The Method of Increments.

The Second Part. [IIa]

[page 53]

Where by some examples it is shown how this method can be applied to mathematical and physical problems.

PROP. XII1. PROB. VIIII.

For some given equidistant terms in a series of quantities, to find [a general formula for] *the sum of the intermediate terms, beyond the nearby terms, in terms of the given distances between each other and their distances from the given initial term.*

Let *a, b, c, d* be the given equidistant terms, and the magnitude of another term is required in general from its given distance from the initial term *a*.
The differences of the given terms are taken, and then the differences of the differences, and thus henceforth, then a final difference is reached, and these differences each have their own signs: $\underset{\cdot}{a}(=b-a,)\ \underset{\cdot}{b}(=c-b,)\ \underset{\cdot}{c}(=d-c,)\ \underset{\cdot\cdot}{a}(=\underset{\cdot}{b}-\underset{\cdot}{a},)\ \underset{\cdot\cdot}{b}(=\underset{\cdot}{c}-\underset{\cdot}{b},)\ \underset{\cdot\cdot\cdot}{a}(=\underset{\cdot\cdot}{b}-\underset{\cdot\cdot}{a}.)$
Now write *x* for any term produced in the series, and for the distance of the same term from the initial term write *z*, and the increment of *z*, $\underset{\cdot}{z}$, is equal to the given distance between the given terms *a, b, c, d,* and [the sum of] all the terms of the series can be expressed in general [at least in an approximation] from the equation
$x = A + Bz + Cz\overset{\backslash}{z} + 3Dz\overset{\backslash}{z}\overset{\backslash\backslash}{z}$. Then (by *Prop*. 1) the difference between two values of *x* to the distance in turn from $\underset{\cdot}{z}$ is expressed by the equation: $\underset{\cdot}{x} = B\underset{\cdot}{z} + 2Cz\underset{\cdot}{z} + 3D\underset{\cdot}{z}z\overset{\backslash}{z}$, and in the same way the difference of the differences is expressed by the equation:
$\underset{\cdot\cdot}{x} = 2C\underset{\cdot}{z}^2 + 2.3D\underset{\cdot}{z}^2 z$, and the third order difference by the equation : $\underset{\cdot\cdot\cdot}{x} = 2.3D\underset{\cdot}{z}^3$. But when $z = 0$, then $x = a$, $\underset{\cdot}{x} = \underset{\cdot}{a}$; $\underset{\cdot\cdot}{x} = \underset{\cdot\cdot}{a}$; $\underset{\cdot\cdot\cdot}{x} = \underset{\cdot\cdot\cdot}{a}$; hence from these equations, A = *a*,

$B\underset{\cdot}{z} = \underset{\cdot}{a}$; $2C\underset{\cdot}{z}^2 = \underset{\cdot\cdot}{a}$, $2.3D\underset{\cdot}{z}^3 = \underset{\cdot\cdot\cdot}{a}$, and thus $A = a,\ B = \frac{\underset{\cdot}{a}}{\underset{\cdot}{z}},\ C = \frac{\underset{\cdot\cdot}{a}}{2\underset{\cdot}{z}^2},\ D = \frac{\underset{\cdot\cdot\cdot}{a}}{2.3\underset{\cdot}{z}^3}$, and from that

$$x = a + \frac{\underset{\cdot}{a}}{\underset{\cdot}{z}}z + \frac{\underset{\cdot\cdot}{a}}{2\underset{\cdot}{z}^2}z\overset{\backslash}{z} + \frac{\underset{\cdot\cdot\cdot}{a}}{2.3\underset{\cdot}{z}^3}z\overset{\backslash}{z}\overset{\backslash\backslash}{z}.\ \textit{Q.E.I.}$$

[The starting-end or initial term is *a*, and three different differences can be formed from the given starting values *a, b, c,* and *d*. Assume $x = A + Bz + Cz\overset{\backslash}{z} + 3Dz\overset{\backslash}{z}\overset{\backslash\backslash}{z}$, or $x = A + Bz + Cz(z - \underset{\cdot}{z}) + 3Dz(z - \underset{\cdot}{z})(z - 2\underset{\cdot}{z})$. Then $x + \underset{\cdot}{x} = A + B(z + \underset{\cdot}{z}) + C(z + \underset{\cdot}{z})z + 3Dz(z + \underset{\cdot}{z})(z - \underset{\cdot}{z})$; and hence $\underset{\cdot}{x} = B\underset{\cdot}{z} + 2Cz\underset{\cdot}{z} + 3D\underset{\cdot}{z}z(z - \underset{\cdot}{z})$; similarly, $\underset{\cdot}{x} + \underset{\cdot\cdot}{x} = B\underset{\cdot}{z} + 2C(z + \underset{\cdot}{z})\underset{\cdot}{z} + 3D\underset{\cdot}{z}(z + \underset{\cdot}{z})z$ giving $\underset{\cdot\cdot}{x} = 2C\underset{\cdot}{z}^2 + 2.3D\underset{\cdot}{z}^2 z$, and likewise for $\underset{\cdot\cdot\cdot}{x} = 2.3D\underset{\cdot}{z}^3$.]

This equation accurately agrees with the given terms *a, b, c*, *d* themselves and then approximately with the intermediate and further terms. But when more terms are given [*i. e*., in addition to *a, b,c,* and *d*], it is understood that this approximation of the equation to the sum of the values of all the terms is closer, both for intermediate and more distant terms. Whereby if the series of given terms is continued to infinity, the equation finally coincides with the values of all the terms, so of the intermediate as with those further away. Hence if a rule is given for the equidistant terms to be formed in some series [p. 55], by this proposition is given [a formula] expressing the values of an infinite series, of all the intermediate, and finally of the whole series.

EXAMPLE.

Let an example of this proposition be a series of terms where the equidistant terms are always in a continued geometrical ratio, and in this series *a* and *a* + *ab* are the [first] two terms at the separation $\underset{\cdot}{z}$; then from the nature of this expression, all the terms are at the same distance $a, a + ab, a \times \overline{1+b}|^2, a \times \overline{1+b}|^3, \&c.$ [which we would now write as $a, a(1+b), a(1+b)^2, a(1+b)^3, \&c.$] and the first differences of these terms are: $ab, ab \times \overline{1+b}|, ab \times \overline{1+b}|^2\ \&c.$; the second differences are : $ab^2, ab^2 \times \overline{1+b}|, ab^2 \times \overline{1+b}|^2\ \&c.$ and the third differences are : $ab^3, ab^3 \times \overline{1+b}|, ab^3 \times \overline{1+b}|^2\ \&c.$; and thus henceforth. Hence, according to the idea of this proposition : $a = a, \underset{\cdot}{a} = ab, \underset{\cdot\cdot}{a} = ab^2, \underset{\cdot\cdot\cdot}{a} = ab^3, \&c.$; and thus if *z* is the distance of some term *x* from the term *a,* then it is given by : $x = a + \frac{abz}{1\underset{\cdot}{z}} + \frac{ab^2 z\overset{\backslash}{z}}{1.2\underset{\cdot}{z}^2} + \frac{ab^2 z\overset{\backslash}{z}\overset{\backslash\backslash}{z}}{1.2.3\underset{\cdot}{z}^3} + \&c.$ But in this case : $\frac{x}{a} = \overline{1+b}|^{\frac{z}{\underset{\cdot}{z}}}$; hence $\overline{1+b}|^{\frac{z}{\underset{\cdot}{z}}} = 1 + \frac{bz}{1\underset{\cdot}{z}} + \frac{b^2 z\overset{\backslash}{z}}{1.2\underset{\cdot}{z}^2} + \frac{b^2 z\overset{\backslash}{z}\overset{\backslash\backslash}{z}}{1.2.3\underset{\cdot}{z}^3} + \&c$. This series coincides with Newton's theorem for finding the powers of a binomial.

This theorem can also be investigated in as follows : Let $\overline{a+b}|^{n} = a^{n} + xba^{n-1} + vb^{2}a^{n-2} + zb^{3}a^{n-3}\ \&c.$ Then multiply the series by $a + b$ and it becomes $\overline{a+b}|^{n+1}$, or according to our 'next-term' notion, we write $\overline{a+b}|^{n}_{/}$, equal to

$$a^{n}_{/} + xba^{n-1}_{/} + vb^{2}a^{n-2}_{/} + zb^{3}a^{n-3}_{/} + \&c.$$
$$\quad +1 \qquad\quad +x \qquad\quad +v$$

[Thus, the first row increases the power of a from n to $n + 1$, while the second row adds 1, x, and v to the coefficients $ba^{n-1}_{/}, b^{2}a^{n-2}_{/}$, and $b^{3}a^{n-3}_{/}$, which now become $ba^{n}, b^{2}a^{n-1}$, and $b^{3}a^{n-2}$] Hence with $\underset{\bullet}{n}=1$; by the method of increments, $\underset{\bullet}{x}=1$, and hence $x=\frac{n}{1}$; $\underset{\bullet}{v}=x(=n,)$ and thus $v=\frac{n\grave{n}}{1.2}$; $\underset{\bullet}{z}=v(=\frac{n\grave{n}}{1.2})$; thus $z=\frac{n\grave{n}\grave{\grave{n}}}{1.2.3}$; and so on; (where all the variables are taken as pure, since everything should vanish when $n = 0$,) hence

$\overline{a+b}|^{n} = a^{n} + \frac{n}{1}ba^{n-1} + \frac{n\grave{n}}{1.2}b^{2}a^{n-2} + \frac{n\grave{n}\grave{\grave{n}}}{1.2.3}b^{3}a^{n-3}\ \&c.$, that is

$\overline{a+b}|^{n} = a^{n} + \frac{n}{1}\frac{b}{a}\mathrm{A} + \frac{n-1}{2}\frac{b}{a}\mathrm{B} + \frac{n-2}{3}\frac{b}{a}C + \&c.$ truly with the proper signs written for A, B, C, etc, for the individual terms of the series.

PROP. XIV. PROB. IX.

For some given method of forming the terms in a series of quantities, to find the sum of the terms for some equidistance of the terms.

If the sum x of the terms is sought, with the same term increased by one more, it becomes $x+\underset{\bullet}{x}$. Whereby with the law of the formation of the terms given, if x is made equal to the nearest term to be added, the sum x is given in general terms, by Prop. 10. [p. 57.] Which, in order that it shall become equal to the sum sought, the value of the same whole taken away, which is produced when the sum sought is zero.

EXAMPLE I.

Let an example of a series of equidistant terms be $a, a+b, a+2b, a+3b, \&c.$ In this series I always write z for the final term, and $\underset{\bullet}{z}=b$, and the end term, $z+\underset{\bullet}{z}$ is to be added next to the sum sought, or $z_{/}$; and thus $\underset{\bullet}{x}=z_{/}$. Hence on returning to the whole quantities, the sum $x=\frac{zz_{/}}{2\underset{\bullet}{z}}+\mathrm{A}$. [Thus, $n=\frac{z}{\underset{\bullet}{z}}, a+(n+1)b=z_{/}$ and $x=\frac{n(2a+(n+1)b)}{2}$ in modern terms.] But this sum must be made equal to zero, when the initial term a is to be added on, that is , when $z_{/}=a$, and thus $z=a-\underset{\bullet}{z}$; hence from the sum $\frac{zz_{/}}{2\underset{\bullet}{z}}+\mathrm{A}$, take the sum $\frac{\overline{a-\underset{\bullet}{z}}\times a}{2\underset{\bullet}{z}}+\mathrm{A}$, the remainder $\frac{z z_{/} - \overline{a-\underset{\bullet}{z}}\times a}{2\underset{\bullet}{z}}$ is the sum required, that is, (with b put back in place of $\underset{\bullet}{z}$)

$x = \frac{z\times\overline{z+b} - \overline{a-b}\times a}{2b}$.

