(Ndv - \frac{mmvydx}{nn} + \frac{mmvdx}{y}) : M = -yxdx : M$$

seu

$$Ndv - \frac{mmvydx}{nn} + \frac{mmvdx}{y} = -yxdx,$$

quae aequatio generaliter integrari potest, quandoquidem litterae N & y sunt functiones datae ipsius x & litera v unius tantum dimensionis est.

Corollarium 1.

§. 9. Quum velocitates sint in ratione reciproca amplitudinum, patet fore altitudinem, quae velocitati aquae effluentis respondet, $= \frac{mm}{nn} v$, quae proin si vocetur z, erit

$$nnNdz - mmzydx + \frac{mmnnzdx}{y} = -mmyxdx.$$

Corollarium 2.

§. 10. Si foramen sit valde parvum, ratione amplitudinum vasis, fit $n = 0$, totaque aequatio abit in hanc

$$-mmzydx = -mmyxdx \text{ vel } z = x;$$

tunc igitur aqua ea constanter effluit velocitate, qua ad altitudinem supremae superficiei usque ascendere possit, quem solum casum Geometrae hactenus fuerunt recte assecuti: valetque haec propositio pro omnibus vasis utcunque formatis: at cum foramen non ut infinite parvum consideratur, nequaquam negligenda est vasis figura. Notari tamen potest, quod nisi foramen sit amplissimum, sine notabili admodum errore idem ut infinite parvum considerari possit.

Corollarium 3.

§. 11. Cum fluidum non est ubique idem, simili modo instituendus est calculus, inquirendo nimirum tum in incrementum *ascensus potentialis* fluidi compositi, tum in *descensum actualem*, eaque inter se aequando. Quod si autem foramen sit valde parvum, per se patet, quod etiam calculus ostendit, fore ut fluidum velocitate exillat altitudini debita tali, ut si vas ad eandem altitudinem liquore eodem, qui exilit, repletum sit, eandem pressionem latera foraminis sustineant.

Scholium Generale.

§.12. Priusquam Corollaria specialiora ex theoria nostra deducamus circa motum fiuidorum ex vasis cylindricis, conveniet hic examinare, quousque hypotheses assumtae cum rei natura conspirent & quaenam aliae intervenire possint causae, quarum in computo nullam rationem habuimus, motum fiuidum diminuentes.

Quod primo attinet ad Principium *conservationis virium vivarum* seu *perpetuae aequalitatis inter ascensum potentialem descensumque actualem*, nihil hic video, quod ei notabili impedimento esse possit, si modo a frictionibus, tenacitate, aëris resistentia hujuscemodique aliis obstaculis mentem abstrahamus. Saepe quidem fit, ut principium istud non sine limitatione adhiberi possit, quod in sequentibus ostendemus, nempe cum particulae aquae motu singulae diverso feruntur, quo fit ut singulis momentis aliquid de motu, vel si mavis de *ascensu potentiali*, perdatur. Sed in praesenti casu nihil simile accidit, quandoquidem omnes particulae similiter fere moventur & praesertim, quando foramen est valde parvum, motus particularum internarum fere nullus est, nihilque adeo inde detrimenti venire potest. Alterum autem principium, quo assumitur velocitatem cujuslibet particulae eam esse, quae respondet inversae rationi amplitudinum, duplici quidem laborat incommodo, *primo* nempe, quod motus circa latera vasis tardior paulo sit quam in medio nec proin omnes particulae eidem amplitudini vasis respondentes aequali velocitate ferantur, & *secundo,* quod aqua a fundo non admodum remota motum, quem principium hoc postulat, habere non possit: Utrumque autem nullum sensibilem errorem post se trahit, quando in hoc problemate simplici figura vasis interna nihil fere ad motum aquae effluentis attineat. Ex eadem ratione intelligitur non multum diversum esse posse motum aquae sub alia quacunque directione effluentis, quia scilicet motus aquae internus in ima vasis parte tantum diversus fit, haecque diversitas nullius momenti fere esse potest. Apparet ergo hypotheses, quibus calculus nostri hujus Problematis innititur, ita convenire cum natura quaestionis, ut error inde nullus sensibus perceptibilis oriri possit. At vero impedimenta supra memorata, attritus, tenacitas fiuidi aliaque similla majoris efficaciae sunt, praesertim cum foramen, per quod fiuida exiliunt, perquam exiguum, aut altitudo aquae supra foramen admodum magna, aut denique tubus valde gracilis est, qua de re experimenta plurima extant apud Mariottum in *Tract. de mot. aquarum.* Jam vero progredior ad examinandum motum aquarum ex vasis cylindricis per foramina cujuscunque magnitudinis effluentium. Vasa autem compendii & elegantioris solutionis causa considerabimus verticaliter posita.

De his quae pertinent ad effluxum aquarum ex cylindris verticaliter positis, per lumen quodcunque, quod est in fundo horizontali.

§.13. Geometrae, quibus de aquis ex vase erumpentibus sermo fuit, considerare potissimum solent cylindros verticaliter positos: Igitur haud abs re erit ex theoria nostra generali consectaria illa, quae huc pertinent, deducere. Sit amplitudo cylindri ad amplitudinem foraminis ut m ad n*;* altitudo aquae supra foramen, cum fluxus incipit, $= a$ *;* altitudo aquae residuae $= x$, altitudo velocitati aquae internae debita $= v$ *;* erit in aequatione canonica paragraphi octavi $y = m$, $N = mx$ (per§. 6), quae adeoque abit in hanc aequationem,

$$mxdv - \frac{m^3}{nn}vdx + mvdx = -mxdx,$$

vel

$$\left(1 - \frac{mm}{nn}\right)vdx + xdv = -xdx;$$

multiplicetur haec posterior aequatio per $x^{-\frac{mm}{nn}}$, ut habeatur

$$\left(1 - \frac{mm}{nn}\right)x^{-\frac{mm}{nn}}vdx + x^{1-\frac{mm}{nn}}dv = -x^{1-\frac{mm}{nn}}dx.$$

Potest jam haec aequatio integrari: observanda autem est in integratione constantis additio, talis nempe, ut a fluxus initio, id est, cum $x = a$, sit velocitas fluidi nulla, ipsaque proin v partier $= 0$: ita vero oritur:

$$x^{1-\frac{mm}{nn}}v = \frac{nn}{2nn - mm}\left(a^{2-\frac{mm}{nn}} - x^{2-\frac{mm}{nn}}\right)$$

vel

$$v=\frac{nna}{2nn-mm}\left(\left(\frac{a}{x}\right)^{1-\frac{mm}{nn}}-\frac{x}{a}\right).$$

§. 14. Ex hac igitur aequatione cognoscitur altitudo generans velocitatem aquae *nn* internae; ubi notari meretur, si vas sit amplissimum, mox posse censeri $v=\frac{nn}{mm}x$, postquam scilicet vel tantillum descendit aqua, id est, statim ac *x* paulo minor est quam *a*. Regula haec fallit notabiliter tantum circa primum motus initium & si primum istud motus elementum consideratur (quo nempe altitudo $a-x$ ut infinite parva censeri potest) indicat aequatio, esse tunc $v=a-x$. Unde sequitur, in omni cylindro, quodcunque fuerit foramen, aquam internam instar corporum libere cadentium accelerari ab initio motus. Si vero motus aliquantulum continuet, eo minus fallet haec Regula, quo majus fuerit foramen, & quo altior est aqua in tubo; si porro desideretur altitudo ea, quae velocitati aquae effluentis respondeat, quam § 9. posuimus $=z$, erit $z=\frac{mm}{nn}v$, seu

$$z=\frac{mma}{2nn-mm}\left(\left(\frac{a}{x}\right)^{1-\frac{mm}{nn}}-\frac{x}{a}\right).$$

