PROP. VII. THEOR. III.

There are two variable quantities z & x, of which z is regularly increased by the given increment $\underset{\cdot}{z}$*, and* $n\underset{\cdot}{z} = v$, $v - \underset{\cdot}{z} = \overset{\backslash}{v}$, $\overset{\backslash}{v} - z = \overset{\backslash\backslash}{v}$, *and thus henceforth. Moreover, I say that in the time z increases to z + v, x increases likewise to become* $x + \underset{\cdot}{x}\frac{v}{1\underset{\cdot}{z}} + \underset{\cdot\cdot}{x}\frac{v\overset{\backslash}{v}}{1.2\underset{\cdot}{z}^2} + \underset{\cdot\cdot\cdot}{x}\frac{v\overset{\backslash}{v}\overset{\backslash\backslash}{v}}{1.2.3\underset{\cdot}{z}^3} + \&c.$

[p. 22]

DEMONSTRATION:

x	$\underset{\cdot}{x}$	$\underset{\cdot\cdot}{x}$	$\underset{\cdot\cdot\cdot}{x}$	$\underset{\cdot\cdot\cdot\cdot}{x}$	&c.
$x+\underset{\cdot}{x}$	$\underset{\cdot}{x}+\underset{\cdot\cdot}{x}$	$\underset{\cdot\cdot}{x}+\underset{\cdot\cdot\cdot}{x}$	$\underset{\cdot\cdot\cdot}{x}+\underset{\cdot\cdot\cdot\cdot}{x}$	&c.	
$x+2\underset{\cdot}{x}+\underset{\cdot\cdot}{x}$	$\underset{\cdot}{x}+2\underset{\cdot\cdot}{x}+\underset{\cdot\cdot\cdot}{x}$	$\underset{\cdot\cdot}{x}+2\underset{\cdot\cdot\cdot}{x}+\underset{\cdot\cdot\cdot\cdot}{x}$	&c.		
$x+3\underset{\cdot}{x}+3\underset{\cdot\cdot}{x}+\underset{\cdot\cdot\cdot}{x}$	$\underset{\cdot}{x}+3\underset{\cdot\cdot}{x}+3\underset{\cdot\cdot\cdot}{x}+\underset{\cdot\cdot\cdot\cdot}{x}$	&c.			
$x+4\underset{\cdot}{x}+6\underset{\cdot\cdot}{x}+4\underset{\cdot\cdot\cdot}{x}+\underset{\cdot\cdot\cdot\cdot}{x}$	&c.				
&c.					

The successive values of this quantity are to be collected together by addition x, $x+\underset{\cdot}{x}$, $x+2\underset{\cdot}{x}+\underset{\cdot\cdot}{x}$, $x+3\underset{\cdot}{x}+3\underset{\cdot\cdot}{x}+\underset{\cdot\cdot\cdot}{x}$, etc., as it is apparent from the operation expressed in the adjoining table. But the numerical x coefficients of the terms $x, \underset{\cdot}{x}, \underset{\cdot\cdot}{x}, \&c.$ are formed in the same way from these values, and these are the coefficients of the corresponding terms in the power of the binomial. And (by a Theorem of *Newton*) if the index of the power is n, the coefficients are $1, \frac{n}{1}, \frac{n}{1}\times\frac{n-1}{2}, \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}$, etc. Hence by the time that z increases to $z+n\underset{\cdot}{z}$, that is $z + v$, x becomes equal to the series $x+\frac{n}{1}\underset{\cdot}{x}+\frac{n}{1}\times\frac{n-1}{2}\underset{\cdot\cdot}{x}+\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\underset{\cdot\cdot\cdot}{x}+\&c.$

But $\frac{n}{1} = \left(\frac{n\underset{\cdot}{z}}{\underset{\cdot}{z}} = \right)\frac{v}{\underset{\cdot}{z}}$, $\frac{n-1}{2} = \left(\frac{n\underset{\cdot}{z}-\underset{\cdot}{z}}{2\underset{\cdot}{z}} = \right)\frac{\overset{\backslash}{v}}{2\underset{\cdot}{z}}$, $\frac{n-2}{3} = \left(\frac{n\underset{\cdot}{z}-2\underset{\cdot}{z}}{3\underset{\cdot}{z}} = \right)\frac{\overset{\backslash\backslash}{v}}{3\underset{\cdot}{z}}$, &c. [p. 23] Hence in the time in which z has increased to $z + v$, x has increased to become $x + \underset{\cdot}{x}\frac{v}{1\underset{\cdot}{z}} + \underset{\cdot\cdot}{x}\frac{v\overset{\backslash}{v}}{1.2\underset{\cdot}{z}^2} + \underset{\cdot\cdot\cdot}{x}\frac{v\overset{\backslash}{v}\overset{\backslash\backslash}{v}}{1.2.3\underset{\cdot}{z}^3} + \&c.$

COROLL. I.

And with $\underset{\cdot}{z}, \underset{\cdot}{x}, \underset{\cdot\cdot}{x}, \underset{\cdot\cdot\cdot}{x}, \&c.$ remaining the same, by changing the sign of v, in the time in which z has decreased to $z - v$, in the same time x has decreased to $x - \underset{\cdot}{x}\frac{v}{1\underset{\cdot}{z}} + \underset{\cdot\cdot}{x}\frac{v\overset{\backslash}{v}}{1.2\underset{\cdot}{z}^2} - \underset{\cdot\cdot\cdot}{x}\frac{v\overset{\backslash}{v}\overset{\backslash\backslash}{v}}{1.2.3\underset{\cdot}{z}^3} + \&c.$ or by using our notation for the converses of $\overset{\backslash}{v}, \overset{\backslash\backslash}{v}, \&c.$, which become $-\underset{\backslash}{v}, -\underset{\backslash\backslash}{v}, \&c.$, the series becomes $x - \underset{\cdot}{x}\frac{v}{1\underset{\cdot}{z}} + \underset{\cdot\cdot}{x}\frac{v\underset{\backslash}{v}}{1.2\underset{\cdot}{z}^2} - \underset{\cdot\cdot\cdot}{x}\frac{v\underset{\backslash}{v}\underset{\backslash\backslash}{v}}{1.2.3\underset{\cdot}{z}^3} + \&c.$

COROLLARY II.

If for the evanescent increments, the fluxions of the proportionals themselves are written, now with all the $\overset{//}{v}, \overset{/}{v}, v, \underset{/}{v}, \underset{//}{v}$, & c. equal to the time z uniformly flows to become $z + v$, and x becomes, $x + \dot{x}\frac{v}{1\dot{z}} + \ddot{x}\frac{v^2}{1.2\dot{z}^2} + \dddot{x}\frac{v^3}{1.2.3\dot{z}^3} + \&c.$ or by changing the sign of v, by which z decreases to $z - v$, x decreases to become : $x - \dot{x}\frac{v}{1\dot{z}} + \ddot{x}\frac{v^2}{1.2\dot{z}^2} - \dddot{x}\frac{v^3}{1.2.3\dot{z}^3} + \&c.$

[Note : Newton's notation for fluents and their fluxions, where the lines slope the other way, are now used rather than Taylor's own notation for the analogous case of a string of finite differences in this and similar cases. This is, of course, what is now called Taylor's Theorem, as becomes apparent by writing the expansion in modern terms. If we set $z = 0$ initially for convenience, and introduce the function $f(z)$ rather than x, which by common usage now means something else for us, then the expansion is equivalent to $f(v) = f(0) + vf'(0) + \frac{v^2}{2!}f''(0) + ...$ It appears to have been found by Stirling originally, and publicised by Maclaurin.]

[p. 24]

PROP. VIII. PROB. V.

For a given [incremental]*equation involving the increments of some other variable x besides the uniformly increasing z; the value of x is to be found for a given z by a series of an infinite number of terms.*

By the first Proposition whereby all the increments of a proposed equation are to be given to infinity. Then if $\underset{n}{x}$ is the smallest of the increments of x in a proposed equation given by these equations [recall Taylor's usage of increment size discussed earlier], then all the increments less than $\underset{n}{x}$ are given by the superior increment $\underset{n}{x}$ itself.

Let a, & $c, \underset{\cdot}{c}, \underset{\cdot\cdot}{c}, \underset{\cdot\cdot\cdot}{c}$, & c. be certain corresponding values of z, & $x, \underset{\cdot}{x}, \underset{\cdot\cdot}{x}, \underset{\cdot\cdot\cdot}{x}$, & c. and by the same equations all the following terms $\underset{n}{c}, \underset{n+1}{c}, etc$ given are expressed by the terms preceding $\underset{n}{c}$ itself. Thus if for z , $a + v$ is written, x is given by

$x = c + \underset{\cdot}{c}\frac{v}{1\underset{\cdot}{z}} + \underset{\cdot\cdot}{c}\frac{v\overset{\backslash}{v}}{1.2\underset{\cdot}{z}^2} + \underset{\cdot\cdot\cdot}{c}\frac{v\overset{\backslash}{v}\overset{\backslash\backslash}{v}}{1.2.3\underset{\cdot}{z}^3} + \&c.$ (per Prop. 7.) Where the coefficients of the terms $c, \underset{\cdot}{c}, \underset{\cdot\cdot}{c}, \underset{\cdot\cdot\cdot}{c}$, *et*c. the number of which is *n*, give all the conditions of the problem.

SCHOLIUM.

When x has been made from some whole number of positive powers of z itself, with the vanishing of the smaller increments after a certain number of terms, the series is thus interrupted and comes to an end. Let $\underset{\cdot}{x}z - x + 1 = 0$ be a given equation & let $\underset{\cdot}{z} = 1$. Then

by taking an increment, the equation becomes $\underset{\cdot\cdot}{x}z+\underset{\cdot\cdot}{x}=0$. [p. 25] [Note that is involves replacing $\underset{\cdot}{x}$ by $\underset{\cdot}{x}+\underset{\cdot\cdot}{x}$, z by $z+\underset{\cdot}{z}$, and x by $x+\underset{\cdot}{x}$, and taking away the original expression.]
But this can only happen if $\underset{\cdot\cdot}{x}=0$; otherwise z can indeed be determined by the equation z + 1= 0. Hence if $a + v$ is written for z, and c & $\underset{\cdot}{c}$ are themselves the values of x & $\underset{\cdot}{x}$ when $v = 0$, then always $x=c+\underset{\cdot}{c}v$, that is, (for $\underset{\cdot}{c}$ I write the value of this found from the proposed equation [for which $\underset{\cdot}{x}=\frac{x-1}{z}=\frac{c-1}{a}$]) $x=c+\frac{c-1}{a}v$, that is (when for v I write the value $z-a$ of this) $x=1+\frac{c-1}{a}z$.

For series produced in this way, after a certain number of terms, by observation from analogy, generally the coefficients can be found for as far as I wish to calculate beyond. And sometimes series can be found that can be compared with known series, which are produced from known finite expressions : whereby by substituting these finite expressions in place of the series, from which arrangement integrals are given by a finite number of terms. Let a fluctional equation be

$\ddot{x}z+n\ddot{x}x-\dot{x}-\dot{x}^2=0$, where for $\dot{z}$ is written 1. Hence $\ddot{x}=\frac{\dot{x}+\dot{x}^2}{z+nx}$ or (for $z + nx$ I write y). And by continued calculation it is found [Recall that this fluxional equation was used as a model in discussing Prop. III earlier. Here Taylor solves the fluxional equation by using a known formula for the higher differences; he was unable to find such a formula for the other equation.]

: $\dddot{x}=\left((2-n)\times\frac{\dot{x}^2+\dot{x}^3}{y^2}=\right)(2-n)\frac{\dot{x}}{y}\ddot{x}$, $\overset{4}{x}=(3-2n)\frac{\dot{x}}{y}\dddot{x}$, $\overset{5}{x}=(4-3n)\frac{\dot{x}}{y}\overset{4}{x}$, & thus henceforth. Where

(for $a + nc$ I write p [in place if y]) by this *Prop*. [Coroll. II with $\dot{z}=1$.] :

$x=c+\dot{c}v+\frac{\ddot{c}v^2}{2}+\frac{(2-n)\ddot{c}\dot{c}v^3}{1.2.3p}+\frac{(2-n)(3-n)\ddot{c}\dot{c}v^4}{1.2.3.4p}+\&\text{c.}$, that is

$x=c+\dot{c}v+\frac{\ddot{c}p^2}{n\dot{c}^2}\times\frac{n}{p}\times\frac{1}{2p}\dot{c}^2v^2+\frac{n}{p}\times\frac{1}{2p}\times\frac{(2-n)}{3p}\dot{c}^3v^3+\frac{n}{p}\times\frac{1}{2p}\times\frac{(2-n)}{3p}\times\frac{(3-2n)}{4p}\dot{c}^4v^4+\&\text{c.}$ But the numbers n, 1, $2-n$, $3-2n$, &c. are produced by the continued subtraction of the number $n-1$. Whereby let $p = n-1$, and the series is :

$\frac{n}{p}\times\frac{1}{2p}\dot{c}^2v^2+\frac{n}{p}\times\frac{(2-n)}{3p}\dot{c}^3v^3+\&c.=(1+\dot{c}v)^{\frac{n}{n-1}}-1-\frac{n}{n-1}\dot{c}v$. Thus put $p = n-1$, [p. 26] that is make $a = n-1-nc$, and x is given by the finite equation :

$x=c+\dot{c}v+\ddot{c}\frac{(n-1)^2}{n\dot{c}^2}\times\overline{(1+\dot{c}v)^{\frac{n}{n-1}}-1-\frac{n}{n-1}\dot{c}v}$; where $\ddot{c}=\frac{\dot{c}+\dot{c}^2}{n-1}$, and $v = z-n+1+nc$; and c & $\dot{c}$ are unknowns that are to be determined by the two conditions of the problem. Again in this equation x and z emerge interchanged between themselves as z and x in the equation $\ddot{x}x+\dot{x}\dot{x}+n\ddot{x}z+\dot{x}\dot{z}=0$, that we added in *Lem*. 1. Truly in that order in finding this finite expression (indeed with the transformation sought there in vain) to be transformed by the third Proposition. The amount of use of this proposition can also be agreed upon from this example. But with the fluent *z* moving uniformly, where in the proposed equation, fluxions of the second, third order, or depending on *x* itself, if by this proposition you wish to find the value of z itself from the given *x*, the equation will be transformed in the same way.

Occasionally by other transformations finite expressions can be found. Let $4\dot{x}^3 - 4x^2 = (1+z^2)\dot{x}^2$ be an equation (also that we added in *Lem*. 1.) Put $x = v^\theta y^\lambda$, then by substituting this value *x*, and thus with the resultant value of the fluction $\dot{x}$, by seeking the most simple form of the equation to be determined by the indices θ and λ, and the value of one of *v* or *y*.

Hence therefore let $\dot{x} = \overline{\vartheta\dot{v}y + \lambda\dot{y}v} \times v^{\theta-1}y^{\lambda-1}$, and (by substitution)

$4v^{3\theta}y^{3\lambda} - 4v^{2\theta}y^{2\lambda} = \overline{1+z^2}|^2 \times \overline{\vartheta\dot{v}y + \lambda\dot{y}v}|^2\, v^{\theta-2}y^{\lambda-2}$. I put λ = - 2 and

$v^\theta - y^2 = \overline{\vartheta z y - \dot{y}v}\ |^2$, that is $v^\theta = v^2z^2y^2 + 1 - 2\vartheta zvy + v^2y^2$. And then put $\vartheta = 1$, in order that $\vartheta z^2 + 1 = v$, by which arrangement the equation can be divided by *v* [p. 27]
$x = y^2 - 2zy\dot{y} + v\dot{y}\dot{y}$. Hence by taking fluxions,
$0 = 2y\dot{y} - 2y\dot{y} - 2z\dot{y}\dot{y} - 2zy\ddot{y} + \dot{v}\dot{y}\dot{y} + 2v\dot{y}\ddot{y}$, that is (for the value of $\dot{v}$ I write 2z)
$-2z\dot{y}\ddot{y} + 2v\dot{y}\ddot{y} = 0$. Hence either $\ddot{y} = 0$, or $-zy + v\dot{y} = 0$, from which is found $y^2 = v$, and thus $x = (vy^{-1} =)\,1$; which is indeed a singular solution of the problem. But if we make $\ddot{y} = 0$, then for *z* and *y* and $\dot{y}$, with the concurrent values written as $o, a,$ and $\dot{a}$, it is found that $\dot{a} = \sqrt{1-aa}$, and thus $y = a + \sqrt{1-aa}\,z$ (by this Proposition) and hence
$x = (v^\theta y^\lambda =)\frac{1+z^2}{(a+\sqrt{1-a^2}z)^2}$.

As often as the value of this given *z* can be made equal to zero, and without the number of terms of the series made infinite, a series is produced in the simpler form rising by powers of *z*. And in this case the series can be assumed in general terms to be expressing the value of *x*; in which the coefficients are later determined by comparison of the terms, to the standard of the following example.

