HYDRODYNAMICS SECTION THIRTEEN.

Concerning the reaction of fluids flowing out of vessels, and with the impulse of the same after they have flowed out, on planes which they meet.

§. 1. Water, while being ejected from a vessel, acts on the vessel from which it flows out, in a similar way as the ball from a cannon or gun from which it explodes: clearly it repels the vessel ; and indeed now Newton has noted the same in *Princ. Math. Phil. Nat.* first ed., *p.* 332, and from that the ascent of cannon balls is deduced correctly, which may be placed in the barrel with gunpowder and carbon in proportion ; indeed the flammable material projects balls to a height, while it blows out a little through the touch hole. [Bernoulli indirectly mentions here his own experiments with cannon fired vertically, discussed previously.]

But neither did the author cited generally treat the proof well enough concerning the business of the impulse, nor truly did he give a measure of the outcome (since that was not from his own design). Indeed in short, in the two later editions he passed over that in silence : [In fact, the fire in Newton's chambers had destroyed part of his calculations regarding this experiment; see the Koyré & Cohen edition of the *Principia*, pp. 778-779 for details] moreover he considered *that repulsive force to be equal to the weight of a cylinder of water, the base of which shall be the opening sending out the water and of which the height shall be equal to the height of the surface of the water above the opening.* Indeed this measure rightly was deduced from the belief, as was then favored at the time of Newton about the velocity of water flowing out of a vessel, while it asserted water might be able to ascend to half the height of the surface by its velocity.

But as the falseness of this proposition is no longer unknown to anyone, and thus thence indeed any other defect may be picked up, although it may seem true enough at first appearance.

§. 2. At first we will consider the matter in the simplest case, where surely we may consider the water to flow out horizontally from a vessel of infinite cross-section. Moreover I have shown the force of repulsion from the initial flow not all to be present at once, unless inasmuch as the whole velocity shall be present in the water flowing out, thus so that if the vessel were not of infinite cross-section, the force of repulsion slowly may increase together with the velocity of the water flowing out slowly increasing, or even may decrease according to the nature of the circumstances: But at first we will not concerned with momentary changes, by requiring the flow from an infinite vessel to be made equably. And thus the force of repulsion may be defined best, if it may be asked, thus what force is required according to the motion being produced : Hence truly in the end, not only will it be according to the velocity of the water, but also with regard to the quantity of that flowing out; but the amount depends partially on the size of the opening, partially on the contraction of the jet, which latter is variable : However we have seen in Sect. IV the whole can be avoided, if yet there shall be certain, a section of the jet with the maximum contraction or attenuation for the opening being considered, and then I say *the force of repulsion to be equal to the weight of a cylinder of water, the vase of which shall be the opening sending out the water* (that is, the horizontal section of the jet maximally contracted) *and the height of which shall be equal to twice the height of the water in the vessel above the hole or more accurately, to twice the height owed to the velocity of the water flowing out.* Therefore if there shall be no contraction of the jet, because it is zero when the water may flow out through a short tube, the repulsion will be twice or nearly twice as great, as was defined by Newton.

§. 3. So that we may show this proposition, here a certain principle of mechanics is required to be considered, the use of which also I have tested often in solving other questions : here is the principle :

If a body from rest will have acquired the same velocity from a variable motion - producing pressures in whatever directions, and the individual pressures may be multiplied by their time increments, the sum of all the products will be the same always, that is, if the pressure shall be $= p$*, the time increment* $= dt$*,* $\int pdt$ *will be constant.* I have presented this matter more clearly in *Comment. Acad. Imp. Petrop. Book.* I, *page* 132.

§. 4. Now we may consider a cylinder as if of infinite cross-section, from which water flows out horizontally with a uniform velocity, by removing from the action, the weight which acts on the particles, after they have now flowed out, thus so that the individual particles may begin to move horizontally and uniformly ; but the particles may be accelerated and they may experience a pressure, as long as the maximum degree of speed is not yet present, and this will be obtained when they arrive at the place of the maximum contraction of the jet ; this is the reason, as I have said the section of the same jet is required to be considered to be the opening of the efflux as it were. The cross-section of this section shall be $=1$, and where the water may have a velocity corresponding to the height *A:* a cylinder of water flowing out may be considered, which may have 1 for a base and *L* for the length*:* if the time may be expressed by the distance divided by the velocity, the velocity due to the height *A* being expressed by $\sqrt{2A}$, and the time of the flow by $\frac{L}{\sqrt{2A}}$ From these premises we may investigate the motive pressure, which shall be able in the time $\left(\frac{L}{\sqrt{2A}}\right)$ to communicate to the cylinder *L* a velocity $\sqrt{2A}$: let that pressure be $= p$: therefore it may be considered for the calculation to have been performed in a shorter time *t* and the cylinder given the velocity *v;* there will be [*i.e.* in a form of Newton's second law of motion] $dv = \frac{pdt}{L}$ and $v = \frac{pt}{L}$, hence $p = \frac{Lv}{t}$; now there may be put $\sqrt{2A}$ for v and $\frac{L}{\sqrt{2A}}$ for t and there will be

$$p = \left(L\sqrt{2A}\right) : \left(\frac{L}{\sqrt{2A}}\right) = 2A.$$

Therefore the water pressure acting for the efflux is equal constantly to the weight of a cylinder of water, of which the base shall be the opening transmitting the water defined above and the height of which is equal to double the height corresponding to the velocity of the water flowing out: and the reaction is so great also, which repels the vessel. Q.E.D.

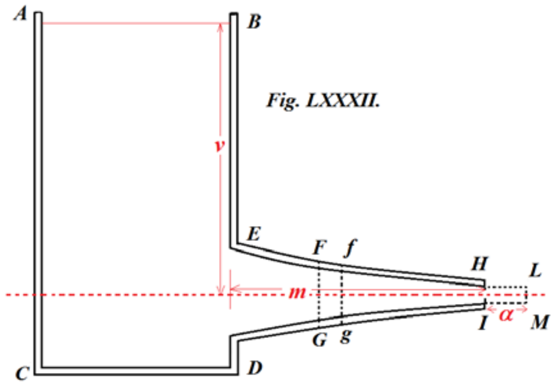
§. 5. The same is shown if water may not flow out through an opening, but through a horizontal cylindrical tube with a uniform velocity, or even through some tube of any kind of unequal cross-section: that also can be demonstrated directly, if the pressure required may be expressed well by individual drops, so that these may accept a due increase or decrease in the velocity.

§. 6. The height, which we have called *A*, indeed differs a little in experiments from the height of water above the efflux of the opening, especially if the water may flow out from a vessel with a very large cross-section through a simple opening, and that not excessively small: but the opening efflux more often differs notably from the section of the minimum jet, which as it were we may consider the orifice transmitting the water; that amount of water flowing out in a given time with its velocity compared in experiments indicates this.

Hence it arises that our proposition §.3, addressed by the usual experiments, does not differ much from Newton's proposition set out in §.1 according to experiment; if truly everything acting may be avoided, which can produce contraction of the jet and which can diminish the velocity, the repelling force according to our theory becomes no more than twice as great, than that which was defined by Newton, and then also such as may be confirmed by experiments.

But so that we may put the matter plainly in the light, we will pursue that now more generally, and we will put that to the test, so that we will determine the repelling force from the start of the flow, while the velocity may be changing continually : for our first theorem had no place other than remained with the velocity invariant. So that there we may be more understandable in the question, with that being handled with a little more intricacy, indeed here it will help to be forewarned more generally.

§. 7. The *quantity of motion* is made from the velocity into the mass: if the velocities shall be unequal, the *absolute quantity of motion* will be had, if the individual particles may be multiplied by their respective velocity and the sum of the products may be taken. The *quantity of motion* [*i.e.* momentum] is generated from the motive pressures acting for a given time, and the effect arising from the cause is required to be agreed on equally: Therefore the sum of the motive pressures multiplied by their time increments is required to be estimated from the quantity of motion arising. And because any motive pressure reacts in the vessel, from which the water flows out, the total repelling force for any instant equals the new quantity of motion divided by the increment of the time, in which it is generated. From these forewarnings I proceed to the question itself.



§. 8. The vessel *ACDB* (Fig. 82) shall be of infinite cross-section and to that the horizontal tube *EHID* attached , the cross-sections of which may be placed unequal in any manner: the cross-section of the opening *HI* shall be $=1$, the length of the tube $=m$; the velocity at any variable point in $HI=\sqrt{2v}$, or such, which may be owed to the height *v*: I say in the first place, the absolute quantity of motion contained in the tube to be equal to $m\sqrt{2v}$, that is, such as if the tube were cylindrical and its cross-section might be regarded as the equal of the orifice *HI*, because surely the velocity of any incremental layer *FGgf* is inversely proportional to the mass.

Now truly we imagine in a given infinitely small time the infinitesimal column *HLMI* to leap out through the opening *HI*, the length of which *HL* or *IM* we may put $=\alpha$: the mass of this infinitesimal column $=\alpha$, and it will have the quantity of motion $=\alpha\sqrt{2v}$: but in the same time the mass of water contained in the pipe has acquired the quantity of motion $\frac{mdv}{\sqrt{2v}}$ (for it was $m\sqrt{2v}$); therefore the absolute quantity of motion generated in the infinitesimal time is $=\alpha\sqrt{2v}+\frac{mdv}{\sqrt{2v}}$, truly this if it may be divided by the same time increment (which is expressed by $\frac{\alpha}{\sqrt{2v}}$), the pressure sought repelling the vessel will be had, as we saw in §. 7, which therefore if it may be called *p*, will be

$$p=\left(\alpha\sqrt{2v}+\frac{mdv}{\sqrt{2v}}\right):\frac{\alpha}{\sqrt{2v}},$$

or

$$p=2v+\frac{mdv}{\alpha}.$$

(α) Thence it is apparent the final definition of the question depends on the ratio which exists between dv and α; truly we have defined this generally in Section three, yet with no attention made of the impediments, which are due for this case. And therefore here the shape of the tube also will contribute a little.

(β) Again it follows, if the flow is considered to be made uniform, *p* will be constantly $= 2v$, because then $dv = 0$: Truly this agrees with that, which we showed in §.5. Truly while the flow takes increases (which indeed it makes noticeably, and that for a long enough time, if the channel *EI* were longer), the vessel experiences one after another repelling force.

(γ) *dv* to α always has a real ratio : therefore at no time is the repelling force zero, thus so that from the first time the flow is repelling the vessel, even if then hardly any water may flow out on account of the small velocity of the same. In truth, as the use of our general rule may be apparent to anyone, we will apply that now to a special case, by attributing to the tube *EHID* a cylindrical shape of cross-section 1.

§. 9. Therefore if the tube may be considered cylindrical with the whole opening at *HI* with the other positions and denominations retained, the *vis viva* of the water contained in the tube $= mv$; the increment of which $= mdv$, to which be adding the *vis viva* of the small column *HIMI* or αv, and the sum of these being made equal to the factor from the height, which the surface of the water *AB* has above the orifice *HI*, and which we will call *a*, and from the mass increment α. Therefore there is $mdv + \alpha v = \alpha a$, from which this becomes $\frac{dv}{\alpha} = \frac{a-v}{m}$. But with this value substituted into the equation of the above paragraph it becomes

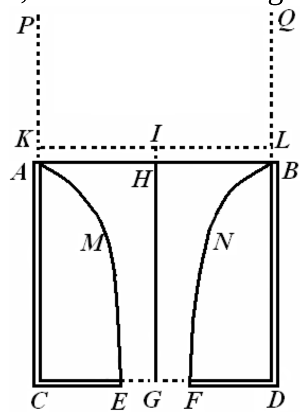
$$p = a + v,$$

from such I deduce the conclusions.

(α) The length of the tube contributes nothing to the reaction force, which the vessel may sustain, if the velocity may be considered the same, because the letter *m* has vanished from the calculation ; but this length enables (just as previously and in the above we have shown well enough) the velocities may take faster or slower increments; so that indeed were the tube longer, there the water is accelerated more slowly and vice versa, thus so that in an instant it may acquire its maximum degree of speed from rest, if the length of the tube were zero; but if this same tube were of infinite length, the water would not be able to acquire its final degree of speed unless after a notably infinite time.

(β) Therefore it can happen with no change in the height of the water, so that with the loss of any amount of water however small there shall be a noticeable force of reaction, and that may endure as it pleases; and indeed that can be obtained in two ways, both by elongating the tube, as well as also by obstructing the opening more, before the water will have reached a noticeable degree of velocity; yet the first method puts a free flow of water through the tube : indeed with the retardation in the flow of the water from external impediments, never to be avoided in much elongated tubes, the repelling force also is reduced.

(γ) Here in a few words it may be allowed to mention another proposition from the *Princ. Math. Phil. Nat.* 2nd edit. of Newton : Here the author may have changed his opinion about the velocity of water flowing out from a vessel presented in the first edition of the work cited, and these, if the water may be ejected upwards, he had acknowledged to rise to the whole height of the surface of the water in the second edition, presented in such words in the *second book*, *Proposition* 36, *coroll.* 2: *And the force, by which the whole motion of the water streaming forth can be generated, is equal to the weight of the cylindrical column of water, the base of which is the opening EF, and the height 2GI or 2CK.* This opinion was attacked by me and others at one time, and by others again to be confirmed. But now after I have thought about this theory of the motion of water, the difference of opinion is seen by me to be divided, so that when the water reaches a constant speed, which indeed is Newton's hypothesis, then that force may be defined correctly by the height 2*GI*, but from the start of the motion, when the velocity is essentially zero, the force may correspond to the simple height *GI*, and soon with the increased velocity likewise the force may increase agitating the water for the efflux, and at last it may rush out according to that magnitude, as Newton designated. Now these are clear to everyone, because the force generating the motion of the water, which Newton is discussing, cannot be equal to the force of repulsion, which we have seen to be equal to $a+v$. Also the illustrious Riccati, with whom I had an argument about this matter, was asked, *from where may that force arise agreeing with twice the height of water, since with the opening obscured the same threatening drop evidently may be urged to appear corresponding to the simple force of the height,* replied, *to be distinguishing the state of rest from the state of motion.*



§. 10. If the tube inserted into the vessel shall not be cylindrical, a calculation thus will be required to be put in. The cross-section of the channel at *FG* or fg shall be $= y$, the distance of the layer *FGgf* from the opening $ED = x$, and the other denominations may be retained: the *vis viva* of the water contained in the tube shall be $= v\int\frac{dx}{y}$, and its increment $= dv\int\frac{dx}{y}$, to which, as has been done in the preceding § , the *vis viva* of the small column *HLMI* , αv may be added, and there will be

$$dv\int\frac{dx}{y}+\alpha v=\alpha a;$$

thus from which there arises

$$\frac{dv}{\alpha}=(a-v):\int\frac{dx}{y},$$

with which value substituted into the equation in §. 8 there becomes

$$p=2v+m(a-v):\int\frac{dx}{y}.$$

Therefore since in the uniform flow of water there shall be $v = a$, then again there shall be $p = 2a$. Certainly as long as the flow of water is being accelerated, the motion of the water in the vessel *ACDB* near the opening *DE*, from which in this whole work we have disregard the force, is not to be ignored here : but it is not possible to determine the correct motion itself, nor therefore the expression squared which I gave accurately for the repelling force, if the water has not yet been assumed to flow uniformly, but when the water does flow equably, the expression prevails most accurately.

§. 11. Thus after we have shown for the uniform efflux of water, the repelling force always to be equal to the weight of a water cylinder placed above the opening, and at twice the height of the water rushing out, it pleases to show indirectly by *reduction to the absurd* [*reductio ad absurdum*], that the truth of this paradox can be seen through well enough and to be ignorant of the mechanical rules of this proposition.