§.15. Cum *n* est $=m$, id est, cum nullum est fundum, apparet ex ipsa rei natura, aquam instar corporum gravium libere cadere atque accelerari; id ipsum autem indicat etiam aequatio; fit enim in hac positione $z=a-x$. Si vero foramen est vel uti infinite parvum ratione amplitudinis vasis, quem casumjam supra consideravimus, ponendum est $n=0$, & tunc fit $z=x$, quod indicat, aquam ea constanter effluere velocitate, qua ad totam aquae altitudinem ascendere possit. Denique cum $mm=2nn$, prodit $z=\frac{mm}{0}(x-x)$, ex quo valore cum nihil cognosci possit, descendendum est ad aequationem differentialem §.13, quae nunc haec est:

$$-vdx+xdv=-xdx,$$

vel

$$\frac{xdv-vdx}{xx}=-\frac{dx}{x},$$

quae integrata cum debitae constantis additione dat

$$\frac{v}{x}=\log\frac{a}{x},$$

vel

$$v=x\log\frac{a}{x},$$

aut
$$z = 2v = 2x\log\frac{a}{x}.$$

§.16. Velocitas aquae effluentis ab initio crescit posteaque decrescit, estque alicubi maxima, nempe eo in loco, quo aqua descendit ad altitudinem

$$a:\left(\frac{mm-nn}{nn}\right)^{nn:(mm-2nn)};$$

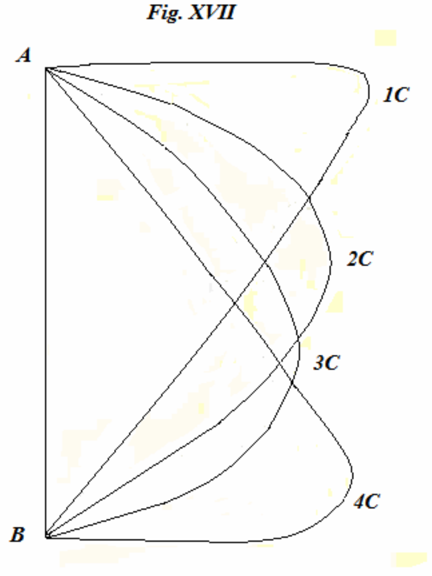
id quoque experientia edoctus indicavit Mariottus in *Tract. de motu aquarum, part.* 3, *disc.* 3, *exp.* 5, ipsaque velocitas maxima talis est, quae debetur altitudini

$$\frac{mma}{mm-2nn}\times\left(\left(\frac{nn}{mm-nn}\right)^{nn:(mm-2nn)}-\left(\frac{nn}{mm-nn}\right)^{(mm-nn):(mm-2nn)}\right),$$

quae quantitas reducta fit

$$=\frac{mma}{mm-nn}\times\left(\frac{nn}{mm-nn}\right)^{nn:(mm-2nn)}.$$

Intelligitur ex istis formulis tempus, quo velocitas a nihilo in maximam vertitur, plane imperceptibile esse, quando foramen vel mediocriter parvum tubusque non admodum longus est: notabile autem fieri, cum res secus se habet, quod videmus in fontibus salientibus, ad quos aquae per longos tractus vehuntur; haec vero quae ad tempora pertinent, magis in sequenti sectione explicabuntur, atque simul ostendetur, quam parum aquae ex vasis amplissimis ejiciatur, priusquam maxima velocitate effluant.
Natura velocitatum melius intelligitur ex apposita Figura decima septima, in qua si *AB* repraesentet totam altitudinem fluidi supra foramen ab initio fluxus, expriment curvae *A1CB, A2CB, A3CB, A4CB* scalas altitudinum respondentium, ad quas fluidum effluens sua velocitate ascendere possit in diversis foraminum magnitudinibus: nempe scala accedet ad figuram A1CB, si foramen habeat exiguam rationem ad vasis amplitudinem, & ad figuram A2CB, cum assumitur fundum majori lumine perforatum; & si jam ratio foraminis sit ad amplitudinem vasis ut 1 ad $\sqrt{2}$, erit scala illa ut *A3CB* (quo in casu minor fit maxima velocitas quam in quocunque alio, estque nominatim ea quae debetur altitudini $\frac{2a}{c}$, intelligendo per *c*

Fig. XVII


numerum c cujus Logarithmus est unitas, id est, altitudini paulo minori quam $\frac{3}{4}a$) ac denique erit scala ut *A4CB* cum fere nihil fundi superest.

§. 17. Jam vero exemplo quodam illustrabimus, quod supra §.10 indicatum fuit, nempe nisi foramen sit amplissimum, posse id sine valde sensibili errore in calculo considerari ut infinite parvum, atque adeo assumi, $z = x$ ut §§.10 & 15 dictum fuit. Videtur id tantum apud nonnullos Auctores valuisse, ut censuerint, nullam magnitudinis in foramine rationem unquam esse habendam, quantumvis magnum ponatur foramen, quae res certe ridicula est: saltem nemo hactenus quod sciam magnitudinem foraminis pro hoc negotio recte consideravit. Ponamus igitur cylindrum, cujus diameter quadrupla tantum sit diametri foraminis, cujusmodi magna foramina in instrumentis hydraulicis raro occurrere solent, & fingamus superficiem aquae per centesimam partem descendisse tantum totius altitudinis initialis (descendisse autem aliquantulum assumo, quia a primo initio motus aquae nullus inesse potest, nedum tantus, ut aqua effluens ad totam altitudinem ascendere motu suo possit); hae positiones faciunt $m = 16\text{n}$ & $mm = 256nn$, atque $x = \frac{99}{100}a$, unde prodit

$$z = \frac{128}{127}\left(\frac{99}{100} - \left(\frac{99}{100}\right)^{255}\right)a = \frac{92}{100}a,$$

quae quidem aliquantulum differt a quantitate x, seu $\frac{99}{100}a$, sed tamen non multum admodum, fitque differentia multo minor, cum minus est foramen, & paullo magis descendit superficies aquae. Igitur differt haec Theoria a vulgari potissimum circa fluxus initium, quo minor est motus, quam statutum fuit: e contrario circa fluxus finem majori velocitate aqua ejicitur, quam secundum principia solita deberet.

§.18. Hactenus consideravimus motum aquae a propria sua gravitate ortum; ponamus nunc vi aliena aquam ejectam fuisse praeter vim gravitatis, talemque aquae effluenti communicatam fuisse velocitatem, qua ad altitudinem multo majorem ascendere possit, quam si sola aquae gravitas motum produxisset; dein subito vim illam alienam evanescere, & aquam sibi relinqui; id autem si fiat, experientia docet citissime aquae velocitatem decrescere & mox talem esse, ut notabiliter non superet velocitatem eam, quae ex sola aquae gravitate oritura fuisset. Ita videmus fieri aliquando in fontibus salientibus (de cujus rei causa vera atque mensura alibi dicam) ut aquae ad triplam vel quadruplam majoremve altitudinem assiliant, quam est ordinaria; quod cum ita contingit, saltus iste protinus cessat solitamque altitudinem, quantum id sensibus percipi potest, non excedit: loquor autem de tubis foraminibus non valde magnis perforatis; nam cum foramen est aliquanto majus, non ita cito decrescit aquae saltus. Jam itaque examinabimus, quousque theoria cum istis phaenomenis conveniat, accuratasque mensuras eorum, quales inde sequuntur, subjungemus. Ut vero rem generaliter prosequamur, ponemus rursus amplitudinem cylindri ad amplitudinem foraminis ut m ad n: aquam ea explodi velocitate qua assurgere possit ad altitudinem α , eoque ipso temporis puncto altitudinem aquae supra foramen esse $= a$, cujus sola gravitas nunc aquam expellat; deinde descendere superficiem aquae in Cylindro per altitudinem verticalem $a-x$, ita ut altitudo residua sit $=x$ & tunc velocitatem aquae ejectae talem esse, quae debeatur altitudini *z*. His ita positis utemur aequatione generali differentiali §. 9, quae haec est

$$nnNdz - mmzydx + \frac{mmnnzdx}{y} = -mmyxdx$$

(ubi rursus, ut §.13 indicatum fuit, est $y=m$ & $N=mx$) quaeque in casu nostro particulari talis fit

$$\left(1-\frac{mm}{nn}\right)zdx + xdz = -\frac{mm}{nn}xdx,$$

quae multiplicata per $x^{-\frac{mm}{nn}}$ posteaque sic integrata, ut posita $x=a$ fiat $z=\alpha$, dabit aequationem desideratam finalem

$$z = \left(\frac{mm}{2nn-mm} + \frac{\alpha}{a}\right)x^{\frac{2nn-mm}{nn}} \times x^{\frac{mm-nn}{nn}} - \frac{mm}{2nn-mm}x$$

vel
$$z = \frac{mma}{2nn-mm}\left(\left(\frac{\alpha}{a}\right)^{1-\frac{mm}{nn}} - \frac{x}{a}\right) + \left(\frac{x}{a}\right)^{\frac{mm-nn}{nn}}\alpha$$

quae altitudo si comparetur cum illa, quae paragrapho 14 indicata fuit, invenitur excessus unius super alteram