Let the equation be $\ddot{x} - \dot{x}z - 2x = 0$. Put $x = \mathrm{A} + \mathrm{B}z + \mathrm{C}z^2 + \mathrm{D}z^3 + \mathrm{E}z^4 + \&c.$ Then on taking the fluxions, the equations are : $\dot{x} = \mathrm{B} + 2\mathrm{C}z + 3\mathrm{D}z^2 + 4\mathrm{E}z^3 + \&c.$ and $\ddot{x} = 2\mathrm{C} + 6\mathrm{D}z + 12\mathrm{E}z^2 + \&c.$ From which with the values of $x, \dot{x}, \ddot{x}$ written in the equation, and with the terms arranged according to the powers of z :

$$
\begin{array}{l}
2\mathrm{C} + 6\mathrm{D}z + 12\mathrm{E}z^2 + \&c. = 0, [\ddot{x}\text{ series;}] \\
-2\mathrm{A} - 2\mathrm{B} - \quad 2\mathrm{C} \qquad \&c. = 0, [-2x\text{ series;}] \\
\qquad -\mathrm{B} - \quad 2\mathrm{C} \qquad \&c. = 0. [-\dot{x}z\text{ series.}]
\end{array}
$$

In this equation (the quantity *z* is not to be determined by some affected equation of some other rules of analysis, which in this case

is absurd, since by hypothesis z is a variable quantity, and can always be taken as you please)
[p. 28.] all the terms ought to vanish by themselves, by the values of the coefficients A, B, C, &c. Hence through the first term, C = A, by the second D = $\frac{1}{2}$ B, by the third E = $\frac{1}{3}$C = $\frac{1}{3}$A, and thus henceforth; hence $x = \mathrm{A} + \mathrm{B}z + \mathrm{A}z^2 + \frac{1}{3}\mathrm{B}z^3 + \frac{1}{4}\mathrm{A}z^4 + \&c.$

When you wish to proceed by this method through the assumption of some general form of series, it is often difficult to find that form, especially if for the coefficients you desire there is a need for indeterminates to be left, as advised by the conditions of the problem. Newton sought certain particular series by the extractions of roots from affected equation, and he elaborates on the method, of finding the forms of series of this kind, by arranging the terms in parallelograms. We explain this method in the following proposition (made easier to understand in this way).

PROP. IX. PROB. VI.

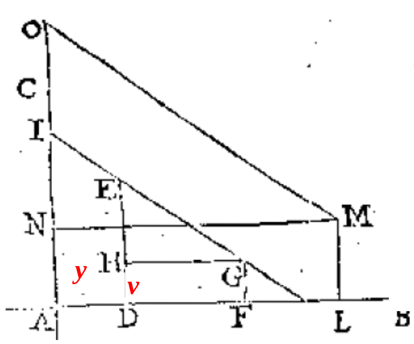
For a given fluxional equation involving only two fluents z and x and their fluxions, and for which z flows uniformly with fluxion 1*; it is required to find the forms of the series by which it is possible to express the value of x in terms of the increase in powers of z.*

The form of the general series sought is $\mathrm{A}z^{\vartheta} + \mathrm{B}z^{\vartheta+\eta} + \mathrm{C}z^{\vartheta+2\eta} + \&c.$, and in a given special case the degrees of the indices $\vartheta \,\&\, \eta$ are to be determined. The series should be of such a kind that, with all the terms of the proposed equation converted into [associated] series, then with the values of x and its fluxions substituted for in terms of this series and with its fluxions in turn expressed by series, it should come about that all the terms of the series can be disposed of between each other in the following manner : for, by comparison of the terms of the same degree z , the coefficients A, B, C, D, &c can be determined, either all of them, or some number of them. [p. 29] In order to do this, there are two requirements. In the first place, it is necessary that the indices of the powers of z, in the series, that have arisen by substitution from the terms of the proposed equation, should all lie in the same series of an arithmetical progression; for if there are always solitary terms present, then either nothing can be determined from them, or all the coefficients are made zero. In the second place, it is also required, for a series arising in this way, that the indices of at least two of the first terms should be equal to each other, so that the first coefficient A can be determined [from a simple equation independent of the other coefficients] : for the coefficient A cannot be given by a solitary term at the start of the equation, for a comparison of the terms cannot be set up in order, or perhaps some other quantity is given in the equation from which A is equal to zero, by which it can be agreed that the changed order of the series can be found.

From these premises, if a general term of some proposed equation is given, along with a separate [numerical] coefficient, by $z^{\mu}\, x^{\alpha}\, \dot{x}^{\beta}\, \ddot{x}^{\gamma}\, \dddot{x}^{\delta}\ \&c.$, then the first term of the series arising from this term, (also supposed to have a separate numerical coefficient) will be of the form $z^{\mu+[\alpha+\beta+\gamma+\delta+\&c]\times\vartheta-\beta-2\gamma-3\delta-\&c.}$, and if π is written in general for the index of this first term, then this series solution with separate numerical coefficients has the form $.....z^{\pi}.....z^{\pi+\eta}.....z^{\pi+2\eta}.....z^{\pi+3\eta}......\&c.$, thus as all the members of the series originate from terms of the equation raised to the power of z to the π. [*i. e.* z^{π} is a common factor.] This is easily understood from a little attention to the formation of the fluxions of the series $\mathrm{A}z^{\vartheta}+\mathrm{B}z^{\vartheta+\eta}+\mathrm{C}z^{\vartheta+2\eta}+\&c.$, and to the generation of the series from the terms of the proposed equation by the multiplications of the fluxions of this kind in turn. Hence, as now has been said, all the indices π fall in the same series in arithmetical progression, with a common difference η, and at least two values of indices π at the start of the transformed equation should be equal to each other. For which either everything is the smallest if η is positive, or everything is greatest if η is negative. Hence, by running through all the terms of the proposed equation, the number π is found for the individual terms, $\mu+[\alpha+\beta+\gamma+\delta+\&c]\times\vartheta-\beta-2\gamma-3\delta-\&c.$ [p. 30] (for $\alpha+\beta+\gamma+\delta+\&c.$ I write y, & for $\mu-\beta-2\gamma-3\delta-\&c$, I write v,) or $y\vartheta+v$. Then $y\vartheta+v$, & $\grave{y}\vartheta+\grave{v}$, are two numbers of this kind.

[Note that y is the sum of all the indices of x and of all its derivatives in a given term of the equation, while v is the sum of the indices of z and all its derivatives for the same term in the series expansion, while ϑ is the constant index in the original expansion. Thus, each term gives rise to a number, and two equal numbers corresponds to two terms with the same index, which can ϑ can then be found.]

Then if these two numbers [*i.e.* equations formed from the indices of different terms at the start of the expansion] can be equated, and hence ϑ determined, then the $y\vartheta+v$ values of all the numbers arise from that correctly determined value ϑ, largest or smallest. Moreover, this result follows from the following artifice.



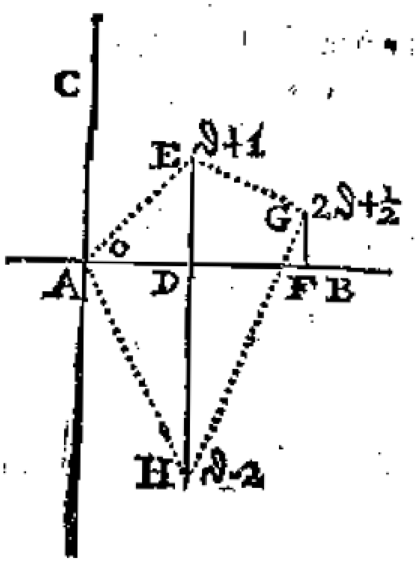
Draw two right lines AB and AC of indefinite length, and (with some line taken of unit length) on AB (to the right if y is positive, but to the left if y is negative) take AD = y, and by drawing DE parallel to AC, in that (again if v is positive, or otherwise if contrary) take DE = v, and the number $y\vartheta+v$ is found at the point E. [*i. e*. the co-ordinates y and v are plotted for a given term in the series, and the gradient found.] With all the numbers $y\vartheta+v$ placed at points in this way, the outer [*i. e.* the largest or smallest of the indices] of these shall be the two points E & G, thus in order that all the remaining points lie on the same part of the line EG. Then two numbers equated between the points E and G determine the value of the index ϑ. For GF is drawn parallel to CA and crosses AB at F, and M is a different point at which another number π is found, and ML is drawn parallel to CA and crossing AB in L ; GE crossing AC at I, and MO is drawn parallel to this line crossing AC in O ; GH and MN are drawn parallel to AB and crossing DE and AC in H and N. Then the numbers π found at the points E, G, and M are

$\mathrm{AD}\times\vartheta+\mathrm{DE}, \mathrm{AF}\times\vartheta+\mathrm{FG}, \mathrm{AL}\times\vartheta+\mathrm{LM},$ (from the construction). Whereby if the numbers at E and G are made equal, then $\vartheta=\frac{\mathrm{DE}-\mathrm{FG}}{\mathrm{AF}-\mathrm{AD}}$, that is $\vartheta=\frac{\mathrm{HE}}{\mathrm{HG}}$, or (on account of the similar triangles EHG, ONM) $\frac{\mathrm{NO}}{\mathrm{MN}}$. [p. 31] Hence now the number $\mathrm{AL}\times\vartheta+\mathrm{LM}$ at the point M is $\mathrm{AL}\times\frac{\mathrm{NO}}{\mathrm{MN}}+\mathrm{LM}$, that is AO; and in the same way the equal numbers at E and G make [the same number] AI. Thus if the points E and G are always the outer points, in order that the point I thus falls either below or above all the points O, then AI will be – that is the number π at E or at G – less or greater by some other number AO as you please from some other point M. Hence by the position of the point I with respect to the points O, the sign of π is determined; certainly it is positive when I lies below O, and negative if contrary to this. And hence it is easily agreed that η is the greatest of the common divisors of AI and of all the AO; for all the other π values do not lie in the same series of arithmetic proportionality, as has now been said should be the case.

Hence in this manner with all the numbers π in the plane accounted for, if the rule is applied to the two outer points E and G, the index ϑ is given, and thus, the sign of the index η. Then η itself is found by taking the greatest common divisor of all the numbers π arising from the values of ϑ now found. Thus the form of the series sought will be given. *Q.E.I.* Moreover, the sign of ϑ is positive when GE subtends the angle CAB, and negative when it subtends the supplement of this angle.

Let an example of this procedure be given for the equation $1+zx-z^{\frac{3}{2}}x\dot{x}-\ddot{x}=0$.
[Recall that $y=\alpha+\beta+\gamma+\delta+\&c.$ and $v=\mu-\beta-2\gamma-3\delta-\&c$.]
By running through the terms of this equation, in the first term 1, $\mu=0=\alpha=\beta=\gamma=\&c.$

Hence the first number π (or $y\vartheta+v$) is 0. In the second term *ax,* the indices are $\mu=1=\alpha;\beta=0=\gamma=\&c.$; hence the second number π is $\vartheta+1$. In the third term $z^{\frac{3}{2}}x\dot{x}$, the indices are $\mu=\frac{3}{2};\alpha=1=\beta;\gamma=0=\delta=\&c.$, thus the third π is $2\vartheta+\frac{3}{2}$. And then in the final term $\ddot{x}$, they are $\mu=0=\alpha=\beta;\gamma=1;\delta=0=\&c.$, and hence the final π is $\vartheta-2$. [p. 32]



And thus with AB and AC drawn, the position of the first point A, or of the first π is 0. Take the abscissa AD = 1, and the ordinate parallel to AC is DE = 1; E is the place of the second π, or $\vartheta+1$. Take the abscissa AF = 2, and the ordinate FG = $\frac{1}{2}$, and the position of the third π is G, or $2\vartheta+\frac{1}{2}$. Then take AD = 1, and with the ordinate DH = - 2, and H is the position of the third number $\vartheta-2$.

Now with straight lines drawn through all the outer points, all the points of the trapezium AHGEA are included [each corresponding to the index of a term in the differential or fluxional equation, which are then made equal in turn to determine particular values of ϑ, and from the other indices the common ratio can be deduced for the arithmetical progression]. From the equations between the numbers 0 & $\vartheta - 2$ at the ends of the line AH, $\vartheta = 2$ is given, and all the [possible] numbers π become [for the other points] 0, $0\,(= \vartheta - 2 = 0,)3(= \vartheta + 1,)$ & $\frac{9}{2}(= 2\vartheta + \frac{1}{2},)$ of which the two smallest are equal to zero, and the greatest common divisor is $\frac{3}{2}$; whereby in this case, $\eta = \frac{3}{2}$.

For the equal numbers taken $\vartheta - 2$ & $2\vartheta + \frac{1}{2}$ in the ends of the length HG, gives $\vartheta = \frac{-5}{2}$, and all the numbers π become $\frac{-9}{2}, \frac{-9}{2}, \frac{-3}{2}, 0;$ of which the two smallest equal ones are $\frac{-9}{2}$, and the greatest common divisor is $\frac{-3}{2}$; whereby in this case, $\eta = \frac{3}{2}$.

If $2\vartheta + \frac{1}{2}$ & $\vartheta + 1$ are made equal to each other, then $\vartheta = \frac{1}{2}$, and all the numbers are $\frac{3}{2}, \frac{3}{2}, 0, \frac{-3}{2};$ of which the greatest equal numbers are $\frac{3}{2}$, and the common divisor is $\frac{3}{2}$; whereby in this case $\eta = \frac{-3}{2}$.

Hence if $\vartheta + 1 = 0$, then $\vartheta = -1$, and all the numbers are $0, 0, \frac{-3}{2}, -3,$ of which there are two equal maximum numbers 0, and the greatest common divisor is $\frac{3}{2}$; whereby in this case $\eta = \frac{-3}{2}$. [p. 33]

Therefore the series for x can become :

either 1. $x = \mathrm{A}z^2 + \mathrm{B}z^{\frac{7}{2}} + \mathrm{C}z^5 + \&\mathrm{c}.$

or 2. $x = \mathrm{A}z^{-\frac{5}{2}} + \mathrm{B}z^{-1} + \mathrm{C}z^{\frac{1}{2}} + \&\mathrm{c}.$

or 3. $x = \mathrm{A}z^{\frac{1}{2}} + \mathrm{B}z^{-1} + \mathrm{C}z^{-\frac{5}{2}} + \&\mathrm{c}.$

or 4. $x = \mathrm{A}z^{-1} + \mathrm{B}z^{-\frac{5}{2}} + \mathrm{C}z^{-4} + \&\mathrm{c}.$

In the third case the analysis is done as set out below [to determine the numerical coefficients]. [p. 34]

Proposed equation.	$1+zx-z^{\frac{3}{2}}x\dot{x}-\ddot{x}=0.$	
Assumed equation.	$x=\mathrm{A}z^{\frac{1}{2}}+\mathrm{B}z^{-1}+\mathrm{C}z^{-\frac{5}{2}}+\&c.$	
Fluxions.	$\dot{x}=\frac{1}{2}\mathrm{A}z^{-\frac{1}{2}}-\mathrm{B}z^{-2}-\frac{5}{2}\mathrm{C}z^{-\frac{7}{2}}-\&c.$ $\ddot{x}=-\frac{1}{4}\mathrm{A}z^{-\frac{3}{2}}+2\mathrm{B}z^{-3}+\frac{35}{4}\mathrm{C}z^{-\frac{9}{2}}+\&c.$	
Transformed equations.	zx $-z^{\frac{3}{2}}x\dot{x}$ $1-\ddot{x}$	$\mathrm{A}z^{\frac{3}{2}}+\mathrm{B}+\mathrm{C}z^{-\frac{3}{2}}+\&c.$ $-\frac{1}{2}\mathrm{A}^2+\frac{1}{2}\mathrm{AB}+2\mathrm{AC}+\mathrm{B}^2+\&c.$ $+1+\frac{1}{4}\mathrm{A}-\&c. \qquad =0.$
Coefficients found.	1. 2. 3. &c.	$\mathrm{A}-\frac{1}{2}\mathrm{A}^2=0$ Hence A = 2 or A = 0. $\mathrm{B}+\frac{1}{2}\mathrm{AB}+1=0,\ \mathrm{B}=-\frac{1}{2};\quad \mathrm{B}=-1.$ $\mathrm{C}+2\mathrm{AC}+\mathrm{B}^2+\frac{1}{4}\mathrm{A}=0,\ \ \mathrm{C}=-\frac{3}{20},\ \mathrm{C}=-1.$ &c. &c. &c
Values of the roots x.	1.	$x=2z^{\frac{1}{2}}-\frac{1}{2}z^{-1}-\frac{3}{20}z^{-\frac{5}{2}}+\&c.$
	2.	$x=-z^{-1}-z^{-\frac{5}{2}}+\&c.$

[p. 35]

For this equation, as you see, there are two series that express the value of x, produced by two values of A in the equation $\mathrm{A}-\frac{1}{2}\mathrm{A}^2=0$, and of these series the second is of the same form as in the final case; whereby through one application, both series are found, so the fourth case as well as the second. Truly indeed by the analysis set up for the second case in the same way also the series is found for the first case. Hence, by only two analysis, all the series are found. But here, this shall only be the case when η is the same in both series; and when one root of A in the first equation found for comparison of the terms is zero. Moreover, there can be more roots of A in that equation, for anyone with talent, in a proposed equation; and for as many roots as there are for A, as many series should be given by the individual analysis.