Towards this end we will consider water flowing out vertically from a cylinder, by withdrawing the mind somewhat from impediments to the velocity of the water by removing a little from that contraction of the jet, which can be avoided. For the hole may correspond to the vertical tube, such as is seen in Fig. 76; everything may be had themselves as was said in Sect. XII §. 13: the water may have an equitable flow ; the walls of the vessel and of the channel may be understood to be without weight; the height of the cylinder may be put $=a$, and the height of the tube $=b$, with the height $CF=x$, with the cross-section at $E=1$; the cross-section will be at $F=\frac{\sqrt{(a+b)}}{\sqrt{(a+x)}}$ and at $C=\frac{\sqrt{(a+b)}}{\sqrt{a}}$; finally the cross-section of the cylinder $=M$. With these in place we seek the weight of all the water *ABCE*: we express the weight of water *ABC* by *Ma* and thus the weight of the water $CE=2a+2b-2\sqrt{(aa+ab)}$; therefore the weight of all the water $ABCE=Ma+2a+2b-2\sqrt{(aa+ab)}$: Therefore thus with still water to be placed in the vessel and in the tube, the force required to suspend the water is

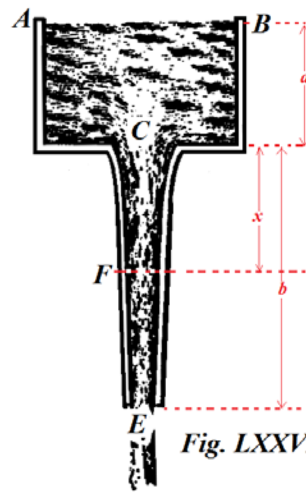


Fig. LXXVI

$$=Ma+2a+2b-2\sqrt{(aa+ab)}.$$

Truly now we will find the similar force when the water flows out through *E* with its whole velocity (by which surely it can rise to the height $a+b$) : moreover this will be found, if the repelling force may be taken away from the first force: Therefore if this repelling force may be put in place, as we have stated, $=2a+2b$, the force experienced by the flow being held in place $=Ma-2\sqrt{(aa+ab)}$.

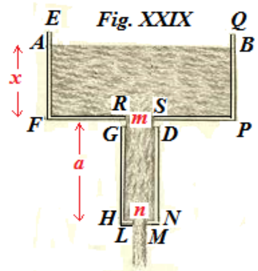
But truly imagine the tube *CE* to be missing, and the suspending force will be by our same rules, while the water burst out through the opening *C*, again $=Ma-2\sqrt{(aa+ab)}$, thus, because the weight of the water *ABC* is *Ma* and because the cross-section of the opening *C* is $\frac{\sqrt{(a+b)}}{\sqrt{a}}$, which multiplied by twice the height *a* gives $2\sqrt{(aa+ab)}$. Therefore our estimate of the repelling forces shows, the suspending force experienced by the outflow of the water is the same, as if either there were no tube, or if it were present and should have some length, only the tube may be had described in figure §. 13 of Sect. XII: and the agreement of this and the necessity of the identity appears also without calculation from that nature of the matter, when the tube thus formed makes no change to the water flowing through, since the jet of water at once leads to the same figure which the tube has, as long as the water stays in contact. But if we may consider another force, at no time generally will we obtain that consensus between the suspending forces: Thus for example, if according to common opinion we may call the repelling force, often named, to be equal to the simple weight of the cylinder, the repelling force will be, while water is imagined to flow out through the channel *CE* from the vessel *ACB*, $=a+b$; and this force if subtracted from the weight of the whole column of water *ABCE* or from $Ma+2a+2b-2\sqrt{(aa+ab)}$, leaves $Ma+a+b-2\sqrt{(aa+ab)}$, which is the required force for suspending the system *ABCE*, while the water flows out : But we have seen this force ought to be the same, if the channel *CE* shall be absent : But then the suspension force $=Ma-\sqrt{(aa+ab)}$, because the weight of the water *ABC* is $=Ma$ and the repelling force by the hypothesis is the simple cylinder of the opening *C* superimposed to the height a. Therefore according to this hypothesis there shall be always $Ma+a+b-2\sqrt{(aa+ab)}=Ma-\sqrt{(aa+ab)}$ or $a+b=\sqrt{(aa+ab)}$, which is absurd.

A similar absurdity may be shown, if the jet may be considered to ascend vertically upwards: and this may be taken out in vain as an exception to the common firmly held belief, that the jet flowing out of *CE* cannot be imagined as continuous, unless the water at the same time may be imagined to be held together somehow (for otherwise the jet is going to be broken up into droplets soon after the orifice), and the tenacity of the jet changes the state: for surely neither the velocities of the water may be changed by the mutual cohesion of the water in *CE*, nor do the walls of the channel *CE* experience any pressure, just as I have shown in Sect. XII §. l3, so I may remain silent whether the cohesion of the water arises from this tenacity, because of some magnetic property, or whether by some mutual attraction, by virtue of which the centre of gravity can acquire neither a greater or lesser velocity in any system.

But this again clearly has no place in vertically ascending jets, the exception of the opposing arguments, since water may remain there continually, as if there were no tenacity or mutual attraction of the water present.

[Thus Bernoulli suggests here that the normal laws of motion enable the jet to stay intact as it rises, at least initially, and it does not require some extra force to prevent it degenerating into droplets. We should note however, the inadequate understanding Daniel Bernoulli had regarding surface tension, derived from his father's wrong explanation of the reason why water climbs up capillary tubes, as mentioned at the beginning of this book, and the lack of understanding about viscosity.]

But I should be able to confirm our opinion in an infinite number of other ways and by particular examples, if I might wish to pursue these for a longer time. Thus for example in Fig. 29, described in Sect. V §.4 , if the height shall be $\mathrm{NS}=1$, the opening $LM=1$, and the opening $RS=2$, there will be $PB=\frac{1}{3}$, the repelling force which arises from the efflux through *RS*, $=2\times\frac{2}{3}=\frac{4}{3}$, and I can show the repelling force, which is produced from the efflux of water from the simple cylinder *RN* through *LM,* also to be $=\frac{4}{3}$, and thus the total repelling force is $=\frac{8}{3}$, which makes exactly twice [the ratio of] the water cylinder standing on the opening *LM* to the height $NS+PB$. But such an agreement received produces hardly anything from other false theories, thus so that there shall be no more doubt concerning ours, unless to be ignorant of these inner matters : Truly so that, as I have said, if I wish to show that the repelling force of the water from the outflow through *LM* from the simple cylinder *RN* to be $=\frac{4}{3}$, it is required that the repelling force may be defined, when the water flow out from the vessel with some given finite unchanging velocity : Truly so that I do not become more longwinded in this matter, I leave that to be effected by others ; nor that now any further great amount of work may need to be done ; I go on [to other matters].



§. 12. The demonstrations which we have given so far prevail only for right tubes, in which certainly the motive force of any drops whatsoever, and from thence arises the repelling force, act together between each other, and have a common direction: But when the tubes inserted into the vessel are curved, through which the water flows out, another kind of demonstration is required to be used: therefore so that we may omit nothing in this new argument, hence also we will provide instruction for the case: nor will it be, because it might be regretted because of the labour, since thence the true laws of the pressures will be apparent, which nature may follow not only in these cases, but also in many others.

[For the following section, we may recall that the formula for the radius of curvature or osculation is normally written as :

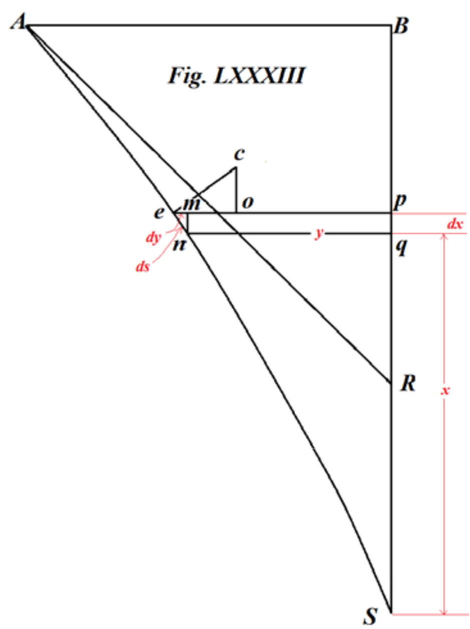
$$\frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}=\frac{\left(dx^2+dy^2\right)^{\frac{3}{2}}}{dxddy}=\frac{ds^3}{dxddy}=\frac{-ds^3dy}{dx^2ddx}=\frac{-dsdy}{ddx};$$

where $ds^2=dx^2+dy^2=const.$, then $dxddx+dyddy=0$;

with $ds=dx$ and $dy=0$ at the point on the circle with the same curvature.]

§.13. And thus we may take for an infinite vessel a small tube to be implanted of a certain uniform cross-section, but curved following some kind of curvature *AS* (Fig. 83) thus so that *A* shall be the point of the insertion, *S* the place of the efflux. Tangents may be drawn at *A* and *S*, namely *AR* and *SB*, and *AB* shall be perpendicular to *SB*; the velocity of the water flowing through the tube shall be uniform and such as may be owed to the height *A;* the cross-section of the tube everywhere shall be $=1$. *I say the total repelling force always to be taken in the direction SB to be* $=2A$, and this alone to be present.

For the sake of the demonstration the infinitely close perpendiculars *nq*, *ep* may be drawn to *SB*; *nm* parallel to the same *SB;* let $Sq=x,\ qp=dx,\ qn=y,\ em=dy$: the radius of osculation at *en* $=\frac{-dsdy}{ddx}$, with the elements *en* which I call *ds* taken for constants ; thus the centrifugal force of the elementary column of water taken between *e* and *n* is required to be determined: the weight of the column is $=ds$ (because its base $=1$ and the height $=ds$) and if the radius of osculation becomes $=2A$, by the theorem of Huygens the centrifugal force of the particle equals the weight of the same, and the centrifugal forces are with the other quantities equal in the inverse ratio of the radii : therefore the centrifugal force of the elemental column $=\frac{-2Addx}{dy}$:



[The centrifugal force on an element of fluid is given by

$$F=\frac{mV^2}{r}=\rho fds\cdot\frac{V^2}{r}=\rho fds\cdot\frac{-2gA}{dsdy}\cdot ddx\rightarrow -2A\frac{ddx}{dy},$$

where $f=$ cross-section of tube $=1$, $V=\sqrt{2gA}$, where A is the height due to the velocity V, the density $\rho=1$, and the acc. due to gravity $g=1$.]

this equation may express the centrifugal force perpendicular to the curve along *ec*, and *co* may be drawn parallel to *BS* itself: the force *ec* may be resolved into *oc* and *eo;* there will be (of account of the similarity of the triangles *eoc* & *nme)* (as *ds* is constant)

$$\text{the force } oc = \frac{-2Addx}{ds},$$

$$\text{the force } eo = \frac{-2Adxddx}{dyds}$$

$$= \frac{2Addy}{ds}.$$

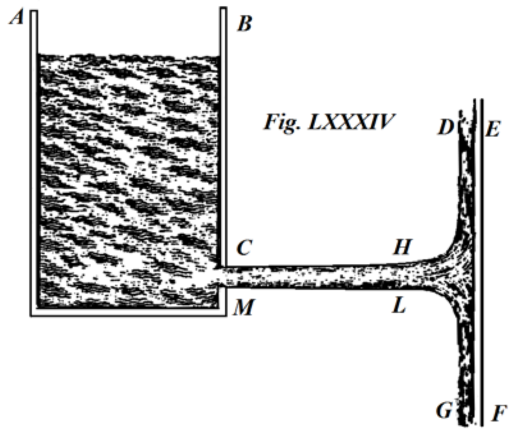
But the force of the element *oc* acts only in the direction *SB,* while the other *eo* is to be ignored along this direction: the integral of the element of the force *oc* is taken with such a constant, so that the integral may vanish together with the abscissa: this integral is $= 2A - \frac{2Adx}{ds}$, because at *S* there is $dx = ds$. Now so that the force may be had in the direction of the tangent for the whole tube, putting $\frac{RB}{RA}$ for $\frac{dx}{ds}$, therefore the whole force along the tangent $SB = 2A - \frac{2A \times RB}{RA}$. Truly this arises from the centrifugal force of any droplet : but another force remains to be considered ; truly while the water flows continually into the tube from a vessel with an infinite cross-section with a uniform velocity corresponding to the height *A*, the vessel is repelled along the direction *RA* by the force *2A* (per §. 4), with which resolved into the tangent along *SB* and along the perpendicular to that along *BA*, the first $\frac{2A \times RB}{RA}$ will be considered along; and because it has a common direction with the force $2A - \frac{2A \times RB}{RA}$ arising from the centrifugal force and in the manner defined, the same may be added : and thus the sum $2A - \frac{2A \times RB}{RA} + \frac{2A \times RB}{RA}$ or *2A* expresses the repelling force along the direction *SB.*

Again as we may demonstrate the force of repulsion along no other direction, we may return to the force of the element *eo,* as we have seen $= \frac{2Addy}{ds}$, of which the integral $= \frac{2A \times RB}{RA}$, which is cancelled out exactly by the repelling force *2A* along the direction *RA,* after this duly has been resolved. Q. E. D.

§. 14. This simplicity of the most general theorem, truly by which the repelling force may be indicated $= 2A$ in the contrary direction to the water flowing out constantly, this can be the *argument to the man*, as it is called, for its integrity, for those who either cannot follow our reasoning or do not care to examine our reasoning with sufficient attention. Truly if you may put the repelling force of the water $= A$, flowing from a vessel with an infinite cross-section into a tube along the direction *AR* , you see the system to be repelled in the direction *SB* by a force which shall be $2A - \frac{2A \times RB}{RA}$, which is absurd or the formula itself seems to indicate to me. Neither in this opinion shall the force in a direction perpendicular to the first be zero : For the force must be pressed back along the direction *BA* by the force $\frac{A \times RB}{RA}$, which again to me is absurd and its falsity I have known by experiment, in the case in which the angle *ARS* was right and $AB = AR$.

Many other theorems can be obtained according to this argument in its totality, by extension to be elicited and demonstrated for the flow of water not yet uniform and for that through some unequal tube, which may be attended to in a manner similar to those which were considered in §. 8. Truly because there is no time to go through the individual cases, I progress initially to the examination of another equal force in the opposite direction, that truly which the jet flowing out exerts on a plane, while it may strike that at right angles.

§. 15. Many have commented on the impulse of water jets striking a plane, and many experiments have been undertaken. I too concerning this matter have given certain ones in the *Comm. Acad. Sc. Petrop. book.* 2. Experiments are set out in Mariotti's book *Traité du movement des eaux,* in *Hist. Acad. Sc.* written by *du Hamel, p.* 48, and others. Indeed they do not agree very well, yet most indicate at first glance to consider the pressure of the jet of water flowing uniformly to be equal to the weight of a cylinder of water, of which the base shall be the opening, through which the water flows out, and of which the height shall be equal to the height of the water above the opening: And to this opinion most, indeed all have adhered and until now do adhere, because with other experiments also, especially those accustomed to be taken concerned with the motion of a ball in a resisting medium, it agrees remarkably well: I have followed the same myself, although several were held in doubt, as cited in *Comm. Petrop.;* nor have I hesitated in this work itself, which I have at hand, truly in the §§. 31, 32 of Section IX, to use that instead as an example. Indeed truly with more diligent thought about the matter, and with new principles used, and likewise with other new kinds of experiments set up, clearly at last I have seen that common opinion about the impulse of the jet of water requires to be changed in the same manner, just as for Newton with the repelling force, evidently so that in place of the opening, the section of the contracted jet may be considered and in place of the height there may be used twice the height corresponding to the actual velocity of the water : For I have shown, the force of repulsion I set out in §. 2, generally is equal to the impulse of the jet, if this may be incident wholly perpendicular to the plane : thence it follows the impulse of the jet to be greater, where the contraction of the jet were smaller, and plainly with this contraction vanished, and likewise the water with its whole velocity, as they can become in theory, for the eruptions, then the impulse will be twice as great as is generally decided : because truly both the velocity always drops a little and with the jet often contracted to almost half, it is a fact that most experiments in a cylinder prove the simple height seen was required for that impulse to be estimated. But I may wish to observe properly, with solitary jets to be discussed by me here only, which whole planes ward off, but not for fluids embracing bodies in making the same impetus all around, such as with winds or rivers: for I say these two kinds of impetus, which authors at present confuse, are required to be properly distinguished between each other, on account of the reasons being expounded briefly below.