$$= \left(\frac{x}{a}\right)^{\frac{mm-nn}{nn}}\alpha;$$

unde jam omnia ea confirmantur Phaenomena, quae modo indicata fuerunt; excessus enim iste, cum *m* numerus est multo major quam *n*, insensibilis statim fit, postquam aqua vel tantillum descendit, id est, post brevissimum temporis spatium, nunquam tamen omnis evanescit, quam diu durat fluxus, & denique eo notabilior continue est, quo magis ratio numeri *m* ad *n* ad aequalitatem accedit. Fuerit v. gr. diameter tubi decies major diametro foraminis, expellaturque aqua vi tali, ut velocitate sua assilire possit ad altitudinem quae sit quadrupla altitudinis *a* seu aquae supra foramen, quaeritur ad quam altitudinem sua velocitate aqua effluens ascendere poterit, postquam per millesimam partem ipsius *a* superficies aquea descendent in tubo, si interea aqua sola propria gravitate ad effluxum solicitetur, dein quaenam similis altitudo futura fuisset, si aqua nullum motum ab initio habuisset: est autem
$m=100\text{n}, mm=10\,000nn,\ x=\frac{999}{1000}a,\ \alpha=4a$, unde in priori casu fit

$$z=\left(\frac{10\,000}{9998}\left(\frac{999}{1000}-\left(\frac{999}{1000}\right)^{9999}\right)+4\left(\frac{999}{1000}\right)^{9999}\right)a,$$

sive

$$z=\frac{99915}{100\,000}a+\frac{18}{100\,000}a,$$

in posteriori casu autem fit

$$z=\frac{99915}{100\,000}a$$

ex quo exemplo patet, quam exiguus & plane insensibilis sit excessus prioris altitudinis supra alteram, & quam cito diminuatur jactus ille aqueus, quandoquidem tota mutatio fiat, dum superficies aquae per millesimam partem altitudinis *a* descendit, quod tempus in machinis hydraulicis solitis non potest non esse admodum breve. Tum etiam confirmatur, quod supra paragrapho 17 dictum fuit, esse scilicet proxime $z=x$, quando foramen est vel mediocriter parvum, cum in praesenti casu, ubi motus a quiete incipit, differentia inter *z* & *x* sit tantum quindecim centies millesimarum partium ipsius altitudinis *a;* quoniam interim paululum major est altitudo *z* quam *x*, patet ad majorem altitudinem ascendere posse aquam effluentem, postquam aliquantisper effluxit aqua, quam est altitudo aquae supra foramen.

§. 19. Postquam sic ex Theoria nostra generali deduximus, quae motum fluidorum ex cylindris verticaliter positis spectant, jam etiam considerabimus tubos oblique positos, qui praelongi esse solent in fontibus salientibus. In his enim id singulare est, quod acceleratio motus non ita repente fiat, veluti cum cylindri sunt verticales, atque sic liceat sensibus percipere consensum Theoriae cum motu aquarum reali.

§. 20. Fingamus canalem utcunque incurvum, sed tamen cylindricum, cujus amplitudo habeat rursus ad amplitudinem foraminis rationem *m* ad *n*. Incipiat motus a quiete, sitque altitudo verticalis aquae supra foramen ab initio motus $=a$; effluxerit certa aquae quantitas, ponaturque altitudo verticalis aquae residuae supra foramen $=x$, longitudo canalis, quae eo ipso momento plena est, $=\xi$, habeatque tunc aqua interna (cujus singulas particulas motu axi canalis parallelo ferri hic assumo) velocitatem, quae respondeat altitudini *v;* his ita positis, si simili ratiocinio utamur quo supra, quaerendo nimirum incrementum *ascensus potentialis* dum guttula effluit, uti paragrapho 6 fecimus, idemque ponendo = *descensui actuali*, obtinetur nunc talis aequatio

$$\xi dv-\frac{mm}{nn}vd\xi+vd\xi=-xd\xi,$$

sive

$$\left(1-\frac{mm}{nn}\right)vd\xi+\xi dv=-xd\xi,$$

cujus integralis, quod patet multiplicatis terminis per $\xi^{-\frac{mm}{nn}}$, , haec est

$$v=\xi^{\frac{mm}{nn}-1}\int -x\xi^{-\frac{mm}{nn}}d\xi.$$

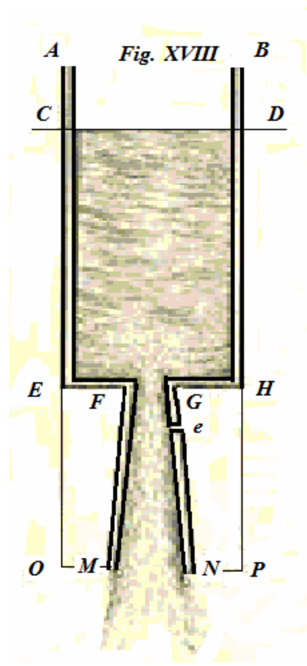
Fuerit v. gr. canalis rectus & ita inclinatus versus horizontem, ut sinus anguli intercepti inter utrumque sit ad sinum totum ut 1 ad *g*, erit $\xi=gx$ *;* unde

$$v=\frac{nna}{2nn-mm}\left(\left(\frac{a}{x}\right)^{\frac{nn-mm}{nn}}-\frac{x}{a}\right),$$

quae aequatio cum non differat ab aequatione §. 13 pro cylindris verticalibus data, sequitur in utroque casu velocitates aquae easdem esse, postquam descensus verticales superficiei aquae iidem sunt: Igitur accelerationes in locis homologis utrobique similes sunt ratione altitudinum verticalium, & hoc tantum discriminis intercedit, quod in canali inclinato lentius fiant, idque in ratione ut 1 ad *g:* facile igitur sensibus percipi poterunt hae accelerationes in canalibus valde inclinatis, quae in verticalibus ob nimiam mutationum celeritatem non possunt. Coeterum patet per se ex eo, quod frictiones a longitudine tubi augeantur, non posse non velocitates inde diminui, ad quod animum advertent ii, quibus experimenta hac de re instituere animus erit.

De effluxu aquarum ex cylindris verticaliter positis, qui in alios tubos strictiores pariter verticales desinunt.

§. 21. Constat experientia, inter duos cylindros omnino aequales similiterque positos, quorum alterius foramini tubus strictior respondeat, hunc citius depleri, qui tubum appensum habet, & quidem eo citius, quo magis tubus a loco insertionis versus extremitatem amplitudine crescit, quae pluribus exposuit D. 'sGravesande in *Phys. Elem. Math., lib.* 2, *cap.* 8. Totam rem sequenti Problemate comprehendemus.



Problema.

§. 22. Fuerit vas cylindricum *AEHB* (Fig. 18) verticaliter positum perforatum in *FG,* quo lumine communicet cum tubo conico *FMNG,* per cujus demum orificium *MN* aquae effluant. Quaeritur velocitas superficiei aqueae *CD,* postquam a quiete descendit per *AC* vel *BD.*

Solutio.

Sit altitudo aquae supra *MN* initialis, nempe $NG+HB=a$, altitudo superficiei aqueae in situ *CD* supra *MN*, id est, $NG + HD = x$; longitudo tubi annexi seu $NG = b$; amplitudo orificii $MN = n$; amplitudo orificii $FG = g$, amplitudo cylindri superioris $= m$; sit velocitas superficiei aqueae in *CD* talis quae debeatur *bmm* altitudini *v*, erit in aequatione generali §. 8 $y = m$ & $N = m(x-b) + \frac{bmm}{\sqrt{gn}}$, quae substitutiones instituto calculo conformes esse patebunt cum§. 6; reliquae autem positiones eaedem sunt quae ante. Abit igitur aequatio paragraphi 8 in hanc

$$m(x-b)dv + \frac{bmm}{\sqrt{gn}}dv - \frac{m^3vdx}{nn} + mvdx = -mxdx, \quad \text{quae porro}$$

divisa per *m* factoque $x - b + \frac{bm}{\sqrt{gn}} = d$, dat

$$\left(1 - \frac{mm}{nn}\right)vdz + zdv = -zdz - bdz + \frac{mbdz}{\sqrt{gn}},$$

quae multiplicata per $z^{-\frac{mm}{nn}}$ facit

$$\left(1 - \frac{mm}{nn}\right)z^{-\frac{mm}{nn}}vdz + z^{1-\frac{mm}{nn}}dv = -z^{1-\frac{mm}{nn}}dz - bz^{-\frac{mm}{nn}}dz + \frac{mbz^{-\frac{mm}{nn}}dz}{\sqrt{gn}},$$

post cujus integrationem addita constante *C* oritur

$$z^{\frac{nn-mm}{nn}}v = C - \frac{nn}{2nn-mm}z^{\frac{2nn-mm}{nn}} - \frac{nnb}{nn-mm}z^{\frac{nn-mm}{nn}} + \frac{mnnb}{(nn-mm)\sqrt{gn}}z^{\frac{nn-mm}{nn}}$$

in quo valor quantitatis. constantis *C* ex eo definitur quod ab initio fluxus (cum nempe $x = a$ sive $z = a - b + \frac{mb}{\sqrt{gn}}$) sit $v = 0$ quia non potest motus oriri in instanti temporis puncto; hinc igitur fit

$$C = \left(\left(a - b + \frac{mb}{\sqrt{gn}}\right) \times \frac{nn}{2nn-mm} + \frac{mb\sqrt{gn} - mnnb}{(nn-mm)\sqrt{gn}}\right) \times \left(a - b + \frac{mb}{\sqrt{gn}}\right)^{\frac{nn-mm}{nn}}.$$