EXAMPLE II.

In the same way, if the final term in the series is $z z_{/}$, and this first term is that for which $z = a$, that is $a\times\overline{a+\underset{\bullet}{z}}$, with the term to be added next, after the last, proving to be $z_{/} z_{//}$, then $\underset{\bullet}{x} = z_{/} z_{//}$; and hence $x = \frac{z z_{/} z_{//}}{3\underset{\bullet}{z}} - \frac{\overline{a-\underset{\bullet}{z}}\times a\times\overline{a+\underset{\bullet}{z}}}{3\underset{\bullet}{z}}$. And in same way one is permitted to proceed to the summations of terms, in which there are more factors $z\times z_{/}\times z_{//}\times \&c.$

[p. 58.]

EXAMPLE III.

The sum of the terms is to be found, the final term of which is always $\frac{1}{z z_{/}}$, and of which the first term is that for which $z = a$. In this case the term to be added to the last term of the sum sought is $\frac{1}{z_{/} z_{//}}$. Whereby it is accomplished that $\underset{\bullet}{x} = \frac{1}{z_{/} z_{//}}$, then by regressing to the whole sum, it follows that $x = A - \frac{1}{\underset{\bullet}{z} z_{/}}$. But when the next term to be added, from the start, is $\frac{1}{a\times\overline{a+\underset{\bullet}{z}}}$, that is, when $z_{/} = a$, the sum sought should be equal to zero, and hence $A = \frac{1}{\underset{\bullet}{z} a}$ and thus $x = \frac{1}{\underset{\bullet}{z} a} - \frac{1}{\underset{\bullet}{z} z_{/}}$.

COROLLARY.

And hence the sum of all the terms of the series is given :
$\frac{1}{a.a+z} + \frac{1}{a+z.a+2z} + \frac{1}{a+2z.a+3z} + \&c.$ to infinity. For in this series continued indefinitely the divisor z in the final term is infinite. Hence with the term $\frac{1}{\underset{\bullet}{z} z_{/}}$ disappearing, the value of the sum is $x = \frac{1}{\underset{\bullet}{z} a}$.

And in the same way one can proceed to sums of terms in which there are more divisors $z, z_{/}, z_{//}, \&c.$

[p. 59]

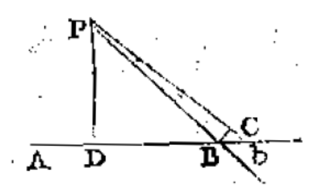
PROP. XV. PROB. X.

To find fluxions from given geometrical figures.

Just how this is to be accomplished is better understood from examples, as will be apparent from these precepts.

EXAMPLE I.

Hence the first example in the figure ABP, where from the position given for the line AB, and from the point P, the ratio is sought of the fluxion of the line AB, and of distance PB.

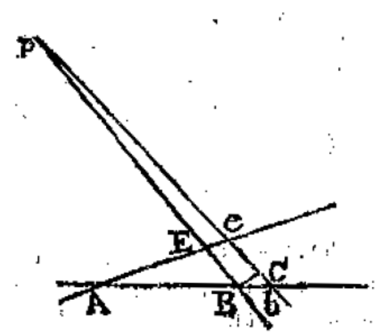


" The line PB is moved from its own place PB to the new place P*b*. In P*b,* take PC = PB, and to AB is drawn PD thus, in order that the angle *b*PD is equal to the angle *b*BC; and on account of the similar triangles *b*BC, *b*PD, the increase Bb to the increase C*b*, is as P*b* to D*b*. Now P*b* can be returned to its own former place PB, in order that the increases vanish, and the final ratio of evanescence, that is the final ratio P*b* to D*b*, is that which is as PB to DB, for the existing right angle PDB, and therefore the fluxion of AB to the fluxion PB is in this ratio.

[Thus, a point moves along AB so that the increase in length B*b* in a given time has a corresponding increase in length of PB in the same time is *b*C; from the similar triangles, the ratio of the increases *b*C to B*b* is equal to DB to PB. The same result follows analytically.]

EXAMPLE II.

The line PB revolving around the given pole P, cuts the two given lines AB and AE in B and E : the proportion of the fluxions of these lines AB and AE [to each other] is sought.



The revolving line PB progresses from its own [initial] position PB to a new position P*b*, cutting the line AB in the point *b*, [p. 60] and the line AE in the point *e*, and the line BC is drawn parallel to the line AE, crossing P*b* in C, and B*b* is to BC as A*b* is to A*e*, and BC is to E*e* as PB is to PE, and from the ratios taken together, B*b* is to E*e* as A*b* × PB is to A*c* × PE.

[B*b*/BC = A*b*/A*e*, and BC/E*e* = PB/PE; hence B*b*/E*e* = (A*b* × PB)/(A*c* × PE).]

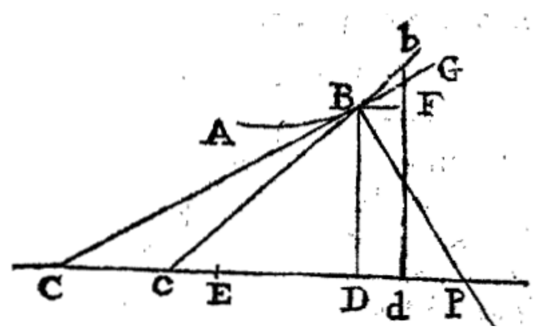
Now the line P*b* can return to its previous position PB, and the vanishing increase B*b* is to the vanishing increase E*e* as AB × PB to AE × PE, thus the fluxion of the line AB to the fluxion of the line AE is in this ratio.

Thus, on the revolving line PB, any curved lines you please, cut at the points B & E at the given positions, and the moving lines AB and AE are now tangents to these curves in the section with the points B & E; the ratio of the fluxion of the curve to which AB is the tangent to the fluxion of the curve that has AE as tangent is as AB × PB to AE ×PE. Since this also is the case if the line PB is always a tangent from any given position to a curve with a movable point P."

These two examples from the works of N*ewton* are required.

EXAMPLE III.

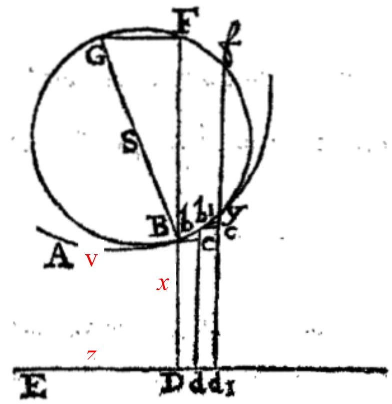
AB is any curve in the position given, and from some point B of this curve the line BD is drawn cutting the line ED in the position given at the angle given at D; the fluxion of the abscissa ED is sought, of the ordinate DB, and of the curve AB.



The ordinate BD can be moved from its place BD to the new position *bd*, and BF is drawn parallel to ED, and crossing *bd* in F, and through the points *b* and B, *b*B is drawn crossing ED in *c*, and from the point B the tangent is drawn [p. 61] crossing ED and *db* in C and G. Then on account of the similar triangles BF*b* and *c*DB, BF : F*b* : B*b* : : *c*D : DB : *c*B. [*i. e.* BF = k.*c*D, F*b* = k.DB, and B*b* = k.*c*B for some constant k.] Now the ordinate *bd* can be returned to its first position BD, and since the line *c*B is now coincident with the tangent CB, the vanishing abscissa ED and the ordinate DB of the curve AB are increased between themselves, as the sides of the vanishing triangle BFG, or of the triangle CDB similar to that; therefore the flexions of the lines ED, DB, and of the curve AB are in this ratio. And if the angle BDE is right, with the normal BP drawn to the curve crossing ED in P, on account of the similar triangles BFG and BDP, the same flexions will be between these sides as the sides of the triangle BDP.

Hence from a given ratio the tangents can be drawn for any proposed curve, the proportion will be given of the fluxion of the abscissa, of the ordinate, and of the curve ; and in turn from the given proportion of the fluxion of the abscissa, and of the ordinate, the proportion will be given between the sub-tangent CD, the ordinate DB, and the tangent itself CB; as from the proportion between the ordinate BD, the subnormal DP, and the normal BP. Moreover the proportion of the fluxion is given (per *Prop*. 1.) from the defining equation, and the relation between the abscissa ED and the ordinate DB. Whereby through such a proportion the tangents and normals can be drawn for all curves.

EXAMPLE IV.

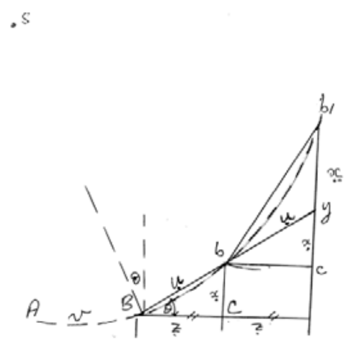


If some curve AB is given, and the proposition is to find the radius of curvature at the point B, that is, the radius of the circle with the same curvature as that which the curve AB has at the point B.

Draw three equidistant ordinates BD, *bd*, *b*1*d*1, cutting the given line ED at right angles at D, *d,* and *d*1; and draw BC and *bc* parallel to ED, crossing *bd,* and *b*1*d*1 at C and *c*; and draw B*b* meeting *b*1*d*1 in *y*, [p. 62.] and a circle with centre S passes through these three points B, *b, b*1, crossing DB and *d*1*b*1 [extended]in F and *f*, and draw the diameter BG, and FG. Moreover, the abscissa ED = *z*, with the ordinate DB = *x*, and with the [arc length of the] curve AB = *v*. Then just as in the Method of Increments it follows that

D*d* = $\underset{\cdot}{z}$ (=*dd*1= BC = *bc*) ; C*b* = $\underset{\cdot}{x}$(= *cy*) & *yb*1 = $\underset{\cdot\cdot}{x}$.

Also let B*b* = (*by* =) $\underset{\cdot}{u}$. Then from the nature of the circle [for both products are equal to the length of the tangent squared from *y* to the circumference, from similar triangles]:



$yb1\times yf = yB\times yb$, that is $yf = \dfrac{2\underset{\cdot}{u}\times\underset{\cdot}{u}}{\underset{\cdot\cdot}{x}}$. But with the coincidence of the points B, *b,* and *b*1, the vanishing arc for the circle B*bb*1 and of the curve AB is in common, therefore in this case a circle with centre S is described with the same curvature as the curve at the point B.
Hence the increments vanish, and now with *yf* coinciding with BF, and by making $\dfrac{\underset{\cdot}{u}\underset{\cdot}{u}}{\underset{\cdot\cdot}{x}} = \dfrac{\dot{v}\dot{v}}{\ddot{x}}$, then BF $=\dfrac{2\dot{v}\dot{v}}{\ddot{x}}$. [Note that the passage to the limit involves increments becoming differentials.]

But from the coincident points B and *b*, BF is to BG, as BC is to B*b*, that is, as $\dot{z}$ to $\dot{v}$.

[$\dfrac{\mathrm{BF}}{\mathrm{BG}} = \dfrac{\dot{z}}{\dot{v}}$ and BF $=\dfrac{2\dot{v}\dot{v}}{\ddot{x}}$]. Hence BG $=\dfrac{2\dot{v}^3}{\dot{z}\ddot{x}}$, and the radius of curvature BS = $\dfrac{\dot{v}^3}{\dot{z}\ddot{x}}$.