§. 16. Thus I assess in the reasoning of the jet of water : I put the water to flow out horizontally with a constant velocity from the vertical cylinder with the infinite cross-section *ABM* (Fig. 84) through the opening in the side *CM*, and the jet to impinge perpendicularly on the plate *EF:* thus I can see easily, because the particles following those before are impeded so that they shall not be able to rebound, the individual particles are going to be deflected to the side, and thus with a motion parallel to the plate *EF* (but only if this were large enough, so that the whole jet may be taken to be dispersed in some manner) or almost not quite : And because everything is in a *state of permanence*, it is allowed to imagine, the plate *EF* to be attached to the vessel, and the jet surrounded by the sides *CHDGLM*, thus, thus so that the water may be agreed to flow out from the vessel *ABCHDEFGLM* through the circular gap *DEGF*. If it were thus, we have shown in §.13 the droplets flowing out at *DE* produce a certain repelling force along *EF;* but likewise it is apparent the force at *GF* to be repelling opposite to the former, thus so that this class of repelling forces shall not be required to be attended to. Because truly according to the direction of the lamina *EF* or attached perpendicular to the cylinder *BC*, we have shown at the end of the same §.13, along that direction plainly there is to be no repulsion: Therefore however much the lamina *EF* is propelled, the cylinder is repelled just as much. And that is what I had wished to demonstrate : And now thence it follows, *the pressure of the water jet, which runs into the lamina totally, to be just as great as the weight of the cylinder of water, the cross-section of the jet may have for a base (after it has acquired this uniform cross-section) and for the height twice the height owed to the velocity of the water (after this similarly has been made uniform).*

§.17. Without doubt there shall be many, for whom this proposition plainly may seem to be suspect and contrary to experiments: I may wish to consider these carefully, the experiments accepted up to the present by no means correspond accurately to the common rules, and in many cases our rule differs a little from the common one, although in theory they shall be especially diverse: besides also I wish to remind people about these in the proceedings, other experiments being set up by me, which each confirm exactly my opinion, and plainly refute the old! The experiments undertaken by me I will review at the end of the section. The manner of being demonstrated with which I have made use, perhaps also will be considered with a little accuracy; but I have another direct demonstration, which depends on a new mechanical property now observed by me, and which I shall communicate here, because both the said demonstration thence is somewhat easier to deduce, and also because the same will be able to be expended for other uses: Moreover, it is established thus.

If a body may be moving with a uniform velocity, but its direction changes continually from whatever causes and acting in whatever manner, while it has acquired a direction perpendicular to the first direction, and if the individual pressures deflecting the body may be resolved into two classes, the one parallel to the first direction, the other perpendicular; and finally if the individual parallel pressures may be multiplied by their times ; I say the sum of the products to be constantly the same, and indeed equal to that, which prevails whether the whole motion to be generated from rest or the whole generated to be absorbed.

When we use this favorable dynamical proposition in our present problem, the lamina *EF* is required to be considered, which changes the direction of the same by its reaction in water, until it should be made perpendicular to the first: Therefore with the aid of this favorable proposition of the preceding paragraph it will be shown in the same manner, as we have used in §.4 for determining the repelling force with the aid of the principle I have established in §.3. Therefore this true idea may be seen, as we must grasp mentally the impulse of the water: moreover it considers the individual drops of water to rebound along the direction of the lamina to the side, to which nature I have observed the water always to conform : yet I also saw some drops but not many to bounce backwards; but these produce a greater pressure, than the ones which are deflected to the side: And from that itself I am led, so that I may believe more firmly, if a jet of water may impinge on the plate obliquely with a great impetus, for example at an angle of thirty degrees, the pressure thence arising to be more than half of this, which arises from the same jet impinging normally, since following the ordinary rules exactly half the force must be exercised : the reason for occurrence is, because in the oblique impulse more particles shall be able to rebound, than in the direction, indeed almost all, if the velocity were great. But if all may be considered to rebound thus, so that the angle of incidence shall be equal to the angle of reflection, then each of the impulses will be agreed to be the same. Here the optimum pressure of the water is to be estimated in the manner, which depends on the latter reasoning.

§.18. Again it follows from the aforementioned favorable assertion to be understood correctly, the same effect to arise from the pressures, either the plate may deflect the water to the sides, or the cause may be imagined to be the absorption of all the motion, which the particles of water have acquired having left the cylinder.

Thence it is understood what shall happen, if the opening *CM* (Fig. 85) through which the water may flow out from the cylinder *ABM*, were it immersed in other still water in the vessel *PQFE* : truly the cylinder *ABM* will be pushed back towards *PQ* within the vessel *PQFE*, if this may not stick together with the cylinder; but if the vessel were joined together, the system would experience no prevailing pressure ; for as much as the pressure from the outflow of water towards *PQ*, just as much also arises from the opposite pressure towards *EF* from the continual destruction of the motion, which the particles leaving the cylinder acquire.

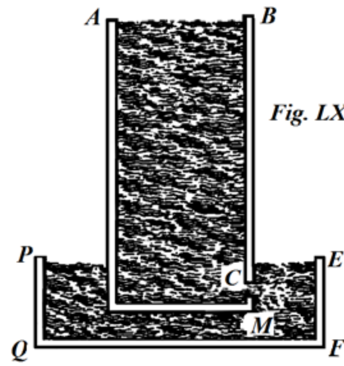


Fig. LXV

§.19. I have spoken about the pressure of the jet which the lamina intercepts, even if all the water spreads out: I come to another kind of impulse of the water, which clearly laminas sustain from a fluid on all sides: but I do not consider this able to be defined *absolutely*, because the individual particles striking the lamina may be deflected in any way. Truly if the deviation of any particle may be considered to be known, the solution of the question will not be difficult, with the theorem changed a little, as we have used in §.17, and with that returned more generally, truly to such: *if the angle of the change in the direction in the motion of the body were not right, but less than right; then also the sum of the products will be smaller (from what was mentioned before) in the ratio as the versed sine of the change of direction to the whole sine.*

[If a body departs from the orifice horizontally with the speed V along the x-axis, and reaches the lamina at some acute angle α to this axis, with velocity components $V_x = V\cos\alpha$ and $V_y = V\sin\alpha$, then the impulse along the x-axis will be $mV(1-\cos\alpha)$; *i.e.* in the ratio of the versed sine to the whole sine $(1-\cos\alpha):1$.]

Therefore for any droplet to be investigated, however great the direction of the motion be forced to change to be running from the obstacle or lamina. But in theories of this kind definitions can scarcely be shown to be accurate; nor do experiments approve theorems accustomed to be shown in this matter ; just as because the force of a river impinging directly against a disc shall be greater by a factor of two with the force of the same stream against a sphere of the same diameter, and which are similar: because moreover the amount of the pressure for a sphere, such as is usually given by authors, may agree well enough with the experiments conducted by Newton and others, and recounted in *Princ. Math. Phil. Nat.*, that with all the cases considered carefully, I assess to be fortuitously attributed.

In Book II *et seq* of the *Comm. Acad. Sc. Petrop.*, [CP II pp. 304-342], I have given theorems which were produced for the theoretical consideration of motion in resisting media, as well as various physical observations. Nor therefore shall I repeat these here, although they may pertain to our setup here; there is no time to tarry longer over these contemplations concerned with hydrodynamics: Therefore I hasten to the end. This new theorem about the reaction and impulse of fluids, which received overturned the opinion of all the authors at that time in a matter of great importance, which I have pursued in a single dissertation, which in its own time will be inserted into the *Com. Acad. Sc. Imp. Petropol.* [CP III pp.214-229], and I have confirmed the same with unquestionable experiments. I come now to another argument, worthy of the Geometers with minimum attention.

§. 20. At some time it crossed my mind, with regard to the force arising from the repulsion of the fluids, to be able from that while they were being ejected - and from each here I have set out the greater part - I have been considering to be usefully applied to setting up a new method of sailing : nor indeed do I see, what may stand in the way no less, by which large ships might be able to move forwards by this method without sails or oars, as water continually may be raised to a height and flowing out through holes in a deeper part of the ship, on requiring to be made that the direction of the water flowing out is observed towards the stern. Truly lest anyone should laugh at the outset about this belief, as exceedingly absurd, from that concern our argument itself will be sought out more accurately and reduced to a calculation: for it is able to be most useful and to be most abundant with many geometrical inquiries.

I shall begin from that, from which then it will be apparent, under which circumstances the maximum success ought to be expected from this new method of sailing.

§. 21. Therefore it is to be noted, a ship is slowed down continually by drawing up water on account of the inertia of the same, when the same velocity is shared with that as with which the ship bears, and while it shall be shared, the ship from the reaction of the water may be forced backwards, and likewise from the same outflow it may be sent forwards. This same coming together of contrary actions puts a limit on the force propelling ships obtained from a given absolute potential: for unless the first action be present (about which I may admit the truth I did not consider for a long time) it may be possible *by the work of men however small, a force propelling the ship however great, would be obtained,* which I show thus.

I have shown in Section nine (see especially §.26), the work of men expended in raising water, which I designate by the name *absolute potential,* is to be estimated from the product of the quantity of water by the height raised, thus so that for argument's sake all the following measures, both four cubic feet to be raised to a height of sixteen feet and sixteen cubic feet raised to a height of four feet, shall be with the same labour: Now again I say a uniform force to be present propelling ships forwards, as long as the fluids flow out with an equal velocity, which force shall be required to be estimated from the amount of water flowing out and from the square root of the height of the water put into the above vessel : let the amount of water flowing out in a given time be $=Q$, its height $=A$; the size of the opening with the water bursting out may be considered proportional to the quantity $\frac{Q}{\sqrt{A}}$ for the same time: but truly the repelling force, which here moves the ship forwards, is equal to the size of the opening multiplied by twice the height of the water (per §. 4), that is, equal to the amount $\frac{Q}{\sqrt{A}}\times 2A$ or $2Q\sqrt{A}$. From the comparison of each proposition it follows the labour of the men in raising the water drained to be to the force propelling the ship thence required to be obtained, as QA to $2Q\sqrt{A}$ or as $\sqrt{A}$ to some constant quantity : therefore when the height to which the water may be raised is smaller, there a greater the force moving the ship forwards is obtained from the same labour, *thus so that by the labour of the men any small amount of the force driving the ship shall be able to be obtained greater.* Truly also the inertia of the water, which is being drawn off (with regard to which we have spoken from the start of this paragraph), retarding the ships there may hold a greater ratio to the force propelling the ship, when the height *A* is assumed smaller, towards which the mind properly is required to turn towards here.

§. 22. It is clear from the preceding paragraph, the height to which the water is raised to be from that class, which is a maximum somewhere. Truly so that the maximum height for the proposed benefit may be determined, other questions first offer themselves requiring to be examined.

Problem.

A ship may be considered to be progressing with a uniform velocity, which is generated by [the equivalent of] a free fall through a height *B*, and the water may be imagined to flow uniformly into the ship, such as in the form of rain, and indeed with such an amount, as great as will supply the needs of a cylinder constantly full to the height *A* through an opening of size M, with all foreign hindrances removed. How much resistance the ship may experience perpetually is sought from this and the inertia of the same by the uniform inflow of the water.

Solution.

Some time is assumed *t*, which if it may be estimated from the distance which the inflowing fluid traverses with its velocity, divided by the same velocity, then the velocity is required to be expressed by $\sqrt{2A}$, and the amount of water inflowing in the time t is equal to a cylinder constructed above the base *M* of length $t\sqrt{2A}$: truly that amount, while it is carried away from the ship in the time *t*, assumes a velocity due to the height *B* and expressed by $\sqrt{2B}$: and thus the uniform force is sought, which shall be able in the time *t* to communicate to the water cylinder [of mass] $Mt\sqrt{2A}$ and a velocity $\sqrt{2B}$, and that force itself on account of the reaction, which acts on the ship in turn, may be agreed to be equal to the resistance sought. Let the aforementioned force $= p$, and there may be considered to be given the velocity *v*, and the water cylinder $Mt\sqrt{2A}$ at the time θ, and there will be

$$dv = \frac{pd\theta}{Mt\sqrt{2A}},$$

and

$$v = \frac{p\theta}{Mt\sqrt{2A}}:$$

and now there may be put $\sqrt{2B}$ for v & t for θ and there will be

$$\sqrt{2B} = \frac{p}{M\sqrt{2A}}:$$

or

$$p = 2M\sqrt{AB}.$$

Therefore the resistance sought is equal to the weight of a cylinder of water, of which the base shall be equal to the opening *M* and of which the length is equal to twice the mean proportional between the heights A & B.

Problem.

§. 23. In the ship there shall be a cylinder of height *A* above the surface of the sea, through the opening of which at the same surface put a cylinder and kept filled constantly with water of cross-section *M*, the water may flow out towards the stern without other impediment : to determine the force propelling the ship constantly.

Solution.

The force propelling the ship is equal to the reaction of the water while it flows out, or with the force of repulsion arising, diminished as in the preceding paragraph from the inertia of the water, which is being drawn off constantly. The repelling force is equal, by paragraph four of this section, to *2MA* and this moves the ship forwards: the other force which retards the ship is by the preceding paragraph $= 2M\sqrt{AB}$. Therefore the absolute force moving the boat forwards $= 2MA - 2M\sqrt{AB}$.

Corollary.

§. 24. If the ship may have no velocity, the force acting on the ship will be $= 2MA$ *;* and if the ship may be moving with the same velocity with which the water may be flowing out in the opposite direction, there becomes $B = A$ and then the ship is propelled by no force. Therefore even if the ship may be moving quite freely on the sea, yet it would not acquire from the action of the water, which is drawn off continually and flows out lower down, with a greater velocity than that, by which the water flows out, not because the water flows out with a uniform motion from the vessel with a smaller repelling force than from the unmoved vessel, but because then the inertia of the resisting water may produce an equal force of repulsion.

Problem.

§. 25. With the given power of the workers, who raise the water, and with the height given to which the water is raised, to find the cross-section of the opening of the efflux and the repelling force.
[Note that Bernoulli uses the word *potential* here as power or the rate of doing work; in the original Latin it means power in a military sense, meaning that the Roman army was capable of doing something, which is how the word is normally interpreted, and has come to us as potential energy, which Bernoulli calls the *actual potential*.]

Solution.

The power of such workers shall be, by which the number N cubic feet of water shall be able to be raised to a height of one foot in one second, which power the number of workers can exert by a force designated by $\frac{5}{4}N$, from the second experiment adjoined to Section nine. The height to which the water shall be raised continually $= A$ expressed in feet, the cross-section of the opening in square feet $= M$: the number of cubic feet of water, which the workers with a given power are able to raise to the height A per second, $= \frac{N}{A}$ (by §. 22 Sect. IX): therefore an orifice of this cross-section is required to be constructed, in order that in a single second this number of cubic feet of water shall be able to flow out through that, if flowing most freely. But we may accept in place of seconds of time, the time which the body spends while it falls freely from a height A : this time is expressed here by $\frac{1}{4}\sqrt{A}$ [*i.e.* from $s = \frac{1}{2}gt^2$] (for the sake of a neater calculation put in place, a body falling from rest resolves 16 feet in one second), and in this time the number of cubic feet of water is designated by $\frac{N}{A} \times \frac{1}{4}\sqrt{A}$ or $\frac{N}{4\sqrt{A}}$ must flow out: but actually *2MA* flows out, truly a cylinder of water of which the base is M and the length of which N makes twice the height A: therefore there is $\frac{N}{4\sqrt{A}} = 2MA$; from which the cross-section of the opening becomes

$$M = \frac{N}{8A\sqrt{A}}.$$

Moreover, the repelling force shall be equal to *2MA* or $= \frac{N}{4\sqrt{A}}$

Scholium.

§. 26. In any ship the water is raised to one height or another, so that by the same power it is raised to a different height, as by the same power, which is expended in raising water, the maximum force may be obtained in moving the ship forwards, and two items are required for that most useful height requiring to be defined for a certain number of workers.

Firstly, so that for instance the velocity the ship may acquire from a given power becomes known: on account of this postulate, we may consider the ship with a force, which shall be equal to the weight of one cubic foot of water or around 72 pounds, to acquire the velocity, which may be generated from the fall of the weight through a height *C*, and thereafter because always we will express all experiments in the measures of feet, the weight of a cubic foot of water is required to be expressed by [the number] one.

Secondly, by assuming as known the relation between the speed of the ship and the power propelling the ship: this generally is set up to have the velocities in the square root ratio of the propelling forces ; indeed experiments do not confirm this hypothesis exactly with slow motions; yet meanwhile we may agree that being preferred before the rest. If anyone may wish to this matter under other hypothesis, this can be done in the same manner as we use now to put the calculation in place.

Problem.

§. 27. To find the height, to which water is required to be raised continually, by putting in place the most useful, truly such, so that by the same power being used in raising the water the maximum force may be generated in moving the ship forwards.

Solution.

All the denominations used in this argument may be kept : before all, the speed of the ship being examined, or the height corresponding to this velocity, we will call *B*. Truly because the velocities of the ship may be put to be proportional to the roots of the potentials acting on the ship, the heights of the velocities will be proportional to the potentials themselves. Therefore such an analogy is required to be put in place.