Ex his quidem aequationibus definiuntur omnia; quia vero calculus fit paullo prolixior, nisi amplitudo vasis superioris indicata per *m* tanta sit, ut possit ratione amplitudinum *g*

& *n* infinita censeri, hunc solum considerabimus casum, idque eo magis quod error notabilis inde non oriatur, etsi mediocris sit magnitudinis numerus $\frac{m}{n}$ aut $\frac{m}{g}$.

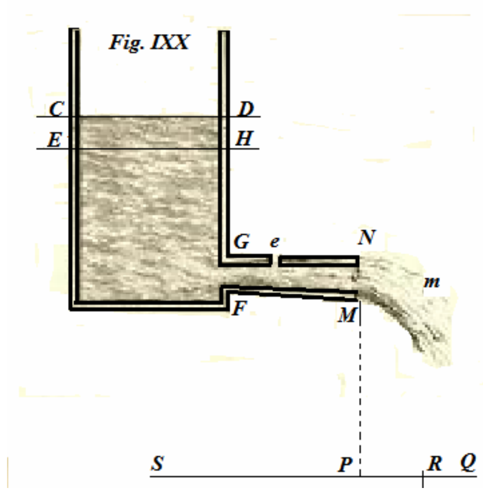
§. 23. Quod si proinde ponamus $m=\infty$, simulque utamur prima aequatione differentiali proximi paragraphi, atque in hac ponatur $v=\frac{nn}{mm}s$, ut sic inveniatur ex valore litterae *s* altitudo ad quam aqua per orificium *MN* effluens sua velocitate ascendere possit, erit primo

$$\frac{nn}{m}(x-b)ds+\frac{bnn}{\sqrt{gn}}ds-msdx+\frac{nn}{m}sdx=-mxdx$$

& quia $m=\infty$ atque facile praevidetur rationem fore finitam inters & *x*, atque inter *ds* & *dx*, haec eadem aequatio mutabitur rejectis terminis rejiciendis rursus in hanc

$-msdx=-mxdx$ vel $s=x$, quod pariter paragr.10 jam fuit demonstratum. E re vero duxi id de novo hic demonstrare, quia casus praesens diversus videri poterat ab illo, de quo in praefato paragrapho dicitur. His intellectis non opus est pluribus explicare Phaenomena circa hanc rem§. 21 Auctore 'sGravesande indicate ; patet enim, aquam non aliter effluere per vas compositum *AEFMNGHB*, quam per vas *simplex AOMNPB*, cum nempe orificium MN est valde parvum, atque hinc majorem esse velocitatem superficiei aqueae *CD*, quam si per vas *AEFGHB* aquae effluerent, posito orificio $MN=FG$, multoque magis si *MN* fuerit majus quam *FG*, quod fit cum tubus versus inferiora amplitudine crescit: attamen observari debet, ab initio motus aquam tardius descendere, quam sic definitum fuit, nec regulam istam prius locum habere quam superficies *CD* per spatiolum aliquod descenderit, quod tamen brevi fit tempore: mutationes, quae ab initio motus fiunt in hoc casu, examinabimus in sectione sequente.

§. 24. Eodem modo computus esset instituendus, si vasi, quod semper nunc amplitudinis infinitae ponimus, implantatus esset tubulus non verticalis sed horizontalis, veluti in fig. 19, aut sub alia directione qualicunque, semper autem reperietur aquas per orificium *MN* mox, postquam superficies aquae in vase principali aliquantulum descendit, ea proxime effluere velocitate, quae respondeat altitudini istius superficiei supra orificium; inde liquet quod manentibus tam altitudine aquae supra tubulum *GN*, quam ipso orificio *FG*, augeatur quantitas aquae dato tempore effluens ab aucta amplitudine orificii *MN:* Sic igitur



demonstratum hic dedimus, quod dictum fuit in fine§. 4 Sect. 1 Frontinum experientia fuisse edoctum, nempe, *plus debito aquae erogari per calicem legitimae tum mensurae tum positionis, cui statim fistulae amplioris moduli subjectae sint.* Et quidem quantitates aquae caeteris paribus erogari deberent ipsis orificiis *MN* proxime proportionales, nisi multa essent impedimenta, quae hanc quantitatem valde diminuant, de quibus proxime dicam: facere possunt haec impedimenta, ut admodum parum fluxus aquarum promoveatur ab aucto orificio extremo; semper tamen promovebitur aliquantum.

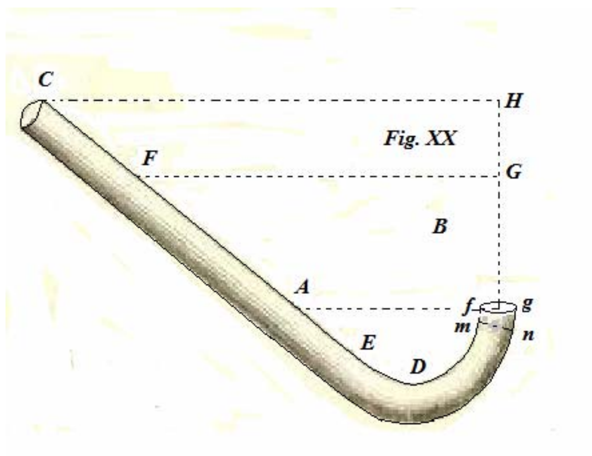
§. 25. Ex praemissis liquet velocitatem, qua superficies aquae *CD* in utroque, de quo diximus, casu descendit caeteris paribus pendere ab amplitudine orificiorum *MN;* haec autem ea innituntur hypothesi, quod aqua lateribus tubulorum *GN* ubique adhaereat & pleno orificio *MN* effluat, quae hypothesis locum amplius habere non posset, si nimium orificium istud augeretur. Dein patet quoque, cum aquae per tubum verticalem in Fig.18 effluunt, earum fluxum accelerari a longitudine hujus tubi aucta: posset tamen haec quoque ita augeri, ut tandem aquae desinant esse continuae in tubo, quin potius in columnas dividantur, quod fiet, si tubus longitudinem habeat plus quam triginta duorum pedum aut minorem etiam, si simul amplitudine crescat versus *MN;* ita si orificium *MN* duplum sit orificii alterius *FG*, non poterit longitudo major esse quam octo pedum, sine periculo subsecuturae aquarum separationis in suprema tubi parte, quam rem alibi demonstrabo : sed est alia insuper causa praeter nimiam tubi longitudinem, quae aquae separationem producere potest, nempe quod altitudo aquae *CEHD* minor sit, quam ut sat cito in tubum irrumpere possit, quo fit, ut aër una cum aqua simul sperne influat, dum superficies aquae formam cataractae seu infundibuli cavi assumit, sic ut non totum orificium *FG* aqua obtegatur; Haec quidem res facit, ut aqua minori copia effluat, non autem ut minori velocitate, quod posterius putavit Auctor quidam Italus, nomine Carolus Fontana, qui hac de re lingua sua vernacula ita scripsit : *mà se non vi fosse,* inquit, *tant'acqua, che bostasse a mantenere piena detta canna, l'acqua attraherà l'aria dentro di se in tanta quantitii, quanto gli mancherà l'acqua intermettendosi fra l' acqua da ogni banda; mà la velocità dell' acqua mancherà tanto, quanto sarà l' altezza di tutta l' aria raccolta insieme che sarà in essa canna.* Rationem ejus, quod dixi, non inde velocitatem aquae diminui posse, quilibet perspicit ex eo, quod alias non posset *ascensus potentialis* esse aequalis *descensui actuali* poteritque res facili experimento confirmari, incurvata tubi extremitate *MN*, ut aquae horizontaliter effluant, & ex amplitudine jactus velocitas aquae dignosci possit. Quomodo autem pro lubitu fieri possit, ut nullis mutatis aliis circumstantiis aër aquis circa summitatem tubi misceatur, sic habe: fiat nempe parvulum foramen in tubo haud procul ab orificio *FG* (Fig. 18 & 19) quod si autem durante aquae fluxu digito obturaveris istud foraminulum, aquae transfluent purae, & si removeris digitum, mox aër per foraminulum idem irrumpet seque cum aqua praeterfluente miscebit. His intellectis facile erit rationem reddere Phaenomenorum, quae in caminis seu fumi ductibus observantur, fumus enim altum petit, quia aëre levior est, quod constat experimentis de fumo in vacuo, ubi descendisse visus fuit, sumtis: idem igitur est de fumo ascendente, quod de aqua descendente: haec autem in fig.18 eo celerius effluit per orificium *MN*, quo amplius est, & quo humilius positum: ergo etiam fumus eo celerius caminum transibit, eoque magis ignis in foco accendetur, quo altius ducetur caminus, & quo magis superiora versus divergit, si modo non nimis divergat; quod utrumque experientia confirmat; ipse deinde insuper expertus sum, si caminus alicubi perforetur, tantum abesse, ut fumus per foramen istud exitum tentet, quin potius aër magno impetu irruat, seque fumo miscens per caminum ascendat, non secus atque aërem per foraminulum *e* in tubum *FGNM* (Fig.18 & 19) irrumpere indicavimus. Ita vero fumus minori certe copia, aut saltem difficilius ascendet ignisque remittet.