In this analysis it is to be observed in the second place, that all the coefficients A, B, C, &c. are completely determined by the comparison of the terms. Whereby series found in this way are all particular solutions, and are unable to be adapted to the conditions of a problem, on account of the lack of undetermined coefficients.

SCHOLIUM.

Sometimes when the index θ is a positive whole number, the first terms in the series expressing the fluxions of x vanish : for the coefficients of these first terms are produced by the continued multiplication of A in the numbers θ, θ – 1, θ – 2, &c. In this case often it shall be that as the term vanishes, that should be one of the terms in the beginning of the transformation of the equation, by which it is agreed finally that the series is impossible. But if the terms disappear in the start of the fluxions, and only two terms remain at the start of the transformed equation, the series still will be given ; which here is for the rest of the outstanding terms, since in that there will be some undetermined coefficients, by which it will be possible to apply some conditions to the problem. Indeed by a similar vanishing of the terms in producing the fluxions, there are sometimes other series expressing the root[p. 36], the smallest of which can be found by this proposition.

Since concerning the equation for the particular roots that we now discuss, it is permitted also to observe along the way; truly that if v is a quantity from given and from some variables composed in some way, and the equation can be reduced to such a form, in order that all the term involves either v itself, or some increment, the equation for the particular solution of the problem will be $v = 0$. If in the equation transformed in this way v itself is involved, the equation $v = 0$ contains no undetermined coefficient; and thus, this solution will be especially particular ; specially if v only involves integral amounts. If the equation transformed does not change v, but $\dot{v}$, the equation $v=0$ contains one undetermined coefficient. If the same equation neither contains v nor $\dot{v}$, but $\ddot{v}$, the equation $v = 0$ contains two undetermined coefficients : and in the kind where more of the missing terms $v,\dot{v},\ddot{v},\&c.$ fall short in the transformed equation, the solution of the problem by the equation $v = 0$ will therefore be more general.

Concerning these when v only involves integrals, it is advantageous to find the most general solution to the problem, through v and its increments the remaining variables can be removed, and then by seeking the root v by some method is now drawn. Thus in the equation $x-\frac{2}{z}-\dot{x}z=0$, for $x-\frac{1}{2}$ I write v, this gives $v-\dot{v}z=0,$ that is $\frac{v\dot{z}-\dot{v}z}{v^2}=0.$ But the fluent of $\frac{v\dot{z}-\dot{v}z}{v^2}$ is $\frac{x}{v}$: whereby for any invariable quantity I write that A will be $\frac{z}{v}=\text{A}$, that is $\frac{z}{x-\frac{1}{2}}=\text{A}$.

[p. 37]

LEMMA II.

If x is given from some given z by some analytical equation, of some certain number of dimensions; also some increment of x, $\underset{n}{x}$ *, is given from z by an equation of the same number of dimensions.*

For whatever number the dimensions of x are in the proposed equation, there is the same number of roots (certainly also impossible to be counted.). But the individual roots have their own increments. Whereby there are just as many roots of the increments of any

$\underset{n}{x}$ as there are roots of the whole quantity x itself; and thus there is given in each case an equation of the same dimensions. *Q.E.D.*

COROLLARY.

Hence with the proposed equation defining the relation of any individual increments $\underset{n}{x}$ in terms of the known variable x, if it is possible to give the whole variable x, from the given $\underset{\cdot}{z}$ by an equation with a finite number of terms, and it is given by an equation in which x increases through just as many dimensions as $\underset{n}{x}$ increases in the proposed equation.

PROP. X. PROB. VII.

From a given equation of one dimension, by which the value of any individual increment $\underset{n}{x}$ is defined; to find the value of the integral quantity x itself in a number of finite terms, if it is possible to happen.

If it is possible for the relation of x to be given to known quantities in a finite number of terms, to be given by an equation of one dimension (by the *Cort*e *Lem*.2.) The solution thus sought is by trying to find a quantity to which $\underset{n}{x}$ is equal, from which it is agreed that it can be reduced to the form of increments of some known quantity of the same order. For if that is the case, the root x is given in a finite number of terms, making that equal to the integrated or whole-variable counterpart of the expression, in a finite number of terms. But if this does not happen, then the problem is not solvable.

With fluxions so many fluents can be given in a finite number of terms found by the Newtonian Quadrature of Curves. And sometimes expressions of this kind can readily be found by the two following Propositions.

PROP. XI. THEOR. IV.

The fluent of $\dot{r}s$ can be expressed in terms of either the series

$$\boxed{\dot{r}s} = rs - \overset{\prime}{r}\dot{s} + \overset{\prime\prime}{r}\ddot{s} - \overset{\prime\prime\prime}{r}\dddot{s} + \&c.,or\ \boxed{\dot{r}s} = \dot{r}\overset{\prime}{s} - \ddot{r}\overset{\prime\prime}{s} + \dddot{r}\overset{\prime\prime\prime}{s} - \&c.$$

The theorem is investigated in the following way. Let the fluent sought be $rs + p$, that is, it shall be given by :

$\boxed{\dot{r}s} = rs + p.$ Then by taking the fluxions, it becomes $\dot{r}s = \dot{r}s + r\dot{s} + \dot{p}$, *i. e.* $\dot{p} = -r\dot{s}$:

and thus :

$p = \boxed{-r\dot{s}}$, and hence $\boxed{\dot{r}s} = rs - \boxed{r\dot{s}}$. Then the following expression becomes :

$\boxed{r\dot{s}} = \overset{/}{r}\dot{s} + q$; and by taking fluxions : *i. e.* $r\dot{s} = r\dot{s} + \overset{/}{r}\ddot{s} + \dot{q}$; [p.39] *i. e.* $\dot{q} = -\overset{/}{r}\ddot{s}$; and hence

$q = -\boxed{\overset{/}{r}\ddot{s}}$; hence $\boxed{\dot{r}s} = rs - \overset{/}{r}\dot{s} + \boxed{\overset{/}{r}\ddot{s}}$. And by repeating these operations in the same way, it is found that

$\boxed{\dot{r}s} = rs - \overset{/}{r}\dot{s} + \overset{//}{r}\ddot{s} - \boxed{\overset{//}{r}\dddot{s}} = rs - \overset{/}{r}\dot{s} + \overset{//}{r}\ddot{s} - \overset{///}{r}\dddot{s} + \boxed{\overset{///}{r}\ddddot{s}}$ + &c.

[The theorem follows by repeated integration by parts, and closely resembles a theorem of Johan. Bernoulli. It is assumed that this procedure can be carried out indefinitely in general. Thus, in the first place, in modern notation, the integration by parts of the integral $\int s(t)dr(t) = \int s[dr/dt]dt = sr - \int r[ds/dt]dt$; the final term is now integrated by parts in the same manner : $\int r[ds/dt]dt = [ds/dt]\int r'dt' - \int\int r(t')dt'[d^2s/dt^2]dt$; and the process is continued to generate further terms at will, and the terms can be added to give the series shown.]

When this theorem is to be applied in a particular case, some other fluxion $\dot{w}$ can be selected, and in the computation of the fluents $\dot{r}, \ddot{r}, \dddot{r}$,&c. or $\dot{s}, \ddot{s}, \dddot{s}$,&c. the fluent is compared first with some other, that is multiplied by $\dot{w}$, and the fluent of the product is to be taken for the adjoining fluent sought. In the same way, in the computation of the fluxions $\dot{r}, \ddot{r}, \dddot{r}$,&c. or $\dot{s}, \ddot{s}, \dddot{s}$,&c. the quotient with another fluxion is taken, that makes use of $\dot{w}$, and similarly for the fluxion of the quotient, by making use of $\dot{w}$ taken for the nearby fluxion sought. Moreover this extra fluxion $\dot{w}$ is thus to be taken in order that the terms can be as simple as possible. [Thus, integration by parts can often be avoided by the appropriate choice of an integrating factor.]

For the series found by this theorem is adapted in two ways [for two forms of the series are given] for the conditions of the problem, that is, for a given single value of the fluent sought there corresponds a known value of the variable. Here in the first place all the fluents $r, \overset{/}{r}, \overset{//}{r}$,&c. or $s, \overset{/}{s}, \overset{//}{s}$,&c. are to be taken as pure variables, without the addition of any constant terms, and then a constant [of integration] can be added on later to the series found according to that condition to be determined. Likewise it follows for all the fluents produced by the first method, thus they are to be corrected by the addition of constant terms, [p. 40] in order that everything vanishes together, (and thus the whole series vanishes,) when the variable is given some certain value [as one expects from a definite integral].

In addition, when some term $\dot{s}, \ddot{s}, \dddot{s}, \&c.$ is equal to zero, the first series comes to an end abruptly, and this gives the fluent a finite number of terms. And likewise in the other series, when some term of $\dot{r}, \ddot{r}, \dddot{r}, \&c.$ disappears.

[The interested reader can refer to L. Feigenbaum's article : *'Brook Taylor and the Method of Increments,* Arch. Hist. Exact Sci. ***34*** (1-2) (1985), 1-140' for a more in-depth account of this proposition, which has at least a resemblance to a theorem published by Jonan. Bernoulli in 1695 in the *Acta. Erud.*, and which led to a charge of plagiarism by one of Bernoulli's students, which according to Feigenbaum was not truly justified, though Taylor should have expressed his indebtedness to Bernoulli for the conception of his more general and perhaps more useful proposition, generalised even more in the following proposition. The following examples show how the proposition works.]

EXAMPLE I.

Let the fluxional equation be given : $x\dot{x} = -z\dot{z}$,
[in which we recognise the circle $x^2 + z^2 = a^2$]
and it is proposed to find the fluent of $\dot{z}x$.
[*i. e.* the integral in the modem sense, or the quantity of which the fluxion is $\dot{z}x$]
In this case, $\dot{z}$ is taken for $\dot{r}$, and x is taken for s, and most conveniently we can make $\dot{w} = z\dot{z}\ (= -x\dot{x})$. With this agreed upon,

$$r = z,\ \overset{/}{r} = \left(\boxed{\dot{w}r} = \right) \frac{z^3}{3},\ \overset{//}{r} = \left(\boxed{\dot{w}\overset{/}{r}} = \right) \frac{z^5}{5.3},\ \overset{///}{r} = \left(\boxed{\dot{w}\overset{//}{r}} = \right) \frac{z^7}{7.5.3},$$ and thus henceforth;

[Use is made of continued integration by parts in the formulas for this proposition, however, by a careful choice of the independent variable, this can sometimes be reduced to a simple integration, essentially by a change of variable. Thus,
$\overset{/}{r} = \int r dw = \int z.\frac{dw}{dz} dz = \int z.zdz = \frac{z^3}{3}$, etc.]
likewise

$$\dot{r} = \left(\frac{\dot{z}}{\dot{w}} = \right) \frac{1}{z},\ \ddot{r} = \left(\frac{\dot{r}}{\dot{w}} = \right) \frac{-1}{z^3},\ \dddot{r} = \left(\frac{\ddot{r}}{\dot{w}} = \right) \frac{3}{z^5},$$ and thus henceforth.

[In this case, $\dot{r} = \frac{\dot{z}}{\dot{w}} = \frac{dz}{dw} = \frac{1}{z}$; where the increments $\dot{w} = dw$ and $\dot{z} = dz$ form a differential ratio. Subsequently, $\ddot{r} = \frac{\dot{r}}{\dot{w}} = \frac{d(\frac{1}{z})}{dw} = \frac{d(\frac{1}{z})}{dz}.\frac{dz}{dw} = \frac{-1}{z^3}$; etc.]

Also $s = x$,

$\dot{s} = \left(\frac{s}{\dot{w}} = \right) \frac{-1}{x}$, $\ddot{s} = \left(\frac{\dot{s}}{\dot{w}} = \right)\frac{-1}{x^3}$, $\dddot{s} = \left(\frac{\ddot{s}}{\dot{w}} = \right)\frac{-3}{x^5}$, and thus henceforth
[we can consider *s* as starting from an infinitesimal value *δs*; hence
$\dot{s} = \frac{ds}{dw}\left(= \frac{\delta x}{-x\delta x} = \right)\frac{-1}{x}$, etc.; and $\overset{/}{s} = \int - x.xdx = \frac{-x^3}{3}$, etc.]; likewise

$\overset{/}{s} = \left(\boxed{\dot{w}s} = \right)\frac{-x^3}{3}$, $\overset{//}{s} = \left(\boxed{\dot{w}\overset{/}{s}} =\right)\frac{x^5}{5.3}$, $\overset{///}{s} = \left(\boxed{\dot{w}\overset{//}{s}} = \right)\frac{-x^7}{7.5.3}$, and thus henceforth. Hence, from the first series $rs - \overset{/}{r}\dot{s} + \overset{//}{r}\ddot{s} - \overset{///}{r}\dddot{s} + \&c.$: $\boxed{\dot{z}x} = zx + \frac{z^3}{3x} - \frac{z^5}{5.3x^3} + \frac{z^7}{7.5.3x^3} - \&c.$ And by the other series $\dot{r}\overset{/}{s} - \ddot{r}\overset{//}{s} + \dddot{r}\overset{///}{s} - \&c.$, it is $\boxed{\dot{z}x} = \frac{-x^3}{3z} + \frac{x^5}{5.3z} - \frac{x^7}{7.5.3z^5} + \&c.$

[p. 41]

In these series the fluents $r, \overset{/}{r}, \overset{//}{r}, \&c.$ and likewise $s, \overset{/}{s}, \overset{//}{s}, \&c.$ are taken as independent variables; whereby the series are fitted to the condition of the problem by the addition of some constant. Accordingly for the first of these series the area of a circle is shown, adjacent to the sine *x* and to the cosine *z*, and by the other series the area of the complement of the square is shown with the negative sign : Which thus is that area lying next to the abscissa *z* [and with *x*] produced beyond the ordinate. [Thus, the first integral can be the area of some segment of the circle $x^2 + z^2 = a^2$, given by $\int_a^b xdz$ and expressed by the given infinite series, where $x = a\sin\theta$, $z = a\cos\theta$, etc.]

EXAMPLE II.

Let $x = a + bz^n$, and the fluent of $\dot{z}z^{\theta-1}x^{\lambda-1}$ is to be found. In this case, if we make $\dot{r} = \dot{z}z^{\theta-1}$ and $s = x^{\lambda-1}$ then the most convenient choice [for the independent variable]will be $\dot{w} = \dot{x} = nb\dot{z}z^{n-1}$. Hence by taking pure fluents, the first series will be found to be

$$r = \frac{z^\theta}{\theta}.$$

$$\overset{/}{r} = \left(\boxed{\dot{w}r} = \right)\frac{nbz^{\vartheta+n}}{(\theta+n)\theta} = \frac{nbz^n}{\theta+n.}r.$$

$$\overset{//}{r} = \left(\boxed{\dot{w}\overset{/}{r}} = \right)\frac{n^2b^2z^{\theta+2n}}{(\theta+2n)(\theta+n)\theta} = \frac{nbz^n}{\theta+2n}\overset{/}{r}.$$

$$\overset{///}{r} = \left(\boxed{\dot{w}\overset{//}{r}} = \right) \frac{n^3 b^3 z^{\theta+3n}}{(\theta+3n)(\theta+2n)(\theta+n)\theta} = \frac{nbz^n}{\theta+3n} \overset{//}{r}. \text{ and thus henceforth.}$$

Likewise $\dot{r}, \ddot{r}, \dddot{r}, \&c.$ in the same manner as in the previous example . The fluents $s, \overset{/}{s}, \overset{//}{s}, \&c.$ can be found in the following way.

$\overset{/}{s}$ is the fluent of $\dot{w}s$, *i. e.* of $\dot{x}x^{\lambda-1}$. The pure fluent of this is $\frac{x^\lambda}{\lambda}$, whereby in order that $\overset{/}{s}$ vanishes when $x = d$, then by taking away $\frac{d^\lambda}{\lambda}$ the fluent becomes

$$\overset{/}{s} = \frac{x^\lambda}{\lambda} - \frac{d^\lambda}{\lambda},$$

$$\overset{//}{s} = \frac{x^{\lambda+1}}{(\lambda+1)\lambda} - \frac{d^\lambda x}{1.\lambda} + \frac{d^{\lambda+1}}{(\lambda+1).1},$$

$$\overset{///}{s} = \frac{x^{\lambda+2}}{(\lambda+2)(\lambda+1)\lambda} - \frac{d^\lambda x^2}{2.1.\lambda} + \frac{d^{\lambda+1}x}{1.1(\lambda+1)} - \frac{d^{\lambda+2}}{2.1.(\lambda+2)} \text{ and thus henceforth.}$$

The other ratio for the formation of the rest of the nearby terms can now indeed be agreed upon.

SCHOLIUM.