[In modern terms, it is easy to show that this is equivalent to the curvature of the function $y = y(x)$; $(1+(dy/dx)^2)^{3/2}/(d^2y/dx^2)$, where $\dot{z} = dx; \ddot{x} = d^2y;$ and $\dot{v}^3 = (dx^2+dy^2)^{3/2}$.]

If *p* is the [length of the]normal to the curve intercepted between the point B and the axis ED, then $\dot{v} = \dfrac{p\dot{z}}{x}$ (by *Ex*. 3.) [in which *p* = BP, *x* = BD, and the angle DBP is θ; or from the extra diagram, $\cos\theta = \dfrac{\dot{z}}{\dot{v}} = \dfrac{x}{p}$]

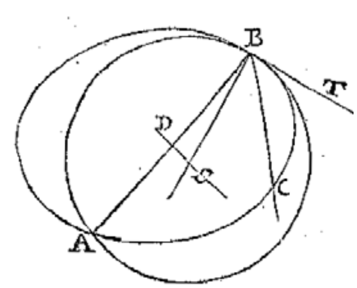
Hence the radius of curvature is also : BS = $\frac{p^3 \dot{z}^2}{x^3 \ddot{x}}$.

And these are made for z flowing uniformly. But if you want to find an expression for the same radius of curvature where v flows uniformly, through the equation $\dot{v}\dot{v} = \dot{x}\dot{x} + \dot{z}\dot{z}$ [p.63.] (hitherto with z the uniform fluent) it was $2\dot{v}\ddot{v} = 2\dot{x}\ddot{x}$, that is $\ddot{x} = \frac{\dot{v}\ddot{v}}{\dot{x}}$, and hence BS $= \frac{\dot{v}^2 \dot{x}}{\ddot{v}\dot{z}}$. Now since v becomes the uniform fluent, for $\ddot{v}$ write $\ddot{v} = -\frac{\ddot{z}\dot{v}}{\dot{z}}$, or by ignoring the sign, as it only indicates $\dot{z}$ to decrease with the increase of ED, when the curve is convex towards the axis, as is shown in this diagram, BS = $\frac{\dot{v}\dot{x}}{\ddot{z}}$. An example of the use of this proposition arises in some conic section, by considering $ax^2 = dz^2 - da^2$; where d is the semi-parameter to the semi-axes a, in which z is taken as the abscissa.[This is the hyperbola $\frac{z^2}{a^2} - \frac{x^2}{(d/a)a^2} = 1$] Then with the uniform fluent z, $2ax\dot{x} = 2az\dot{z}$, and thus

$\dot{x} = \frac{dz\dot{z}}{ax}$: and again by taking fluxions [and substituting]:

$$\ddot{x} = \left(\frac{d\dot{z}\dot{z}ax - a\dot{x}dz\dot{z}}{a^2x^2} = \frac{da^2x^2\dot{z}^2 - d^2az^2\dot{z}^2}{a^3x^3} = \right) - \frac{d^2\dot{z}^2}{x^3}.$$ Hence

$$\text{BS} = \left(\frac{p^3\dot{z}^2}{x^3\ddot{x}} = \right) \frac{-p^3}{d^2}$$; where the sign – only indicates that the centre S falls below the point B, that we have asked for above B. Hence on the whole conic section, the radius of curvature is the fourth proportional from the semi-parameter to whatever axis, and with the normal to the curve ending on the same axis. Whereby the radius of curvature to the extremity of the axis is equal to the semi-diameter to the same axis. For a is the semi-axis to the end of which the radius of curvature is sought, and d the semi-parameter of the same; and p is the ratio of the normal end to the curve to the other axis. [p.64] But in the vertex of the diameter a , $p = a$; whereby in this case the radius of curvature is d. Moreover this expression for the radius of curvature in conic sections was first found by the most distinguished *Newton*.

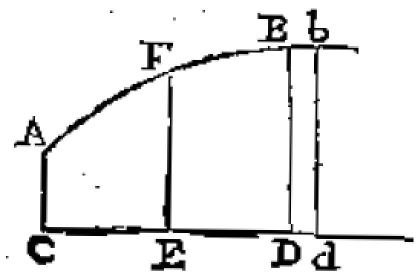


Now the remaining description for a conic section is able to determine the same radius of curvature by geometry from the following construction.

Let ABC be some given conic section, and the radius of curvature is sought at the point B. Draw the tangent BT, and the BS perpendicular to it, and BC is drawn parallel to either of the axes, and the angle CBA is made equal to the angle CBT, and BA crosses the curve again at A. Then bisect AB in D, and erect the perpendicular DS crossing BS in S, then S is the centre of the circle of osculation. The demonstration of this is very easy. [For these latter examples, it seems appropriate to refer the interested reader to an older text, such as *Elements of Analytical Geometry*, Gibson & Pinkerton (1911). The *Mathematical Works of Isaac Newton*, Vol.1,in the Sources of Science series (1964), also deals with the radius of curvature of conics and special curves, in a slightly different way from Taylor's presentation.]

PROP. XVI. PROBLEM XI.

To square all curves.



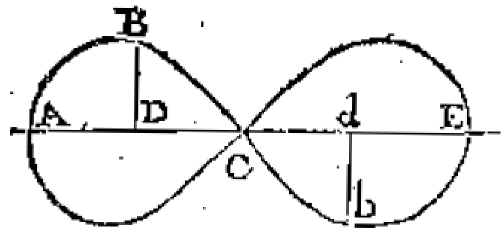
Let AB be a curve to be squared, the abscissa of which is CD, and the ordinate DB. The ordinate may be moved from its own place BD to a new place *bd*; and the area BD*db* is the increase in the area corresponding to the increase of the abscissa D*d*. [p. 65] The ordinate *bd* can then be returned to its previous location BD, and the greatest area BD*db* equal to BD × D*d*; whereby if the abscissa CD is *z*, and the ordinate BD is called y, the fluxion of the area is equal to $\dot{z}y$. Thus with the fluent of $\dot{z}y$ found (per Prop. 10) if the fluent arising from the abscissa CE can be taken away by the fluent arising from the abscissa CD, or with the unknown constant in the expression of the fluent being determined, by making the fluent arising from the abscissa CE equal to zero, the given area FEDB is described by the motion of the ordinate from EF to DB.

EXAMPLE I.

Let the abscissa from the given point C to the end position CD = *z*, and with the ordinate DB = y, [in the same diagram] and with some line taken for unity, let $y = z^{n-1}$. Then the fluxion for the area is equal to $\dot{z}z^{n-1}$, the fluent of which is $\frac{z^n}{n} + A$. Let some given abscissa be CE = *a*, then with the fluent $\frac{a^n}{n} + A$ taken from the fluent $\frac{z^n}{n} + A$, (or by making $\frac{a^n}{n} + A = 0$, and thus $A = -\frac{a^n}{n}$) then $\frac{z^n}{n} - \frac{a^n}{n}$ is equal to the area next to the abscissa ED.

When *n* is a negative number, as now $y = z^{-n-1}$, the area EFBD $= \frac{z^{-n}}{-n} - \frac{a^{-n}}{-n}$, that is $\frac{1}{na^n} - \frac{1}{nz^n}$. If now the abscissa *z* is made infinite, with the term $\frac{1}{nz^n}$ vanishing, then the area EFBD becomes equal to $\frac{1}{na^n}$, adjoining to the abscissa beyond the ordinate EF produced to infinity. And always when the sign of the fluent is opposite to the sign of the fluxion, the area is expressed by the fluent for the abscissa produced beyond, adjoining the ordinate. For in contrast it is shown that the fluent becomes smaller as the abscissa is increased, and vice versa. [p. 66]

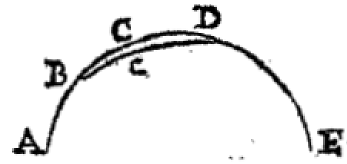
EXAMPLE II.



Let ABC*b* be the curve to be squared, of which the abscissa AD is considered to be *z*, and the ordinate DB of this to be applied is $\overline{1-z}\times\overline{2z-z^2}|^{\frac{1}{2}}$. The fluxion of the area for this curve is $\dot{z}\overline{1-z}\times\overline{2z-z^2}|^{\frac{1}{2}}$. The fluent of this, taken free from any correcting constants, is $\frac{1}{3}|\overline{2z-z^2}|^{\frac{3}{2}}$. Moreover this fluent is equal to zero either when $z = 0$, or when $z = 2$. Thus on the axis AD with AE taken equal to 2, the area bounded by this fluent is expressed either [by referring] to the point A or to the point E. When $z < 1$, (or AC, with the point C considered to lie between A and E) in the first case the area is ADB, in the second case it is the difference between the areas C*b*EC and CDBC. But when z (=A*d*) > 1, in the first case the area is the difference between the areas ABCA and C*db*C; in the second case it is the area *db*Ed.

The positions of areas of this kind are gathered together from the signs of the ordinates of the expressions and the areas. In the present case, by considering $z < 1$, both the ordinates and the areas are positive; whereby with the abscissa *z* increased, the area sought is increased; which hence is either ABD, or the difference between the areas CE*b*C and CBDC. But when $z > 1$, the ordinate and the area having opposite signs, for in this case the expression for the ordinate $\overline{1-z}\times\overline{2z-z^2}|^{\frac{1}{2}}$ is a negative quantity, for the expression of the area is $\frac{1}{3}|\overline{2z-z^2}|^{\frac{3}{2}}$ always considered to be positive; hence with the increase of the abscissa *z*, thus the area expressed is decreased, either the difference between the areas ABC and CD*b*, or by the area *d*E*b*. Moreover the sign of the area *d*E*b* is positive since as it is adjacent to the abscissa *d*E, as for a negative ordinate *db* [p. 67.]
[Taylor presents the curve $y^2 = x^2(1-x^2)$, if C is made the origin rather than A; he confronts the difficulty of finding the area between the axis and the positive roots of his expression for this curve, by considering the positive section ABC and the negative section C*b*E, with origin at A. It is of interest to note that he chooses as origin the point A rather than C, indicating that at this time there was still a reluctance to use negative numbers.]

LEMMA III.

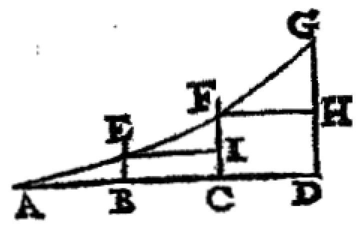


If the figure corresponding to some line ABCD is such that, as constructed from the whole line, it has is a maximum or minimum, which is greater or lesser than the figure assumes when some other line of a similar form is put in place; also the figure formed from any such part of the same line BCD will be either greater or lesser than that assumed by some other figure, if some other line BcD is put in place of BCD.

For if the part constructed corresponding to the line B*c*D is greater or lesser than the part corresponding to the line BCD; then jointly the whole makes a total greater or lesser corresponding to the line AB*c*DE, which is greater or lesser than that corresponding to the line ABCDE, which is contrary to the hypothesis.

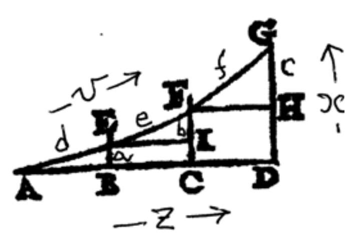
[Thus, the curve passes through a maximum or a minimum according to some condition placed on the curve; any other shorter interval containing the max. or min. behaves in the same way. This lemma is the lead-in to an extended set of results relating the behavior of a curve at a turning point to the curvature of the arc at such a point.]