Caeterum duae sunt potissimum causae, altera aliena altera naturae rei propria, quae motum aquae valde retardare possunt in Fig.18 & 19. Prior est adhaesio aquae ad latera tubi, & altera, quod cum tubus amplitudine crescit velocitas aquae nullibi sibi constans in quovis tubi loco mutetur, quae mutatio si oriri censeatur ab impulsibus infinite parvis aquae velocius motae in aquam minus velociter motam, apparet singulis momentis ab impulsibus his corporum mollium aliquid de *ascensu potentiali* perdi, unde necessario aquarum effluxus notabiliter diminuitur

§. 26. Loco ultimo nunc dicam quaedam de vasis recurvis, ex quibus aquae non omnes effluunt: brevitatis autem gratia canalem considerabimus cylindricum, & cujus quidem pars, quam superficies aquea non transgreditur, sit recta.

Problema.

Sit nempe canalis cylindricus *CEDB* (Fig. 20) cujus pars *CE* quanta sufficit est recta, reliqua *EDB* utcunque incurvata; fuerit canalis totus aqua plenus effluxura per foramen *B*, perveneritque superficies aquae ex *C* in *F*, quaeritur altitudo respondens velocitati aquae in *F*. Solutio. Ducantur verticalis *BH* & horizontales *CH, FG, AB*, sitque sinus anguli *HCE* ad sinum totum ut 1 ad *g:* Jam vero si rem recte perpendamus, videbimus contineri problema praesens in altero generaliori, quod supra paragrapho 20 tractavimus, ubi habuimus hanc aequationem:



$$v = \xi^{\frac{mm}{nn}-1}\int -x\xi^{-\frac{mm}{nn}}d\xi.$$

ubi pro nostro casu praesente intelligitur per v altitudo quaesita respondens velocitati superficiei aqueae in situ *F*, per ξ longitudo *BDEF* & per x altitudo *BG*, atque per $\frac{m}{n}$ index rationis inter amplitudines tubi & foraminis *B:* Quod si vero dicatur longitudo . ~ $a\,BDA = \alpha$, erit $x = \frac{\xi-\alpha}{g}$, unde nunc habetur

$$v = \xi^{\frac{mm}{nn}-1}\int -\left(\frac{\xi-\alpha}{g}\right)\xi^{-\frac{mm}{nn}}d\xi.$$

Indicetur longitudo totius canalis *BDEC* per β, & erit

$$\int -\left(\frac{\xi-\alpha}{g}\right)\xi^{-\frac{mm}{nn}}d\xi = \frac{nn\alpha}{g(nn-mm)}\left(\xi^{\frac{nn-mm}{nn}}-\beta^{\frac{nn-mm}{nn}}\right)$$
$$-\frac{nn}{g(2nn-mm)}\left(\xi^{\frac{2nn-mm}{nn}}-\beta^{\frac{2nn-mm}{nn}}\right),$$

atque proinde

$$v=\frac{nn\alpha}{g(nn-mm)}\left(1-\left(\frac{\beta}{\xi}\right)^{\frac{nn-mm}{nn}}\right)-\frac{nn\xi}{g(2nn-mm)}\left(1-\left(\frac{\beta}{\xi}\right)^{\frac{2nn-mm}{nn}}\right).$$

Q.E.I.

§. 27. Quoniam hae aequationes sunt paullo prolixiores, non immorabimur generali earundem contemplationi, consideraturi potius casus istos particulares, qui calculum abbreviant, nec ultima ista aequatione definiri possunt.

Si operculum in *B* omne abesse ponamus, fit $m=n$ & (quod seorsim pro hoc pariter atque altero casu mox dicendo erui debet)

$$v=\frac{\beta-\xi+\alpha\log\xi-\alpha\log\beta}{g}$$

tuncque velocitas maxima est in *A*, nominatimque talis, quae respondet altitudini

$$\frac{\beta-\xi+\alpha\log\xi-\alpha\log\beta}{g}.$$

Denique punctum *E* maximo respondens descensui obtinetur ope hujus aequationis,

$$\xi-\alpha\log\xi=\beta-\alpha\log\beta.$$

Alter casus seorsim subducendus calculo est, cum $mm=2nn$, ubi oritur

$$v=\frac{\alpha\xi-\alpha\beta-\xi\beta\log\xi+\xi\beta\log\beta}{g\beta}$$

atque si capiatur, posito *c* pro numero, cujus logarithmus est unitas, $\xi=c^{\frac{\alpha-\beta}{\beta}}\beta$, determinabitur sic locus maximae velocitatis, cujus altitudo generatrix est $=\left(c^{\frac{\alpha-\beta}{\beta}}\beta-\alpha\right):g$ dum maximus descensus, qui proportionalis est toti aquae effluenti, definitur faciendo

$$\alpha\xi - \alpha\beta - \xi\beta 1\log\xi + \xi\beta \log\beta = 0.$$

Non dubito, quin haec ad amussim experientiae essent responsura, si modo adhaesio aquae ad latera tubi motum non retardaret; puto tamen, eventum experimentorum talem esse posse, ut intelligenti, qui horum impedimentorum rationem habeat, satis ostendant propositionum veritatem.

§. 28. Ultimo loco communicabo veram solutionem phaenomeni alicujus, quod primo aspectu valde videtur paradoxon. Postquam enim ex omnibus hactenus dictis luculenter apparet fieri non posse, ut aquae multo majori velocitate effluant quam qualis altitudini aquae supra foramen debetur (possunt tamen aliquanto majori, praesertim si foramina sunt magna, *confer ea quae dixi de velocitatibus maximis* §. 16), multis mirum fortasse videbitur, *contingere aliquando in fontibus salientibus, ut aqua ad temporis momentum jactum faciat longe altiorem, quam secundum regulas nostras fieri posse videtur.* Verum tantum abest, ut hae inde aliquid roboris perdant, quin potius egregie confirmentur. Solutio autem paradoxi in eo consistit, quod nos hactenus aquas consideraverimus continuas, & nullo vacuo aëreo separatas: Recteque observavit D.us De la Hire non fieri hujusmodi saltus irregulares, nisi aër una cum aqua tubum prope scaturiginem fuerit ingressus, quod saepe fieri indicavi §. 25. Iste vero aër simul cum aqua fertur usque ad orificium effluxus, per quod mox erumpit: id dum fit, massa aquea impetum acquirit, qui in expellendas aquas solus impenditur, hocque pacto enormem jactum producit. Hanc phaenomeni causam mox clarius una cum debitis mensuris explicabo, postquam praemisero verba, quae hac de re extant *in Histor. Acad. Reg. Sc. Paris. ad an. 1702. On voit quelques fois,* dicitur in loco citato, *1' eau qui sort par un ajutage saillir trois ou quatre fois plus haut que ne lui permet la hauteur du réservoir, aussi se remet-elle bien vite à la hauteur, que lui prescrivent les loix de l'hydrostatique. Mais comment a-t-elle pu en sortir en un instant? Msr. De la Hire l'attribue à de l'air enfermé dans la conduite, qui ayant été pressé* & *mis en ressort par l 'eau, qui descendoit toujours, s 'est debandé contre celle qui montoit & lui a donné cette vitesse momentanée.*

Recte itaque animadvertit Dn. De la Hire aëri saltum deberi, dubiumque nullum est quin veram rationem, qua aër id producere possit, fuisset eruturus, si phaenomenon, quod obiter attigit, attentius considerasset, facile utique perspecturus, aërem inter medias aquas nullam sustinere pressionem, nisi super incumbentis aquae (imo ne hanc quidem in aquis fluentibus, uti inferius in Sect. XII demonstrabo) nec adeoque aërem compressum fortius expellere posse aquam sibi praecedentem, quam si sui loco aqua esset. Ego quidem praevidi (quod facillimo experimento saepe postea sum expertus) non esse aquam ante aërem positam solito altius assurgentem, sed illam, quae aërem sequitur, quod nunc clarius faciam.