Just as the weight of one cubic foot [of water] shall be to the height *C* (cf. §. 26) thus the force acting on the ship or $2MA-2M\sqrt{AB}$ (see §. 23) shall be to the height corresponding to the velocity, which hence will be $B=2MC\times(A-\sqrt{AB})$. Hence truly we have called the height B. And thus there shall be

$$B=2MC\times(A-\sqrt{AB}).$$

Thence the force acting on the ship will be $=\frac{B}{C}$, and thus proportional to the height *B*, because *C* is a constant quantity: therefore both the force moving the ship forwards as well as the height corresponding to the velocity of the ship likewise shall become maxima: Therefore if for the present set up the quantity $2MA-2M\sqrt{AB}$ may be differentiated, which expresses the force propelling the ship, it will be able to consider $dB=0$. Truly in the first place as it may be established by differentiation it is necessary to substitute the value for M from §.25 [i.e. $M=\frac{N}{8A\sqrt{A}}$.], and then the force moving the ship forwards $=\frac{N}{4\sqrt{A}}-\frac{N\sqrt{B}}{4A}$, in which the letter N is constant, truly the letters B and A are variable. Now its differential may be taken, by making $dB=0$, and that becomes $=0$; and thus there will be found $A=4B$.

[On differentiating: $\frac{d}{dB}\left(\frac{1}{\sqrt{A}}-\frac{\sqrt{B}}{A}\right)=-dA\left(\frac{1}{2A\sqrt{A}}+\frac{\sqrt{B}}{A^2}\right)+\frac{dB}{2A\sqrt{B}}=0.$ When $dB=0$ there becomes $\frac{1}{2A\sqrt{A}}+\frac{\sqrt{B}}{A^2}=0$, or $A=4B$.]

Therefore the force moving the ship forwards is a maximum when the height, to which the water is raised, is four times the height corresponding to the velocity of the ship.

$A=4B$ found above may be put into the formula $B=2MC\times\left(A-\sqrt{AB}\right)$ and there will be found : $M=\frac{1}{4C}$, and because (by §. 25) $M=\frac{N}{8A\sqrt{A}}$, then there becomes

$A=\left(\frac{1}{2}NC\right)^{\frac{2}{3}}$, [*i.e.* the maximal height to which the water is pumped corresponding to the velocity of the ship;]

and

$B=\frac{1}{4}\left(\frac{1}{2}NC\right)^{\frac{2}{3}}$. [*i.e.* the maximal height corresponding to the speed of the water.]

Corollary.

§. 28. If the orifice according to the precepts of the preceding paragraph, through which the water flows out from a channel below towards the stern, may be fitted out with a cross-section $\frac{1}{4C}$, that is such, which it may itself have to a cross-section of one square foot, just as a measure of one foot shall have to the height quadrupled corresponding to the velocity of the ship, moving along by a force of 72 pounds, it then comes about that the ship may be carried along with half the velocity of that by which the water flows out, and the repelling force of its water flowing out will be half of this, and thus half the effect may be lost from the inertia of the same water, which is raised continually.

[*i.e.* if $M=\frac{1}{4C}=1$ then $C=\frac{1}{4}$; and $A=\left(\frac{1}{2}NC\right)^{\frac{2}{3}}=\frac{N^2}{4}$; and

the max. condition gives $A=4B$ and $B=\frac{N^2}{16}$;

giving the speeds $v_A=\frac{N}{2}$ and $v_B=\frac{N}{4}$.]

Scholium.

§. 29. Thus after we have shown, how this method of sailing may be put in place with the greatest usefulness, now again I think that idea requires to be illustrated by an example of such, because with the nature of that matter I do not believe it will be bad to be agreed more or less so that it may be apparent likewise, that such an event may be going to take place.

We will consider the trireme, the common galley, with 260 oarsmen: we may consider this galley to be drawn by a weight of one cubic foot of water of 72 pounds per second to complete a distance of two feet, [in non-metric units this is around 150 ft.lb. per sec or approx. 0.4 hp; 1 hp is 550 ft.lb. per sec.] the generating height of which velocity is indicated by $C=\frac{1}{16}$, [*i.e.* $v^2=2gs$ gives here $4=64s$, or $s=C=\frac{1}{16}$;] with a body freely falling from rest completing 16 *ft.* in the first second. Again because 260 workers are used, of whom any by the force according to the second experiment pertaining to Sect. IX is able to raise four fifths parts of a cubic foot to a height of one foot in one second, there will be $N=\frac{4}{5}\times 260=208$ [~ 27.6hp.] Therefore the opening becomes, through which the water may flow in, of a cross-section 4 square feet :
[optimally, $M=\frac{1}{4C}=\frac{16}{4}=4sq.ft.$] and the workers can put the water into a channel to be kept above the opening raised to a height of approximately $3\frac{1}{2}$ *ft.* which is indicated by the letter *A*, [for $A=\left(\frac{1}{2}NC\right)^{\frac{2}{3}}=\left(\frac{1}{2}\times 208\times\frac{1}{16}\right)^{\frac{2}{3}}=3.5\,ft.$] and if you take the fourth part of this height [as $A=4B$,optimally] you will have $B=\frac{7}{8}$ *ft.* thus so that the ship will be sailing along with such a velocity, as acquired by a weight falling freely through a height of $\frac{7}{8}$ *ft.*; [*i.e.* $\sqrt{56}\sim 7.5\,ft./sec$] thus the ship therefore may complete a distance of $7\frac{1}{2}$ *ft. per sec*. and 27000 *ft. per hour*, that is, more than two Gallic miles per hour: a ship by rowing scarcely can obtain so great a velocity or indeed not at all.

Now truly I will construct a calculation from another hypothesis, which I trust an understanding of nautical matters will not be going to disapprove excessively: since it squares up with many observations which I myself have made at sea : I assume the expanse of sails to have an area of 1600 square feet perpendicular to the keel of the galley, and these to be receiving the wind by direct impact, which traveled through a distance of 18 *ft. per second*, the ship truly thus in the same direction completed 6 feet in one second . Thus the wind ran into the sails with a respective speed of 12 ft.: the force of this wind I estimate = to a weight of $\frac{9\ \times 1600}{850}$, or nearly 17 cubic feet of water. [This formula arises from the momentum transfer per second :

$$[F = \Delta p / \Delta t = 2\times\rho\times Q\times V = 2\times\rho\times A\times V\times V / g = 2\times\left(V^2 / g\right)\times\rho\times A$$

$$= 2\times\left(144/32\right)\times 1600\times 1/850 \simeq \frac{9\times 1600}{850}];$$

assuming a completely inelastic collision, and the final result expressed as a weight in pounds, on dividing by *g*; in which case, where 850 is the relative density of water to air.]

Thus if they shall be, it follows the ship from the raising of the water of 260 workers to be able to be propelled with that velocity, which traverses $6\frac{1}{2}$ feet in one second.

An estimate not very different from these follows, which de Chazelles has published in *Comm. Acad. Reg. Sc. Paris.* for the year 1702, p. 98. Truly in order that they may be applied correctly according to our set-up, it is required to be noted, in the force propelling the trireme it is not required to be estimated from the force of the oarsman on the oars, but from the force, which the ends of the oars submerged exert against the water. So that we may define this approximately, these are to be observed initially. The oarsmen were to have used 260 with all the oars: in the first minute 24 strokes of the oars were made (*palades* in French) : the whole disturbance of the oars can be resolved into three motions, which I consider of the same duration, and of these one alone moves the trireme forwards: in this manner the trireme was carried forwards with a velocity, by which a distance of $7\frac{1}{5}$ *ft.* was accomplished in one second, the part of the oar within the ship was 6 feet and outside the ship 12 feet : moreover the surface of all the oars (in French *les pales* or blades), which were thrust against the water, gathered together de Chazelles made to be 130 square feet: again it was observed the internal end of the individual oars in their motions describe a distance of six feet : and because whatever motion was completed in the time of $\frac{60}{24}$ of a second consisting of the three motions, which I consider to be completed in the same time, however it is apparent the retraction of the oar to made in the time $\frac{20}{24}$ or $\frac{5}{6}$ of a second, and in this time the inner end of the oar completes a distance of 6 feet. Again, on account of the length the surface of the oars, which is impelled against the water, requiring to be agreed not with the whole to be at a distance of 12 feet: therefore I put that to be distant by 10 feet, surely as if the part of the oar should jut out a length 10 feet beyond the ship: the end of this part described 10 feet in a time of $\frac{5}{6}$ one second: truly because the trireme has that velocity, by which it accomplishes six feet in the same time, it is agreed, the ends of the oars to be thrust against the water with that respective velocity, by which it may describe 4 feet in a time of $\frac{5}{6}$ *sec.* : therefore the force propelling the trireme is equal to the force which the water exerts on a surface of 130 square feet, if it may run into that with a velocity, by which it may complete 4 *ft*. in a time of $\frac{5}{6}$ *sec.* : this force following common estimation I found besides properly to equal the weight of 40 *cubic ft.* of water; truly this force will not be applied continually, but only in the time in which the oars are being drawn back: therefore two thirds of this force are to be removed, thus so that the force which continually propels the trireme, at last is required to be agreed equal to the weight of $13\frac{1}{3}$ *cubic ft.* of water.

Thence it follows, if the velocities of the ship may be put to follow the square root ratio of the propelling forces, because this same trireme shall have a velocity from the impulse of a single cubic foot of water, by which it shall be possible to complete approximately two feet in one second; which hypothesis is the same as that, which we have used in the first place, thus so that thence on the contrary it may follow the trireme is going to acquire a velocity from that [method of] sailing, from which it may be possible to complete $7\frac{1}{2}$ feet in a second, which velocity is a little greater than that, which the trireme was given by the strongest 260 oarsmen.

With these matters well considered I hesitate, whichever kind of sailing shall be preferred, either by rowing or by raising water; the success of each I would believe to be almost equal, and for certain I dare to affirm, if a ship may be moved forwards less from the raising of water, the failing to be small: but perhaps it will be moved forwards more. Meanwhile I have no doubt, why this new idea of sailing may appear empty and laughable by those ignorant about the matter. Truly I feel otherwise and I wish to start thinking about the following.

[Even if Bernoulli's method of propulsion did work better, the design could have suffered from the severe inconvenience of raising the centre of gravity of the ship above the metacentric height, meaning it would be unstable in water and liable to capsize.]

In the first place. Because water generally is able to be raised conveniently in all ships, where plainly oars may be unable to be used, so thus by this new method of sailing very heavy warships also, for which they are used in navel battles, so that they shall be able to be driven as it pleases with the lack of all wind.

Second. Because such an example may be had in theory, the driving forces or propellants to be given, which I have said are able to be intrinsic: by this example ingenious people will be awakened towards understanding other principles of motion and these requiring to be perfected and used for sailing.

Third. Because the labour of men is able to raise water in many ways other than can come about by the use of oars : truly there are natural things endowed with a remarkable and almost unbelievable strength and these are to be prepared for a reasonable price besides what can be effected by the labors of man: the uses of these especially can be put into service for the shortest passages in a clear and quiet season. By virtue of this same natural method implanted in things, with reference to the effects thence obtained and the measures I have gone into in Sect. X, §.40, and following: but in the first place I wish that §. 43 be attended to, where anyone with a natural ingenuity for understanding machines would be happy, and where they should be excited to be testing the completion of this matter.

Fourth. Because not only other purely mechanical savings can be used in a manner similar to that which was given in §. 27, truly with the help of these from the same labour the effect in moving ships forwards increases significantly: Truly now not everything is handled following the true nature of the matter.

Experiments in Section thirteen.

So that the repelling force may become known correctly by experiment, a vessel will be able to be uses which may have a parallelepiped form and of which the weight to be taken both empty as well as full of water, and afterwards the ratio between the cross-section of the vessel and the cross-section of the opening to be investigated, which must be in the side of the vessel, and just as the ratio between the heights of the water above the opening and above the base must be taken: Thence it will be allowed to deduce the ratio between the weight of the vessel filled with water and the weight of the cylinder of water placed vertically above the opening. Again from the observed cross-section, the velocity of the water jet will be obtained: from that, if likewise you may bring together equally the amount of water flowing out in a give time, you can deduce the cross-section of the contracted jet, that you will be able to compare with the cross-section of the opening.

With all these investigated the vessel likewise carefully may be suspended from a long thread, so that it may not have another motion, than what shall be opposite to the direction of the water flowing out. Then finally from the efflux of the water it may be conceded and the thread will be observed if it moves away from the vertical, and from the angle of inclination the repelling force will be known and with that measured, as we have indicated , it will be comparable.

Experiment 1.

When I did all four, as now I have indicated, and it was seen that our rule in §. 2 was confirmed correctly: yet I was not able then with sufficient time to set up the experiment accurately, nor did I repeat it later.

Experiment 2.

At another time I tested the matter in another way: truly the vessel from which all the required measures were obtained was in the stern of a small ship I had put in place filled with water: the boat floating in a trough of water : Then with the water flowing out from the vessel (yet thus so that is did not push against the boat) the boat moved forwards in the opposite direction: I investigated the velocity of the boat from the distance traversed in a straight line in a given time. Then I investigated how much weight were to be attached to the weight of the boat, so that by being acted on by that weight it would acquire the same velocity. Then with the comparison put in place of this weight with the weight of the cylinder of water of the given diameter, I say that our theorem to be confirmed correctly.

Experiment 3.

With the outflow from the vessel of the boat superimposed on the boat, this generally remained without moving: That indicated the impulse of the jet to be equal to the repelling force, as I have shown in §§.16 and 17. Then also if the jet of water struck the fixed plane fixed to the boat this also stood unmoved, because in return it approves the equality of the impulse and of the repelling force: but if the jet struck the plane obliquely, the boat obtained a certain motion, but smaller.

Finally if the water flowing out from the boat was being intercepted thus, so that the opening for the water was submerged in still water, it persisted similarly without the boat moving, with the lesson, because the same force may arise from the jet, whether it happens that it may be held together with all its motion, or that it may be turned round by a right angle, as was shown in §.18; the equality between the repelling force and the force of the jet of water incident normally on the plane I have confirmed most exactly by other ways. Moreover I have confirmed this strength of our theory conforms with experiment and to be contrary to the common opinion with greater exception to everything up to the present because with Emanuel Koenig, my cousin Nicolas Bernoulli and my father present in my house I have set up with some much confidence, so that with all the measures of the forces of the jet of water taken, how much it was going to become, even if never before with the experiment done by me, I predicted everything precisely. All this with the new principles of mechanics elicited I have communicated to the *Petersburg Academy of Sciences,* in the *Commentaries* of which they will be inserted at some time.

Experiment 4.

Also so that the falsity of the received rule may be shown concerning both the repelling force as well as the impulse of the water, I have used such a vessel as shown in Figure 86, with a channel of constant cross-section *AB* inserted and with a curvature, the direction of which was horizontal at *A* and vertical at *B*: I saw clearly the vessel was not repelled horizontally ; therefore by §.14 the rule, which corresponds to the definition for a simple cylinder in that place, is false.

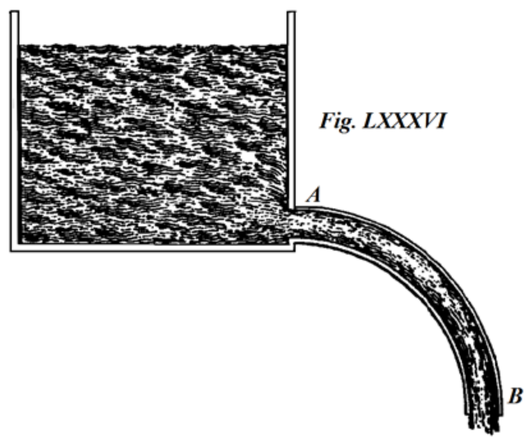


Fig. LXXXVI

FINIS.

HYDRODYNAMICAE SECTIO DECIMA TERTIA.

De reactione fluidorum ex vasis effluentium eorundemque, postquam effluxerunt, impetu in plana quibus occurrunt.

§. 1. Aquae dum ex vase ejiciuntur simili agunt modo in vas, ex quo effluunt, quo globus in tormentum bellicum aut sclopetum, ex quo exploditur: vas nempe retropellunt; & id quidem jam annotavit Newtonus in *Princ. Math. phil. nat.* edit. prim., *p.* 332, recteque inde deducit ascensum pilarum, quae pulvere pyrio carbone temperato implentur; materia enim inflammata, dum per foramen paullatim expirat, pilas in altum projicit.