1. A proposed fluxion can be resolved into the factors $\dot{r}$ & $s.$ in various ways and many different series can arise. Thus the fluxion now proposed : $\dot{z}z^{\theta-1} \times \overline{a+bz^n}|^{\lambda-1}$, can also be written as $\dot{z}z^{\theta+\lambda n-n-1} \times \overline{b+az^{-n}}|^{\lambda-1}$. Where, if for $\dot{r}$, s & $\dot{w}$, are taken $\dot{z}z^{\theta+\lambda n-n-1}, \overline{b+az^{-n}}|^{\lambda-1}$, & $-na\dot{z}z^{-n-1}$, and x is written for $a+bz^n$, the same fluent is expressed by the following series :

$$\boxed{\dot{z}z^{\theta-1}x^{\lambda-1}} = \frac{z^\theta x^{\lambda-1}}{\theta+\lambda n-n} + \frac{\lambda n-n}{\theta+\lambda n-2n}.\frac{a}{x}\mathrm{A} + \frac{\lambda n-2n}{\theta+\lambda n-3n}.\frac{a}{x}\mathrm{B} + \frac{\lambda n-3n}{\theta+\lambda n-4n}.\frac{a}{x}\mathrm{C} + \&c.$$

$$\boxed{\dot{z}z^{\theta-1}x^{\lambda-1}} = \frac{z^\theta x^{\lambda}}{\lambda na} + \frac{\theta+\lambda n}{\lambda n+n}.\frac{x}{a}\mathrm{A} + \frac{\theta+\lambda n+n}{\lambda n+2n}.\frac{x}{a}\mathrm{B} + \frac{\theta+\lambda n+2n}{\lambda n+3n}.\frac{x}{a}\mathrm{C} + \&c.$$

Where the letters A, B, C, &c. are written for all the terms with their signs in the respective series.

2. In the investigation of the Theorem it is found that $\boxed{\dot{r}s} = rs - \boxed{\overset{/}{r}\dot{s}.}$ Hence if for $\dot{r}$ & s is taken

$\dot{z}z^\theta$ & $\overline{a+bz^n}|^\lambda$, and for $x = a+bz^n$ is written z, then $\boxed{\dot{z}z^\theta x^\lambda} = \frac{z^{\theta+1}x^\lambda}{\theta+1} - \boxed{\frac{\lambda bn\dot{z}}{\theta+1}z^{\theta+n}x^{\lambda-1}},$ that is,

$$\boxed{\dot{z}z^{\theta+1}x^{\lambda-1}} = \frac{z^{\theta+1}x^\lambda}{\theta+1} - \frac{\lambda bn}{\theta+1}\boxed{\dot{z}z^{\theta+1}x^{\lambda-1}}.$$

Thus for some given fluent $\dot{z} z^{\theta+1} x^{\lambda-1}$, the fluent of $\dot{z} z^{\theta+n} x^{\lambda-1}$ is also given. Hence if for some n successive whole numbers, either positive or negative, the fluent of one is given $\dot{z} z^{\theta+n} x^{\lambda-n}$, then the fluents of all $\dot{z} z^{\theta+n} x^{\lambda-n}$ are also given.

3. The same flux can also be written thus $\dot{z} z^{\theta+\lambda n} \times \overline{b+az^{-n}}|^{\lambda}$: where if now $\dot{z} z^{\theta+\lambda n}$ is taken for r, and $\overline{b+az^{-n}}|^{\lambda}$ is taken for s :

$$\boxed{\dot{z} z^{\theta} x^{\lambda}} = \frac{z^{\theta+\lambda n-1} \times \overline{b+az^{-n}}|^{\lambda}}{\theta+\lambda n+1} + \frac{\lambda na}{\theta+\lambda n+1} \boxed{\dot{z} z^{\theta+\lambda n-n} \times \overline{b+az^{-n}}|^{\lambda-1}}$$, that is [p. 46]

$$\boxed{\dot{z} z^{\theta} x^{\lambda}} = \frac{z^{\theta+1} \times x^{\lambda}}{\theta+\lambda n+1} + \frac{\lambda na}{\theta+\lambda n+1} \boxed{\dot{z} z^{\theta} x^{\lambda-1}}$$. Thus with the flux of $\dot{z} z^{\theta} x^{\lambda}$, also the flux of $\dot{z} z^{\theta} x^{\lambda-1}$ is given, and vice versa. Thus with the same index θ remaining, if it is continued to be made smaller , or to be increased by unit amounts of λ, for the given fluent of one $\dot{z} z^{\theta} x^{\lambda}$, the fluents of all $\dot{z} z^{\theta} x^{\lambda}$ are given. And by these two cases taken together, if some successive whole numbers are written by σ and τ, either positive or negative, with θ and λ remaining, then if the fluent of one of these fluxions $\dot{z} z^{\theta+\sigma n} \times \overline{a+bz^{n}}|^{\lambda+\tau}$ is given, also the fluents of all the fluxions come about in the same way. And in the same way it is permitted to go to comparison of the fluents, when the quantity within the root brackets is of the third, fourth, or of a greater nomination. But then these are more elegantly produced by the most illustrious Newton in his *Quadratura Curvarum*.

PROP. XII. THEOR. V.

Let n be the index of the order of the fluent $Q = \overset{\backslash}{\dot{r}} s$, *for example, if n = 2, then* $\overset{n}{Q} = \overset{//}{Q}$, *if n = 0, then* $\overset{n}{Q} = \overset{0}{Q} = Q$, *if* n = -1, *then* $\overset{n}{Q} = \overset{-1}{Q} = \dot{Q}$, *and thus for the remaining quantities; then* $\overset{n}{Q} = \overset{\overset{\backslash}{n}}{r} s - \frac{n}{1}.\overset{n}{r}\dot{s} + \frac{n}{1}.\frac{/}{2}\overset{\overset{n}{/}}{r}\ddot{s} - \frac{n}{1}.\frac{/}{2}.\frac{//}{3}\overset{\overset{n}{//}}{r}\dddot{s} +$ &c. *When* n = 1, *and thus henceforth* (*from our notation*) $\overset{\backslash}{n} = n - \underset{\bullet}{n}$; $\underset{/}{n} = n + \underset{\bullet}{n}$; $\underset{//}{n} = \underset{/}{n} + \underset{\bullet}{n}$. [p. 47]

[This is a generalisation of the previous proposition, where the argument Q is integrated n times for positive indices n. The term $\overset{/}{r}$ means the function r has been integrated once, $\overset{//}{r}, \overset{///}{r}$, etc. that it has been integrated twice and three times. On the other hand, $\overset{n}{r}, \overset{\overset{n}{/}}{r}$, and $\overset{\overset{n}{//}}{r}$ indicate that the function r has been integrated n, n + 1, and n + 2 times; while

$\overset{\backslash}{\overset{n}{r}}$ indicates that r has been integrated $n-1$ times. Thus, the rising and falling slopes of the small lines indicate an increase of decrease in an index or number of some kind.]

When $n = 1$, the theorem is the same as the preceding one; and hence the form of the series is found. For by investigation thus, the coefficients are found to be :
$1, \frac{n}{1}, \frac{n}{1}\times\frac{\underset{/}{n}}{2}, \frac{n}{1}\times\frac{\underset{/}{n}}{2}\times\frac{\underset{//}{n}}{3}$, &c. [*i. e.* $1, \frac{n}{1}, \frac{n}{1}\times\frac{n+1}{2}, \frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}$, etc. are all unity.]

Let the coefficients to be found be $x, v, y, w, \&c.$ and by their increments $\underset{\bullet}{x}, \underset{\bullet}{v}, \underset{\bullet}{y}, \underset{\bullet}{w}, \&c.$ to be increased to $\underset{/}{x}, \underset{/}{v}, \underset{/}{y}, \underset{/}{w}, \&c.$ Thus if $\overset{n}{Q} = x\overset{\backslash}{\overset{n}{r}}s + v.\overset{n}{r}\dot{s} + y\overset{\underset{/}{n}}{r}\ddot{s} + w\overset{\underset{//}{n}}{r}\dddot{s} + \&c.$, then the next integration is : $\overset{\underset{/}{n}}{Q} = \underset{/}{x}\overset{n}{r}s + \underset{/}{v}.\overset{\underset{/}{n}}{r}\dot{s} + \underset{/}{y}\overset{\underset{//}{n}}{r}\ddot{s} + \underset{/}{w}\overset{\underset{///}{n}}{r}\dddot{s} + \&c.$ Now if the coefficients $x, v, y, w, \&c.$ are of the desired value, maintaining the same values $\underset{/}{x}, \underset{/}{v}, \underset{/}{y}, \underset{/}{w}, \&c.$, and if the fluxion of the new series is taken to reduce the series to the original series. Then by taking the fluxions first in *r*, then in *s*, [and adding as in the total derivative] it follows that
$\overset{n}{Q} = \underset{/}{x}\overset{\backslash}{\overset{n}{r}}s + (\underset{/}{v} + \underset{/}{x}).\overset{n}{r}\dot{s} + (\underset{/}{y} + \underset{/}{v})\overset{\underset{/}{n}}{r}\ddot{s} + (\underset{/}{w} + \underset{/}{y})\overset{\underset{//}{n}}{r}\dddot{s} + \&c.$ Hence by comparing the terms of this series with the related terms of the first series, then $\underset{/}{x}(= x + \underset{\bullet}{x}) = x$; and thus ; $\underset{\bullet}{x} = 0$. Hence x is a constant. But when $\underset{\bullet}{x} = 0$, $x = 1$; whereby [the coefficient in the first term of the expansion] $x = 1$ always. By comparing the second terms in the two series, it follows that $\underset{/}{v} + x(= v + \underset{\bullet}{v} + 1) = v$; and hence $\underset{\bullet}{v}(= -1) = -\underset{\bullet}{n}$; and hence $v = \frac{-n}{1} + a.$
[p. 48.] But when $n = 0$, $v = 0$; whereby $a = 0$ and $v = \frac{-n}{1}$. By comparing the third terms, $\underset{/}{y} + \underset{/}{v}(= y + \underset{\bullet}{y} - \underset{/}{n}) = y$; $\underset{\bullet}{y} = \underset{/}{n}$ $(=n + 1)$ and hence $y = \frac{n}{1}\frac{\underset{/}{n}}{2} + b$. But when $n = 0$, $y = 0$; hence $b = 0$, and hence $y = \frac{n}{1}\frac{\underset{/}{n}}{2}$. In the same manner, $\underset{/}{w} + \underset{/}{y}(= w + \underset{\bullet}{w} + \frac{\underset{/}{n}}{1}\frac{\underset{//}{n}}{2}.) = y$; and thus $\underset{\bullet}{w} - \frac{\underset{/}{n}}{1}\frac{\underset{//}{n}}{2}$, and hence $w = -\frac{n}{1}\frac{\underset{/}{n}}{2}\frac{\underset{//}{n}}{3}$. And hence by going on indefinitely, the remaining coefficients are found, and all as are shown in the theorem.

EXAMPLE.

From this arrangement, the fluent $Q = \dot{z} z^{\theta-1} \times \overline{a+bz^n}\,|^{\lambda-1}\ (= \dot{z} z^{\theta-1} x^{\lambda-1})$ can be written in any form you please, following from example 2 of Prop. XI:

either 1.

$$\overset{n}{Q} = \frac{\overset{\overset{\backslash\ \backslash}{n\ n}}{nb}\, z^{\theta+\overset{\backslash}{n}x} x^{\lambda-1}}{\theta.\theta+n.\theta+2n.\theta......\theta+\overset{\backslash}{n}\,n} + \frac{n-\lambda n}{\theta+nn}.\frac{nbz^n}{1.x}\mathrm{A} + \frac{2n-\lambda n}{\theta+\underset{/}{n}\,n}.\frac{\underset{/}{n}\,bz^n}{2.x}\mathrm{B} +$$

$$\frac{3n-\lambda n}{\theta+\underset{//}{n}\,n}.\frac{\underset{//}{n}\,bz^n}{3.x}C + \&c.$$

or 2.

$$\overset{n}{Q} = \frac{z^{\theta-x} x^{\lambda+\overset{\backslash}{n}}}{\lambda.\lambda+1.\lambda+2.......\lambda+\overset{\backslash}{n}.nb} + \frac{n-\theta}{\lambda n+nn}.\frac{nx}{1.bz^n}\mathrm{A} + \frac{2n-\theta}{\lambda n+\underset{/}{n}\,n}.\frac{\underset{/}{n}\,x}{2.bz^n}\mathrm{B} +$$

$$\frac{3n-\theta}{\lambda n+\underset{//}{n}\,n}.\frac{\underset{//}{n}\,x}{3.bz^n}C + \&c.$$

or 3. [p. 49.]

$$\overset{n}{Q} = \frac{\overline{-na}|^{\overset{\backslash}{n}} z^{\theta-\overset{\backslash}{n}x} x^{\lambda-1}}{\theta-n+\lambda n.\theta-n+\lambda n-n\theta+.......\lambda-n+\lambda n-\overset{\backslash}{n}\,n} +$$

$$\frac{\lambda n-n}{\theta-n+\lambda n-nn}.\frac{na}{1.x}\mathrm{A} + \frac{\lambda n-2n}{\theta-n+\lambda n-\underset{/}{n}\,n}.\frac{\underset{/}{n}\,a}{2.x}\mathrm{B} + + \& c$$

or 4.

$$\overset{n}{Q} = \frac{-z^{\lambda+\overset{\backslash}{n}} z^{\theta+\overset{\backslash}{n}\lambda x-\overset{\backslash}{n}x}}{\lambda.\lambda+1.\lambda+2.....\lambda+\overset{\backslash}{n}\,a} + \frac{\theta+\lambda n}{\lambda n+nn}.\frac{nx}{1a}\mathrm{A}$$

$$\frac{\theta+\lambda n+n}{\lambda n+\underset{/}{n}\,n}.\frac{\underset{/}{n}\,x}{2a}\mathrm{B} + \&c$$

Obviously in the first two series, for $\dot{w}$ take $nb\dot{z}z^{n-1}$, and by making $\overset{/}{Q} = \boxed{\dot{w}Q}$, $\overset{//}{Q} = \boxed{\dot{w}\overset{/}{Q}}$, and thus henceforth; and in the two final forms of the series for $\dot{w}$ take $-na\dot{z}z^{-n-1}$

and similarly by making $\overset{/}{Q} = \boxed{\dot{w}Q}$, $\overset{//}{Q} = \boxed{\dot{w}\overset{/}{Q}}$, and thus henceforth.

Moreover by these series both the fluxions and the fluents of Q are shown. Thus if $n = -1$, the series gives the value $\overset{/}{Q}$, if $n = -2$, the series gives the value $\overset{//}{Q}$, and so on.

But in that case, where the sign of the number n is changed, the coefficient of the first term is found with certain difficulty. Therefore an example of the method is here set out for finding the first term of the series for the value $\overset{//}{Q}$. The coefficient of this term, taken as $\overset{\overset{\backslash\,\backslash}{n\,n}}{nb}$, is $\frac{1}{\theta.\theta+n.\theta+2n.\theta......\theta+\overset{\backslash}{n}n}$. Moreover, the greatest divisor of this coefficient should be [p. 50.] $\theta+\dot{n}n$. Whereby as the coefficient is to be found when $\overset{\backslash}{n}$ is a negative number, in place of $\frac{1}{\theta.\theta+n.\theta+2n.\theta......\theta+\overset{\backslash}{n}n}$ I write $\frac{...\theta-4n.\theta-3n.\theta-2n.\theta-n}{....\theta-4n.\theta-3n.\theta-2n.\theta-n.\theta,\theta+n\theta,\theta+2n\theta.\&c}$. Then by rejecting all the divisors after $\theta+\overset{\backslash}{n}n$, when $\overset{\backslash}{n}=1$, the coefficient is $\frac{etc.\theta-2n.\theta-n}{etc.\theta-2n.\theta-n.\theta.\theta+n}$, that is $\frac{1}{\theta.\theta+n}$; when $\overset{\backslash}{n}=0$, the coefficient is $\frac{etc.\theta-2n.\theta-n}{etc.\theta-2n.\theta-n.\theta}$, that is $\frac{1}{\theta}$. And by the same argument, when $\overset{\backslash}{n}=-1$, the coefficient is $\frac{etc.\theta-2n.\theta-n}{etc.\theta-2n.\theta-n}$, that is 1; when $\overset{\backslash}{n}=-2$, the coefficient is $\frac{etc.\theta-2n.\theta-n}{etc.\theta-2n}$, that is $\theta-n$; when $\overset{\backslash}{n}=-3$, the coefficient is $\overline{\theta-2n}.\overline{\theta-n}$; and hence forth. Thus, now if m is the index of the fluxion sought of Q, that is m is written for $-n$,

$\overline{\theta-mn}.\overline{\theta-m\overset{\backslash}{n}}.\overline{\theta-m\overset{\backslash\backslash}{n}}.....\overline{\theta-n}$ is the numerical coefficient of the terms of the first series sought.

SCHOLIUM.