LEMMA IV.



On the line AB in the given position, four equidistant points are taken A, B, C, D, etc. and the normals BE, CF, DG are erected; the lines AE, EF, FG are drawn through the four ends E, F, G, and FH is drawn parallel to AB crossing DG in H, and EI is drawn parallel to AD crossing FC in I. From the given points A & G, and from the sum of the lines AE, EF, FG, the ratio of the fluxions of the lines BE and GH is sought, when the whole figure vanishes, and AEFG becomes the element of a curve.

[We imagine that the figure is composed of infinitesimal lengths: AB, BC, and CD are fixed increments (corresponding to $\dot{z}$), as also is AG, the sum of the arc lengths, but BE and AE, EI and IF, and FH and GH are variables. It is required to find the ratio $\dot{c}:\dot{a}$]



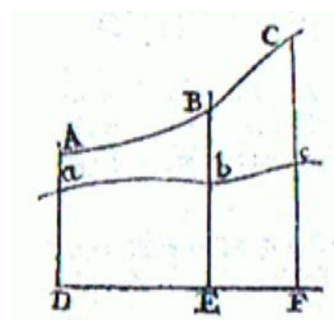
Let BE = *a*, IF = *b*, HG = *c*, AE = *d*, EF = *e*, FG = *f*. Then on account of the given sums $a+b+c\,(=DG)$ and $d+e+f$, by taking [p. 68] fluxions, $\dot{a}+\dot{b}+\dot{c}=0,(\text{or}\,\dot{b}=-\dot{a}-\dot{c},)$ likewise $\dot{d}+\dot{e}+\dot{f}=0.$ But on account of the given AB, EI, FH, and the right angles at B, I, and H, it follows that $\dot{d}=\frac{a\dot{a}}{d},\dot{e}=\frac{b\dot{b}}{e}(=\frac{-b\dot{a}-b\dot{c}}{e},)$; and $\dot{f}=\frac{c\dot{c}}{f}$. [Thus, $d^2=a^2+AB^2$; giving $d\dot{d}=a\dot{a}$, etc.]

Whereby $\frac{a\dot{a}}{d} - \frac{b\dot{a}}{e} - \frac{b\dot{c}}{e} + \frac{c\dot{c}}{f} = 0,$ that is $\dot{c} : \dot{a} :: \frac{b}{e} - \frac{a}{d} : \frac{c}{f} - \frac{b}{e}$. But when AEFG is the element of a curve, if x is the ordinate and v the element of [length of] the curve : then $a = \dot{x}, b = \dot{x} + \ddot{x}, c = \dot{x} + 2\ddot{x} + \dddot{x}, d = \dot{v}, e = \dot{v} + \ddot{v}, f = \dot{v} + 2\ddot{v} + \dddot{v};$ and thus $\dot{c} : \dot{a} :: \frac{\dot{x}+\ddot{x}}{\dot{v}+\ddot{v}} - \frac{\dot{x}}{\dot{v}} : \frac{\dot{x}+2\ddot{x}+\dddot{x}}{\dot{v}+2\ddot{v}+\dddot{v}} - \frac{\dot{x}+\ddot{x}}{\dot{v}+\ddot{v}}$; or, if y is written for $\frac{\dot{x}}{\dot{v}}$, then $\dot{c} : \dot{a} :: \dot{y} : \dot{y} + \ddot{y}$. For if $\frac{\dot{x}}{\dot{v}} = y,$ then $\frac{\dot{x}+\ddot{x}}{\dot{v}+\ddot{v}} = y + \dot{y},$ and $\frac{\dot{x}+2\ddot{x}+\dddot{x}}{\dot{v}+2\ddot{v}+\dddot{v}} = y + 2\dot{y} + \ddot{y}.$

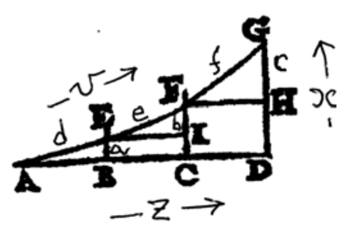
[These last ratios correspond to the sines of the angles of the three incremental arcs: they are read from the diagram rather than calculated.]

PROP. XVII. PROBLEM. XII.

With the line DE put in place, and with the perpendicular DA drawn, a curve ABC is cut at the point A, the ordinate of this curve is the perpendicular BE; abc is another curve, and the perpendicular ordinate of this curve Eb is composed in some manner from the common abscissa DE, along with the ordinate BE [p. 69.], *and from the arc AB. The form of the curve abc is sought from the curve ABC, when the area DabE is to be the maximum of all possible areas described by the ordinates bE in this manner, from the given base DF, with the ordinates DA and FC, and the length of the curve intercepted ABC.*



[The theorem establishes in a general way how a curve *abc* with a turning point can be made from a given section ABC of another curve, from its coordinates *z* and *x*, and the arc length *v*. A fluctional equation is established and solved under four conditions. The the curve *abc* has a maximum value somewhere on DF, and it is establised that the slope of abc rises and falls on opposite sides of the turning point.]



The abscissa DE = *z*, the ordinate BE = *x*, the [length of the] curve AB = *v*, and the ordinate E*b* = P. By *Lem*. 3 the same property is agreed upon for any part of the same given curve. Therefore the point A is the end of the ordinate BE for the present figure, and AEFG is a small part of the curve sought, and with P considered to be the ordinate *b*E pertaining to the point A (see figure for *Lem*. 4 , [repeated here opp.]), let $\overset{/}{P}$ be the similar ordinate pertaining to the point E, and with $\overset{//}{P}$ the third ordinate *b*E pertaining to the point F [Thus, the line *b*E sweeps through the length DF, and the end point traces out the part of the curve shown on the second small diagram, along AG]. Then the small [incremental] part of the area D*ab*E corresponding to the small part of the curve AEFG is $\dot{z}P+\dot{z}\overset{/}{P}+\dot{z}\overset{//}{P}$ [*i. e.* three incremental rectangles, where $P, \overset{/}{P}, \text{and } \overset{//}{P}$ represent three successive ordinate values of E*b* necessary to define the curvature], which since it must be an extreme value, the fluxion of this is zero (by the method of max. and min.) [*i.e.* $\dot{z}P+\dot{z}\overset{/}{P}+\dot{z}\overset{//}{P}=0$ (*); note the different method of treating the turning point of the variable, as the function notation was not yer in use.] Moreover, the fluxions are to be estimated only from the motions of the points E and F up and down. Hence by considering $\ddot{z}$ and $\dot{\mathrm{P}}$ equal to zero in the equations [obtained from the fluxion of (*)], on account of the absence of the motion of the points A, B, C, D, it follows that the fluxion is given by $\dot{z}\dot{\overset{/}{P}}+\dot{z}\dot{\overset{//}{P}}=0$, that is $\dot{\overset{/}{P}}+\dot{\overset{//}{P}}=0$.

[*i. e.* the gradients at the points E and F are equal and opposite.]

In the generation of these points, $\dot{P} = Q\dot{z} + R\dot{x} + S\dot{v}$. (**). [That is, the motion of a general point P on ABC is now considered, that would now be expressed as the total derivative of the function P = P(*z, x, v*), or

$dP = dP/dz.dz + dP/dx.dx + dP/dv.dv = Q\dot{z}+R\dot{x}+S\dot{v}$.]

Then for the values of the constants Q, R, and S pertaining to the points E and F, by writing Q, R, S, and $\overset{/}{Q}, \overset{/}{R}, \overset{/}{S}$ for those, and with [the appropriate] $\dot{z}$, $\dot{x}$, and $\dot{v}$ written for the values of $\dot{\overset{/}{P}}$ & $\dot{\overset{//}{P}}$ for the motions of the ends of EB and FC, as thus designated in *Lemma* 4, $\dot{\overset{/}{P}}=R\dot{a}+S\dot{d}$, and $\dot{\overset{//}{P}}=-\overset{/}{R}\dot{c}-\overset{/}{S}\dot{f}$, or with $\dot{d}$ & $\dot{f}$ replaced by the values $\frac{a\dot{a}}{d}$ and $\frac{c\dot{c}}{f}$;

$\dot{\overset{/}{P}}=R\dot{a}+S\frac{a\dot{a}}{d}$, and [p. 70.] $\dot{\overset{//}{P}}=-\overset{/}{R}\dot{c}-\overset{/}{S}\frac{c\dot{c}}{f}$.

[That is, for $\dot{\overset{/}{P}}+\dot{\overset{//}{P}}=0$ for the *abc* curve, the same ratio as for the ABC curve is used with a sign inversion, and the increments are taken from the fixed ends A and G: we now proceed without the sign inversion] Hence: $\dot{c}:\dot{a}::R+S\frac{a}{d}:\overset{/}{R}+\overset{/}{S}\frac{c}{f}$. But, (by *Lem*. 4)

$\dot{c}:\dot{a}::\dot{y}:\dot{y}+\ddot{y}$. Whereby (as in that lemma, for $\frac{a}{d}$ or $\frac{\dot{x}}{\dot{v}}$ I write *y*, and thus also

$\frac{c}{f}=y+2\dot{y}+\ddot{y}$.) the ratio here becomes : $R+Sy:\overset{/}{R}+\overset{/}{S}\times\overline{y+2\dot{y}+\ddot{y}}::\dot{y}:\dot{y}+\ddot{y}$. For

$\overset{/}{R}$ and $\overset{/}{S}$ write $\mathrm{R}+\dot{R}$ and $\mathrm{S}+\dot{\mathrm{S}}$, and the ratio becomes

$R+Sy:R+\dot{R}+Sy+2S\dot{y}+S\ddot{y}+\dot{S}y+2\dot{S}\dot{y}+\dot{S}\ddot{y}::\dot{y}:\dot{y}+\ddot{y}$, and on dividing

[in this context this means taking 1 from both sides of a ratio and simplifying; in this case R + S*y* and $\dot{y}$ are removed from the left and right denominators of the ratio,]

$R+Sy:\dot{R}+2S\dot{y}+S\ddot{y}+\dot{S}y+2\dot{S}\dot{y}+\dot{S}\ddot{y}::\dot{y}:\ddot{y}$, or (in the first consequence, rejecting the evanescent $S\ddot{y}, 2\dot{S}\dot{y}, \& \dot{S}\ddot{y}$), $R+Sy : \dot{R}+2S\dot{y}+\dot{S}y :: \dot{y} : \ddot{y}$ that is $\dot{R}\dot{y}-R\ddot{y}+\dot{S}y\dot{y}+2S\dot{y}\dot{y}-Sy\ddot{y}=0,$ (***)This equation of the fluxions cannot be reduced [*i. e.* resolved]; whereby with the values of $y, \dot{y}, \ddot{y}$ written in terms of $\dot{x}, \ddot{x}, \dddot{x}$ and $\dot{v}, \ddot{v}, \dddot{v}$: the expression becomes $\dot{R}\ddot{v}\dot{v}^2 - R\dddot{x}\dot{v}^2 + 3R\ddot{x}\ddot{v}\dot{v} + \dot{S}\ddot{x}\dot{x}\dot{v} + 2S\ddot{x}^2\dot{v} - S\dddot{x}\dot{x}\dot{v} + S\ddot{x}\dot{x}\ddot{v} = 0.$