Sed nec satis generaliter pro rei momento argumentum pertractavit citatus auctor (cum id ex ipsius instituto non erat) nec veram rei mensuram dedit. Imo in duabus editionibus posterioribus id prorsus silentio praeteriit: putavit autem *vim illam repulsionis esse aequalem ponderi cylindri aquei, cujus basis sit orificium aquas transmittens* & *cujus altitudo sit aequalis altitudini superficiei aqueae supra foramen.* Recte quidem haec mensura deducitur ex opinione, quam tunc temporis fovebat Newtonus circa velocitatem aquae ex vase effluentis, dum statueret aquam ad dimidiam superficiei altitudinem sua velocitate ascendere posse.

Prouti autem hujus propositionis falsitas nemini amplius nunc ignota est, ita & alterius defectum inde quivis facile colliget, quamvis prima fronte satis verisimilis.

§. 2. Considerabimus primo rem in casu simplicissimo, quo nempe aquas ex vase infinitae amplitudinis horizontaliter effluere ponemus. Habeo autem demonstratum repulsionis vim non statim a fluxus initio totam adesse, nisi quatenus & ipsa velocitas in aquis effluentibus tota adsit, ita ut si vas non sit infinitae amplitudinis, vis repulsionis una cum velocitate aquarum effluentium sensim sensimque crescat, aut etiam decrescat pro circumstantiarum natura: Ab his autem mutationibus momentaneis animum primo abstrahemus, fluxum ex vase infinito fieri aequabilem ponendo. Atque sic optime definietur vis repulsionis, si inquiratur, quaenam sic vis ad motum producendum requisita: Hunc vero in finem non solum ad velocitatem aquae effluentis, sed & ad illius quantitatem erit respiciendum; quantitas autem pendet partim a magnitudine orificii, partim a contractione venae, quae posterior variabilis est: Vidimus quidem in Sect. IV posse totam evitari; si tamen quaedam sit, erit sectio venae maxime contractae sive attenuatae ceu orificium considerandum & tunc dico fore *vim repulsionis aequalem ponderi cylindri aquei, cujus basis sit orificium aquas transmittens* (id est, sectio venae horizontalis maxime contractae) *& cujus altitudo sit aequalis duplae altitudini superficiei aqueae supra foramen vel accuratius, duplae altitudini velocitati aquae effluentis debitae.* Igitur si nulla sit venae contractio, prouti nulla est, cum per tubulum brevem aquae effluant, repulsio duplo aut fere duplo major erit, quam a Newtono definita fuit.

§. 3. Ut hanc propositionem demonstremus, considerandum hic erit principium aliquod Mechanicum cujus usum in aliis etiam quaestionibus solvendis saepe expertus sum: principium hoc est:

Si corpus a quiete velocitatem eandem per pressiones motrices directas utcunque variabiles acquisiverit, atque singulae pressiones in tempuscula sua multiplicentur, erit summa omnium productorum semper eadem, id est, si pressio sit $= p$ *, tempusculum* $= dt$ *, erit* $\int pdt$ *constans.* Hanc rem clarius exposui in *Comment. Acad. Imp. Petrop. tom.* I, *pag.* 132.

§. 4. Ponamus jam cylindrum infinitae veluti amplitudinis, ex quo aquae horizontaliter effluant velocitate uniformi, abstrahendo ab actione, quam gravitas exerit in particulas, postquam jam effluxerunt, ita ut singulae horizontaliter & uniformiter moveri pergant; particulae autem accelerantur pressionemque patiuntur, quamdiu maximus velocitatis gradus nondum adest, huncque obtinent cum ad locum venae maxime contractae pervenerunt; haec est ratio, quod sectionem venae ibidem conceptam ceu orificium effluxus considerandum esse dixi. Sit amplitudo istius sectionis $=1$, habeantque ibi aquae velocitatem quae debeatur altitudini *A:* ponatur, cylindrum aquae effluxisse, qui pro base habeat 1 & pro longitudine *L:* si tempus exprimatur per spatium divisum per velocitatem, erit velocitas altitudini *A* debita exprimenda per $\sqrt{2A}$, tempusque fluxus per $\frac{L}{\sqrt{2A}}$ His praemissis indagabimus in pressionem motricem, quae possit tempore $\left(\frac{L}{\sqrt{2A}}\right)$ cylindro *L* communicare velocitatem $\sqrt{2A}$: sit illa pressio $= p$: putetur brevioris calculi ergo egisse tempore *t* cylindroque dedisse velocitatem *v;* erit $dv=\frac{pdt}{L}$ & $v=\frac{pt}{L}$, hince $p=\frac{Lv}{t}$; ponatur jam $\sqrt{2A}$ pro *v* & $\frac{L}{\sqrt{2A}}$ pro *t* atque erit

$$p=\left(L\sqrt{2A}\right):\left(\frac{L}{\sqrt{2A}}\right)=2A.$$

Est igitur pressio aquam ad effluxum constanter sollicitans aequalis ponderi cylindri aquei, cujus basis sit orificium aquas transmittens supra definitum & cujus altitudo sit aequalis duplae altitudini velocitati aquae effluentis debitae: & tanta quoque est reactio, quae vas repellit. Q. E. D.

§. 5. Eadem est demonstratio si aquae non per orificium sed per tubum horizontalem cylindricum velocitate uniformi effluant, aut etiam per tubum utcunque inaequaliter amplum: posterius id directe demonstrari etiam potest, si bene exprimatur pressio requisita in singulis guttis, ut hae debita velocitatum incrementa aut decrementa suscipiant.

§. 6. Altitudo, quam vocavimus *A*, parum quidem differt in experimentis ab altitudine aquae supra orificium effluxus, praesertim si aquae ex vase valde amplo per orificium simplex, idque non admodum parvum effluant: differt autem saepius notabiliter orificium effluxus a sectione minima venae, quam nos ceu orificium aquas transmittens consideramus; id quantitas aquae dato tempore effluentis cum velocitate sua comparata in experimentis indicat.

Hinc fit ut propositio nostra §. 3 ad experientiam vocata ordinaria non multum discrepet ab propositione Newtoni §. 1 exposita; si vero omnia sollicite evitentur, quae contractionem venae producere & quae velocitatem diminuere possunt, vis repellens secundum theoriam nostram fiet tantum non duplo major, quam quae a Newtono fuit definita & tunc talis etiam experimentis confirmatur.

At ut rem plane in apricum ponamus, eam generalius nunc prosequemur, idque tentabimus, ut vim repellentem a fluxus initio, dum velocitates continue mutantur, determinemus: neque enim primum nostrum theorema aliter quam cum velocitas invariata manet locum habet. Ut in quaestione hac paullo intricatiore pertractanda eo intelligibiliores simus, hic quaedam generaliora praemonuisse juvabit.

§. 7. *Quantitas motus* est factum ex velocitate in massam: si velocitates sint inaequales, habebitur *quantitas motus absoluta,* si singulae particulae per suam respective velocitatem multiplicentur productorumque summa accipiatur. *Quantitas motus* generatur a pressionibus motricibus dato tempore urgentibus & effectus causae est aequalis censendus: Igitur summa pressionum motricium per sua tempuscula multiplicatorum aestimanda est ex genita quantitate motus. Et quia quaelibet pressio motrix reagit in vas, ex quo aquae effluunt, erit tota vis repellens pro quovis momento aequalis novae quantitati motus divisae per tempusculum, quo generatur. His praemonitis ad quaestionem ipsam progredior.

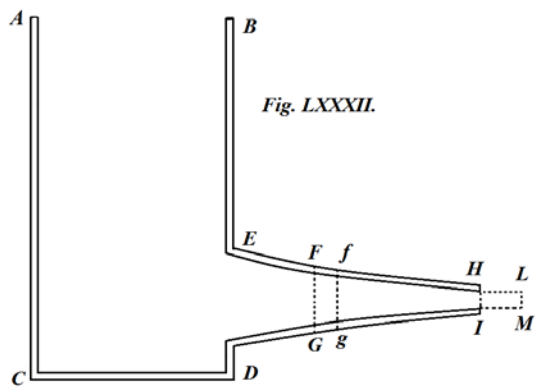


Fig. LXXXII.

§. 8. Sit igitur vas infinitae amplitudinis *ACDB* (Fig. 82) eique horizontaliter infixa fistula *EHID,* cujus amplitudines utcunque inaequales ponuntur: amplitudo orificii *HI* fuerit $=1$, longitudo fistulae $=m$*;* velocitas utcunque variabilis in $HI=\sqrt{2v}$, seu talis, quae debeatur altitudini *v*: dico primo, fore quantitatem motus absolutam aquae in fistula contentae aequalem $m\sqrt{2v}$, id est, talem ac si fistula esset cylindrica suaque amplitudine orificium *HI* exaequaret, quia nempe cujuslibet strati *FGgf* velocitas est massae reciproce proportionalis.

Jam vero fingamus dato tempusculo infinite parvo exilire per orificium *HI* columellam *HLMI,* cujus longitudinem *HL* vel *IM* ponemus $=\alpha$: erit massa hujus columellae $=\alpha$, habebitque quantitatem motus $=\alpha\sqrt{2v}$: sed eodem tempore massa aquae in fistula contentae acquisivit quantitatem motus $\frac{mdv}{\sqrt{2v}}$ (habuit enim $m\sqrt{2v}$); est igitur quantitas motus absoluta dato tempusculo genita $=\alpha\sqrt{2v}+\frac{mdv}{\sqrt{2v}}$, haec vero si dividatur per idem tempusculum (quod exprimendum est per $\frac{\alpha}{\sqrt{2v}}$), habebitur, ut vidimus §. 7, pressio quaesita vas repellens, quae proinde si vocetur *p*, erit

$$p=\left(\alpha\sqrt{2v}+\frac{mdv}{\sqrt{2v}}\right):\frac{\alpha}{\sqrt{2v}},$$

sive

$$p=2v+\frac{mdv}{\alpha}.$$

(α) Apparet inde ultimam definitionem quaestionis pendere a ratione quae intercedit inter $dv\,\&\,\alpha$; hanc vero in Sectione tertia generaliter definivimus, nulla tamen impedimentorum, quae debentur casui, facta attentione. Igitur & figura fistulae hic aliquid confert.

(β) Sequitur porro, si fluxus uniformis factus ponatur, esse *p* constanter $=2v$, quia tunc $dv=0$: Id vero conforme est cum eo, quod demonstravimus §. 5. Donec vero fluxus incrementa accipit (quod quidem facit notabiliter, idque diu satis, si canalis *EI* longior fuerit), vas aliam atque aliam patitur vim repellentem.

(γ) Habet *dv* ad α semper rationem realem: ergo vis repellens nunquam est nulla, sic ut a primo fluxus tempore vas repellatur, etiamsi tunc aquae fere nullae effluant ob exiguam earundem velocitatem. Verum, ut usus regulae nostrae generalis unicuique pateat, eam nunc ad casum specialem applicabimus, tribuendo fistulae *EHID* figuram cylindricam amplitudinis 1.

§. 9. Si igitur fistula ponatur cylindrica tota aperta in *HI* retentis caeteris positionibus & denominationibus, erit *vis viva* aquae in fistula contentae $=mv$; hujus incrementum $=mdv$, cui addenda *vis viva* columellae *HIMI* seu αv, eorumque summa aequalis facienda facto ex altitudine, quam habet superficies aquae *AB* supra orificium *HI*, quamque vocabimus *a*, & ex massula α. Est igitur $mdv+\alpha v=\alpha a$, unde hic fit $\frac{dv}{\alpha}=\frac{a-v}{m}$. Isto autem valore substituto in aequatione superioris paragraphi fit

$$p=a+v,$$

unde talia deduco consectaria.

(α) Longitudo fistulae nihil ad vim repellentem, quam vas sustinet, tribuit, si velocitas eadem ponatur, quia littera *m* e calculo evanuit; facit autem haec longitudo (sicuti in superioribus satis superque demonstravimus) ut velocitates citiora aut lentiora incrementa capiant; quo longior enim fuerit fistula, eo tardius accelerantur aquae & vicissim, sic ut in instanti a quiete maximum suum celeritatis gradum acquirant, si longitudo fistulae nulla fuerit; at si infinitae fuerit eadem haec fistula longitudinis, aquae nonnisi post tempus infinitum notabilem celeritatis gradum acquirere possunt.
(β) Fieri igitur potest non mutata aquarum altitudine, ut dispendio aquarum quantumvis parvo vis repellens notabilis sit, eaque pro lubitu duret; & id quidem duplici obtineri potest modo, tum prolongando fistulam, tum etiam obturando saepius orificium, antequam aquae notabilem velocitatis gradum attigerint; prior tamen modus liberum aquarum fluxum per fistulam ponit: retardato enim ab impedimentis extrinsecis, in praelongis fistulis nunquam vitabilibus, aquarum fluxu, diminuitur quoque vis repellens.

(γ) Liceat hic paucis attingere verbis propositionem aliquam ex *Princ. math. phil. nat.* edit. 2 Newtoni: Auctor hic postquam sententiam suam de velocitate aquarum ex vase effluentium in prima citati operis editione exhibitam mutasset, easque, si verticaliter sursum ejiciantur, ad integram superficiei aquae altitudinem ascendere agnovisset in editione secunda, talia subjecit verba in *libro secundo, propos.* 36, *coroll.* 2: *Vis qua totus aquae exilientis motus generari potest, aequalis est ponderi cylindricae columellae aquae, cujus basis est foramen EF* (vid. fig. Newt.) *& cujus altitudo est* 2*GI vel* 2*CK.* Ista sententia a me olim & ab aliis fuit impugnata, ab aliis rursus confirmata. Nunc autem postquam hanc aquarum motarum theoriam meditatus sum, lis ita dirimenda mihi videtur, ut cum aquae ad motum uniformem pervenerint, quae quidem hypothesis est Newtoni, tunc recte altitudine 2*GI* vis illa definiatur, sed ab initio fluxus, ubi velocitas adhuc nulla est, vis simplici altitudini *GI* respondeat, moxque crescente velocitate simul vis aquam ad effluxum animans crescat, & tandem ad eam magnitudinem exsurgat, quam Newtonus assignavit. Haec nunc sunt unicuique obvia, quia vis motum aquae generans, de qua Newtonus loquitur, non potest non esse aequalis vi repellenti, quam vidimus esse aequalem $a+v$. Recte etiam Ill. Riccatus, cum quo mihi de hoc argumento res erat, interrogatus, *unde vis illa duplae aquarum altitudini conveniens oriri possit, cum obturato orificio gutta eidem imminens vi simplicis altitudinis urgeri manifeste appareat,* respondit, *distinguendum esse statum quietis a statu motus* .

§. 10. Si fistula vasi implantata non sit cylindrica, calculus ita erit ponendus. Sit amplitudo canalis in *FG* vel $fg=y$, distantia strati *FGgf* ab orificio $ED=x$, retineanturque caeterae denominationes: erit *vis viva* aquae in fistula contentae $=v\int\frac{dx}{y}$, ejusque incrementum $=dv\int\frac{dx}{y}$, cui, ut in § praecedente factum est, addatur *vis viva* columellae *HLMI* seu αv, eritque

$$dv\int\frac{dx}{y}+\alpha v=\alpha a;$$

unde sic oritur

$$\frac{dv}{\alpha}=(a-v):\int\frac{dx}{y},$$

quo valore substituto in aequatione §. 8 fit

$$p=2v+m(a-v):\int\frac{dx}{y}.$$

Igitur cum in fluxu aquarum uniformi sit $v = a$, erit tunc rursus $p = 2a$. Caeterum quamdiu aquarum fluxus acceleratur, motus aquae in vase *ACDB* orificio *DE* proximae, a quo in toto hoc opere animum abstraximus, hic non est negligendus: determinari autem recte motus iste non potest, nec igitur accurate quadrat expressio quam dedi pro vi repellente si aquae nondum uniformiter fluere ceperint, sed cum aequabiliter fluunt aquae valet expressio accuratissime.

§. 11. Postquam sic demonstravimus pro effluxu aquarum uniformi, vim repellentem semper esse aequalem ponderi cylindri aquei foramini superinstructi & ad duplam aquae altitudinem exsurgentis, lubet id etiam indirecte demonstrare per *deductionem ad absurdum*, ut & regularum mechanicarum ignari propositionis hujus satis paradoxae veritatem perspiciant.