Now it is permitted to go on to find the integral of an infinite number of terms, the particular increments of which are given by equations of higher gradients. But since in these cases the solution can only be found from exceedingly large constructions, I have thought that it is not worthwhile to expand further on these matters of no future use. Quadratic equations are reduced to simple equations by the extraction of roots, and cubic equations are resolved by Cardans's rule, and equations of many dimensions also can be resolved by the removal of intermediate terms. Whereby if you have a mind to become involved in a task of great labour, with all the intermediate terms removed, then the solution can be found from the preceding propositions. Moreover the following observation can only lessen so much labour a little, truly that the value of the increment in an equation with a given affection, the coefficient of the following term is similar to the increment of coefficient of following term in the equation defining the value of the integral itself. Whereby there is a risk involved with the coefficient of the following term, if this is not resolved into an integral with a finite number of terms, and the attempt to finding a solution in a finite number of terms is frustrated in the remainder of the equation.

With the principles of the method of increments and the method of fluxions briefly explained, it remains in the other part of this little book that we demonstrate with some examples how much use can be made of these methods in the solution of certain more difficult problems

PROP. VII. THEOR. III.

Sint z & x quantitates duae variables, quarum z uniformiter augetur per data incrementa $\underset{\cdot}{z}$*, & sit* $n\underset{\cdot}{z} = v$, $v - \underset{\cdot}{z} = \overset{\backslash}{v}$, $\overset{\backslash}{v} - z = \overset{\backslash\backslash}{v}$, *& sic porro. Tum dico quod quo tempore z cresendo sit z + v, x item crescendo fiet* $x + \underset{\cdot}{x}\frac{v}{1\underset{\cdot}{z}} + \underset{\cdot\cdot}{x}\frac{v\overset{\backslash}{v}}{1.2\underset{\cdot}{z}^2} + \underset{\cdot\cdot\cdot}{x}\frac{v\overset{\backslash}{v}\overset{\backslash\backslash}{v}}{1.2.3\underset{\cdot}{z}^3} + \&c.$

[p. 22]

DEMONSTRATIO:

$$\begin{array}{llllll}
x & \underset{\cdot}{x} & \underset{\cdot\cdot}{x} & \underset{\cdot\cdot\cdot}{x} & \underset{\cdot\cdot\cdot\cdot}{x} & \&c. \\
x+\underset{\cdot}{x} & \underset{\cdot}{x}+\underset{\cdot\cdot}{x} & \underset{\cdot\cdot}{x}+\underset{\cdot\cdot\cdot}{x} & \underset{\cdot\cdot\cdot}{x}+\underset{\cdot\cdot\cdot\cdot}{x} & \&c. & \\
x+2\underset{\cdot}{x}+\underset{\cdot\cdot}{x} & \underset{\cdot}{x}+2\underset{\cdot\cdot}{x}+\underset{\cdot\cdot\cdot}{x} & \underset{\cdot\cdot}{x}+2\underset{\cdot\cdot\cdot}{x}+\underset{\cdot\cdot\cdot\cdot}{x} & \&c. & & \\
x+3\underset{\cdot}{x}+3\underset{\cdot\cdot}{x}+\underset{\cdot\cdot\cdot}{x} & \underset{\cdot}{x}+3\underset{\cdot\cdot}{x}+3\underset{\cdot\cdot\cdot}{x}+\underset{\cdot\cdot\cdot\cdot}{x} & \&c. & & & \\
x+4\underset{\cdot}{x}+6\underset{\cdot\cdot}{x}+4\underset{\cdot\cdot\cdot}{x}+\underset{\cdot\cdot\cdot\cdot}{x} & \&c. & & & & \\
\&c. & & & & &
\end{array}$$

Valores successivi ipsius per additionem continuam collecti sunt x, $x+\underset{\cdot}{x}$, $x+2\underset{\cdot}{x}+\underset{\cdot\cdot}{x}$, $x+3\underset{\cdot}{x}+3\underset{\cdot\cdot}{x}+\underset{\cdot\cdot\cdot}{x}$, &c. ut patet per operationem in tabula annexa expressam. Sed in his valoribus x coefficientes numerales terminorum $x, \underset{\cdot}{x}, \underset{\cdot\cdot}{x}, \&c.$ eodem modo formantur, ac coefficientes terminorum correspondentium in dignitate binomii. Et (per Theorema *Newtonianum*) si dignitatis index sit n, coefficientes erunt $1, \frac{n}{1}, \frac{n}{1}\times\frac{n-1}{2}, \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}, \&c.$ Ergo quo tempore z crescendo sit $z+n\underset{\cdot}{z}$, fiet x aequalis seriei

$x+\frac{n}{1}\underset{\cdot}{x}+\frac{n}{1}\times\frac{n-1}{2}\underset{\cdot\cdot}{x}+\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\underset{\cdot\cdot\cdot}{x}+\&c.$

Sed sunt $\frac{n}{1}=\left(\frac{n\underset{\cdot}{z}}{\underset{\cdot}{z}}=\right)\frac{v}{\underset{\cdot}{z}},\ \frac{n-1}{2}=\left(\frac{n\underset{\cdot}{z}-\underset{\cdot}{z}}{2\underset{\cdot}{z}}=\right)\frac{\overset{\backslash}{v}}{2\underset{\cdot}{z}},\ \frac{n-2}{3}=\left(\frac{n\underset{\cdot}{z}-2\underset{\cdot}{z}}{3\underset{\cdot}{z}}=\right)\frac{\overset{\backslash\backslash}{v}}{3\underset{\cdot}{z}}$, & c. [p. 23] Proinde quo tempore z crescendo sit $z + v$, eadem tempore x crescendo fiet

$$x+\underset{\cdot}{x}\frac{v}{1\underset{\cdot}{z}}+\underset{\cdot\cdot}{x}\frac{v\overset{\backslash}{v}}{1.2\underset{\cdot}{z}^2}+\underset{\cdot\cdot\cdot}{x}\frac{v\overset{\backslash}{v}\overset{\backslash\backslash}{v}}{1.2.3\underset{\cdot}{z}^3}+\&c.$$

COROLL. I.

Et ipsis $\underset{\cdot}{z}, x, \underset{\cdot}{x}, \underset{\cdot\cdot}{x}, \underset{\cdot\cdot\cdot}{x}$, & c. iisdem manentibus, mutato signo ipsius v, quo tempore z descrescendo sit $z - v$, eodem tempore x decrescendo fiet $x-\underset{\cdot}{x}\frac{v}{1\underset{\cdot}{z}}-\underset{\cdot\cdot}{x}\frac{v\overset{\backslash}{v}}{1.2\underset{\cdot}{z}^2}-\underset{\cdot\cdot\cdot}{x}\frac{v\overset{\backslash}{v}\overset{\backslash\backslash}{v}}{1.2.3\underset{\cdot}{z}^3}-\&c.$ vel juxta notationem nostram $x-\underset{\cdot}{x}\frac{v}{1\underset{\cdot}{z}}+\underset{\cdot\cdot}{x}\frac{v\underset{/}{v}}{1.2\underset{\cdot}{z}^2}-\underset{\cdot\cdot\cdot}{x}\frac{v\underset{/}{v}\underset{//}{v}}{1.2.3\underset{\cdot}{z}^3}+\&c.$ ipsis $\overset{\backslash}{v}, \overset{\backslash\backslash}{v}$, &c. conversis in $-\underset{/}{v}, -\underset{//}{v}$, & c.

COROLL. II.

Si pro Incrementis evanescentibus scribantur fluxiones ipsis proportionales, factis jam omnis $v, \overset{\backslash}{v}, \overset{\backslash\backslash}{v}, \underset{/}{v}, \underset{//}{v}$, & c. aequalibus quo tempore z uniformiter fluendo sit $z + v$ fiet x, $x+\dot{x}\frac{v}{1\dot{z}}+\ddot{x}\frac{v^2}{1.2\dot{z}^2}+\dddot{x}\frac{v^3}{1.2.3\dot{z}^3}+\&c.$ vel mutatio signo ipsius v, quo tempore z descrescendo sit $z - v$, x descrescendo fiet $x-\dot{x}\frac{v}{1\dot{z}}+\ddot{x}\frac{v^2}{1.2\dot{z}^2}-\dddot{x}\frac{v^3}{1.2.3\dot{z}^3}+\&c.$

[p. 24]

PROP. VIII. PROB. V.

Data Aequatione praeter uniformiter crescentem z involvente quotvis incrementa alterius variabilis x; invenire valorem x ex dato z per seriem terminorum numero infinitam.

Per Propositionem primam quare aequationis propositae incrementa omnia in infinitum. Tum si sit $\underset{n}{x}$ infimum incrementum ipsius x in aequatione proposita per has aequationes dubuntur omnia incrementa $\underset{n}{x}$ & inferiora expressa per incrementa ipsa $\underset{n}{x}$ superiora. Sint a, & $c, \underset{\cdot}{c}, \underset{\cdot\cdot}{c}, \underset{\cdot\cdot\cdot}{c}$, & c. certi quidam valores correspondentes ipsorum z, & $x, \underset{\cdot}{x}, \underset{\cdot\cdot}{x}, \underset{\cdot\cdot\cdot}{x}$, & c. atque per easdem aequationes dabuntur omnes termini $\underset{n}{c}, \underset{n+1}{c}$, & sequentes expressi per terminos praecedentes ipsum $\underset{n}{c}$. Unde si pro z scribatur $a + v$, dabitur x per $x=c+\underset{\cdot}{c}\frac{v}{1\underset{\cdot}{z}}+\underset{\cdot\cdot}{c}\frac{v\overset{\backslash}{v}}{1.2\underset{\cdot}{z}^2}+\underset{\cdot\cdot\cdot}{c}\frac{v\overset{\backslash}{v}\overset{\backslash\backslash}{v}}{1.2.3\underset{\cdot}{z}^3}+\&c.$ (per Prop. 7.) Ubi terminorum coefficientes $c, \underset{\cdot}{c}, \underset{\cdot\cdot}{c}, \underset{\cdot\cdot\cdot}{c}$, & c. quorum numerus est n, dabuntur per totidem conditiones Problematis.

SCHOLIUM.

Ubi est x compositum aliquod ex dignitatibus integris affirmativis ipsius z, evanescentibus incrementibus inferioribus, post certum numerum terminorum series abrumpetur & fiet finita. Sit aequatio $\underset{.}{x}z - x + 1 = 0$ & sit $\underset{.}{z} = 1$. Tum capiendo incrementa, fiet $\underset{..}{x}z + \underset{..}{x} = 0$. [p. 25]

Sed hoc fieri nequit nisi sit $\underset{..}{x} = 0$; alias enim determinaretur z per aequationem $z + 1 = 0$. Ergo si pro z scribatur $a + v$, & sint c & $\underset{.}{c}$ ipsorum x & $\underset{.}{x}$ valores quando $v = 0$, erit semper $x = c + \underset{.}{c}v$, hoc est, (pro $\underset{.}{c}$ scripto ipsius valore per aequationem propositam invento) $x = c + \frac{c-1}{a}v$, hoc est (pro v scripto ipsius valore $x - a$) $x = 1 + \frac{c-1}{a}z$.

In seriebus hoc modo prodeuntibus post aliquot terminos ex observata analogia plerumque possunt inveniri coefficientes sequentes absque ulteriori culculo. Et possunt nonnunquam series inventae comparari cum aliis seriebus cognitis, quae producuntur a cognitis expressionibus finitis : quare vice serierum substitutis istis expressionibus finitis, eo pacto dabuntur integrales in terminis numero finitis. Sit aequatio fluxionalis

$\ddot{x}z + n\ddot{x}x - \dot{x} - \dot{x}^2 = 0$, ubi pro $\dot{z}$ scribitur 1. Hinc sit $\ddot{x} = \frac{\dot{x}+\dot{x}^2}{z+nx}$ vel (pro $z + nx$ scripto y). Et per calculum continuatu invenietur

$\dddot{x} = \left((2-n)\times\frac{\dot{x}^2+\dot{x}^3}{y^2} = \right)(2-n)\frac{\dot{x}}{y}\ddot{x}$, $\overset{4}{x} = (3-2n)\frac{\dot{x}}{y}\dddot{x}$, $\overset{5}{x} = (4-3n)\frac{\dot{x}}{y}\overset{4}{x}$, & sic porro. Quare (pro a + nc scripto p) per hanc *Prop*. erit $x = c + \dot{c}v + \frac{\ddot{c}v^2}{2} + \frac{(2-n)\ddot{c}\dot{c}v^3}{1.2.3p} + \frac{(2-n)(3-n)\ddot{c}\dot{c}v^4}{1.2.3.4p} + \&\text{c.}$ hoc est

$$x = c + \dot{c}v + \frac{\ddot{c}p^2}{n\dot{c}^2}\times\frac{n}{p}\times\frac{1}{2p}\dot{c}^2v^2 + \frac{n}{p}\times\frac{1}{2p}\times\frac{(2-n)}{3p}\dot{c}^3v^3 + \frac{n}{p}\times\frac{1}{2p}\times\frac{(2-n)}{3p}\times\frac{(3-2n)}{4p}\dot{c}^4v^4 + \&\text{c.}$$ Sed numeri n, 1, $2 - n$, $3 - 2n$, &c. producuntur per continuam subductionem numeri $n - 1$. Quare si sit $p = n - 1$, erit series $\frac{n}{p}\times\frac{1}{2p}\dot{c}^2v^2 + \frac{n}{p}\times\frac{(2-n)}{3p}\dot{c}^3v^3 + \&c. = (1+\dot{c}v)^{\frac{n}{n-1}} - 1 + \frac{n}{n-1}\dot{c}v$.

Pone itaque $p = n - 1$, [p. 26] hoc est fiat $a = n - 1 - nc$, atque dabitur x per aequationem finitam $x = c + \dot{c}v + \dot{c}\frac{(n-1)^2}{n\dot{c}^2}\times\overline{(1+\dot{c}v)^{\frac{n}{n-1}} - 1 - \frac{n}{n-1}\dot{c}v}$; ubi est $\ddot{c} = \frac{\dot{c}+\dot{c}^2}{n-1}$, atque $v = z - n + 1 + nc$; atque incogniti c & $\dot{c}$ determinandi sunt per duas conditiones Problematis. Porro in hac aequatione x & z subeunt vices ipsorum z & x in aequatione $\ddot{x}x + \dot{x}x + n\ddot{x}z + \dot{x}\dot{z} = 0$, quam adduximus in *Lem*. 1. Eam vero in ordine ad inventionem hujus expressionis finitae (absque isthac transformatione quidem frustra quaesitae) transformavi per Propositionem tertiam. Cujus Propositionis usus quantus sit constat etiam ex hoc exemplo. Sed & fluente uniformiter z, ubi in aequatione proposita involuntur fluxiones secundae, tertiae, vel sequentes ipsius x, si per hanc Propositionem cupis invenire valorem ipsius z ex data x, erit aequatio eodem modo transformanda.

Nonnunquam per alias transformationes inveniuntur expressiones finitae. Sit aequatio $4\dot{x}^3 - 4x^2 = (1+z^2)\dot{x}^2$ (quam etiam adduximus in *Lem*. 1.) Pono $x = v^{\theta} y^{\lambda}$, deinde substituto hoc valore *x*, & valore fluxionis $\dot{x}$ inde resultante, quaerendo aequationis formam simplicissimam determino indices θ & λ, & valorem unius *v* vel *y*.

Hinc ergo sit $x = \overline{\vartheta \dot{v} y + \lambda \dot{y} v} \times v^{\theta-1} y^{\lambda-1}$, atque (per substitutionem)

$4v^{3\theta} y^{3\lambda} - 4v^{2\theta} y^{2\lambda} = \overline{1+z^2}|^2 \times \overline{\vartheta \dot{v} y + \lambda \dot{y} v}|^2 \; v^{\theta-2} y^{\lambda-2}$. Pono λ = - 2 & sit

$v^{\theta} - y^2 = \overline{\vartheta z\, y - \dot{y} v}\;|^2$, hoc est $v^{\theta} = v^2 z^2 y^2 + 1 - 2\vartheta z v y + v^2 y^2$. Pono denique $\vartheta = 1$, utque sit $\vartheta z^2 + 1 = v$, quo pacto aequatio divisa per *v* sit [p. 27] $x = y^2 - 2zy\dot{y} + v\dot{y}\dot{y}$. Unde capiendo fluxiones sit $0 = 2y\dot{y} - 2y\dot{y} - 2z\dot{y}\dot{y} - 2zy\ddot{y} + \dot{v}\dot{y}\dot{y} + 2v\dot{y}\ddot{y}$, hoc est (pro $\dot{v}$ scripto ipsius valore 2z) $-2zy\ddot{y} + 2v\dot{y}\ddot{y} = 0$. Hinc sit vel $\ddot{y} = 0$, vel $-zy + v\dot{y} = 0$, invenietur $y^2 = v$, adeoque $x = (vy^{-1} =) 1$; quae est singularis quaedam solutio Problematis. Sed si fiat $\ddot{y} = 0$, pro ipsorum *z, y*, & $\dot{y}$ valoribus concurrentibus scriptis $o, a, \& \dot{a}$ invenietur $\dot{a} = \sqrt{1-aa}$, adeoque erit $y = a + \sqrt{1-aa}\,z$ (per hanc Propositionem) & inde $x = (v^{\theta} y^{\lambda} =) \frac{1+z^2}{(a+\sqrt{1-a^2}z)^2}$.