Sit igitur in Figura vigesima aquae ductus *CADB* cylindricus, ut esse solet, isque totus aqua plenus, praeter particulam *mnB* aëre plenam. Ducantur lineae horizontalis & verticalis *CH* & *HB:* ponamus brevitatis ergo aëris gravitatem prae gravitate aquae nullam censeri posse, ita ut transitus aëris per orificium *B* nihil resistat fluxui aquae, quamvis de caetero facile foret inertiae aëris rationem habere, nisi calculi prolixitatem evitare vellemus in re, ubi nullam quaerimus praecisionem. Sit longitudo canalis *CADf* vel *CADm* (ponimus enim differentiolam *mf* aëre repletam valde parvam) $=\beta$; mf vel $ng=\delta$; $HB=\alpha$; amplitude tubi $=m$; amplitude orificii $B=n$ *;* denique demus aquae, cum superficies est in *mn*, nullum esse motum, quaesituri altitudinem velocitati debitam, quam superficies *mn* habet, cum pervenit in situm *fg*; sit ista altitudo $=v$, erit *ascensus potent.* omnis aquae eo ipso momento pariter $=v$ *: descensus actualis* autem est per §. 7 = tertiae proportionali ad totam massam aquae, particulam aquae *mngf* & altitudinem verticalem *HB*, id est, $=\frac{\delta}{\beta}a$; est igitur $v=\frac{\delta}{\beta}a$. Haec quidem altitudo dicto citius minuitur statim atque aqua per orificium *B* fluere cogitur, quod demonstravi §.18; sed primo tamen temporis puncto aqua servabit motum quem acquisivit, & sic guttula orificio proxima ejicietur velocitate, quae debeatur altitudini $\frac{mm\delta}{nn\beta}a$. Potest autem haec altitudo non solum esse tripla aut quadrupla ipsius *a*, sed & quantumcunque magna: ego certe cum tubis vitreis pro lubitu jactus feci decies aut vigesies altiores ipsius *a;* fuerit v. gr. $\beta=100$ pedum, $\delta=$uni pollici, diameter autem tubi decupla diametri, quam orificium habet; & erit $\frac{mm\delta}{nn\beta}=\frac{10\,000}{1200}$ ita ut in his circumstantiis prima guttula assilire demta aëris resistentia debeat ad altitudinem plus quam octies majorem altitudine solita *a.* Sunt coeterum multa impedimenta eaque maximi momenti, quae jactus enormes cohibeant; perditur nempe aliquid de motu ab impulsu superficiei aqueae *mn* in latera *fg*, dein etiam ab ingenti attritu quem aqua per foraminulum, quod parvulum esse debet, tam celeriter lata patitur: multum etiam abest, quominus aqua *CADm* omni sua celeritate moveatur ob adhaesionem aquae ad latera tubi, quae adhaesio in tam longo tractu valde notabilis est.

Interim veram hanc esse solutionem phaenomeni nullum potest esse dubium, istique solutioni experimenta quae feci in omni extensione satisfaciunt. Dein hac theoria etiam recte solvitur alterum phaenomeni momentum, quod nempe jactus iste sit quasi momentaneus, postque brevissimum tempusculum ad sensus non major solito: ita in praesenti, quem modo finximus, casu si per regulam §.18 paullo mutatam (ibi enim de vasis verticaliter positis tantum dicitur) exploremus, quantum aquae effluere debeat ut jactus non amplius millesima parte (quae utique observari in hujusmodi experimentis minime potest) superet jactum solitum, cum ab initio fuerit eodem octies major, invenimus tam parvam esse illam quantitatem, ut tempus, quo tota ejicitur, nullo modo percipi possit.

Experimenta quae ad Sectionem III pertinent.

Praenotanda.

Plurima quidem sunt in hac Sectione eaque fere praecipua, quae vix ad experimenta revocari *immediate* possunt; etenim cum Auctores hactenus motum in fluidis effluentibus alium non consideraverint, quam qui fiunt per foramina valde parva, cumque proin nova sit theoria quam dedimus pro amplitudinibus foraminum qualibuscunque, haec ipsa est, cujus confirmatio maxime juvaret. At non video, quomodo in Cylindris verticalibus, de quibus potissimum egimus, velocitas aquae effluentis observari possit, praesertim cum foramen est valde amplum (secus enim ex tempore depletionis aliquod de velocitatibus judicium ferri potest). Haec ita perpendens cogitavi demum scopo nostro inservire posse paragraphos 16 & 20, in quorum priore determinata fuit velocitas maxima aquae effluentis ex cylindris verticaliter positis, in altero autem demonstratum est, eundem esse motum ex cylindris oblique positis & verticalibus, si utrobique altitudines verticales similes assumantur: Commode igitur utemur cylindris oblique positis, ut ex maxima amplitudine jactus aquei possit velocitas maxima aquae seu altitudo eidem debita experimento haberi: & hac quidem ratione accurate velocitas illa maxima, qualis revera est, explorari potest, etiamsi foramina sint quantumlibet magna, quae proin si convenire observetur cum regulis nostris, de integra theoria dubium superesse nullum poterit. Priusquam vero rem ipsam aggrediar, praemittendum erit theoremamechanicum, quod sequitur.

Lemma.

Sit *AB* (Fig. 21) linea verticalis, *BD* horizontalis; linea autem *AD* directionem habeat qualemcunque, sub cujus directione corpus in *A* projectum intelligatur, arcum describens parabolicum AC, cujus nempe tangens in *A* est recta *AD*, erit altitudo $=\dfrac{BC^2\times AD^2}{4AB\cdot BD\cdot CD}$ debita velocitati, qua corpus in *A* projectum fuit, = atque si *AD* fuerit horizontalis sive angulus *BAD* rectus, erit eadem illa altitudo $=\dfrac{BC^2}{4AB}$.

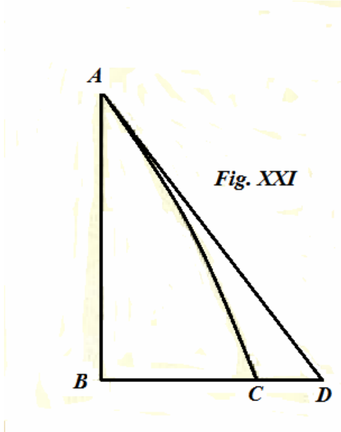


Fig. XXI

Jam vero quae mihi observata fuerint exponam.

De velocitatibus maximis fluidorum per foramina valde ampla effluentium. Ad §§. 16 & 20.

Experimentum primum.

Tubum cylindricum *FA* (Fig. 22) longitudinis quatuor pollicum oblique ad horizontem posui, in eoque situ firmavi; erat autem amplitudo tubi ad amplitudinem luminis in *A* ut 2 ad 1, & quidem diameter tubi praeter propter septem lineas exaequabat; dein mensuris acceptis in particulis aequalibus linearum *FE, AB* & *BD* (quarum lex ex ipsa figura per se patet) illas inveni 81, 619, & 740.