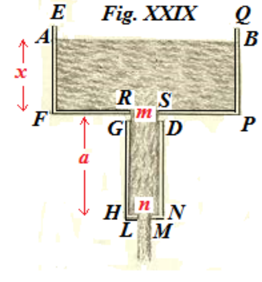
Hunc in finem considerabimus aquas verticaliter defluentes ex cylindro, abstrahendo animum ab impedimentis velocitati aquarum aliquid derogantibus & ab illa contractione venae, quae vitari potest. Foramini respondeat tubus verticalis, qualis conspicitur Fig. 76; habeant se omnia, ut in Sect. XII §. 13 dictum fuit: aquae habeant fluxum aequabilem; latera vasis & canalis gravitate carere intelligantur; altitudo cylindri ponatur $=a$, & altitudo fistulae $=b$, altitude $CF=x$, amplitudo in $E=1$; erit amplitude in F $=\frac{\sqrt{(a+b)}}{\sqrt{(a+x)}}$ & in $C=\frac{\sqrt{(a+b)}}{\sqrt{a}}$; denique ponatur amplitudo cylindri $=M$. His positis quaeremus pondus omnis aquae *ABCE*: exprimemus pondus aquae *ABC* per *Ma* & sic erit pondus aquae $CE=2a+2b-2\sqrt{(aa+ab)}$; ergo pondus omnis aquae $ABCE=Ma+2a+2b-2\sqrt{(aa+ab)}$: Sic igitur posito aquas stagnare in vase & fistula, vis requisita ad suspendendam aquam est

$$=Ma+2a+2b-2\sqrt{(aa+ab)}.$$

Jam vero indagabimus vim similem cum aquae per *E* tota sua velocitate (qua nempe ad altitudinem $a+b$ ascendere possunt) effluunt: haec autem habebitur, si a priori vi subtrahatur vis repellens: Si proinde haec vis repellens ponatur, ut nos statuimus, $=2a+2b$,erit vis aquas durante fluxu suspendens $=Ma-2\sqrt{(aa+ab)}$.

At vero finge abesse tubum *CE*, & erit per easdem nostras regulas vis suspensoria, dum aquae per orificium *C* erumpunt, rursus $= Ma - 2\sqrt{(aa+ab)}$, ideo, quia pondus aquae *ABC* est *Ma* & quia amplitudo foraminis *C* est $\frac{\sqrt{(a+b)}}{\sqrt{a}}$, quae multiplicata per duplam altitudinem *a* dat $2\sqrt{(aa+ab)}$. Monstrat igitur nostra virium repellentium aestimatio, vim suspensoriam durante aquarum effluxu eandem esse, sive nulla sit fistula, sive adsit & quamcunque habeat longitudinem, modo fistula figuram habeat §. 13 Sect. XII descriptam: atque hujus consensus & identitatis necessitas apparet quoque sine calculo ex ipsa rei natura, quando fistula ita formata nullam in aquis transfluentibus facit mutationem, cum vena aquae sua sponte eandem figuram induit, quam habet fistula, quamdiu aquae cohaerent. Sed si aliter vim repellentem aestimemus, nunquam consensum illum inter vires suspensorias generaliter obtinebimus: Ita v. gr. si secundum sententiam communem dicamus vim repellentem esse aequalem ponderi simplicis cylindri saepe nominati, erit vis repellens, dum aquae per canalem *CE* ex vase *ACB* effluere finguntur, $= a + b$*;* & haec vis si subtrahatur a pondere totius aquae *ABCE* seu $Ma + 2a + 2b - 2\sqrt{(aa+ab)}$, relinquitur $Ma + a + b - 2\sqrt{(aa+ab)}$, quae est vis requisita ad suspendendum systema *ABCE*, dum aquae fluunt: Vidimus autem hanc vim eandem esse debere, si canalis *CE* absit: Sed tunc est vis suspensoria $= Ma - \sqrt{(aa+ab)}$, quia pondus aquae *ABC* est $= Ma$ & vis repellens per hypothesin est simplex cylindrus foramini *C* ad altitudinem *a* superinstructus. Deberet igitur in hac hypothesi semper esse $Ma + a + b - 2\sqrt{(aa+ab)} = Ma - \sqrt{(aa\ +ab)}$ seu $a + b = \sqrt{(aa+ab)}$, quod est absurdum. Similis absurditas demonstrari posset, si vena sursum verticaliter ascendere putetur: & frustra hic exciperetur pro communi sententia firmanda, venam effluentem *CE* fingi non posse tanquam continuam, nisi aliqua aquae tenacitas fingatur simul (alias enim venam mox prae orificio in guttulas abruptum iri), & tenacitatem rei statum permutare: nam profecto nec velocitates aquae a cohaesione mutua aquae in *CE* mutantur nec latera canalis *CE* pressionem ullam sentiunt, sicut demonstravi Sect. XII§. l3, ut taceam cohaesionem aquae non oriri a tenacitate sed ab aliqua virtute magnetica seu a mutua attractione, aqua virtute centrum gravitatis in nullo systemate nec majorem nec minorem velocitatem acquirere potest. Sed haec porro adversariorum exceptio in venis verticaliter ascendentibus nullum plane locum habet, cum aquae ibi continue maneant, si vel nulla aquis insit tenacitas aut mutua attractio.

At possem infinitis aliis modis & exemplis particularibus sententiam nostram confirmare, si hisce diutius insistere vellem. Ita v. gr. in Fig. 29, Sect. V §.4 descripta , si sit altitude $NS = 1$, orificium $LM = 1$, & orificium $RS = 2$, erit $PB = \frac{1}{3}$ vis repellens, quae oritur ab effluxu aquae per *RS,* $= 2 \times \frac{2}{3} = \frac{4}{3}$, & demonstrare possum vim repellentem, quae prodit ab effluxu aquae ex simplici cylindro *RN* per *LM,* esse etiam $= \frac{4}{3}$, & sic



vim repellentem totalem esse $=\frac{8}{3}$, quae praecise facit duplum cylindrum aqueum foramini *LM* ad altitudinem $NS+PB$ insistentem. Talis autem consensus ex aliis theoriis falso receptis minime prodit, ita ut de nostra amplius non possint dubitare, nisi harum rerum penitus ignari: Id vero, quod dixi, vim repellentem aquae ex simplici cylindro *RN* per *LM* effluentis esse $=\frac{4}{3}$, si demonstrare velim, postulat ut vis repellens definiatur, cum aquae ex vase non infinito data velocitate quacunque non variata fluunt: Ne vero prolixior sim in hac re, id aliis efficiendum relinquo; neque id nunc amplius magnam facesset operam; pergo ad alia.

§. 12. Demonstrationes quas adhuc dedimus non valent nisi pro fistulis rectis, in quibus nempe uniuscujusque guttulae vis motrix, indeque oriunda vis repellens, inter se singulae conspirant, communemque habent directionem: at cum fistulae vasi implantatae, per quas aquae effluunt, sunt incurvatae, alius adhibendus est demonstrandi modus: Ut nihil in isto argumento prorsus novo omittamus, hunc quoque casum docebimus: nec erit, quod laboris poeniteat, cum inde verae pressionum leges, quas natura non solum in his casibus, sed & multis aliis sequatur, apparebunt.

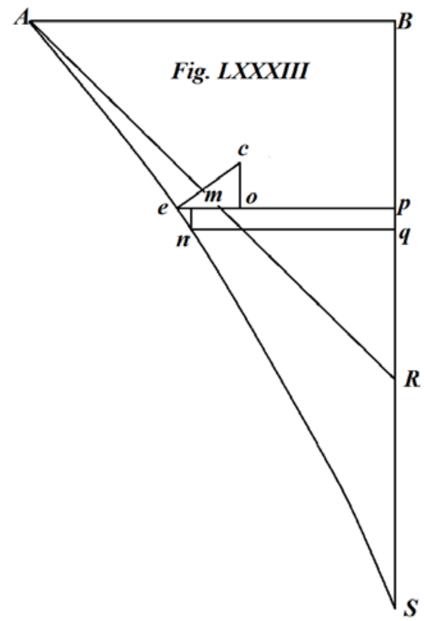


Fig. LXXXIII

§.13. Concipiamus itaque vasi infinito fistulam implantatam esse uniformis quidem amplitudinis, sed incurvatam secundum curvaturam qualemcunque *AS* (Fig. 83) ita ut *A* locus sit insertionis, *S* locus effluxus. Ducantur tangentes in *A* & *S*, nempe *AR* & *SB*, sitque *AB* ad *SB* perpendicularis; fuerit velocitas aquae per fistulam transftuentis uniformis & talis, quae debeatur altitudini *A;* amplitudo fistulae ubique $=1$. *Dico totam vim repellentem in directione SB sumtam fore rursus* $=2A$, hancque solam adfore.

Demonstrationis gratia ducantur infinite propinquae *nq, ep* ad *SB* perpendiculares; *nm* parallela eidem *SB;* sit $Sq=x,\ qp=dx,\ qn=y,\ em=dy$: erit radius osculi in $en=\frac{-dsdy}{ddx}$, sumtis elementis *en* quae vocabo *ds* pro constantibus ; habet autem columella aquae intercepta inter *e* & *n* vim centrifugam sic determinandam: gravitas columellae est $=ds$ (quia basis ejus $=1$ & altitudo $=ds$) atque si radius osculi foret $=2A$, haberetur per theorema Hugenianum vis centrifuga particulae aequalis ejusdem gravitati, & sunt vires centrifugae caeteris paribus in reciproca ratione radiorum: est igitur vis centrifuga columellae $=\frac{-2Addx}{dy}$: exprimatur haec vis centrifuga per *ec* ad curvam perpendicularem, ducaturque *co* ipsi *BS* parallela: resolvatur vis *ec* in *oc* & *eo;* erit (ob similitudinem triangulorum *eoc* & *nme)* (ob *ds* constans)

$$\text{vis } oc = \frac{-2Addx}{ds},$$

$$\text{vis } eo = \frac{-2Adxddx}{dyds} =$$

(ob *ds* constans)

$$= \frac{2Addy}{ds}.$$

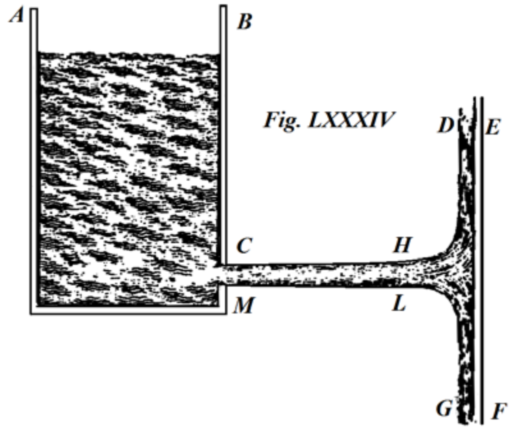
Sed vis elementaris *oc* agit sola in directione *SB,* dum altera *eo* pro hac directione est negligenda: sumatur integrale vis elementaris *oc* cum constanti tali, ut integrale una cum abscissa evanescat: integrale hoc est $= 2A - \frac{2Adx}{ds}$, quia in *S* est $dx = ds$. Nunc ut habeatur vis in directione tangentis pro tota fistula, ponendum est $\frac{RB}{RA}$ pro $\frac{dx}{ds}$, ergo tota vis secundum tangentem $SB = 2A - \frac{2A \times RB}{RA}$. Haec vero oritur a vi centrifuga cujusvis guttulae: sed alia vis superest consideranda; nempe dum aqua ex vase infinite amplo continue in fistulam influit velocitate uniformi respondente altitudini *A*, vas repellitur secundum directionem *RA* vi *2A* (per §. 4), qua resoluta in tangentialem secundum *SB* eique perpendicularem secundum *BA*, prior $\frac{2A \times RB}{RA}$ erit sola consideranda; & quia habet directionem communem cum vi $2A - \frac{2A \times RB}{RA}$ a vi centrifuga oriunda & modo definite, erit eidem addenda : sicque summa $2A - \frac{2A \times RB}{RA} + \frac{2A \times RB}{RA}$ vel *2A* exprimet vim repellentem secundum directionem *SB.*

Ut porro demonstremus sub nulla alia directione vas repelli, recurremus ad vim elementarem *eo*, quam vidimus $= \frac{2Addy}{ds}$, cuius integrale $= \frac{2A \times RB}{RA}$, quae praecise annihilatur a vi *2A* vas repellente secundum directionem *RA*, postquam haec debite resoluta fuit. Q. E. D.

§. 14. Haec theorematis generalissimi simplicitas, qua nempe vis repellens in directione aquis uniformiter effluentibus contraria indicatur constanter $= 2A$, *argumentum* esse potest, quod dicitur *ad hominem* pro ejusdem bonitate, iis qui ratiocinium nostrum aut non assequentur aut examinare non cupient sufficienti attentione. Si vero vim repellentem aquae ex vase infinito in fistulam influentis sub directione AR statuas $= A$, vides systema repelli in directione *SB* vi quae sit $2A - \frac{2A \times RB}{RA}$, quod absurdum esse vel ipsa mihi indicare videtur formula. Neque in hac opinione nulla esset vis in directione ad priorem perpendiculari: Nam vas deberet reprimi secundum directionem *BA* vi $\frac{A\times RB}{RA}$, quod iterum mihi est absurdum & cujus falsitatem experimento cognovi, in casu quo angulus *ARS* erat rectus & $AB = AR$.

Multa alia theoremata pro hoc argumento in tota sua, quam habere potest, extensione, sumto erui & demonstrari poterunt, pro fluxu aquarum nondum uniformi eoque per fistulam utcunque inaequalem, modo simul attendatur ad ea, quae §. 8 monita fuerunt. Quia vero per singula ire non vacat, ad aliam progredior vim examinandam priori sub directione contraria aequalem, illam nempe quam vena effluens in planum exerit, dum in illud perpendiculariter impingit.

§. 15. De impetu venae aqueae in planum impingentis multi commentati sunt, plurimaque sumsere experimenta. Ego quoque hac de re quaedam dedi in *Comm. Acad. Sc. Petrop. tom.* 2. Experimenta extant apud Mariottum in *Tract. de mot. aquarum,* in *Hist. Acad. Se.* conscripta a *D. du Hamel, p.* 48, & alibi. Equidem non admodum conveniunt, plurima tamen indicare prima fronte videntur nisum venae aqueae uniformiter fluentis aequalem esse ponderi cylindri aquei, cujus basis sit foramen, per quod aquae effluunt, & cujus altitudo sit aequalis altitudini aquae supra foramen: Huic sententiae plerique, imo omnes, adhaeserunt & adhuc adhaerent, quia cum aliis quoque experimentis, praesertim quae de globis in medio resistente motis sumi solent, mire convenit: Eandem igitur ipsemet secutus sum, quamvis plura animum suspendebant, in cit. *Comm. Petrop.;* nec haesitavi in ipso hoc opere, quod sub manibus habeo, Sectione nempe IX §§. 31, 32, illa instar exempli uti. Verum enim vero re attentius perpensa, novisque adhibitis principiis, simulque aliis novi generis experimentis institutis, clare tandem vidi communem istam opinionem de impetu venae aqueae eodem modo mutandam esse, sicuti Newtoni de vi repellente, scilicet ut loco orificii consideretur sectio venae contractae & loco altitudinis aquae adhibeatur dupla altitudo velocitati aquarum reali respondens: Demonstratum enim habeo, vim repulsionis §. 2 expositam omnino aequalem esse impetui venae, si haec tota in planum perpendiculariter incidat: sequitur inde impetum venae majorem esse, quo minor fuerit venae contractio, hacque plane evanescente, & aquis simul tota sua velocitate, quam in theoria habere possunt, erumpentibus, tum impetum duplo majorem esse, quam vulgo statuitur: quia vero semper & velocitati aliquid decedit & vena non raro ad dimidium fere contrahitur, factum est ut experimenta pleraque simplam in cylindro altitudinem arguere visa fuerint in impetu illo aestimando. Velim autem probe notetur, de venis solitariis tantum mihi hic sermonem esse, quas plana totas excipiant, non autem de fluidis corpora ambientibus in eademque impetum facientibus, veluti de Ventis aut fluminibus: dico enim hos duplicis generis impetus, quos auctores adhuc confuderunt, probe a se invicem distinguendos esse, ob rationes infra breviter exponendas.