Quoties fieri potest ipsius *z* valor datus aequalis nihilo, nec eo pacto termini seriei redduntur infini, series prodibit in forma simpliciori ascfendens per dignitates ipsius *z*. Et in hoc casu potest assumi series in terminis generalibus exprimens valorem *x*; in qua coefficientes postea determinentur per comparationem terminorum, ad normam sequentis exempli.

Sit aequatio $\ddot{x} - \dot{x} z - 2x = 0.$ Pone $x = \mathrm{A} + \mathrm{B}z + \mathrm{C}z^2 + \mathrm{D}z^3 + \mathrm{E}z^4 + \&c.$ Tum capiendo fluxiones sit $\dot{x} = \mathrm{B} + 2\mathrm{C}z + 3\mathrm{D}z^3 + 4\mathrm{E}z^3 + \&c.$ atque $\ddot{x} = 2\mathrm{C} + 6\mathrm{D}z^2 + 12\mathrm{E}z^2 + \&c.$ Quibus valoribus ipsorum $x, \dot{x}, \ddot{x}$ in aequatione scriptis, & terminis dispositis secundum dignitates ipsius z sit $\left.\begin{array}{llll} 2\mathrm{C} & +6\mathrm{D}z^2 & +12\mathrm{E}z^2 & +\&c. \\ -2\mathrm{A} & -2\mathrm{B} & -2\mathrm{C} & \&c. \\ & -\mathrm{B} & -\mathrm{C} & \&c. \end{array}\right\} = 0.$ In hac aequatione (ne fiat z quantitas determinata per aequationem affectam alicuis ordinis Analytici, quod in hoc casu est absurdum, quoniam ex hypothesi est z quantitatas variabilis, & semper ad libitum fumenda) [p. 28.] debent termini omnes evanescere per se, per valores coefficientium A, B, C, &c. Erit ergo per terminum primum C = A, per secundum D = $\frac{1}{2}$ B, per tertium E = $\frac{1}{3}$C = $\frac{1}{3}$A, & sic porro; unde sit $x = \mathrm{A} + \mathrm{B}z + \mathrm{A}z^2 + \frac{1}{3}\mathrm{B}z^3 + \frac{1}{4}\mathrm{A}z^4 + \&c.$

Ubi hoc modo procedere velis per assumptionem serierum informis generalibus, saepe diffice est istas formas invenire, praesertim si cupis coefficientes quot opus est indeterminatos esse relictos, ut consulatur conditionibus Problematis. Series quasdam particulares quaesit Newtonns per extractiones radicum ab aequatibus affectis, & methodum docet, concinnam sane & elegantem, inveniendi formas hujusmodi serierum, per dispositionem terminorum in parallelogrammis. Quod Artificium (facta levi mutatione) in sequenti propositione explicabimus.

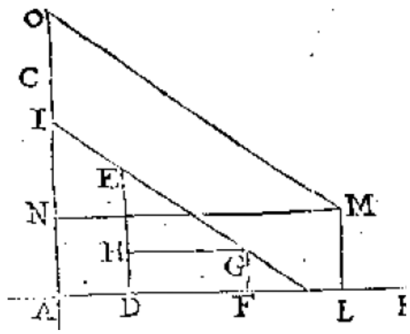
PROP. IX. PROB. VI.

Data aequatione fluxioni duas tantum Fluentes z & x, & earum Fluxiones involvente, quarum z uniformiter fluit per Fluxiones 1; invenire formas Serierum ascendentium per dignitates ipsius z, per quas exprimi possit valor ipsius x.

Seriei quaesitae forma generalis est $\mathrm{A}z^{\vartheta}+\mathrm{B}z^{\vartheta+\eta}+\mathrm{C}z^{\vartheta+2\eta}+\&c.$ & in dato casu speciali determinandi sunt indices dignitatum $\vartheta \,\&\, \eta$. Qui tales esse debent, ut, conversis omnibus terminibus aequationis propositae in series, in iis substituendo valores ipsius x & fluxionum suarum per hanc seriem, & per ejus fluxiones expressos, possint termini omnium serierum hoc modo provenientium ita inter disponi, ut, per comparationem terminorum in quibus sunt eaedem dignatates ipsius z queant determinari coefficientes A, B, C, D, &c. vel omnes, vel quot fieri potest. [p. 29] Ad hoc duo requiruntur. Primum, ut indices dignitatum z, in seriebus ex terminis aequationis propositae per substitutionem provenientibus, omnes cadant in eadem seriem arithemetice proportionalium; alias enim semper essent termini solitarii, per quos vel nihil determinaretur, vel fierent omnes coefficientes aequales nihilo. Secundo etiam requiritur, ut serierum hoc modo provenientium ad minimum duarum terminorum primorum indices inter se aequentur, ut determinetur coefficiens primus A : ne per terminum solitarium in initio aequationis ad comparationem terminorum instituendam ordinatae fiat coefficiens A, vel forte quantitas aliqua data in aequatione proposita, aequalis nihilo; quo pact perturbetur ordo seriei inveniendae.

His praemissis, si terminus aliquis aequationis propositae seposito coefficiente dato sit $z^{\mu}x^{\alpha}\dot{x}^{\beta}\ddot{x}^{\gamma}\dddot{x}^{\delta}\,\&c.$ terminus primiis seriei ab hoc termino provenientis, (seposito etiam coefficiente) erit $z^{\mu+\overline{\alpha+\beta+\gamma+\delta+\&c.}\times\vartheta-\beta-2\gamma-3\delta-\&c.}$, & si pro indice hujus termini scribatur π, series illa (sepositis coefficientibus) hanc habebit formam $.....z^{\pi}.....z^{\pi+\eta}.....z^{\pi+2\eta}.....z^{\pi+3\eta}......\&c.$ ita ut omnes series ex terminis aequationis propositae provenientes ascendant per ipsius z dignitatem π. Hoc est facile intelligitur paululum attendo ad formationes fluxionum $\mathrm{A}z^{\vartheta}+\mathrm{B}z^{\vartheta+\pi}+\mathrm{C}z^{\vartheta+2\pi}+\&c.$ ad geneses serierum ex terminis aequationis propositae per multiplicationes hujusmodi fluxionum in invicem. Ergo per jam dicta, debent omnes π cadere in eandem seriem arithmetice proportionalium, quorum differentia est π, & ad minimum duo π in principio aequationis transformatae debent esse inter se aequales. Qui vel omnium minimi erunt sit π affirmatus, vel omnium maximi si sit π negativus. Ergo percurrendo omnes terminos aequationis propositae ex singulis colligatur numerus π [p. 30] vel

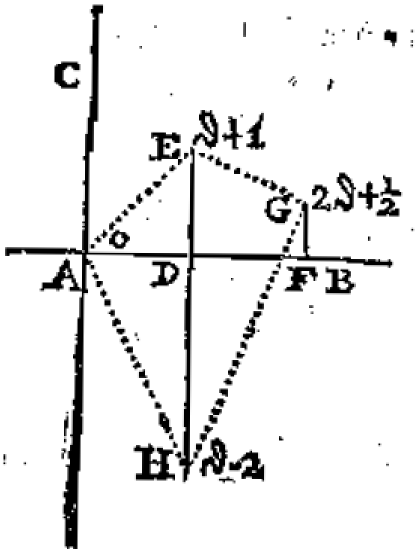
$\mu+\overline{\alpha+\beta+\gamma+\delta+\&c.}\times\vartheta-\beta-2\gamma-3\delta-\&c.$ vel (pro $\alpha+\beta+\gamma+\delta+\&c.$ scripto y, & pro $\mu-\beta-2\gamma-3\delta-\&c$ scripto v,) $y\vartheta+v.$ Sint $y\vartheta+v,$ & $\grave{y}\vartheta+\grave{v},$ duo ex hujusmodi numeris. Tum si hi numeri tales sint, ut facti inter se aequales, & inde determinato ϑ, sint omnium numerorum $y\vartheta + v$ per istum valorem ϑ provenientum vel maximi, vel minimi, recte determinabitur ϑ. Hoc autem sit sequenti artificio.



Duc rectas infinitas AB, AC, & (sumpta aliqua linea pro unitate) in AB (ad dextram si sit *y* affirmativus, sed ad sinistram si sit *y* negativus) sume AD = *y*, & ducta DE ipsi AC parallela, in ea (sursum si sit *v* affirmativus, at deorsum si contra) sume DE = *v*, & collacta numerum $y\vartheta + v$ in puncto E. Omnibus numeris $y\vartheta + v$ in punctis hoc modo dispotis, sint eorum exteriora duo E & G, ita ut puncta reliqua omnia cadant ad easdem partes rectae EG. Tum numeri in punctis E & G inter se facti aequales dabunt valorem indicis ϑ. Duc enim GF parallelam ipsi CA & occurrentem AB in F, & sit M aliud punctum in quo collocatur alius numerus π, & ducatur ML parallela ipsi CA & occurrens AB in L, atque; occurrent GE ipsi AC in I, & ducatur ei parallela MO occurrens AC in O, atque; ducantur GH, MN ipsi AB parallelae & occurrentes DE & AC in H &N. Tum numeri π collacti in punctis E, G, & M erunt $\mathrm{AD}\times\vartheta + \mathrm{DE}, \mathrm{AF}\times\vartheta + \mathrm{FG}, \mathrm{AL}\times\vartheta + \mathrm{LM},$ (per constructionem). Quare si numeri in E & G fiant aequales, erit $\vartheta = \frac{\mathrm{DE}-\mathrm{FG}}{\mathrm{AF}-\mathrm{AD}}$, hoc est $\vartheta = \frac{\mathrm{HE}}{\mathrm{HG}}$, vel (ob similia triangula EHG,ONM) $\frac{\mathrm{NO}}{\mathrm{MN}}$.
[p. 31] Unde jam numerus $\mathrm{AL}\times\vartheta + \mathrm{LM}$ in puncto M sit $\mathrm{AL}\times\frac{\mathrm{NO}}{\mathrm{MN}} + \mathrm{LM}$, hoc est AO; atque; ad eundem modum numeri aequales in E & G fiunt AI. Unde si puncta E & G sint omnium exteriora, adeo ut cadat punctum I vel infra vel supra omnia puncta O, erit AI, hoc est numerus π in E, vel in G, minor, vel major quolibet alio numero AO in puncto quovis alio M. Unde per positionem puncti I respectu punctorum O, determinatur signum ipsius π; quippe qui affirmativus est ubi I cadit infra O, & negativus si contra. Et hinc facile constat esse η maximum divisorem communem ipsius AI & omnium AO; alias enim non caderent omnes π in eandem seriem arithemtice proportionalium, ut per jam dicta fieri debet.

Ergo numeris omnibus π in plano hoc modo disposiis, si applicetur regula ad puncta duo exteriora E & G, dabitur index, ϑ, atqui; signum indicis η. Deinde invenietur ipse η sumendo maximum divisorem communem omnium numerem π provenientium per valorem ϑ jam inventum. Unde dabitur forma seriei quaesitae. *Q.E.I.* Ipsius autem ϑ signum est affirmativum ubi GE subtendit angulum CAB, atque, negativum ubi subtendit ejus complimentum ad duos rectos.

Sit hujus rei exemplum in aequatione $1 + zx - z^{\frac{3}{2}}x\dot{x} - \ddot{x} = 0.$ Percurrendo terminos hujus aequationis, in primo 1 sunt $\mu = 0 = \alpha = \beta = \gamma = \&c.$ Unde primus numerus π (vel $y\vartheta + v$) erit 0. In secundo termino *zx* sunt $\mu = 1 = \alpha; \beta = 0 = \gamma = \&c.$ unde secundus numerus π sit $\vartheta + 1$. In tertio termino $z^{\frac{3}{2}}x\dot{x}$ sunt $\mu = \frac{3}{2}; \alpha = 1 = \beta; \gamma = 0 = \delta = \&c.$ unde tertius π sit $2\vartheta + \frac{1}{2}$..
Denique in ultimo termino $\ddot{x}$ sunt $\mu = 0 = \alpha = \beta; \gamma = 1; \delta = 0 = \&c.$ unde ultimus π sit $\vartheta - 2$.



[p. 32]

Ductis itaque AB & AC, erit punctum A locus numeri π primi, vel 0. Sume abscissam AD = 1, & ordinatam ipsi AC parallelam DE = 1, atq; erit E locus secundi π, vel $\vartheta+1$. Sume abscissam AF = 2, & ordinatam FG = $\frac{1}{2}$, atque erit G locus tertii π, vel $2\vartheta+\frac{1}{2}$. Sume denique AD = 1, & ordinata DH = -2, atque erit H locus numeri $\vartheta-2$.

Jam ductis rectis per puncta omnia exteriora, includentur omnia puncta trapezio AHGEA. Aequatis inter se numeris 0 & $\vartheta-2$ in extremitatibus lateris AH, fiet $\vartheta=2$, & omnes numeri π fient, 0, $0\,(=\vartheta-2=0,)\,3(=\vartheta+1,)$ & $\frac{9}{2}(=2\vartheta+\frac{1}{2},)$ quorum omnium minimi sunt duo aequales 0, & divisor maximus communis est $\frac{3}{2}$; quare in hoc casu est $\eta=\frac{3}{2}$.

Pro numeris aequalibus sumptis $\vartheta-2$ & $2\vartheta+\frac{1}{2}$ in extremitatibus lateris HG, sit $\vartheta=\frac{-5}{2}$, & omnes numeri π fiunt $\frac{-9}{2},\frac{-9}{2},\frac{-3}{2},0;$ quorum minimi sunt duo aequales $\frac{-9}{2}$, & maximus divisor communis est $\frac{3}{2}$; quare in hoc casu est $\eta=\frac{-3}{2}$.

Si fiant $2\vartheta+\frac{1}{2}$ & $\vartheta+1$ inter se aequales erit $\vartheta=\frac{1}{2}$, & numeri omnes erunt $\frac{3}{2},\frac{3}{2},0,\frac{-3}{2}$; quarum maximi sunt duo aequales $\frac{3}{2}$, & divisor communis est $\frac{3}{2}$; quare in hoc casu est $\frac{3}{2}$.

Denique si fiat $\vartheta+1=0$, erit $\vartheta=-1$, & omnes numeri erunt $0,0,\frac{-3}{2},-3,$ quorum maximi sint duo, & divisor maximus communis est $\frac{3}{2}$; quare casu est $\frac{-3}{2}$. [p. 33]

Potest ergo fieri

vel 1. $x=\mathrm{A}z^{2}+\mathrm{B}z^{\frac{7}{2}}+\mathrm{C}z^{5}+\&c.$

vel 2. $x=\mathrm{A}z^{-\frac{5}{2}}+\mathrm{B}z^{-1}+\mathrm{C}z^{\frac{1}{2}}+\&c.$

vel 3. $x=\mathrm{A}z^{\frac{1}{2}}+\mathrm{B}z^{-1}+\mathrm{C}z^{-\frac{5}{2}}+\&c.$

vel 4. $x=\mathrm{A}z^{-1}+\mathrm{B}z^{-\frac{5}{2}}+\mathrm{C}z^{-4}+\&c.$

In casu tertio Analysis se habet ut infra exhibetur. [p. 34]

Aequatio proposita	$1+zx-z^{\frac{3}{2}}x\dot{x}-\ddot{x}=0.$
Aequatio assumpta.	$x=\mathrm{A}z^{\frac{1}{2}}+\mathrm{B}z^{-1}+\mathrm{C}z^{-\frac{5}{2}}+\&c.$
Fluxiones.	$\dot{x}=\frac{1}{2}\mathrm{A}z^{-\frac{1}{2}}-\mathrm{B}z^{-2}-\frac{5}{2}\mathrm{C}z^{-\frac{7}{2}}-\&c.$ $\ddot{x}=-\frac{1}{4}\mathrm{A}z^{-\frac{3}{2}}+2\mathrm{B}z^{-3}+\frac{35}{4}\mathrm{C}z^{-\frac{9}{2}}+\&c.$

Aequatio transformata.	zx $-z^{\frac{3}{2}}x\dot{x}$ $1-\ddot{x}$	$Az^{\frac{3}{2}}+B+Cz^{-\frac{3}{2}}+\&c.$ $-\frac{1}{2}A^2+\frac{1}{2}AB+2AC+B^2+\&c.$ $+1+\frac{1}{4}A-\&c. \quad =0.$
Invenium coefficientium	1. 2. 3. &c.	$A-\frac{1}{2}A^2=0$ Unde A = 2 vel A = 0. $B+\frac{1}{2}AB+1=0,\ B=-\frac{1}{2}; \quad B=-1.$ $C+2AC+B^2+\frac{1}{4}A=0,\ \ C=-\frac{3}{20},\ C=-1.$ &c. &c. &c
Valores Radices x.	1.	$x=2z^{\frac{1}{2}}-\frac{1}{2}z^{-1}-\frac{3}{20}z^{-\frac{3}{2}}+\&c.$
	2.	$x=-z^{-1}-z^{-\frac{5}{2}}+\&c.$

[p. 35]

In hac aequatione, ut vides, duae sunt series exprimentes valorem ipsius x, prodeuntes per duos valores ipsius A in aequatione $A-\frac{1}{2}A^2=0$, & harum serierum secunda est ejusdem formae ac series in casu ultimo; quare per unam hanc Analysin invenitur utraque series, tam casus quarti, quam casus secundi. Quinetiam per Analysin institutam in casu secundo eodem modo simul invenies seriem in casu primo. Unde per duas tantum Analyses series omnes inveniuntur. Sed hoc in eo casu, tantum sit ubi est η idem in duabus seriebus, atque; ubi una radix A in aequatione prima inventa per comparationem terminorum est 0. Possunt autem plures esse radices A in ista aequatione, pro genio cujusvis, aequationis propositae; & quot sunt radicesA, tot dabuntur series per singulas Analyses.