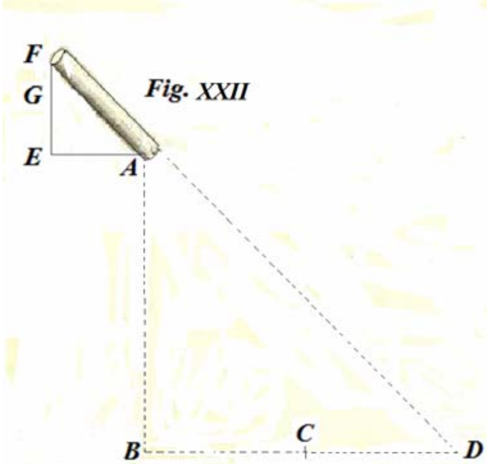


Fig. XXII

His ita praeparatis, tubum aqua replevi, digito interim obturato orificio *A*, eoque confestim remoto aquae brevissimo tempusculo effluxere omnes: observare tamen potui, primas & ultimas propius ad verticalem *AB* quam medias cecidisse; guttas autem longissime projectas incidisse in locum *C* invenique post saepius repetitum experimentum BC particularum, quibus antea usus fueram, 235. Jam vero si per praemissum lemma quaeratur altitudo *EG*, ad quam guttae maxima velocitate ejectae ascendere possint, reperitur $EG=56$ part.; deberet autem vi §§.16 & 20 esse $= 62$, nisi attritus aquae ejusque adhaesio ad latera tubi impedimentum motui afferret: majorem consensum non expectavi.

II. Positis quae prius, diminuto tantum ad dimidium foramine *A*, ita ut amplitudo tubi esset quadrupla amplitudinis ad lumen pertinentis, observavi $BC = 252$; hinc deducitur $EG = 68$ per experimentum; per theoriam autem debuisset esse $= 70$; numeri hi minus differunt quam praecedentes, quia hic multo minus fuit attritus impedimentum ob diminutam velocitatem internae aquae.

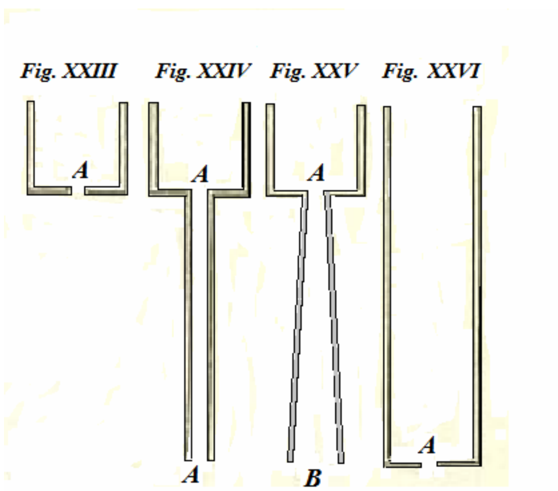
Utrumque autem experimentum egregie profecto theoriam confirmat.

De velocitate aquae ex vase amplissimo erumpentis.

Ad § .17. In isto paragrapho dicimus, si vas sit amplissimum, aquam mox, postquam superficies interna aliquantulum descendit, erumpere velocitate, quae constanter respondeat altitudini aquae supra foramen. Sinas autem sub quacunque directione (neque enim in vasis amplissimis directio venae quicquam velocitatem mutare potest) aquam effluere, & observes quocunque temporis puncto, in quanta distantia ab verticali vena in horizontem impingat, & exinde per praemissam regulam quaere altitudinem velocitati aquae effluentis eo temporis puncto respondentem, sic semper istam altitudinem invenies aequalem altitudini aquae supra centrum foraminis, si modo excipias primas guttulas, quae vi §. 16 minori velocitate effluere debent & acto effluunt: Neque impedimenta, quorum saepius mentionem injecimus, ullam notabilem moram fluxui injicient, si modo diameter foraminis duas aut tres lineas minimum exaequet, & diameter ipsius vasis non sit infra aliquot pollices, & denique altitudo aquae nimia non sit, veluti plurium pedum.

Haec omnia saepe expertus sum, experimenti autem genus nimis est triviale, quam ut prolixe describi mereatur.

De vasis quae sunt tubis verticalibus instructa.



Ad §§. 22 & 23. De his experimenta sumsit Cel. 'sGravesande in *Phys. Elem. Math.* 32, quae repetii; ea vero quae ad rem praesentem faciunt huc potissimum redeunt.

In Figuris nempe 23, 24, 25, 26 sunt singulae aperturae littera *A* notatae inter se aequales, sola *B* existente paullo majore in ratione ut 16 ad 25, amplitudines quoque, ut & altitudines cylindrorum sunt aequales excepto ultimo, cujus longitudo quadrupla est: tubi autem duobus cylindris intermediis annexi triplam habent longitudinem cylindrorum. His igitur vasis aqua repletis de ejus effluxu observatum fuit.

I. Superficiem aquae a principio non citius descendere in Fig. 23 quam Fig. 24; postquam vero utrobique aliquid aquae effluxit, multo celeriorem fieri motum in vase composito quam in simplici; utrumque praemonui in fine §. 23. Res autem melius &

accuratius intelligitur ex aequationibus differentialibus, quas §§. 22 & 23 dedimus, quibus si utamur ad prima motuum incrementa invenienda, tam in cylindro simplici Fig. 23 quam in composito Fig. 24, atque hunc in finem ponamus amplitudines cylindri & tubi esse ut *m* ad *n*, erit incrementum, quod vocavimus *dv* in vase simplici, ad incrementum in vase composito, ut $1+\frac{3m}{n}$ ad 4, adeoque longe majus in isto casu quam in hoc. Si proin primum motum recte percipere liceret, celeriorem statim ilium observaturi essemus, qui fit in cylindro simplici; cum vero in §. 16 & 23 porro demonstratum fuerit, superficiem aquae, postquam paululum descendit in utroque vase, proxime tales esse, quae respondeant altitudinibus $\frac{nn}{mm}x$, intelligendo per *x* altitudines aquae supra orificia, per quae effluit, sequitur mox multo majori velocitate aquam descendere in Fig. 24 quam Fig. 23. Sic igitur Theoria plane convenit cum observatis.

Il. Superficiem aqueam non parum velocius descendere in Figura 26 quam 24, ita ut velocitas in casu Fig. 24 sit quasi media inter casus Figurae 23 & 26. Hic vero rursus patet, primas quidem accelerationes multo tardius fieri in cylindro Fig. 24 quam 26. Hoc igitur respectu ipsa theoria indicat, quod observatum fuit; at certe differentia multum abest, ut tanta inde oriri possit, quantam expertus fui, neque amplius sensibilis esse deberet, postquam utrobique superficies paullulum descendit, per §. 23; debet autem reliquum impedimento tribui, quod ab attritu aquae in Fig. 24 oritur: aqua enim per tubum *AA* magna velocitate fertur, sicque tam ob velocitatem auctam, quam ob amplitudinem vasis diminutam impedimentum motui aquae validissimum offertur.

III. Denique velocissime, si prima temporis puncta excipias, aqueam superficiem descendere in cylindro Fig. 25 & notanter velocius quam in Fig. 26. Id vero conforme est cum his quae §. 23 demonstrata sunt; deberent autem mox post commune motus initium, positis nempe altitudinibus aquae supra orificia effluxus fere aequalibus, velocitates in Figuris 25 & 26 proxime esse ut amplitudines orificiorum *B* & *A*, id est, ut 25 ad 16; & quod minor observetur velocitatum differentia, rursus impedimento frictionis est tribuendum praeter aliam causam in fine §. 25 indicatam.

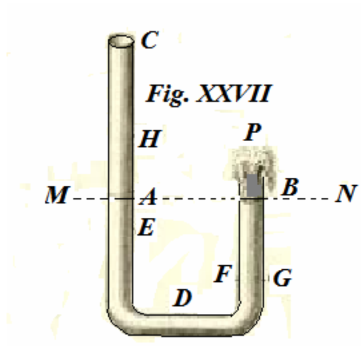
De iisdem vasis, quibus tubi horizontales inseruntur.

Ad §.24. Cum aquae ex vase valde amplo veluti *CDG* (Fig. 19) per tubum horizontalem *GM* ampliorem in extremitate *NM* quam ortu *GF* fluunt, majori velocitate illas ferri per orificium *GF* (si rursus excipias primas guttas) quam si vel tubus abest, vel cylindricus esset. Id etiam Frontinus experientia procul dubio edoctus affirmavit, alii vero moderni negarunt.