§. 16. Ratione venae aqueae sic censeo: aquas velocitate uniformi ex cylindro infinite amplo verticali *ABM* (Fig. 84) per foramen laterale *CM* horizontaliter effluere pono, venamque perpendiculariter impingere in laminam *EF:* ita facile video, quia particulae insequentes priores impediunt ne resilire possint, fore ut singulae ad latera deflectantur, idque motu laminae *EF* (si modo haec sat magna fuerit, ut vena tota quamvis dispersa excipiatur) parallelo vel tantum non tali: Et quia omnia sunt in *statu permanentiae,* fingere licet laminam *EF* vasi esse affirmatam, venamque lateribus *CHDGLM* circumdatam, ita, ut aquae per hiatum circularem *DEGF* effluere ex vase *ABCHDEFGLM* censeri possint. Hoc si ita fuerit, demonstravimus §.13 guttulas in *DE* effluentes vim repellentem quidem producere secundum *EF;* sed simul apparet vim repellentem esse in *GF* priori contrariam, ita ut ad hanc virium repellentium classem hic non sit attendendum. Quod vero ad directionem laminae *EF* vel cylindro *BC* perpendicularem attinet, demonstravimus in fine ejusdem§. 13, sub ea directione plane nullam fieri repulsionem: Igitur tantum lamina *EF* propellitur, quantum cylindrus repellitur. Idque est quod demonstrare volui: Atque inde jam sequitur, *pressionem venae aqueae, quae tota in laminam incurrit, tantam esse quanta pondus cylindri aquei, qui pro base habeat sectionem venae (postquam haec uniformem acquisivit amplitudinem)* & *pro altitudine duplam altitudinem velocitati aquarum (postquam haec similiter uniformis facta est) debitam.*

§.17. Non dubito multos fore, quibus propositio haec plane nova suspecta videatur atque experimentis contraria: Hos vero perpendere velim, experimenta hactenus sumta nequaquam regulae communi accurate respondere, & in plerisque casibus nostram Regulam parum differre a communi, quamvis in theoria maxime sint diversae: tum etiam eos in antecessum monitos cupio, alia me instituisse experimenta, quae singula meam sententiam exactissime confirmant, veteremque plane refellunt! Experimenta a me sumpta in fine Sectionis recensebo. Demonstrandi modus quo usus fui, fortasse etiam parum videbitur quibusdam accuratus; habeo autem aliam demonstrationem directam, quae nova proprietate innititur Mechanica mihi aliquando observata, quamque hic communicabo, tum quia dictam demonstrationem facillime quivis inde deducere, tum etiam quia ad alios usus eandem impendere poterit: Ita autem se habet.

Si corpus movetur velocitate uniformi, directiones autem suas continue mutat a causis quibuscunque & *utcunque agentibus, Donec directionem acquisiverit perpendicularem* *ad primam directionem, sique singulae pressiones corpus deflectentes resolvantur in duas classes, alteram parallelam primae directioni, alteram perpendicularem; denique si pressiones singulae parallelae multiplicantur per sua tempora; dico fore summam productorum constanter eandem,* & *quidem aequalem ei, quae totum motum a quiete generare aut generatum totum absorbere valet.*

Hac affectione dynamica cum utimur in praesenti nostro negotio, consideranda est lamina *EF*, quae sua in aquas reactione earundem directionem mutat, usque dum perpendicularis ad primam facta fuerit: Ergo propositio praecedentis paragraphi ope hujus affectionis eodem modo demonstrabitur, quo usi sumus §. 4 ad determinandam vim repellentem ope principii §. 3 expositi. Haec igitur vera idea videtur, quam de impetu aquarum mente concipere debemus: ponit autem guttas aquae singulas secundum directionem laminae ad latera resilire, a qua indole aquas non recedere semper observavi: vidi tamen etiam guttulas aliquas sed paucas retrorsum resilire; hae autem majorem pressionem producunt, quam quae ad latera deflectuntur: Et eo ipso inducor, ut firmiter credam, si vena aquea magno impetu oblique contra planum impingat, v. gr. sub angulo triginta graduum, pressionem inde orituram plusquam dimidiam ejus, quae a vena eadem directe impingente oritur, cum secundum regulas ordinarias exacte dimidiam vim exerere deberet: ratio ejus rei est, quod in impulsu obliquo plures particulae resilire possint, quam in directo, imo fere omnes, si magna fuerit velocitas. Si autem omnes ita resilire ponantur, ut angulus incidentiae angulo reflexionis aequalis sit, tunc uterque impulsus idem censendus erit. Optimus hic aquarum pressiones aestimandi modus est, qui ratiocinio *a posteriori* innititur.

§.18. Sequitur porro ex praefata affectione probe intellecta, eundem oriri a pressionibus effectum sive lamina aquas ad latera deflectat, sive causa fingatur motum omnem, quem particulae aqueae cylindrum egressae acquisiverunt, absorbens.

Inde intelligitur quid futurum sit, si orificium *CM* (Fig. 85) per quod aquae ex cylindro *ABM* effluunt, aliis aquis in vase *PQFE* stagnantibus submersum fuerit: repelletur nempe cylindrus *ABM* versus *PQ* intra vas *PQFE*, si hoc cum cylindro non cohaereat; at si vasa inter se fuerint firmata, nullam patietur systema pressionem praevalentem; quanta enim est pressio versus *PQ* ab effluentibus aquis, tanta quoque oritur pressio contraria versus *EF* a continua destructione motus, quem particulae cylindrum egressae acquisivere.

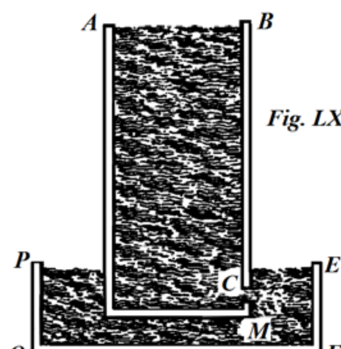


Fig. LXV

§.19. Dixi de pressione venae, quam lamina totam etiamsi expansam excipit: Venio ad alteram speciem impetus aquarum, quem scilicet sustinent laminae fluido undique submersae: puto autem hanc non posse *absolute* definiri, quia singulae particulae in laminam impingentes aliter deflectuntur. Si vero cujuslibet particulae deviatio cognita ponatur, non difficilis erit amplius quaestionis solutio, mutato paullum theoremate, quo §.17 usi sumus, eoque generaliori reddito, nempe tali: *si angulus mutatae in corpore moto directionis non fuerit rectus, sed recto minor; tunc quoque minor erit summa* *productorum (de qua antea sermo fuit) in ratione ut sinus versus mutatae directionis ad sinum totum.*

Igitur pro quavis guttula indagandum esset, quantum directionem motus ab obice seu lamina cursui opposita mutare cogatur. At in theoria hujusmodi definitiones exhiberi accurate vix possunt; nec experientia probat theoremata hanc in rem exhiberi solita; veluti quod conatus fluminis directe contra circulum impingentis duplo sit major conatu ejusdem fluminis contra sphaeram ejusdem diametri, & quae sunt similia: quod autem quantitas pressionis pro sphaera, qualis dari solet ab auctoribus, cum experimentis a Newtono aliisque institutis & in *Princ. math. phil. nat.* recensitis, satis accurate conveniat, id omnibus bene perpensis casui fortuito tribuendum esse censeo.

Theoremata quae ad motum in mediis resistentibus theoretice consideratum faciunt, tum etiam varias observationes physicas dedi in *tom.* II *Comm. Acad. Sc. Petrop. & seqq.* Neque proinde ea hic repetam, quamvis ad institutum nostrum pertineant; diutius meditationibus hisce hydrodynamicis immorari non vacat: Igitur ad finem propero. Novam hanc circa reactionem & impetum fluidorum theoriam, quae receptam ab omnibus adhuc auctoribus opinionem evertit in re magni momenti, singulari Dissertatione prosecutus sum, quae suo tempore *Commentar. Academ. Scient. Imp. Petropol.* inseretur, eandemque indubitatis confirmavi experimentis. Venio nunc ad argumentum aliud, Geometrarum attentione minime indignum.

§. 20. Mentem aliquando subiit, posse ea quae de vi repellente fluidorum, dum ejiciuntur, meditatus fueram. quaeque hic maximam partem exposui, utiliter applicari ad novum instituendum navigationis modum: neque enim video, quid obstet, quo minus maximae naves sine velis remisque eo modo promoveri possint, ut aquae continue in altum eleventur effluxurae per foramina in ima navis parte, faciendo ut directio aquarum effluentium versus puppim spectet. Ne quis vero opinionem hanc in ipso limine rideat, ceu nimis insulsam, e re erit nostra argumentum istud accuratius excutere & ad calculum revocare: utile enim esse potest multisque disquisitionibus geometricis est fertilissimum.

Incipiam ab eo, ex quo deinde apparebit, sub quibus circumstantiis maximus successus a nova ista navigatione expectari debeat.

§. 21. Notandum igitur est, navem ab haustis aquis continue retardari ob inertiam earundem, quando illis eadem velocitas communicatur quacum navis fertur, & dum communicatur, navis a reactione aquarum retrorsum urgetur, simul ac ab earundem effluxu antrorsum premitur. Iste actionum contrariarum concursus limites ponit vi naves propellenti a data potentia absoluta obtinendae: nisi enim actio prior adesset (de qua ut verum fatear diu non cogitavi) posset *labore hominum quantumvis parvo vis naves propellens utcunque magna obtineri,* quod sic demonstro.

In Sectione nona (vide praesertim §. 26) ostendi, laborem hominum in elevandis aquis impensum, quem voce *potentiae absolutae* designo, aestimandum esse ex producto quantitatis aquarum in altitudinem elevationis, ita ut verbi gratia labore secundum omnes mensuras eodem possint & quatuor pedes cubici ad altitudinem sedecim pedum & sedecim pedes cubici ad altitudinem quatuor pedum elevari: Dico nunc porro pressionem uniformem naves antrorsum propellentem adesse, quamdiu fluida velocitate aequali effluunt, quae pressio aestimanda sit ex quantitate aquarum effluentium & ex radice altitudinis aquarum in vase supra foramen positarum: fuerit enim quantitas aquarum dato tempore effluentium $=Q$, altitudo earum $=A$; erit magnitudo foraminis aquas eructantis proportionalis censenda quantitati $\frac{Q}{\sqrt{A}}$ pro eodem tempore: at vero vis repellens, quae hic navem promovet, aequalis est magnitudini foraminis ductae in duplam altitudinem aquarum (per§. 4), id est, aequalis quantitati $\frac{Q}{\sqrt{A}}\times 2A$ seu $2Q\sqrt{A}$. Ex comparatione utriusque propositionis sequitur laborem hominum in elevandis aquis exantlatum esse ad vim naves propellentem inde obtinendam, ut QA ad $2Q\sqrt{A}$ sive ut $\sqrt{A}$ ad quantitatem aliquam constantem: igitur quo minor est altitudo ad quam aquae elevantur, eo major vis naves promovens ab eodem labore obtinetur, *ita ut labore hominum quantumvis parvo vis naves propellens utcunque magna obtineri possit.* Verum etiam inertia aquarum, quae hauriuntur (de qua ab initio hujus paragraphi diximus), naves retardans eo majorem obtinet rationem ad vim naves propellentem, quo minor assumitur altitudo *A*, ad quod animus hic probe est advertendus.

§. 22. Perspicuum est ex praecedente paragrapho, altitudinem ad quam aquae sunt elevandae esse ex earum classe, quae alicubi maximae sunt. Ut vero altitudo maxime ad propositum proficua determinetur, aliae nobis se offerunt quaestiones prius examinandae.

Problema.

Ponatur navis uniformi progredi velocitate, quae generatur lapsu libero per altitudinem *B*, fingaturque aquas continue affluere in navem, veluti sub forma pluviarum, & quidem tanta quantitate, quantam remotis omnibus impedimentis alienis suppeditaret cylindrus constanter plenus ad altitudinem *A* per orificium magnitudinis *M*. Quaeritur quantam resistentiam navis ab isto perpetuo & uniformi aquarum affluxu earundemque inertia patiatur.

Solutio.

Assumatur tempus quodcunque *t*, quod si aestimetur ex spatio, quod fluidum affluens sua velocitate percurrit, diviso per eandem velocitatem, tunc velocitas est exprimenda per $\sqrt{2A}$ & erit quantitas aquae tempore *t* affluens aequalis cylindro super basi *M* constructo longitudinis $t\sqrt{2A}$: ista vero quantitas tempore *t*, dum a nave aufertur, accipit velocitatem debitam altitudini *B* & exprimendam per $\sqrt{2B}$: quaerenda itaque est vis uniformis, quae possit tempore *t* cylindro aqueo $Mt\sqrt{2A}$ communicare velocitatem $\sqrt{2B}$, & erit ista vis ob reactionem, quae in navem reagit, aequalis censenda resistentiae quaesitae. Sit praefata vis $=p$, puteturque dedisse tempore θ velocitatem *v* cylindro aqueo $Mt\sqrt{2A}$ & erit

$$dv = \frac{pd\theta}{Mt\sqrt{2A}},$$

atque

$$v = \frac{p\theta}{Mt\sqrt{2A}}:$$

ponatur jam $\sqrt{2B}$ pro v & t pro θ eritque

$$\sqrt{2B} = \frac{p}{M\sqrt{2A}}:$$

sive

$$p = 2M\sqrt{AB}.$$

Est igitur resistentia quaesita aequalis ponderi cylindri aquei, cujus basis esset aequalis orificio *M* & cujus longitudo aequalis duplae mediae proportionali inter altitudines *A* & *B*.

Problema.

§. 23. Sit in navi cylindrus altitudinis supra superficiem maris *A*, per cujus orificium in eadem superficie positum amplitudinis *M* aquae versus puppim effluant sine ullo impedimento, conserveturque cylindrus aqua constanter plenus: determinare potentiam navem continue propellentem.

Solutio.

Potentia navem propellens est aequalis reactioni aquarum dum effluunt, seu vi repellenti diminutae potentia in praecedente paragrapho definita ab inertia aquarum, quae continue hauriuntur, oriunda. Vis repellens est aequalis, per paragraphum hujus sectionis quartum, *2MA* & haec navem promovet: vis altera quae navem retardat est per praecedentem paragraphum $= 2M\sqrt{AB}$. Est igitur potentia absoluta navem promovens $=$ $= 2MA - 2M\sqrt{AB}$.

Corollarium.

§. 24. Si navis nullam habeat velocitatem, erit vis navem urgens $= 2MA$; atque si navis eadem velocitate movetur qua aquae in plagam contrariam effluunt, fit $B = A$ & tunc navis nulla vi propellitur. Si proinde navis vel liberrime moveretur super mari, non acquireret tamen ab actione aquarum, quae continue hauriuntur inferiusque effluunt, majorem velocitatem quam eam, qua aquae effluunt, non quod aquae ex vase uniformiter moto effluentes vas minori vi quam ex vase immoto repellant, sed quod tunc inertia aquarum resistentiam producat vi repellenti aequalem.

Problema.

§. 25. Data potentia operariorum, qui aquas elevant, & data altitudine ad quam aquae elevantur, invenire amplitudinem foraminis effluxus & vim repellentem.

Solutio.

Sit potentia talis, qua singulis minutis secundis numerus pedum cubicorum aquae N possit ad altitudinem unius pedis elevari, quam potentiam vi experimenti secundi Sectioni nonae subjuncti exerere potest operariorum numerus designandus per $\frac{5}{4}N$. Sit altitudo ad quam aquae continue elevantur $= A$ in pedibus expressa, amplitudo orificii in pedibus quadratis $= M$: erit numerus pedum cubicorum aquae, quem operarii data potentia ad altitudinem A singulis minutis secundis elevare possunt, $= \frac{N}{A}$ (per §. 22 Sect. IX): erit igitur orificium ejus amplitudinis construendum, ut singulis minutis secundis numerus iste pedum cubicorum aquae per id effluere possit, si liberrime effluant. Sumamus autem loco minutorum secundorum tempus, quod corpus insumit, dum libere cadit per altitudinem A : tempus id est hic exprimendum per $\frac{1}{4}\sqrt{A}$ (posito concinnioris calculi gratia corpus a quiete libere cadens intra minutum sec. absolvere 16 *ped.*), & hoc tempore debet effluere numerus pedum cubicorum aquae designandus per $\frac{N}{A}\times\frac{1}{4}\sqrt{A}$ seu $\frac{N}{4\sqrt{A}}$: effluit autem revera *2MA*, nempe cylindrus aqueus cujus basis est M & cujus longitudo N facit duplicem altitudinem A: est igitur $\frac{N}{4\sqrt{A}} = 2MA$*;* unde amplitudo orificii seu

$$M = \frac{N}{8A\sqrt{A}}.$$

Vis autem repellens fit aequalis *2MA* seu $= \frac{N}{4\sqrt{A}}$

Scholium.