[Recall from *Lemma* 4, that $\frac{\dot{x}}{\dot{v}}=y,$ $\frac{\dot{x}+\ddot{x}}{\dot{v}+\ddot{v}}=y+\dot{y},$ and $\frac{\dot{x}+2\ddot{x}+\dddot{x}}{\dot{v}+2\ddot{v}+\dddot{v}}=y+2\dot{y}+\ddot{y}.$

It is useful now to show from these equations that $\dot{y}=\frac{\dot{z}^2}{\dot{v}^3}.$ For

$\dot{y}=\frac{\dot{x}+\ddot{x}}{\dot{v}+\ddot{v}}-\frac{\dot{x}}{\dot{v}}=\frac{(\dot{x}+\ddot{x})\dot{v}-(\dot{v}+\ddot{v})\dot{x}}{(\dot{v}+\ddot{v})\dot{v}}=\frac{\ddot{x}\dot{v}-\ddot{v}\dot{x}}{(\dot{v}+\ddot{v})\dot{v}}$; since $\ddot{x}\dot{x}=\ddot{v}\dot{v},$ from $\dot{x}^2+\dot{z}^2=\dot{v}^2$, and $\dot{z}$ is considered constant, then on substituting $\ddot{v}=\ddot{x}\dot{x}/\dot{v}$, we find $\dot{y}=\frac{\ddot{x}\dot{v}-\ddot{x}\dot{x}^2/\dot{v}}{\dot{v}^2+\ddot{x}\dot{x}}$; leading to $\dot{y}=\frac{\ddot{x}(\dot{v}^2-\dot{x}^2)}{\dot{v}^3+\ddot{x}\dot{x}\dot{v}}\sim\frac{\ddot{x}\dot{z}^2}{\dot{v}^3}$, and $\frac{\dot{y}}{\dot{z}}=\frac{\ddot{x}\dot{z}}{\dot{v}^3}=1/\rho.$ In addition, we can relate these differentials to modern notation. Note initially that for a well-behaved function y = *f*(x) (not italic) in an interval, we can write tanψ = *d*y/*d*x for a point in the interval, and that $\frac{\dot{x}}{\dot{v}}=y$ (italic) in this notation is sinψ = *d*y/*d*s, where we choose to write the modern variables in non-italics in their usual meaning. It follows that $\dot{y}/\dot{z}$ for Taylor's italic y, is equal to our $d(\sin\psi)/dx=\cos\psi.d\psi/dx$; now, $\sec^2\psi.d\psi/d\mathrm{x}=d^2\mathrm{y}/d\mathrm{x}^2$, and hence $d\psi/d\mathrm{x}=\frac{d^2\mathrm{y}/d\mathrm{x}^2}{\sec^2\psi.}$, giving $\dot{y}/\dot{z}=\cos^3\psi.d^2\mathrm{y}/d\mathrm{x}^2=\frac{d^2\mathrm{y}/d\mathrm{x}^2.}{(1+(d\mathrm{y}/d\mathrm{x})^2)^{3/2}}$. Thus, $\dot{y}/\dot{z}$ is the inverse of the radius of curvature ρ at any point where these quantities are defined, and $\ddot{y}$ can be evaluated in the same way.]

And with the help of this equation, together with the equation [from the given curve ABC, from which x and v are measured for a common z with the curve *abc*,] $\dot{v}\dot{v}=\dot{x}\dot{x}+\dot{z}\dot{z}$, (truly for $\mathrm{R}, \dot{\mathrm{R}}, \mathrm{S}, \dot{\mathrm{S}}$, with the values of these expressed by *z, x, v*, and the fluxions of the expressions) the fluents x, and v are given (by *Prop*. 6). Moreover in the resolution of these equations there are four undetermined coefficients, (by *Prop*. 15) two of which are determined by making $v = 0$, and x = AD for the point D, and the remaining two are determined by making v equal to the length ABC, and x = FC for the point C [in the main diagram again].

COROLLARY I.

If the arc v does not enter into the equation for the value of the ordinate P, by considering S = 0 [see (**), then from (***)], $\dot{\mathrm{R}}\dot{\mathrm{y}} - \mathrm{R}\ddot{\mathrm{y}} = 0$. When this equation is compared with the fluxion of n : 2 . *Schol*. *Prop*. 6, [p. 71.] it is found that $\frac{R}{\overset{/}{\dot{y}}} = \frac{a}{\dot{z}}$. [Note that R, which is really dP/dx evaluated at the points stated, is inversely proportional to the curvature ρ, and thus $\dot{P}/\dot{x} = \dot{y}/\dot{z} = 1/\rho$, and where $a = \dot{x}$.] Where I write $\frac{\dot{z}^2 \ddot{x}}{\dot{v}^3}$ for $\overset{/}{\dot{\mathrm{y}}}$ [as evaluated above] to give the value $\mathrm{R}.\frac{\dot{v}^3}{\dot{z}\ddot{x}} = a$. But $\frac{\dot{v}^3}{\dot{z}\ddot{x}}$ is equal to the radius of curvature (by *Prop*. 15, *Ex*.4) whereby in this case the radius of curvature [of the ABC curve] is equal to $\frac{a}{\mathrm{R}}$.

COROLL. II.

With the same equations in place, if in the expression for the ordinate P, z also is absent, then the equation becomes $\mathrm{R}\dot{x} = \dot{\mathrm{P}}$, [$\overset{/}{\dot{P}} = R\dot{a}$ above with S = 0;]from which it is agreed on substituting for R from corollary 1 that the equation becomes $\dot{\mathrm{P}}.\frac{\dot{v}^3}{\dot{z}\dot{x}\ddot{x}} = a.$ In place of $\dot{x}\ddot{x}$ write the value of $\dot{v}\ddot{v}$ itself, [i. e. $\ddot{z} = 0$], and hence it becomes $\dot{\mathrm{P}} = \dot{z}a\ddot{v}\dot{v}^{-2}$. Hence on taking fluents, $P = b - \frac{a\dot{z}}{\dot{v}}$. From which arrangement, in this case the first problem concerning fluxions is recalled.

Also the problem can be solved by the quadrature of the curve. For the value of P does not involve any variables except x. Hence the equation is $a\dot{z} = b\dot{v} - \mathrm{P}\dot{v}$, and thus

$a^2\dot{z}^2 = \overline{b-\mathrm{P}}|^2\dot{v}^2 = \overline{b-\mathrm{P}}|^2 \times \overline{\dot{z}^2 + \dot{x}^2}$, that is

$\dot{z}\sqrt{a^2 + 2b\mathrm{P} - \mathrm{P}^2 - b^2} = \overline{b-\mathrm{P}} \times \dot{x}$,or $\dot{z} = \frac{\overline{b-\mathrm{P}} \times \dot{x}}{\sqrt{a^2 + 2b\mathrm{P} - \mathrm{P}^2 - b^2}}$. Also $\dot{v} = \frac{a\dot{x}}{\sqrt{a^2 + 2b\mathrm{P} - \mathrm{P}^2 - b^2}}$.

Hence by integrating, the curves z and v are given of which the common abscissa is x, [p. 72.] and the ordinates are $z = \frac{(b-\mathrm{P})x}{\sqrt{a^2 + 2b\mathrm{P} - \mathrm{P}^2 - b^2}}$ & $v = \frac{ax}{\sqrt{a^2 + 2b\mathrm{P} - \mathrm{P}^2 - b^2}}$.

[Thus, for a given small interval, for known points a and b chosen in the interval, linear equations for the curve and the arc length can be found in terms of P considered as constant, a, and b.]

COROLLARY III.

If in the expression for the ordinate P, x is missing, then by considering R = 0, the equation becomes $S\dot{y}\dot{y}+2S\dot{y}\dot{y}-S\dot{y}\ddot{y}=0$, From which equation collated with the equation for the fluxion $n:3$. *Schol*. *Prop*. 6, it is found that $S\frac{y^2}{\dot{y}}=\frac{a}{\dot{z}}$. [This can be integrated by parts or by inspection, and the result checked by differentiation, and the constant is assumed as above, to give the appropriate curvature.]

That is (for y and $\dot{y}$ the values $\frac{\dot{x}}{\dot{v}}$ and $\frac{\dot{z}^2\ddot{x}}{\dot{v}^3}$ are written) $S\frac{\dot{x}^2}{\dot{v}^2}\times\frac{\dot{v}^3}{\dot{z}\ddot{x}}=a$. Hence in this case the radius of curvature $(=\frac{\dot{v}^3}{\dot{z}^2\ddot{x}})$ is equal to $\frac{a\dot{v}^2}{S\dot{x}^2}$.

COROLLARY IV.

With the same equations in place, if z is missing as well in the expression for P then $\dot{P}=S\dot{v}$, then $\frac{\acute{P}\dot{x}^2}{\dot{z}\ddot{x}}=a$, that is $\dot{P}=a\dot{z}\ddot{x}\dot{x}^{-2}$. Hence by regressing to fluents, then

$P=b-\frac{a\dot{z}}{\dot{x}}$. Thus also in this case the problem is recalled to the first set of fluxion problems. Also the problem can be solved by the quadratures [integrations] of the curves. For in this case the value of P is not entered into by any variables except v. [p. 72.] Hence the equation is $a\dot{z}=b\dot{x}-P\dot{x}$, and hence $a^2\dot{z}^2=\overline{b-P}|^2\dot{x}^2=\overline{b-P}|^2\times\overline{\dot{v}^2-\dot{z}^2}$, that is :

$\dot{z}=\frac{\overline{b-P}\times\dot{v}}{\sqrt{a^2+b^2-2bP+P^2}}$. Also, $\dot{x}=\frac{a\dot{v}}{\sqrt{a^2+b^2-2bP+P^2}}$. Thus by quadrature, the curves, of which the common abscissa is v, are given and the ordinates give z and x:

$$\frac{(b-P)v}{\sqrt{a^2+b^2-2bP+P^2}},\text{and}\ \frac{av}{\sqrt{a^2+b^2-2bP+P^2}},$$

[There is an algebraic mistake in the original denominators which has been corrected.

Note that the original thought of expressing the curve *abc* in terms of the ABC curve has not been carried through, but is alluded to in the last corollary.]

COROLLARY V.

And hence in turn, for a given curve ABC, the nature of the extreme value [turning point] becomes known. For if the ordinate bE is sought which is composed from the powers of the ordinate BE, then it is given by the equation $P=b-\frac{a\dot{z}}{\dot{v}}$, (*Cor*. 2). And if the ordinate bE is sought which is composed from the powers v, it is found from the equation $P=b-\frac{a\dot{z}}{\dot{x}}$. (*Cor*. 4).

[The rest of the book is devoted to applied mathematics, which we start in IIb.]

METHODUS INCREMENTORUM.

Pars Secunda.

[page 53]

Ubi Exemplis aliquot ostenditur quomodo haec Methodus sit applicanda ad Problemata Mathematica & Physica.

PROP. XII1. PROB. VIIII.

Datis aliquot terminis aequidistantibus in Serie quantitatum, invenire terminos intermedios, & ulteriores quam proxime ex datis eorum distantiis alterutro termino extremo dato.

Sunto *a, b, c, d*, termini aequidistantes dati, & requiratur terminus alius aliquis ex data sua distantia a termino extremo *a*.