In hac Analysi secundo observationum est, quod omnes omnio coefficientes A, B, C, &c. determinantur per comparationem terminorum. Quare series hoc modo inventae sunt omnes particulares, neque accommodari possunt ad conditiones Problematis, ob defectum coefficientium indeterminatorum.

SCHOLIUM.

Nonnunquam ubi index θ est numerus integer affirmativus, evaneseunt termini primi in seriebus exprimentibus fluxiones ipsius x : nam producuntur coefficientes istorum terminorum primorum per continuam multiplicatinem ipsius A in numeros θ, θ – 1, θ – 2, &c. In hoc casu saepe sit ut terminus evanescat, qui debeat esse unus ex terminis in principio aequationis transformatae, quo pacto series ista aliquando sit impossibilis. Sed si evanescent termini in genesi fluxionum, & tamen supersunt termini duo in principio aequationis transformatae, series adhuc dabitur; quae etiam hoc erit caeteris praestantior. quod in ea erunt coefficientes aliquot indeterminati, per quas accomodari potest series ad aliquot conditiones Problematis. Quinetiam per similem evanescentiam terminorum in productione [p. 36] fluxione sunt nonnunquam aliae series radicem exprimentes, quae per hanc propositionem minime inveniuntur.

Quoniam de aequationum radicibus particularibus jam loquimus, libet etiam hoc unum obiter observare; nempe quod si v sit quantitas ex datis & variabilibus quovis modo composita, & possit aequatio ad talem formam reduci, ut omnis terminus involvat vel ipsum v, vel ejus incrementum aliquod, erit aequatio $v = 0$ particularibus solutio Problematis. Si in aequatione hoc mode transformata involvatur ipsum v, aequatio $v = 0$ nullum continebit coefficientem indeterminarum, adeoque; solutio haec erit maxime particularis; praesertim si v integrales tantum involvat. Si aequationem transformaram non ingreditur v, sed $\dot{v}$, aequatio $v=0$ continebit unum coefficientem indeterminatum. Si aequatio eadem non continet nec v, neque $\dot{v}$, sed $\ddot{v}$, aequatio $v = 0$ continebit duo coefficientes indeterminatos: atque; in genere quo plures terminorum superiorum $v, \dot{v}, \ddot{v}, \&c.$ deficiunt in aequatione transforma eo generalior erit solutio Problematis per aequationem $v = 0$.

Ad hace ubi v integrales tantum involvit potest commode inveniri Problematis solutio generalissima, per v & incrementa sua exterminando caeteras variablies, & deinde quaerendo radicem v per methodum aliquam jam traditam. Sic in aequatione $x-\frac{2}{z}-\dot{x}z=0,$ pro $x-\frac{1}{2}$ scripto v, sit $v-\dot{v}z=0,$ hoc est $\frac{\dot{v}z-v\dot{z}}{v^2}=0.$ Sed ipsius $\frac{\dot{v}z-v\dot{z}}{v^2}$ fluens est $\frac{x}{v}$: quare pro quantitate quavis invariabili scripto A erit $\frac{z}{v}=\mathrm{A}$, hoc est $\frac{z}{x-\frac{1}{2}}=\mathrm{A}.$

[p. 37]

LEMMA II.

Si datur x ex dato z per aequationem quamvis analyticam certi cujusvis numeri dimensionum; etiam dabitur ipsius x incrementorum quodvis $\underset{n}{x}$ *ex dato z per aequationem ejusdem numeri dimensionum.*

Nam quot sunt dimensiones ipsius x in aequatione proposita, tot sunt ejusdem radices (quippe etiam impossibiles ad numerando.). Sed singulae radices x sua habent incrementa. Quare tot sunt radices incrementi cujusvis $\underset{n}{x}$ quot sunt radices ipsius integralis x; adeoque. utrumque; dabitur ex dato x per aequationes ejusdem numeri dimensionem. *Q.E.D.*

COROLLARIUM.

Hinc proposita aequatione definiente relationem singularis alicujus increminti $\underset{n}{x}$ ad cognitam variabelem x, si dari potest integralis x, ex dato $\underset{\bullet}{z}$ per aequationem terminorum numero finitam, dabitur per aequationem in qua x ascendit ad tot dimensiones, atque; ascendit $\underset{n}{x}$ in aequatione proposita.

PROP. X. PROB. VII.

Data aequatione unius dimensionis definiente valorem cujusvis incrementi singularis $\underset{n}{x}$; *invenire valorem ipsius integralis x in terminis numero finitis, si fieri potest.*

[p.38]

Si dari potest relatio x ad quantitates cognitas in terminis numero finitis, dabitur per aequationem unius dimensionis (per *Cor. Lem*.2.) Solutio itaque quaerenda est tentando an quantitas, cui sit $\underset{n}{x}$ aequalis, quo pacto reduci potest ad formam incrementi alicujus cogniti ejusdem ordinis. Quod si sit, dabitur radix x in terminis numero finitis, faciendo eam aequalem integrali istius expressionis. Sed si hoc fieri nequit res desperando erit.

In fluxionibus quoties dari possunt fluentes in terminis numero finitis invenientur per finitis invenientur per Quadraturam Curvarum *Newtonianam*. Et nonnunquam commode inveniuntur hujusmodi expressiones per Propositiones duas sequentes.

PROP. XI. THEOR. IV.

Ipsius $\dot{r}s$ *fluens exprimi potest per alterutram exp. seriebus*

$$\boxed{\dot{r}s} = rs - \overset{\prime}{r}\dot{s} + \overset{\prime\prime}{r}\ddot{s} - \overset{\prime\prime\prime}{r}\dddot{s} + \&c., \text{ vel } \boxed{r\dot{s}} = \dot{r}\overset{\prime}{s} - \ddot{r}\overset{\prime\prime}{s} + \dddot{r}\overset{\prime\prime\prime}{s} - \&c.$$

Theorema investigatur ad sequentem modum. Sit fluens quaesita $rs + p$, hoc est, sit $\boxed{\dot{r}s} = rs + p$. Tum capiendo fluxiones erit $\dot{r}s = \dot{r}s + r\dot{s} + \dot{p}$, hoc est $\dot{p} = -r\dot{s}$: adeoque $\boxed{p} = -r\dot{s}$, indeque $\boxed{\dot{r}s} = rs - \boxed{r\dot{s}}$. Itaque secundo fiat $\boxed{r\dot{s}} = \overset{\prime}{r}\dot{s} + q$; & capiendo fluxiones erit $r\dot{s} = r\dot{s} + \overset{\prime}{r}\ddot{s} + \dot{q}$; hoc est [p.39] $r\dot{s} = r\dot{s} + \overset{\prime}{r}\ddot{s} + \dot{q}$; hoc est $\dot{q} = -\overset{\prime}{r}\ddot{s}$; atque $q = -\boxed{\overset{\prime}{r}\ddot{s}}$; adeoque $\boxed{\dot{r}s} = rs - \overset{\prime}{r}\dot{s} + \boxed{\overset{\prime}{r}\ddot{s}}$. Et per operationes repetitas eodem modo invenitur

$$\boxed{\dot{r}s} = rs - \overset{\prime}{r}\dot{s} + \overset{\prime\prime}{r}\ddot{s} - \boxed{\overset{\prime\prime}{r}\dddot{s}} = rs - \overset{\prime}{r}\dot{s} + \overset{\prime\prime}{r}\ddot{s} - \overset{\prime\prime\prime}{r}\dddot{s} + \boxed{\overset{\prime\prime\prime}{r}\ddddot{s}} = \&c.$$

Quando hoc Theorema est applicandum ad casum particularem, eligenda est fluxio aliqua $\dot{w}$, & in computandis fluentibus $\dot{r}, \ddot{r}, \dddot{r}$,&c. vel $\dot{s}, \ddot{s}, \dddot{s}$,&c. cum primum comparuerit fluens aliqua, ea ducenda erit in $\dot{w}$, & producti fluens sumenda erit pro proxima fluente quaesita. Item in computandis fluxionibus $\dot{r}, \ddot{r}, \dddot{r}$,&c. vel $\dot{s}, \ddot{s}, \dddot{s}$,&c. quoties colligitur fluxio aliqua, erit ea applicanda ad $\dot{w}$, & quotientis fluxio similiter applicata ad $\dot{w}$ sumenda erit pro fluxione proxime quaesita. Haec autem fluxio $\dot{w}$ ita sumenda est ut termini sint fieri potest simplicissimi.

Potest etiam series per hoc Theorema inventa dupliciter accommodari ad conditionem Problematis, hoc est, ad datum unum valorem fluentes quaesitae respondentem dato valori variabilis cognitae. Hoc sit primo sumendo omnes fluentes $r, \overset{/}{r}, \overset{//}{r}, \&c.$ vel $s, \overset{/}{s}, \overset{//}{s}, \&c.$ pure, absque ulla correctione per additionem invariabilium, & deinde seriei inventae addendo invariabilem per conditionem istam postea determinandam. Idem sit secundo fluentes omnes, cum primum prodierint, ita corrigendo per additiones invariabilium, [p. 40] ut omnes simul evanescant, (adeoque & series tota evanescat,) quando variabilis data est certi alicujus valoris.

Ad haec quando terminus aliquis $\dot{s}, \ddot{s}, \dddot{s}, \&c.$ aequalis est nihilo, series prima abrumpitur, & fluentem dat in terminis numero finitis. Atque idem sit in serie altera, ubi evanescit terminus aliquis $\dot{r}, \ddot{r}, \dddot{r}, \&c.$

EXEMP. I.

Sit $x\dot{x} = -z\dot{z}$, & propositum sit invenire fluentem ipsius $\dot{z}x$. In hoc casu si pro $\dot{r}$ sumatur $\dot{z}$, & pro s sumatur x, commodissime fiet $\dot{w} = z\dot{z}(= -x\dot{x})$. Et hoc pacto sunt

$$r = z,\ \overset{/}{r} = \left(\boxed{\dot{w}r} = \right) \frac{z^3}{3},\ \overset{//}{r} = \left(\boxed{\dot{w}\overset{/}{r}} = \right) \frac{z^5}{5.3},\ \overset{///}{r} = \left(\boxed{\dot{w}\overset{//}{r}} = \right) \frac{z^7}{7.5.3},$$ & sic porro; item

$$\dot{r} = \left(\frac{\dot{z}}{\dot{w}} = \right) \frac{1}{z},\ \ddot{r} = \left(\frac{\dot{r}}{\dot{w}} = \right) \frac{-1}{z^3},\ \dddot{r} = \left(\frac{\ddot{r}}{\dot{w}} = \right) \frac{3}{z^5},$$ & sic porro. Sunt etiam $s = k$,

$$\dot{s} = \left(\frac{s}{\dot{w}} = \right) \frac{-1}{x},\ \ddot{s} = \left(\frac{\dot{s}}{\dot{w}} = \right) \frac{-1}{x^3},\ \dddot{s} = \left(\frac{\ddot{s}}{\dot{w}} = \right) \frac{-3}{x^5},$$ & sic porro; item

$$\overset{/}{s} = \left(\boxed{\dot{w}s} = \right) \frac{-x^3}{3},\ \overset{//}{s} = \left(\boxed{\dot{w}\overset{/}{s}} = \right) \frac{x^5}{5.3},\ \overset{///}{s} = \left(\boxed{\dot{w}\overset{//}{s}} = \right) \frac{-x^7}{7.5.3},$$ & sic porro. Unde per seriem priorem $rs - \overset{/}{r}\dot{s} + \overset{//}{r}\ddot{s} - \overset{///}{r}\dddot{s} + \&c.$ sit $\boxed{\dot{z}x} = zx + \frac{z^3}{3x} - \frac{z^5}{5.3x^3} + \frac{z^7}{7.5.3x^3} - \&c.$ Et per seriem alteram $\dot{r}\overset{/}{s} - \ddot{r}\overset{//}{s} + \dddot{r}\overset{///}{s} - \&c.$ sit $\boxed{\dot{z}x} = \frac{-x^3}{3.1z} + \frac{z^5}{5.3z} - \frac{z^7}{7.5.3z^5} + \&c.$

[p. 41]

In his seriebus fluentes $r, \overset{/}{r}, \overset{//}{r}, \&c.$ item $s, \overset{/}{s}, \overset{//}{s}, \&c.$ sumantur pure; quare series accommodandae sunt ad conditionem Problematis per additionem quantitatum invariabilium. Porro per harum, serierum primam exhibetur area circulis adjacens sinui x & cosinui z, & per seriem alternam exhibetur ejusdem area complimentum ad quadrantem cum signo negativo : Quod ita sit quoniam area illa adjacet abscissae z ultra ordinatam productae.

EXEMP. II.

Sit $x = a + bz^n$, & invenienda sit fluens ipsius $\dot{z} z^{\theta-1} x^{\lambda-1}$. In hoc casu si fiat $\dot{r} = \dot{z} z^{\theta-1}$, & $z = x^{\lambda-1}$ commodissime sit $\dot{w} = \dot{x} = nb\dot{z} z^{n-1}$. Unde sumendo fluentes pure, ad inveniendam seriem priorem erit

$$r = \frac{z^\theta}{\theta}.$$

$$\overset{/}{r} = \left(\boxed{\dot{w} r} = \right) \frac{nbz^{\vartheta+n}}{\theta+n.\theta} = \frac{nbz^n}{\theta+n.} r$$

$$\overset{//}{r} = \left(\boxed{\dot{w} \overset{/}{r}} = \right) \frac{n^2 b^2 z^{\theta+2n}}{\theta+2n\theta+n\theta} = \frac{nbz^n}{\theta+2n} \overset{/}{r}.$$

$$\overset{///}{r} = \left(\boxed{\dot{w} \overset{//}{r}} = \right) \frac{n^3 b^3 z^{\theta+3n}}{\theta+3n\theta+2n\theta+n\theta} = \frac{nbz^n}{\theta+3n} \overset{//}{r}. \text{ \& sic porro.}$$

Idem $\dot{r}, \ddot{r}, \dddot{r}, \&c.$ iisdem ac in exemplo praecedenti, invenientur fluentes $s, \overset{/}{s}, \overset{//}{s}, \&c.$ modo sequinti.

Est $\overset{/}{s}$ fluens ipsius $\dot{w}s$, i. e. ipsius $\dot{x} x^{\lambda-1}$. Hujus fluens pura est $\frac{x^\lambda}{\lambda}$, quare ut evanescat $\overset{/}{s}$ ubi est $x = d$, hinc dempto $\frac{d^\lambda}{\lambda}$ sit

$$\overset{/}{s} = \frac{x^\lambda}{\lambda} - \frac{d^\lambda}{\lambda},$$

$$\overset{//}{s} = \frac{x^{\lambda+1}}{(\lambda+1)\lambda} - \frac{d^\lambda x}{1.\lambda} + \frac{d^{\lambda+1}}{(\lambda+1).1},$$

$$\overset{///}{s} = \frac{x^{\lambda+2}}{(\lambda+2)(\lambda+1)\lambda} - \frac{d^\lambda x^2}{2.1.\lambda} + \frac{d^{\lambda+1} x}{1.1(\lambda+1)} - \frac{d^{\lambda+2}}{2.1.(\lambda+2)} \text{ \& sic porro.}$$

Caeterum ex terminis jam appositis satis constat ratio formandi reliquos.

SCHOLIUM.

1. Potest fluxio proposita variis modis resolvi in factores $\dot{r}$ & $s.$ unde plerumque oriuntur series diversae. Sic fluxio jam proposita $\dot{z} z^{\theta-1} \times \overline{a + bz^n}\,|^{\lambda-1}$ etiam sic scribi potest $\dot{z} z^{\theta+\lambda n-n-1} \times \overline{b + az^{-n}}\,|^{\lambda-1}$. Ubi si pro $\dot{r}, s$ & $\dot{w}$ sumantur $\dot{z} z^{\theta+\lambda n-n-1}, \overline{b + az^{-n}}\,|^{\lambda-1}$, & $-na\dot{z} z^{-n-1}$, & pro $a + bz^n$ scribatur x, exprimetur eadem fluens per series sequentes;

$$\boxed{\dot{z}z^{\theta-1}x^{\lambda-1}} = \frac{z^{\theta}x^{\lambda-1}}{\theta+\lambda n-n} + \tfrac{\lambda n-n}{\theta+\lambda n-2n}.\tfrac{a}{x}\mathrm{A} + \tfrac{\lambda n-2n}{\theta+\lambda n-3n}.\tfrac{a}{x}\mathrm{B} + \tfrac{\lambda n-3n}{\theta+\lambda n-4n}.\tfrac{a}{x}\mathrm{C} + \&c.$$

$$\boxed{\dot{z}z^{\theta-1}x^{\lambda-1}} = \frac{z^{\theta}x^{\lambda}}{\lambda na} + \tfrac{\theta+\lambda n}{\lambda n+n}.\tfrac{x}{a}\mathrm{A} + \tfrac{\theta+\lambda n+n}{\lambda n+2n}.\tfrac{x}{a}\mathrm{B} + \tfrac{\theta+\lambda n+2n}{\lambda n+3n}.\tfrac{x}{a}\mathrm{C} + \&c.$$

Ubi literae A, B, C, &c. scribuntur pro totis terminis cum suis signis in seriebus respectivis.