§. 26. In quavis nave aquae ad aliam atque aliam altitudinem sunt elevandae, ut eadem potentia, quae in hauriendis aquis insumitur, vis navem promovens maxima obtineatur, & duo requiruntur ad altitudinem illam utilissimam definiendam pro certo operariorum numero. *Primo* ut cognitum fit quamnam velocitatem proposita navis a data potentia acquirat: ratione hujus postulati, ponemus navem a pressione, quae sit aequalis ponderi unius pedis cubici aquae seu circiter 72 librarum, acquirere velocitatem, quae generetur lapsu libero per altitudinem C, & quia deinceps semper in pedibus mensuras omnes exprimemus, erit pondus unius pedis cubici aquae exprimendum per unitatem. *Secundo* pro cognita assumenda est relatio inter celeritates navis & potentias navem propellentes:

statuitur hic vulgo velocitates habere rationem subduplicatam virium propellentium; experimenta quidem hanc hypothesin non exacte confirmant in motibus lentis; interim tamen eam reliquis omnibus praeferendam censemus. Si quis velit rem sub alia hypothesi explorare, is poterit eodem modo, quo nunc utemur, calculum instituere.

Problema.

§. 27. Invenire altitudinem, ad quam aquae continue elevandae sunt, instituto utilissimam, nempe talem, ut eadem potentia in elevandis aquis adhibenda vis navem promovens maxima oriatur.

Solutio.

Serventur denominationes omnes in hoc argumento adhibitae: erit ante omnia inquirenda velocitas navis seu altitudo huic velocitati debita quam vocavimus *B.* Quia vero velocitates navis ponuntur proportionales radicibus potentiarum navem urgentium, erunt altitudines velocitatum ipsis potentiis proportionales. Erit igitur talis analogia instituenda.

Sicuti pondus unius pedis cubici ad altitudinem *C* (conf. §. 26) ita pressio navem urgens seu $2MA-2M\sqrt{AB}$ (vid. §. 23) ad altitudinem velocitati navis respondentem, quae proinde erit $B=2MC\times(A-\sqrt{AB})$. Hanc vero altitudinem vocavimus *B.* Est itaque

$$B=2MC\times(A-\sqrt{AB}).$$

Exinde fit pressio navem urgens $=\frac{B}{C}$, atque adeo proportionalis altitudini *B*, quia *C* est quantitas constans: ergo & pressio navem promovens & altitudo navis velocitati respondens simul fiunt maximae: Igitur si pro praesenti instituto differentietur quantitas $2MA-2M\sqrt{AB}$, quae pressionem navem propellentem exprimit, poterit poni $dB=0$. Prius vero quam differentiatio instituatur oportet pro *M* substituere valorem ejus §. 25, & tunc fit pressio navem promovens $=\frac{N}{4\sqrt{A}}-\frac{N\sqrt{B}}{4A}$, in qua littera *N* est constans, litterae vero *B* & *A* variabiles. Sumatur nunc ejus differentiale, faciendo $dB=0$, idque fiat $=0$; atque sic reperietur $A=4B$.

Est igitur vis navem promovens maxima cum altitudo, ad quam aquae elevantur, est quadrupla altitudinis velocitati navis debitae.

Ponatur in aequatione $B=2MC\times\left(A-\sqrt{AB}\right)$ superius inventa $A=4B$ & reperietur $M=\frac{1}{4C}$, & quia (per§. 25) est $M=\frac{N}{8A\sqrt{A}}$, fit tunc

$$A=\left(\tfrac{1}{2}NC\right)^{\frac{2}{3}},$$

atque

$$B = \frac{1}{4}\left(\frac{1}{2}NC\right)^{\frac{2}{3}}.$$

Corollarium.

§. 28. Si ad praeceptum praecedentis paragraphi orificio, per quod aquae inferius ex canali versus puppim effluunt, concilietur amplitudo $\frac{1}{4C}$, id est, talis, quae se habeat ad amplitudinem unius pedis quadrati, sicuti mensura unius pedis ad altitudinem quadruplam velocitati navis, vi 72 librarum animatae, debitam, fiet tunc ut navis dimidia velocitate feratur ejus qua aquae effluunt & erit vis repellens aquarum effluentium vis vero navem promovens hujus erit dimidia, adeo ut dimidius effectus perdatur ab inertia earundem, quae continue hauriuntur, aquarum.

Scholium.

§. 29. Postquam sic demonstravimus, quomodo utilissime maximoque cum successu iste navigandi modus sit instituendus, nunc porro rem istam exemplo illustrandam esse puto tali, quod cum ipsa rei natura non male convenire crediderim ut simul appareat, qualis praeterpropter eventus futurus sit.

Consideremus triremem, vulgo *galeram*, cum 260 remigibus: ponamus hanc galeram pondere unius pedis cubici aquae seu 72 librarum tractam perficere singulis minutis secundis spatium duorum pedum, cujus velocitatis altitudo genitrix indicata per C est $= \frac{1}{16}$, posito corpus grave libere a quiete decidens primo minuto secundo perficere 16 *ped.* Quia porro 260 operarii adhibentur, quorum quivis vi experimenti secundi ad Sect. IX pertinentis potest singulis minutis secundis quatuor quintas partes pedis cubici ad altitudinem unius pedis elevare, erit $N = \frac{4}{5} \times 260 = 208$. Fiat igitur orificium, per quod aquae effluant, amplitudinis 4 pedum quadratorum: poteruntque operarii aquam in canali supra orificium elevatam conservare ad altitudinem proxime $3\frac{1}{2}$ *ped.* quae indicator litera A, & si sumas hujus altitudinis quartam pattern habebis $B = \frac{7}{8}$ *ped.* adeo ut navis tali velocitate sit ista navigatione progressura, quam grave acquirit lapsu libero per altitudinem $\frac{7}{8}$ *ped.;* sic ergo navis singulis minutis secundis spatium $7\frac{1}{2}$ *ped.* perficiet & singulis horis 27 000 *ped.*, id est, plus duobus milliaribus gallicis: tanta navis velocitas remigatione vix ac ne vix quidem obtineri potest.

Jam vero calculum alia hypothesi superstruam, quam rei nauticae intelligentes non admodum improbaturos esse confido: quadrat enim cum multis, quas ipse super mari feci, observationibus : supponam vela triremis perpendiculariter ad carinam expansa superficiem habere 1600 pedum quadratorum, haecque ventum excipere directe impingentem, qui singulis minutis secundis spatium percurrat 18 *ped.*, navem vero in eadem directione sic singulis minutis secundis spatium perficere 6 pedum. Ita ventus in vela incurret velocitate respectiva 12 pedum: vim istius venti aestimo = ponderi $\frac{9 \times 1600}{850}$ *ped. cub.* aquae, seu fere 17 *ped. cub.* aquae.

Haec si ita sint, sequitur navem ab elevatione aquarum 260 operariorum posse ea velocitate propelli, qua singulis minutis secundis spatium percurrat $6\frac{1}{2}$ pedum.

Aestimatio non admodum diversa sequitur ex iis, quae D. Chazelles habet in *Comm. Acad. Reg. Sc. Paris. ad ann.* 1702, *p.* 98 edit. Paris. Ut vero recte ad institutum nostrum applicari possint, notandum erit, in remigatione vim triremem propellentem non esse aestimandam ex pressione remigum in remos, sed ex pressione, quam remorum extremitates aquis submersae contra aquas exerunt. Ut hanc proxime definiamus, haec prius erunt observanda. Remiges fuere adhibiti 260 totis viribus remigantes: singulis minutis primis remorum impulsus (gallice *palades)* facti sunt 24: integra remorum agitatio tribus absolvitor motibus, quos ejusdem durationis ponam, eorumque unus solus triremem promovet: hoc modo triremis velocitate fuit provecta, qua singulis minutis secundis spatium $7\frac{1}{5}$ *ped.* absolvebat, pars remi intra navem fuit 6 pedum & extra navem 12 pedum: superficies autem (gallice *les pales)* omnium remorum, quae contra aquas impelluntur, in unam collectas D. Chazelles facit 130 pedum quadratorum: notavit porro extremitatem internam remi singulis agitationibus spatium describere sex pedum: & quia quaevis agitatio tempore $\frac{60}{24}$ unius minuti secundi absolvitor simulque ex tribus constat motibus, quos pono tautochronos, apparet quamvis remi retractionem fieri tempore $\frac{20}{24}$ seu $\frac{5}{6}$ unius minuti secundi & hoc tempore extremitas remi interna absolvit spatium 6 pedum. Porro ob longitudinem superficies remorum, quae contra aquas impellitur, non tota est ad distantiam 12 pedum censenda: illam igitur distare ponam 10 pedibus, quasi nempe pars remi extra navem promineret 10 pedes longa: hujus partis extremitas describet 10 pedes tempore $\frac{5}{6}$ unius minuti secundi: quia vero ipsa triremis velocitatem habet, qua eodem tempore sex pedes absolvit, censendum est, remorum extremitates contra aquam impelli velocitate respectiva, qua tempore $\frac{5}{6}$ *min. sec.* 4 pedes describat: igitur vis triremem propellens est aequalis vi, quam aqua contra superficiem 130 pedum quadratorum exereret, si velocitate in illam incurreret, qua tempore $\frac{5}{6}$ *min. sec.* 4 *ped.* absolvat: hanc vim secundum vulgarem aestimationem invenio praeter propter aequalem ponderi 40 *ped. cub.* aquae; ista vero vis non continue applicator, sed tantum eo tempore quo remi retrahuntur: sunt igitur duo trientes istius vis auferendi, ita ut vis quae triremem continue propellat, censenda denique sit aequalis ponderi $13\frac{1}{3}$ *ped. cub.* aquae.

Exinde sequitur, si velocitates navis rationem sequi subduplicatam virium propellentium ponantur, quod eadem haec triremis pondere unius pedis cubici aquae impulsa velocitatem habitura fuisset, qua possit singulis minutis secundis perficere proxime duos pedes; quae hypothesis eadem est cum illa, quam primo loco adhibuimus, ita ut rursus exinde sequatur triremem velocitatem ab ista navigatione acquisituram esse, qua possit perficere singulis minutis secundis $7\frac{1}{2}$ pedes, quae velocitas tantillo major est illa, quae triremi remigatione fortissima 260 remigum data fuit.

Rebus bene perpensis haesito, utrum navigationis genus sit praeferendum, an remigatio, an aquarum elevatio; successum fere aequalem crediderim utriusque, & pro certo affirmare audeo, si minus promoveatur navis ab aquarum elevatione, defectum parvum fore: fortasse autem promovebitur magis. Interim non dubito, quin nova ista navigationis idea harum rerum ignaris vana & ridicula appareat. Ego vero aliter sentio velimque ut animus porro ad sequentia advertatur.

Primo. Quod aquae in omni navium genere, ubi remi plane adhiberi nequeunt, commode elevari possunt, ita ut nova ista navigatione naves etiam bellicae praegraves, quibus in pugnis navalibus utuntur, deficiente omni vento, quo lubet agi possint.

Secunda. Quod sic in theoria exemplum habetur, dari vires motrices sive propellentes, quae dici possunt intrinsecae: Excitabuntur isto exemplo ingenia ad excogitanda hujusmodi alia motus principia eaque magis perficienda & ad navigationis usum adhibenda.

Tertio. Quod multis modis sub levari potest labor hominum in elevandis aquis secus atque fieri potest in remorum usu: sunt nempe res naturales insigni & fere incredibili virtute praeditae eaeque mediocri pretio comparandae, quibus idem quod labore hominum effici potest: harum usus praesertim brevibus trajectibus serena & tranquilla tempestate instituendis inservire posset. De virtute istiusmodi rebus naturalibus insita, de effectibus inde obtinendis horumque mensuris egi in Sect. X, §. 40 & sequentibus: imprimis autem velim ut attendatur ad§. 43, quo omnes quibus ingenium a natura datum fuit felix ad machinas excogitandas, excitari deberent ad rei istius perfectionem tentandam.

Quarto. Quod nonnulla alia compendia pure mechanica adhiberi possint similia illi quod §. 27 datum fuit, quorum nempe ope ab eodem labore effectus in promovendis navibus non parum crescit: Verum non licet jam secundum veram rei indolem omnia pertractare.

Experimenta in Sectionem decimam tertiam.

Ut vim repellentem experimento recte cognoscere liceat, adhiberi poterit vas quod habeat formam parallelopipedi ejusque pondus sumi tam vacui quam aqua pleni, posteaque indagari ratio inter amplitudinem vasis & amplitudinem foraminis, quod in latere vasis esse debet, sicut & ratio inter altitudines aquae supra foramen & supra basin: Inde deducere licebit rationem inter pondus vasis aqua pleni & cylindri aquei foramini verticaliter superincumbentis. Porro ex observata amplitudine jactus habebitur velocitas aquae: ex hac, si simul jungas quantitatem aquae dato tempore effluentem pariter observandam, colliges amplitudinem venae contractae, quam comparare poteris cum amplitudine orificii.

His omnibus exploratis suspendatur vas ex filo praelongo adhibita simul cura, ut alium motum habere non possit, quam qui sit directioni aquarum effluentium contrarius. Tum demum aquis effluxus concedatur & observabitur filum si tum verticalem deserere & ex angulo declinationis cognoscetur vis repellens eaque cum mensuris, quas indicavimus, comparari poterit.

Experimentum 1.

Feci ipse aliquando omnia, ut nunc monui, visumque fuit regulam nostram §. 2 recte confirmari: non potui tamen tum temporis sufficiente accuratione experimentum instituere, nec illud postea repetii.

Experimentum 2.

Alio tempore rem aliter tentavi: vas nempe de quo omnes mensuras requisitas sumseram aqua plenum naviculae imposui in puppi: navicula aquis in alveo innatabat: Deinde aquis ex vase effluentibus (ita tamen ut in naviculam non illiderent) navicula in plagam contrariam progressa est: velocitatem naviculae ex spatio dato tempore percurso rectissime exploravi. Deinde inquisivi quantum ponderusculum naviculae esset appendendum, ut illo pondere sollicitata eandem velocitatem acquireret. Instituta deinde comparatione istius ponderis cum pondere cylindri aquei datae diametri, inde rectissime theoriam nostram confirmari vidi.

Experimentum 3.

Effluentibus aquis ex vase naviculae superimposito in naviculam, haec omnino immota permansit: Id indicat impetum venae aqueae aequalem esse vi repellenti, ut demonstravi §§. 16 & 17. Tum etiam si vena aquea directe impingebat in planum naviculae affixum, haec similiter immota stetit, quod rursus aequalitatem impetus & vis repellentis probat: at si vena oblique in planum incidebat, navicula quidem motum obtinuit sed lentiorem.

Denique si aquae effluentes a navicula excipiebantur, ita ut orificium aquis in navicula stagnantibus esset submersum, similiter absque motu perstetit navicula, documento, quod eadem pressio a vena oriatur, sive fiat ut omnis ejus motus cohibeatur, sive ut ad angulum rectum declinetur, prouti demonstratum fuit §. 18; aequalitatem inter vim repellentem & vim venae aqueae perpendiculariter in planum incidentis plurimis aliis modis exactissime confirmavi. Hanc autem vim theoriae nostrae conformem opinionique omnibus adhuc communi contrariam experimento omni exceptione majore confirmavi, quod praesentibus D. Emanuele Koenig, Patrueli meo Nicolao Bernoullio atque Patre meo in aedibus meis institui tanta cum fiducia, ut acceptis omnibus mensuris pressionem venae aqueae, quanta futura esset, etsi nunquam antea a me capto experimento, omni praecisione praedixerim. Haec omnia novis principiis mechanicis eruta communicavi cum Academia Scientiarum Petropolitana, cujus *Commentariis* aliquando inserentur.

Experimentum 4.

Ut etiam ostenderem falsitatem regulae receptae tum de vi repellente tum de impetu aquarum, adhibui vas quale ostendit Figura 86, instructum canali *AB* uniformis amplitudinis & incurvato, cujus directio in *A* erat horizontalis, in *B* verticalis: vidi vas plane non repelli horizontaliter; ergo per §.14 falsa est regula, quae simplici cylindro ibidem definito adhaeret.

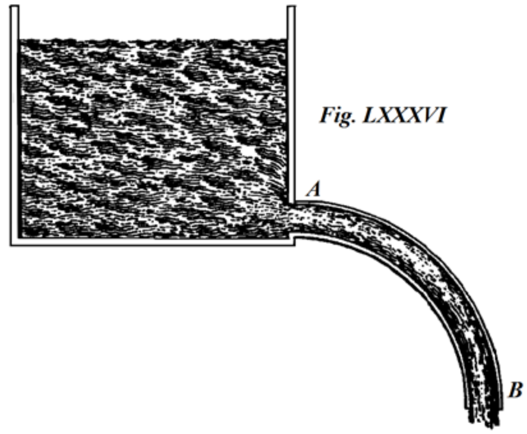


Fig. LXXXVI

FINIS.