Igitur operae pretium duxi rem experimento explorare. Erat autem altitudo vasis, quo usus sum, supra axem tubi $=5\frac{1}{2}$ *poll. Angl.*, longitudo tubi $GN=2$ *poll.* 5 *lin.*, diameter orificii *GF* erat $=3{,}36$ *lin.*, diameter aperturae $MN=5{,}48$ *lin.;* erant proin amplitudines orificiorum ut 3 ad 8 proxime, amplitudo vasis sat magna erat, ut infinita censeri posset prae amplitudine tubi. Volui omnes mensuras allegare, ut quivis experimentum repetere possit. Hoc autem vase aqua repleto observavi amplitudinem jactus, & ex hac postquam omnes mensuras cognovissem requisitas calculum posui de altitudine, quae velocitati aquae transfluentis tum in *GF*, tum in *NM* deberetur: hanc inveni proxime undecim linearum, atque proin alteram $= 6$ *poll.* 6 *lin.* cum duabus nonis lineae partibus, quas easdem altitudines alio etiam experimenti genere inveni. Quoniam autem major est altitudo 6 *poll.* cum $6\frac{2}{9}$ *lin.* quam $5\frac{1}{3}$ *poll.* confirmatur theoria nostra de acceleratione aquae internae ab amplificatione tubi versus extrema, quamvis multum absit, ut duabus potissimum rationibus §. 25 allegatis inductus praemonui, quin tantum revera acceleretur quantum vi§. 24 remotis obstaculis, quorum in calculo nulla ratio habita fuit, deberet.

Ad§. 25. Hoc paragrapho in transitu monui, multis modis fieri posse, ut aër aquae per tubos fluenti misceatur. Inde autem futurum, ut aquae minori copia effluant quidem, sed non minori velocitate, quod utrumque ut experirer primo in tubis *AA* & *AB* (Fig. 24 & 25) non procul ab eorundem origine parvulum utrobique feci foramen; factum est, ut aquae per tubos cum aliquo strepitu ferrentur & turbidae effluerent, superficies autem solito multo lentius descenderet; deinde tubum Figurae 19 pariter aliquantulum perforavi, haud procul a *G*, rursusque observavi, paullo lentius descendere superficiem internam, cujus rei me certum fecit quod numerabam oscillationes alicujus penduli, quibus superficies per datum spatium descendit: ratione autem aquarum effluentium vidi aliquando aquas pleno orificio effluere & tunc aquas solito minus esse pellucidas, jactum autem ordinarium vel ordinario paullo majorem facere; saepissime autem aquam & aërem juxta se ferri, illam in parte tubi inferiore juxta latus *FM*, hunc in superiori juxta *GN* & tunc aquas esse limpidas atque velocitate ejici solito non solum haud minori, sed & multo majori, quod fieri posse haud obscure praevideram. De hac re in sequenti Sectione aliud experimentum majori praecisione institutum apponam. Dabitur autem fortasse alibi locus demonstrandi aquas sufficienti aëris quantitate permistas ea proxime effluere copia, qua effluerent rescisso tubo eo in loco ubi est perforatus, cui rei ipsam quoque experientiam respondere animadverti.

De canalibus recurvis.



Ad§. 27. Ducta in pariete horizontali *MN* (Fig. 27) tubum cylindricum *CDB* totum aqua plenum, cruraque ambo inter se parallela habentem, ita posui, ut extremitas altera *B* horizontalem *MN* raderet, simulque crura essent verticalia, dum interea orificium *C* digito obturabam aquae effluxum sic compescens.

Dein remoto digito observavi altitudinem maximam *BP*, ad quam aquae effluentes ascendebant, aliisque vicibus attendi ad locum *E*, ad quem descendit aquae superficies; feci autem sub duabus diversis circumstantiis experimentum; primo enim loco nullum in *B* posueram operculum; deinde operculum adhibui tali lumine perforatum, quod amplitudinem haberet ratione amplitudinis tubi ut 1 ad $\sqrt{2}$. Interim mensurae tales fuere: $CA = 345$; $ADB = 530$; $BP = 33$; & $AE = 88$ particulis, quarum 375 longitudinem *pedis Lond.* exaequabant. Haec ita fuere in casu priori, in altero autem manentibus reliquis vidi $BP = 64$ & $AE = 54$. Notabo hic in transitu, quod alio explorare cupiens modo maximum descensum *AE*, post finitum experimentum inclinaverim tubum, donec aqua jam jam effluxui per *B* proxima videbatur, quo temporis puncto distantiam mensuravi superficiei a loco *A* antea notato; distantia illa, quam eandem cum maximo descensu *AE* fore putabam, opinione longe minor fuit; unde edoctus fui partem aquae, quae in experimento jam per *B* effluxerat, tubum rursus ingressam fuisse.

His ita observatis, magnitudines *BP* & *AE* calculo quaesivi ad normam §. 27, ponendo *m* primo $= n$, deinde $mm = 2nn$; inveni autem in casu priore $BP = 79$, quae in experimento non superavit 33, maximumque descensum *AE* proxime reperi $= 250$, quem experimentum dedit 88; dein pro casu $mm = 2nn$ oritur *BP* praeter propter dupla illius, quae observata fuit, & $AE = 186$, quae 54 particularum observata fuit.

Enormes has differentias maxima ex parte adhaesioni aquae ad latera tubi tribuo, quae certe adhaesio in hujusmodi casibus incredibilem exercere potest effectum; usus enim sum tubo vix ultra duas lineas in diametro habente, majorem utique consensum experturus cum tubo ampliore. Interim verisimile est, curvaturam tubi in parte inferiore aliquid etiam motui derogare.

Ad §. 28. Eodem tubo recurvo, quem modo descripsi, usus sum: operculum autem posui in *B* minimo foramine pertusum: feci ut totus aqua esset plenus praeter particulam *FGB*, in quo situ aquam detinui ope digiti orificio *C* appositi. Remoto digito descendit aqua, & cum pervenisset in situm *HDB*, guttulae aliquot tanto impetu per foraminulum in *B* fuerunt veluti explosae, ut ad altitudinem plusquam decem pedum ascenderint, quamvis altitudo *HA* altitudinem dimidii pedis vix superaret. Interim ob exiguitatem foraminuli tantam resistentiam offendit aqua dum transiret orificium, ut fracto impetu aqua non solum non ad altitudinem AH ascendent (supra quam tamen remotis omnibus impedimentis assilire paullulum continue debuisset), sed vix guttula una aut altera notabili temporis mora fuerit expressa, ita ut mihi persuadeam, si absque impetu sola aquae pressione naturali tantus jactus producendus fuisset, id fieri non potuisse nisi altitudine minimum centum pedum.

Dein etiam observavi jactum aquae diminui eo magis quo minus ante experimentum relinquitur spatium *GB;* quae omnia theoriae sunt conformia. Mensuras superfluum fuisset sumere, quia ob nimia impedimenta tantus esse utique nequit jactus aquae, quantus illis remotis futurus fuisset. Attamen ut & has convenire cum formulis experimento confirmarem, tubum *CDB* sumsi ampliorem, ut impedimenta adhaesionis maxima parte auferrem, pars *DFB* parvula erat, minorque etiam pars *GB*, quam in experimento ab aqua relinquebam vacuam: ac denique operculum foramine non admodum parvo erat pertusum. Et tunc vidi saltum non multum admodum defecisse ab altitudine $\frac{mm\delta}{nn\beta}a$, quam §. 28 pro hoc negotio dedi, imo memini me praesenti Amico altitudinem saltus recte praedixisse, postquam perpendissem, quantum in calculo praeter propter impedimentis esset dandum.

Similem aquae explosionem *momentaneam* eamque a simili causa oriundam facillime obtinebis cum fontibus, qui aquas per fistulam pleno orificio ejiciunt. Si enim digitum orificio fistulae subito ita apponas, ut pars orificii aperta maneat, protinus aquas magno impetu expelli videbis, moxque tenue aquae filum intra pristinos velocitatis limites reduci. Observabis etiam aquas eo majori impetu atque longius projici quo minus digito relinquas foramen, atque pro eodem relicto foramine, jactum insolitum eo magis protrahi (utut semper brevissimum) fierique oculis sensibiliorem, quo longior est fistula, ita ut in fontibus salientibus, ad quos aquae ex castello per longissimos canales feruntur, si canales non essent admodum ampli & aquae pleno effluerent orificio, non dubito quin sic per notabile temporis spatium vehemens aquae jactus protrahi posset, gradatim ad solitam velocitatem rediturus: Haec omnia conformia sunt cum iis, quae § §. 28 & 18 monita fuerunt.

Experimentum hoc me aliquando & prima quidem vice fecisse memini coram VV. Cel. DD. De Maupertuis & Clairaut, cum quibus antea in sermonem de rebus istis aquariis forte delapsus eram. Quamvis autem hic nullus sit aër, qui accusari possit, revera tamen phaenomenon istud ab eo, quod D. de la Hire observatum fuit, non differt, & utrumque ab eo provenit, quod motus aquae in canali contentae, vel saltem motus istius pars perire non possit, sine ullo inde proveniente effectu, quem ipse enormis aquarum jactus constituit.