2. In investigatione Theorematis inveniebatur $\boxed{\dot{r}s} = rs - \boxed{r'\dot{s}}$. Unde si pro $\dot{r}$ & s sumatur

$\dot{z}z^{\theta}$ & $\overline{a+bz^{n}}|^{\lambda}$, & pro $x = a+bz^{n}$ scribatur z, erit $\boxed{\dot{z}z^{\theta}x^{\lambda}} = \frac{z^{\theta+1}x^{\lambda}}{\theta+1} - \boxed{\frac{\lambda bn\dot{z}}{\theta+1}z^{\theta+n}x^{\lambda-1}}$, hoc est,

$\boxed{\dot{z}z^{\theta+1}x^{\lambda-1}} = \frac{z^{\theta+1}x^{\lambda}}{\theta+1} - \frac{\lambda bn}{\theta+1}\boxed{\dot{z}z^{\theta+1}x^{\lambda-1}}$. Data itaque fluente ipsius $\dot{z}z^{\theta+1}x^{\lambda-1}$, dabitur etiam

fluens ipsius $\dot{z}z^{\theta+n}x^{\lambda-1}$. Unde si pro n sumantur successive numeri quicunque integri, vel affirmati, vel negativi, si datur fluens unius $\dot{z}z^{\theta+n}x^{\lambda-n}$, dabuntur etiam omnium $\dot{z}z^{\theta+n}x^{\lambda-n}$.

3. Potest etiam eadem fluxio sic scribi $\dot{z}z^{\theta+\lambda n} \times \overline{b+az^{-n}}|^{\lambda}$: ubi si jam pro sumatur $\dot{z}z^{\theta+\lambda n}$,

& pro s sumatur $\overline{b+az^{-n}}|^{\lambda}$, erit

$\boxed{\dot{z}z^{\theta}x^{\lambda}} = \frac{z^{\theta+\lambda n-1} \times \overline{b+az^{-n}}|^{\lambda}}{\theta+\lambda n+1} + \frac{\lambda na}{\theta+\lambda n+1}\boxed{\dot{z}z^{\theta+\lambda n-n} \times \overline{b+az^{-n}}|^{\lambda-1}}$, hoc est [p. 46]

$\boxed{\dot{z}z^{\theta}x^{\lambda}} = \frac{z^{\theta+1} \times x^{\lambda}}{\theta+\lambda n+1} + \frac{\lambda na}{\theta+\lambda n+1}\boxed{\dot{z}z^{\theta}x^{\lambda-1}}$. Data itaque fluente ipsius $\dot{z}z^{\theta}x^{\lambda}$, dabitur etiam ipsius $\dot{z}z^{\theta}x^{\lambda-1}$, & vice versa. Manente itaque indice θ, si continu minuatur, vel augeatur λ per unitates, data fluente unius $\dot{z}z^{\theta}x^{\lambda}$, dabuntur fluenes`omnium $\dot{z}z^{\theta}x^{\lambda}$. Et per hos duos casus conjunctos, si pro σ & τ scribantur successive numeri quicunque integri, vel affirmativi vel negativi, manentibus θ & λ, si datur fluens unius cujusvis fluxionis $\dot{z}z^{\theta+\sigma n} \times \overline{a+bz^{n}}|^{\lambda+\tau}$, dabuntur etiam fluentes omnium fluxionum eodem modo provenientium. Et ad eundem modum pergere licet ad comparationem fluentum, ubi quantitas in invinculo radicis est trium, vel quatuor, vel plurium nominum. Sed haec jam elegantius fiunt ab illustrissimo Newtono in Quadratura Curvarum.

PROP. XII. THEOR. V.

Sit n index ordinis fluentis ipsius $Q = \overset{\backslash}{\dot{r}}\, s$ *, verbi gratia n = 2, ut sit* $\overset{n}{Q} = \overset{//}{Q}$*, si n = 0, ut sit* $\overset{n}{Q} = \overset{0}{Q} = Q$*,* $n = -1$*, ut sit* $\overset{n}{Q} = \overset{-1}{Q} = \dot{Q}$*, & sic de caeteris; tum erit*

$\overset{n}{Q} = \overset{\overset{\backslash}{n}}{r}\, s - \frac{n}{1}.\overset{n}{r}\,\dot{s} + \frac{n}{1}.\frac{\underset{/}{n}}{2}\overset{n}{\underset{/}{r}}\,\ddot{s} - \frac{n}{1}.\frac{\underset{/}{n}}{2}.\frac{\underset{//}{n}}{3}\overset{n}{\underset{//}{r}}\,\dddot{s} + \&c.$ *Ubi est n* = 1, & (juxta notationem nostrum) sunt $\overset{\backslash}{n} = n - \underset{\cdot}{n}$; $\underset{/}{n} = n + \underset{\cdot}{n}$; $\underset{//}{n} = \underset{/}{n} + \underset{\cdot}{n}$. et sic porro. [p. 47]

Quando est n = 1, Theorema idem est cum praecenti; unde colligitur forma seriei. Coefficientes autem $1, \frac{n}{1}, \frac{n}{1}\times\frac{\underset{/}{n}}{2}, \frac{n}{1}\times\frac{\underset{/}{n}}{2}\times\frac{\underset{//}{n}}{3}$, &c. sic investigo.

Sint coefficientes quaesiti $x, v, y, w, \&c.$ iidemque incrementis suis $\underset{\cdot}{x}, \underset{\cdot}{v}, \underset{\cdot}{y}, \underset{\cdot}{w}, \&c.$ aucti $\underset{/}{x}, \underset{/}{v}, \underset{/}{y}, \underset{/}{w}, \&c.$ Si ergo sit $\overset{n}{Q} = x\overset{\overset{\backslash}{n}}{r}\, s + v.\overset{n}{r}\,\dot{s} + y\overset{\underset{/}{n}}{r}\,\ddot{s} + w\overset{\underset{//}{n}}{r}\,\dddot{s} + \&c.$ erit proxime

$\overset{\underset{/}{n}}{Q} = \underset{/}{x}\overset{n}{r}\, s + \underset{/}{v}.\overset{\underset{/}{n}}{r}\,\dot{s} + \underset{/}{y}\overset{\underset{//}{n}}{r}\,\ddot{s} + \underset{/}{w}\overset{\underset{///}{n}}{r}\,\dddot{s} + \&c.$ Jam si coefficientes $x, v, y, w, \&c.$ sint justi valoris, manentibus coefficientibus novis $\underset{/}{x}, \underset{/}{v}, \underset{/}{y}, \underset{/}{w}, \&c.$ si capiatur fluxio novae seriei fiet regressio in seriem priorem. Ergo capiendo fluxiones primo in *r*, deinde in *s*, sit

$\overset{n}{Q} = \underset{/}{x}\overset{\overset{\backslash}{n}}{r}\, s + (\underset{/}{v} + \underset{/}{x}).\overset{n}{r}\,\dot{s} + (\underset{/}{y} + \underset{/}{v})\overset{\underset{/}{n}}{r}\,\ddot{s} + (\underset{/}{w} + \underset{/}{y})\overset{\underset{//}{n}}{r}\,\dddot{s} + \&c.$ Unde comparando terminos hujus seriei cum terminis relativis seriei prioris, sit $\underset{/}{x}(= x + \underset{\cdot}{x}) = x$; adeoque; $\underset{\cdot}{x}$ = 0. Proinde est *x* invariabilis. Sed ubi $\underset{\cdot}{x}$ = 0 est *x* = 1; quare est semper *x* = 1. Comparando terminos secondus, sit $\underset{/}{v} + x(= v + \underset{\cdot}{v} + 1) = v$; adeoque $\underset{\cdot}{v}(= -1) = -n$; & inde $v = \frac{-n}{1} + a.$ [p. 48.]
Sed ubi *n* = 0, est *v* = 0; quare est *a* = 0 & $v = \frac{-n}{1}$. Comparando terminos tertios sit $\underset{/}{v} + \underset{/}{v}(= y + \underset{\cdot}{y} - \underset{/}{n}) = y$; & inde $y = \frac{n}{1}\frac{\underset{/}{n}}{2} + b.$ Sed ubi *n* = 0, est *y* = 0; ergo est *b* = 0, atque $y = \frac{n}{1}\frac{\underset{/}{n}}{2}.$ Eodem modo sit $\underset{/}{w} + \underset{/}{y}(= w + \underset{\cdot}{w} + \frac{\underset{/}{n}}{1}\frac{\underset{//}{n}}{2}.) = y$; adeoque $\underset{\cdot}{w} - \frac{\underset{/}{n}}{1}\frac{\underset{//}{n}}{2}$ & inde $w = -\frac{n}{1}\frac{\underset{/}{n}}{2}\frac{\underset{//}{n}}{3}$. Et sic pergendo in infinitum invenientur reliqui coefficientes, omnino ut in Theoremate exhibentur.

EXEMPLUM.

Hoc pacto ipsius $Q=\dot{z}z^{\theta-1}\times\overline{a+bz^{n}}|^{\lambda-1}$ $(=\dot{z}z^{\theta-1}x^{\lambda-1})$ fluens quaevis in genere est

vel 1.

$$\overset{n}{Q}=\frac{\overset{\overset{\backslash}{n}}{n}\overset{\overset{\backslash}{n}}{b}z^{\theta+\overset{\backslash}{n}x}x^{\lambda-1}}{\theta.\theta+n.\theta+2n.\theta......\theta+\overset{\backslash}{n}n}+\frac{n-\lambda n}{\theta+nn}.\frac{nbz^{n}}{1.x}\mathrm{A}+\frac{2n-\lambda n}{\theta+\underset{/}{n}n}.\frac{\underset{/}{n}bz^{n}}{2.x}\mathrm{B}+$$

$$\frac{3n-\lambda n}{\theta+\underset{//}{n}n}.\frac{\underset{//}{n}bz^{n}}{3.x}C+\&c.$$

vel 2.

$$\overset{n}{Q}=\frac{z^{\theta-x}x^{\lambda+\overset{\backslash}{n}}}{\lambda.\lambda+1.\lambda+2.......\lambda+\overset{\backslash}{n}.nb}+\frac{n-\theta}{\lambda n+nn}.\frac{nx}{1.bz^{n}}\mathrm{A}+\frac{2n-\theta}{\lambda n+\underset{/}{n}n}.\frac{\underset{/}{n}x}{2.bz^{n}}\mathrm{B}+$$

$$\frac{3n-\theta}{\lambda n+\underset{//}{n}n}.\frac{\underset{//}{n}x}{3.bz^{n}}C+\&c.$$

vel 3. [p. 49.]

$$\overset{n}{Q}=\frac{\overline{-na}|^{\overset{\backslash}{n}}z^{\theta-\overset{\backslash}{n}x}x^{\lambda-1}}{\theta-n+\lambda n.\theta-n+\lambda n-n\theta+.......\lambda-n+\lambda n-\overset{\backslash}{n}n}+$$

$$\frac{\lambda n-n}{\theta-n+\lambda n-nn}.\frac{na}{1.x}\mathrm{A}+\frac{\lambda n-2n}{\theta-n+\lambda n-\underset{/}{n}n}.\frac{\underset{/}{n}a}{2.x}\mathrm{B}++\&c$$

vel 4.

$$\overset{n}{Q}=\frac{-z^{\lambda+\overset{\backslash}{n}}z^{\theta+\overset{\backslash}{n}\lambda x-\overset{\backslash}{n}x}}{\lambda.\lambda+1.\lambda+2.....\lambda+\overset{\backslash}{n}a}+\frac{\theta+\lambda n}{\lambda n+nn}.\frac{nx}{1a}\mathrm{A}$$

$$\frac{\theta+\lambda n+n}{\lambda n+\underset{/}{n}n}.\frac{\underset{/}{n}x}{2a}\mathrm{B}+\&c$$

Quippe in seriebus duabus primis pro $\dot{w}$ sumpto $nb\dot{z}z^{n-1}$, & facto $\overset{/}{Q}=\boxed{\dot{w}Q}$,

$\overset{//}{Q}=\boxed{\dot{w}\overset{/}{Q}}$, & sic porro; & in seriebus duabus ultimis pro $\dot{w}$ sumpto $-na\dot{z}z^{-n-1}$

& similiter facto $\overset{/}{Q}=\boxed{\dot{w}Q}$, $\overset{//}{Q}=\boxed{\dot{w}\overset{/}{Q}}$, & sic porro.

Per has autem series exhibentur tam fluxiones quam fluentes ipsius Q. Sic si n = -1, series dabit valorem $\overset{/}{Q}$, si n = -2, series dabit valorem $\overset{//}{Q}$, & sic porro.

Sed in hoc casu, ubi mutatur signum numeri n, quaedam est difficultas in inventione coefficientis termini primi. Sit ergo exemplum methodi hoc faciendi in termino primo seriei primae modo exhibitae pro valore $\overset{//}{Q}$. Hujus termini coefficiens, seposito $\overset{\backslash\ \backslash}{\overset{n\ n}{nb}}$, est $\frac{1}{\theta.\theta+n.\theta+2n.\theta......\theta+\overset{\backslash}{n}n}$. Debet autem hujus coefficientis [p. 50.] maximus divisor esse ....[illegible formula] Quare ut inveniatur coefficiens ubi est $\overset{\backslash}{n}$ numerus negativus, vice $\frac{1}{\theta.\theta+n.\theta+2n.\theta......\theta+\overset{\backslash}{n}n}$ scribo $\frac{...\theta-4n.\theta-3n.\theta......-2n\theta}{....\theta-4n.\theta-3n.\theta......-n\theta+n\theta+2n\theta.\&c}$. Tum rejectis omnibus divisoribus post $\theta+\overset{\backslash}{n}n$, quando est $\overset{\backslash}{n}=1$, coefficiens erit $\frac{etc.\theta-2n}{etc.\theta-2n.\theta-n.\theta.\theta+n}$, hoc est $\frac{1}{\theta.\theta+n}$; quando est $\overset{\backslash}{n}=0$, erit coefficiens $\frac{etc.\theta-2n.\theta-n}{etc.\theta-2n.\theta-n.\theta}$, hoc est $\frac{1}{\theta}$. Et eodem argumento ubi $\overset{\backslash}{n}=-1$, erit coefficiens $\frac{etc.\theta-2n.\theta-n}{etc.\theta-2n.\theta-n}$, hoc est 1; ubi $\overset{\backslash}{n}=-2$, erit coefficiens $\frac{etc.\theta-2n.\theta-n}{etc.\theta-2n}$, hoc est $\theta-n$; ubi $\overset{\backslash}{n}=-3$, erit coefficiens $\overline{\theta-2n}.\overline{\theta-n}$; & sic porro. Unde jam si sit m index fluxionis quaesitae ipsius Q, hoc est si pro $-n$ scribatur m, erit $\overline{\theta-mn}.\overline{\theta-\overset{\backslash}{m}n}.\overline{\theta-\overset{\backslash\backslash}{m}n}.....\overline{\theta-n}$ coefficiens numeralis terminali primi seriei quaesitae.

SCHOLIUM.

Pergere jam liceret ad inventionem integralium in terminis numero infinitis, quarum incrementa singularia dantur per aequationes affectas altiorum graduum. Sed quoniam in his casibus solutio quaeri non potest nisi pe calculum valde nimis prolixim, operae pretium non duxi praecepta plura tradere in re nullius usus futura. Aequationes quadraticae revocantur ad aequationes simplices pe extractionem radicis, atque aequationes cubicae resolvuntur pe regulum Cardini, & aequationes plurium dimensionun etiam resolvi possunt per ablationem terminorum intermediorum. Quare si cui animus est rem adeo laboriosam tentare, terminis omnibus intermediis exterminatis, deinde solutio quaeri potest per Propositiones praecendentes. Tantum autem laborem paululum minuere potest haec observatio, nempe quod in aequatione affecta definiente valorem incrementi, termini secundi coefficiens est simile incrementum coefficientis termini secundi in aequatone definiente valorem ipsius integralis. Quare facto periculo in coefficiente termini secundi, si is revocari nequit ad integralem in terminis numero finitis, frustra erit solutionem finitam quaeres in caetera aequatione.

Principiis Methodi Incrementorum & Methodi Fluxionum jam breviter explicatis, superest ut in parte altera hujus opusculi exemplis aliquot ostendamus, quantus sit usus rei in solutione difficiliorum quorundam Problematum.