Sumantur terminorum datorum differentiae, deinde differentiarum differentiae, & sic porro, donec perventum sit ad differentiam ultimam, & sint differentiae illae sub propriis signis $\underset{\cdot}{a}(=b-a,)\ \underset{\cdot}{b}(=c-b,)\ \underset{\cdot}{c}(=d-c,)\ \underset{\cdot\cdot}{a}(=\underset{\cdot}{b}-\underset{\cdot}{a},)\ \underset{\cdot\cdot}{b}(=\underset{\cdot}{c}-\underset{\cdot}{b},)\ \underset{\cdot\cdot\cdot}{a}(=\underset{\cdot\cdot}{b}-\underset{\cdot\cdot}{a}.)$ Jam pro quolibet termino seriei in genere scribe *x*, & pro ejusdem termini distantia a termino *a* scribe *z*, & sit ipsius *z* incrementum $\underset{\cdot}{z}$ aequale distantiae datae inter terminos datos *a, b, c, d,* atque omnes terminos seriei in genere exprimi per aequationem $x = A + Bz + Cz\overset{\backslash}{z} + 3Dz\overset{\backslash}{z}\overset{\backslash\backslash}{z}$. Tum (per *Prop*. 1) differentia inter duos valores ipsius *x* ad distantiam ab invicem $\underset{\cdot}{z}$ exprimetur per aequationem $\underset{\cdot}{x} = B\underset{\cdot}{z} + 2Cz\underset{\cdot}{z} + 3D\underset{\cdot}{z}z\overset{\backslash}{z}$, & hujusmodi differentiarum differentiae exprimentur per aequationem $\underset{\cdot\cdot}{x} = 2C\underset{\cdot}{z}^2 + 2.3D\underset{\cdot}{z}^2 z$, & differentia tertia per aequationem $\underset{\cdot\cdot\cdot}{x} = 2.3D\underset{\cdot}{z}^3$. Sed ubi est $z = 0$, sunt $x = a$, $\underset{\cdot}{x} = \underset{\cdot}{a}$; $\underset{\cdot\cdot}{x} = \underset{\cdot\cdot}{a}$; $\underset{\cdot\cdot\cdot}{x} = \underset{\cdot\cdot\cdot}{a}$; unde per has aequationes sunt $A = a$, $B\underset{\cdot}{z} = \underset{\cdot}{a}$; $2C\underset{\cdot}{z}^2 = \underset{\cdot\cdot}{a}$, $2.3D\underset{\cdot}{z}^3 = \underset{\cdot\cdot\cdot}{a}$, adeoque $A = \overset{\backslash}{a}, B = \frac{\underset{\cdot}{a}}{\underset{\cdot}{z}}, C = \frac{\underset{\cdot\cdot}{a}}{2\underset{\cdot}{z}^2}, D = \frac{\underset{\cdot\cdot\cdot}{a}}{2.3\underset{\cdot}{z}^3}$, & exinde $x = \overset{\backslash}{a} + \frac{\underset{\cdot}{a}}{\underset{\cdot}{z}}z + \frac{\underset{\cdot\cdot}{a}}{2\underset{\cdot}{z}^2}z\overset{\backslash}{z} + \frac{\underset{\cdot\cdot\cdot}{a}}{2.3\underset{\cdot}{z}^3}z\overset{\backslash}{z}\overset{\backslash\backslash}{z}$. *Q.E.I.*

Haec aequatio convenit accurate cum ipsis terminis datis *a, b, c,d*, & quam proxime cum intermediis & ulteribus. Sed quo plures termini dentur, constat eo propius accessuram hanc aequationem ad valores omnium terminorum, tum intermediorum, tum & ulteriorum. Quare si series terminorum datorum continuetur in infinitum, aequatio tandem coincidet accurate cum valoribus terminotum omnium, cum intermediorum, tum ulteriorum. Proinde si detur lex [p. 55] formandi terminos aequidistantes in serie aliqua, per hanc propositione dabitur series infinita exprimens valores omnium terminorum intermediorum etiam & ulteriorum totius seriei.

EXEMPLUM.

Sit hujus rei exemplum in serie terminorum, ubi termini aequidistantes semper sunt in ratione continua Geometrica, in hac serie sint a & $a + ab$ termini duo ad distantiam z; tum ex natura hujus progressionis erunt omnes termini ad eandem distantiam $a, a+ab, a\times\overline{1+b}|^2, a\times\overline{1+b}|^3, \&c.$ differentiae primae hujusmodi terminorum erunt $ab, ab\times\overline{1+b}|, ab\times\overline{1+b}|^2\ \&c.$; differentiae secundae erunt $ab^2, ab^2\times\overline{1+b}|, ab\times\overline{1+b}|^2\ \&c.$; differentiae tertiae erunt $ab^3, ab^3\times\overline{1+b}|, ab^3\times\overline{1+b}|^2\ \&c.$; et sic porro. Unde ad mentem hujus Propositionis erunt $a=a, \underset{\cdot}{a}=ab, \underset{\cdot\cdot}{a}=ab^2, \underset{\cdot\cdot\cdot}{a}=ab^3, \&c.$ adeoque si z sit distantia alicuius termini x a termino a erit $x = a + \frac{abz}{1\underset{\cdot}{z}} + \frac{ab^2 z\overset{\backslash}{z}}{1.2\underset{\cdot}{z}^2} + \frac{ab^3 z\overset{\backslash}{z}\overset{\backslash\backslash}{z}}{1.2.3\underset{\cdot}{z}^3} + \&c.$ Sed in hoc casu est $\frac{x}{a} = \overline{1+b}|^{\frac{z}{\underset{\cdot}{z}}}$, unde sit $\overline{1+b}|^{\frac{z}{\underset{\cdot}{z}}} = 1 + \frac{bz}{1\underset{\cdot}{z}} + \frac{b^2 z\overset{\backslash}{z}}{1.2\underset{\cdot}{z}^2} + \frac{b^3 z\overset{\backslash}{z}\overset{\backslash\backslash}{z}}{1.2.3\underset{\cdot}{z}^3} + \&c$. Coincidit haec series cum Theoremate Newtoniano pro inventione dignitatis Binomii.

Quod Theorema etiam investigari potest ad hunc modum. Sit $\overline{a+b}|^n = a^n + xba^{n-1} + vb^2a^{n-2} + zb^3a^{n-3}\ \&c.$ Tum ducta serie in $a + b$ erit $\overline{a+b}|^{n+1}$, vel jucta notationem $\overline{a+b}|^{\underset{/}{n}}$ aequale

$$\begin{array}{llll} a^{\underset{/}{n}} & +xba^{\underset{/}{n-1}} & +vb^2a^{\underset{/}{n-2}} & +zb^3a^{\underset{/}{n-3}} + \&c. \\ & +1 & +x & +v \end{array}$$

Unde existente $\underset{\cdot}{n}=1$;per Methodum Incrementorum erit $\underset{\cdot}{x}=1$, adeoque $x=\frac{n}{1}$; $\underset{\cdot}{v}=x(=n,)$ adeoque $v=\frac{n\overset{\backslash}{n}}{1.2}$; $\underset{\cdot}{z}=v(=\frac{n\overset{\backslash}{n}}{1.2})$ adeoque, $z=\frac{n\overset{\backslash}{n}\overset{\backslash\backslash}{n}}{1.2.3}$; & sic porro; (ubi sumuntur omnes integrales pure, quoniam debent omnes evanescere ubi est n = 0,) unde sit $\overline{a+b}|^n = a^n + \frac{n}{1}ba^{n-1} + \frac{n\overset{\backslash}{n}}{1.2}b^2a^{n-2} + \frac{n\overset{\backslash}{n}\overset{\backslash\backslash}{n}}{1.2}b^3a^{n-3}\ \&c.$, hoc est $\overline{a+b}|^n = a^n + \frac{n}{1}\frac{b}{a}\mathrm{A} + \frac{n-1}{2}\frac{b}{a}\mathrm{B} + \frac{n-2}{3}\frac{b}{a}C + \&c.$ nempe pro singulis terminis seriei cum propriis signis scriptis A, B, C, &c.

PROP. XIV. PROB. IX.

Data ratione formandi terminos in series quantitatum , invenire aggregatum terminorum quotvis aequidistantium.

Si sit *x* summa quaesita, eadem aucta termino amplius uno erit $x+\underset{\cdot}{x}$. Quare data lege formandi terminos si fiat *x* aequalia terminos si fiat x aequalis termino proxime addendo, dabitur integralis *x* in terminis generalibus, per Prop. 10. [p. 57.] Quae ut fiat aequalis summae quaesitae, demendus est valor ejusdem integralis, qui prodit quando debet summa quaesita esse nihil.

EXEMP. I.

Sit exemplum in serie terminorum aequidistantium, $a, a+b, a+2b, a+3b, \&c.$ In hac serie pro termino ultimo semper scripto z, erit $\underset{\cdot}{z}=b$, atque terminus summae quaesitae proxime addendus erit $z+\underset{\cdot}{z}$, vel $z_{/}$; adeoque erit $\underset{\cdot}{x}=z_{/}$. Unde regrediendo ad integrales erit $x=\dfrac{z z_{/}}{2\underset{\cdot}{z}}+\mathrm{A}$. Debet autem haec summa aequari nihilo, quando terminus proxime addendus est *a*, hoc est, quando est $\underset{\cdot}{z}=a$, adeoque & $z=a-\underset{\cdot}{z}$; adeoque a summa $\dfrac{z z_{/}}{2\underset{\cdot}{z}}+\mathrm{A}$, ablata summa $\dfrac{\overline{a-\underset{\cdot}{z}}\times a}{2\underset{\cdot}{z}}+\mathrm{A}$, residium $\dfrac{z z_{/}-\overline{a-\underset{\cdot}{z}}\times a}{2\underset{\cdot}{z}}$ erit summa quaesta, hoc est, (pro $\underset{\cdot}{z}$ restituto *b*) $x=\dfrac{z\times\overline{z+b}-\overline{a-b}\times a}{2b}$.

EXEMP. II.

Ad eundem modum si terminus ultimus sit $z z_{/}$, atque terminus primus is sit in quo est z = *a*, hoc est $a\times\overline{a+\underset{\cdot}{z}}$, termino post ultimum proxime addendo existente $z_{/} z_{//}$, erit $\underset{\cdot}{x}=z_{/} z_{//}$; adeoque $x=\dfrac{z z_{/} z_{//}}{3\underset{\cdot}{z}}-\dfrac{\overline{a-\underset{\cdot}{z}}\times a\times\overline{a+\underset{\cdot}{z}}}{3\underset{\cdot}{z}}$. Et ad eundem modum pergere licet ad summationes terminorum, in quibus sunt plures factores $z\times z_{/}\times z_{//}\times \&c.$

[p. 58.]

EXEMP. III.

Inveniendum sit aggregatum terminorum, quorum ultimus semper est $\dfrac{1}{z z_{/}}$, & quorum primus is est in quo est z = *a*. In hoc casu terminus summae quaesitae proxime addendus est $\dfrac{1}{z_{/} z_{//}}$. Quare facta $\underset{\cdot}{x}=\dfrac{1}{z_{/} z_{//}}$,deinde regrediendo ad integrales erit $x=A-\dfrac{1}{\underset{\cdot}{z} z_{/}}$. Sed ubi terminus proxime addendus est $\frac{1}{a\times\overline{a+\underset{\cdot}{z}}}$, hoc est, ubi est $z = a,$ debet summa quaesita aequari nihilo; unde sit $A=\frac{1}{\underset{\cdot}{z}\,a}$ adeoque $x=\frac{1}{\underset{\cdot}{z}\,a}-\frac{1}{\underset{\cdot}{z}\,z_{/}}$.

COROLLARIUM.

Et hinc datur summa omnium terminorum $\frac{1}{a.a+z}+\frac{1}{a+z.a+2z}+\frac{1}{a+2z.a+3z}+\&c.$ in infinitum. Nam in hac serie in infinitum continuata in termino ultimo est divisor z infinitus. Proinde evanescente termino $\frac{1}{\underset{\cdot}{z}\,z_{/}}$ in valore x sit $x=\frac{1}{\underset{\cdot}{z}\,a}$.

Et ad eundem modum pergere licet ad summas terminorum in quibus sunt plures divisores $z, z_{/}, z_{//}, \&c.$

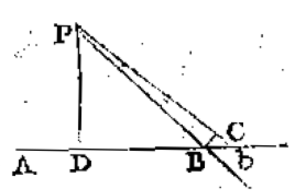
[p. 59]

PROP. XV. PROB. X.

Invenire Fluxiones in datis Figuris Geometricis.

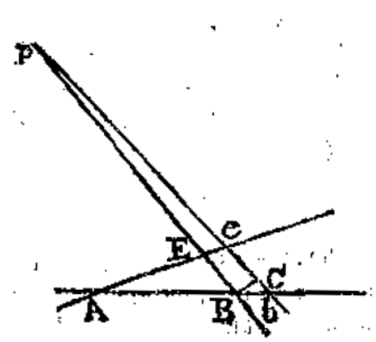
Hoc quomodo sit faciendum exemplis melius, quam praeceptis patebit.

EXEMP. I.



Sit ergo primum exemplum in figura ABP, ubi datis positione recta AB, & puncto P, quaeritur ratio fluxionum rectae AB, & distantiae PB.

" Progrediatur recta PB de loco suo PB in locum novum Pb. In Pb capiatur PC = PB, & ad AB ducatur PD sic, ut angulus *b*PD aequalis sit angulo *b*BC; & ob similitudinem triangulorum *b*BC, *b*PD, erit augmentum B*b* ad augmentum C*b*, ut P*b* ad D*b*. Redeat jam P*b* in locum suum priorem PB, ut augmenta illa evanescant, & evanescentium ratio ultima, id est ratio ultima P*b* ad D*b*, ea erit quae est PB ad DB, existente angulo PDB recto, & propterea in hac ratione est fluxio ipsius AB ad fluxionem ipsius PB.



EXEMP. II.

Recta PB circa datum polum P revolvens secet alias duas positione datas rectas AB & AE in B &

E : quaeritur proportio fluxionum rectarum illarum AB & AE.

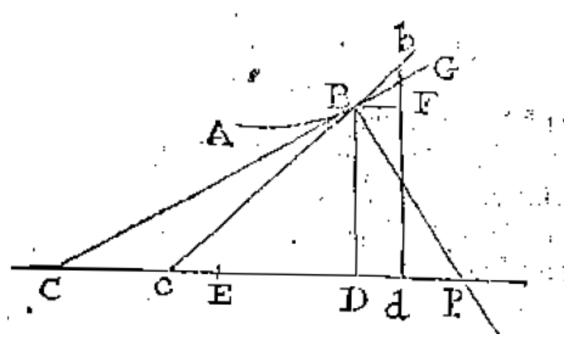
Progrediatur recta revolvans PB de loco suo PB in locum novum P*b* ; rectas AB ,[p. 60] AE in punctis *b* & *e* secantem, & rectae AE parallela BC ducatur ipsi P*b* occurrens in C, & erit B*b* ad BC ut A*b* ad A*e*, & BC ad E*e* ut PB ad PE, & conjunctis rationibus B*b* ad E*e* ut A*b* × PB ad A*c* × PE. Redeat jam linea P*b* in locum suum priorem PB, & augmentum evanscens B*b* erit ad augmentum evanescens E*e* ut AB × PB ad AE × PE, ideoque in hac ratione est fluxio rectae AB ad fluxionem rectae AE.

Hinc in recta revolvens PB lineas quasvis curvas, positione datas secet in punctis B & E, & rectae jam mobiles AB, AE curvas illas tangant in sectionum punctis B & E; erit fluxio curvae quam recta AB tangit ad fluxionem curvae quam recta AE tangit ut AB × PB ad AE × PE. Id quod etiam eveniet si recta PB curvam aliquam positione datam perpetuo tangat mobile P."

Petuntur haec duo exempla ex *Newtonianis.*

EXEMP. III.

Sit AB curva quaevis positione data, & ab ejus puncto quovis B ducatur recta BD secans rectam positione datam ED in angulo dato in D; quaeritur proportio fluxionum, abscissae ED, ordinatae DB, & curvae AB.

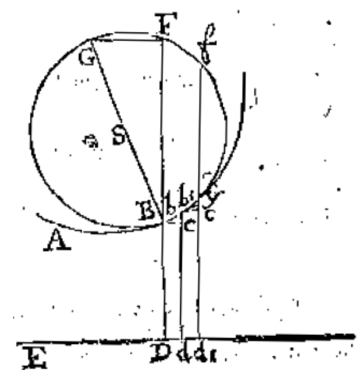


Moveatur ordinata BD de loco suo BD in locum novum *bd*, & ducatur BF parallela ED, & occurrens *bd* in F, & per puncta *b* & B, ducatur bB occurrens ED in *c*, & ad punctum B ducatur tangens [p. 61] occurrens ED, *db* in C & G. Tum ob similia triangula BF*b*, *c*DB, erit BF : F*b* : B*b* : : *c*D : DB : *c*B. Redeat jam ordinata *bd* in locum suum priorem BD, & recta cB jam coincidente cum tangente CB, erunt augmenta nascentia abscissae ED, ordinatae DB, & ipsius curvae AB inter se, ut latera trianguli nascentis BFG, vel ei similis trianguli CDB; ideoque in hac ratione erunt fluxiones rectarum ED, DB, & curvae AB. Et si angulus BDE sit rectus, ducta ad curvam normali BP occurrens ED in P, ob similia BFG, BDP, erunt eaedem fluxiones inter se ut latera trianguli BDP.

Hinc data ratione ducendi tangentes ad curvam aliquam propositam, dabitur proportio fluxionum abscissae, ordinatae, & ipsius curvae; atque vicissum ex data proportione fluxionum abscissae & ordinatae, dabitur proportio inter subtangentem CD, ordinatum DB, & ipsam tangentem CB; ut & proportio inter ordinatam BD, subnormalem DP, & normalem BP. Data autem aequatione definiente relationem inter abscissam ED, &

ordinatam DB, datur proportio fluxionum (per *Prop*. 1.) Quare per istam proportionem duci possunt tangentes & normales ad omnes curvas.

EXEMP. IV.



Sit curva quavis AB, & propositum sit invenire radium curvaturae in puncto B, hoc est, radium circuli cujus curvature eadem sit, quae curvae AB in puncto B.

Duc ordinatas tres aequidistantes BD, *bd, b*1*d*1, secantes rectam positione datam ED ad angulos rectos in D, *d, d*1, & ipsi ED parallalas duc BC, *bc*, occurrens *bd, b*1*d*1, in C & *c*, & duc B*b* occurrentem *b*1*d*1 in *y*, [p. 62.] & per puncta tria B, *b, b*1, transeat circulus cujus centrum sit S, occurrens DB & *d*1*b*1 in F & *f*, & duc diametrum BG, atque FG. Sint autem abscissa ED = $\underset{\cdot}{z}$, ordinata DB = *x*, & curva AB = *v*. Tum juxta Methodum Incrementorum erit D*d* = $\underset{\cdot}{z}$ (=*dd*1= BC = *bc*) ; C*b* = $\underset{\cdot}{x}$ (= *cy*) & *yb*1 = $\underset{\cdot\cdot}{x}$. Sit etiam B*b* = (*by* =) $\underset{\cdot}{u}$. Tum ex natura circuli erit $yb1 \times yf = yB \times yb$, hoc est $yf = \frac{\underset{\cdot}{u} \times \underset{\cdot}{u}}{\underset{\cdot\cdot}{x}}$. Sed coincidentibus punctis B, *b, b*1, erit arcus evanescens B*bb*1 circulo & curvae AB communis, ideoque in hoc casu erit circulus centro S descriptus curvae AB aequicurvus in B. Evanescant itaque incrementa, & jam coincidentu *yf* cum ipso BF, & facto $\frac{\underset{\cdot}{u}\underset{\cdot}{u}}{\underset{\cdot\cdot}{x}} = \frac{\dot{v}\dot{v}}{\ddot{x}}$, erit BF = $\frac{2\dot{v}\dot{v}}{\ddot{x}}$. Sed coincidentibus punctis B & *b*, est BF ad BG, ut BC ad BA, hoc est, ut $\dot{z}$ ad $\dot{v}$. Unde sit BG = $\frac{2\dot{v}^3}{\dot{z}\ddot{x}}$, & radius curvaturae BS = $\frac{\dot{v}^3}{\dot{z}\ddot{x}}$.

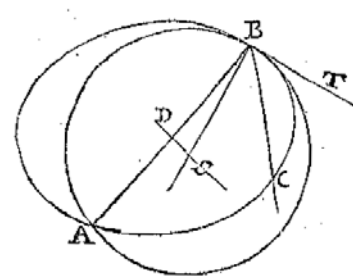
Si sit *p* ad curvam normalis intercepta inter punctum B & axem ED, erit $\dot{v} = \frac{p\dot{z}}{x}$ (per *Ex*. 3.) Unde sit etiam radius curvaturae BS = $\frac{p^3\dot{z}^2}{x^3\ddot{x}}$.

Et haec fiunt fluente uniformiter *z*. Sed si cupis invenire expressionem ejusdem radii ubi fluit uniformiter *v*, per aequationem $\dot{v}\dot{v} = \dot{x}\dot{x} + \dot{z}\dot{z}$ [p.63.] (adhuc fluente uniformiter *z*) fiet $2\dot{v}\ddot{v} = 2\dot{x}\ddot{x}$, hoc est $\ddot{x} = \frac{\dot{v}\ddot{v}}{\dot{x}}$, & inde BS = $\frac{\dot{v}^2\dot{x}}{\ddot{v}\dot{z}}$. Jam ut fluat uniformiter *v*, pro $\ddot{v}$ scribe $\ddot{x} = -\frac{\ddot{z}\dot{v}}{\dot{z}}$, vel neglecto signo quod solum indicat ipsum $\dot{z}$ decrescere crecente ED, quando curva est versus axem convexa, ut in hoc schemate exhibetur, BS = $\frac{\dot{v}\dot{x}}{\ddot{z}}$.

Sit hujus rei exemplum in quavis sectione conica, existente $ax^2 = dz^2 - da^2$; ubi est d semi-parameter ad semi-axem a, in quo sumitur abscissa z. Tum fluente uniformiter z, erit $2ax\dot{x} = 2az\dot{z}$, adeoque $\dot{x} = -\frac{dz\dot{z}}{ax}$: atque iterum capiendo fluxiones $\ddot{x} = \left(\frac{d\dot{z}\dot{z} - a\dot{x}dz\dot{z}}{a^2x^2} = \frac{da^2x^2\dot{x}^2 - d^2az^2\dot{z}^2}{a^3x^3} = \right) - \frac{d^2\dot{z}^2}{x^3}$. Unde sit BS $= \left(\frac{p^3\dot{z}^3}{x^3\ddot{x}} = \right)\frac{-p^3}{d^2}$; ubi signum – tantum indicat centrum S cadere infra punctum B, quod nos quaesivimus supra B. Ergo in omni sectione conica est radius curvaturae quartum proportionale semiparametro ad utrumvis axem, & ad curvam normali terminatae ad eundem axem. Quare ad extremitatem axis est radius curvaturae aequalis ipsi semiparametro ad eundem axem. Sit enim a semi-axis ad cujus extremitatem quaeritur radius curvaturae, & d ejusdem semi-parameter; & sit p ad curvam normalis terminata ad axem alterum, (cujus semiparameter est $\frac{a^{\frac{3}{2}}}{d^{\frac{3}{2}}}$.) Tum erit radius curvaturae $= \frac{p^3d}{a^3}$. [p.64] Sed in vertice diametri a est $p = a$; quare in hoc casu est radius curvaturae d. Hanc autem expressionem radii curvaturae in conisectionibus primus invenit clarissimus *Newtonus.*

Caeterum descripta jam sectione conica, potest idem radius determinari Geometrice per constructionem sequentem.

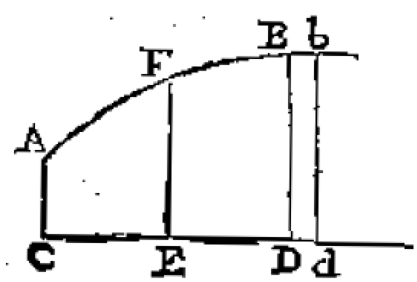
Sit ABC data sectio conica, & quaeratur radius curvaturae ad punctum B. Duc tangentem BT, eique perpendicularem BS, & ducantur BC parallela alterutri axium, atque fiat angulus CBA aequalis angulo CBT, & occurrat BA consectioni in A. Tum bisecta AB in D, & erecta perpendiculari DS occurrente BS in S, erit S centrum circuli osculatori. Hujus demonstratio est perfacilis.



PROP. XVI. PROB. XI.

Curvas omnes quadrare.

Sit AB curva quadranda, cujus abscissa est CD, & ordinata DB. Moveatur ordinata de loco suo BD in locum novum *bd* atque; spatium BD*db* erit augmentum areae respondens abscissae augmento D*d*. [p. 65] Redeat ordinata *bd* in locum suum priorem BD, atque erit ultimo spatium BD*db* aequale BD × D*d*; quare si abscissa CD sit z, & ordinata BD dicatur y, erit fluxio areae aequalis $\dot{z}y$.



Inventa itaque fluente ipsius $\dot{z}y$ (per Prop. 10) si dematur fluens proveniens per abscissam

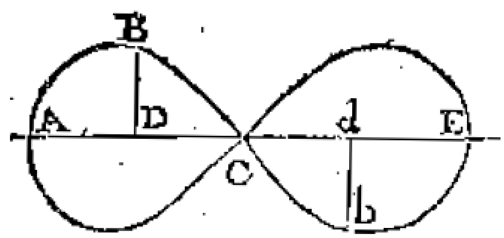
CE a fluente proveniente per abscissam CD, vel invariabilis incognita in expressione fluentis determinetur faciendo fluentem provenientem per abscissam CE aequalem nihilo, dabitur area FEDB descripta per motum ordinatae de EF in DB.

EXEMP. I.

Sit abscissa ad datum punctum C terminata CD = z, & ordinata DB = y, atque sumpta aliqua linea pro unitate sit $y = z^{n-1}$. Tum erit fluxio areae aequalis $\dot{z}z^{n-1}$, cujus fluens est $\frac{z^n}{n} + A$. Sit data aliqua abscissa CE = a, tum dempta fluenta $\frac{a^n}{n} + A$ a fluente $\frac{z^n}{n} + A$, (vel facto $\frac{a^n}{n} + A = 0$, unde fiat $A = -\frac{a^n}{n}$) erit $\frac{z^n}{n} - \frac{a^n}{n}$ aequale areae adjacenti ad abscissam ED.

Ubi est n numerus negativus, ut sit $y = z^{-n-1}$, erit area EFBD $= \frac{z^{-n}}{-n} - \frac{a^{-n}}{-n}$, hoc est $\frac{1}{na^n} - \frac{1}{nz^n}$. Si jam fiat abscissa z infinita, evanescente termino $\frac{1}{nz^n}$ fiet area EFBD adjacens abscissae ultra ordinatam EF in infinitum productae aequalis $\frac{1}{na^n}$. Et semper ubi fluens signum contrarium est signo fluxionis, areae per fluentem expressa adjacet abscissae ultra ordinatam productae. Nam contrarietas monstrat fluentem minui dum augetur abscissa, & vice versa. [p. 66]

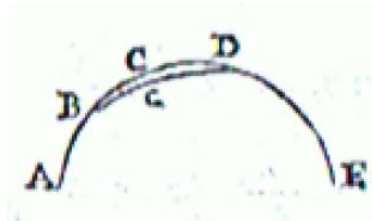
EXEMP. II.



Sit curva quadranda ABCb, cujus abscissa AD existente z, ejus ordinatim applicata DB est $\overline{1-z}\times\overline{2z-z^2}|^{\frac{1}{2}}$. Fluxio areae in hac curva est $\dot{z}\overline{1-z}\times\overline{2z-z^2}|^{\frac{1}{2}}$. Cujus fluens, pure sumpta absque correctione, est $\frac{1}{3}|\overline{2z-z^2}|^{\frac{3}{2}}$.
Est autem haec fluens aequalis nihilo, vel ubi $z = 0$, vel ubi $z = 2$. Itaque in axe AD sumpto AE = 2, terminatur area per hanc fluentem expressa, vel ad punctum A vel ad punctum E. Ubi est $z < 1$, (vel AC, existente puncto C medio inter A & E) in primo casu est area ADB, in secundo casu est differentia inter areas C*b*EC atque CDBC. Sed ubi est z (=A*d*) > 1, in primo casu est area differentia inter areas ABCA, atque C*db*C; in secundo casu est area *db*Ed.

Positiones hujusmodi arearum colliguntur ex signis expressionum ordinatae, atque areae. Sic in casu praesenti, existente z < 1, tam ordinata quam area sunt affirmativae; quare aucta abscissa z, augetur area quaesita; quae proinde est, vel ABD, vel differentia inter areas CEbC atque CBDC. Sed ubi est z > 1, ordinata, atque area signa habens contraria, nam in hoc casu expressio ordinatae $\overline{1-z}\times\overline{2z-z^2}|^{\frac{1}{2}}$ est quantitas negativa, expressione areae $\frac{1}{3}|\overline{2z-z^2}|^{\frac{3}{2}}$ semper existents affirmativa; unde crescente abscissae z decrescit area adeoque exprimitur, vel differentiam inter areas ABC & CD*b*, vel per aream dEb. Areae autem dE*b* signum est affirmativum quoniam adjacet tam abscissae dE, quam ordinatae *db* negativis. [p. 67.]

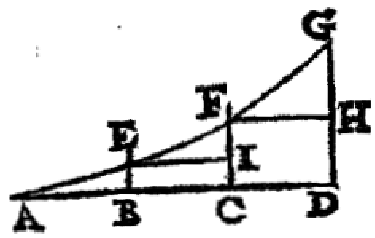
LEMMA III.



Si lineae alicujus ABCDE figura talis sit, ut factum aliquod toti lineae respondens majus sit, vel minus, quam simile factum ubi linea aliam induit figuram; etiam lineae ejusdem partis cujusvis BCD figura a talis erit, ut ei respondens facti pars major sit, vel minor, quam si pars illa BCD aliam quamvis induat figuram BcD.

Nam si facta pars respondens lineae B*c*D major sit, vel minor, quam simile factum respondens lineae BCD; tum conjunctim erit erit totum factum respondens lineae ABcDE majus, vel minus, quam simile respondens Lineae ABCDE, contra hypothesin.

LEMMA IV.



In recta AB positione data sumantur puncta quatuor aequidistantia A, B, C,D, &c erigantur normales BE, CF, DG, per quarum terminos E, F, G ducantur rectae AE, EF, FG, & ducatur FH parallela ipsi AB occurrens DG in H, EI parallela ipsi AD, & occurrens FC in I. Datis punctis A & G, & summa rectarum AE, EF, FG, quaeritur ratio fluxionum linearum BE, GH, quando figura tota evanescit & sit AEFG elementum curvae.

Sunto BE = *a*, IF = *b*, HG = *c*, AE = *d*, EF = *e*, FG = *f*. Tum ob datas summas $a+b+c(=DG)$ & $d+e+f$, capiendo [p. 68] fluxiones erit

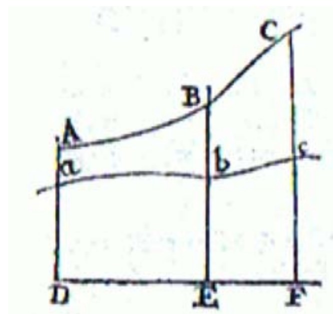
$\dot{a}+\dot{b}+\dot{c}=0,$ (hoc est $\dot{b}=-\dot{a}-\dot{c},$) item $\dot{d}+\dot{e}+\dot{f}=0.$ Sed ob datas AB, EI, FH, & ob angulos rectos in B, I, & H, sunt $\dot{d}=\frac{a\dot{a}}{d}, \dot{e}=\frac{b\dot{b}}{e}(=\frac{-b\dot{a}-b\dot{c}}{e},)$ atque; $\dot{f}=\frac{c\dot{c}}{f}$. Quare est

$\frac{a\dot{a}}{d}-\frac{b\dot{a}}{e}-\frac{b\dot{c}}{e}+\frac{c\dot{c}}{f}=0,$ hoc est $c:a::\frac{b}{e}-\frac{a}{d}:\frac{c}{f}-\frac{b}{e}$. Sed ubi AEFG est elementum curvae, si sit ordinata *x* & curva *v*, erit

$a=\dot{x}, b=\dot{x}+\ddot{x}, c=\dot{x}+2\ddot{x}+\dddot{x}, d=\dot{v}, e=\dot{v}+\ddot{v}, f=\dot{v}+2\ddot{v}+\dddot{v}$; adeoque

$\dot{c} : \dot{a} :: \frac{\dot{x}+\ddot{x}}{\dot{v}+\ddot{v}} - \frac{\dot{x}}{\dot{v}} : \frac{\dot{x}+2\ddot{x}+\dddot{x}}{\dot{v}+2\ddot{v}+\dddot{v}} - \frac{\dot{x}+\ddot{x}}{\dot{v}+\ddot{v}}$; vel si pro $\frac{\dot{x}}{\dot{v}}$ scribatur y, erit $\dot{c} : \dot{a} :: \dot{y} : \dot{y}+\ddot{y}$. Nam si $\frac{\dot{x}}{\dot{v}} = y$, erit $\frac{\dot{x}+\ddot{x}}{\dot{v}+\ddot{v}} = y+\dot{y}$, atque $\frac{\dot{x}+2\ddot{x}+\dddot{x}}{\dot{v}+2\ddot{v}+\dddot{v}} = y+2\dot{y}+\ddot{y}$.

PROP. XVII. PROB. XII.



Detur positione recta DE, & ducta perpendiculari DA, per punctum A transeat curva ABC cujus ordinata perpendicularis est BE; atque sit abc alia curva cujus ordinata perpendicularis Eb quovis modo dato componitur ex abscissa communi DE, ordinata BE [p. 69.] *, & curva AB. Quaeritur forma curvae ABC, quando area DabE est omnium arearum per ordinatas bE hoc modo provenientes descriptarum maxima, ex data basi DF,ordinatis DA, FC, & longitudine curvae interceptae ABC.*

Sit abscissa DE = z, ordinata BE = x, curva AB = v, atque ordinata Eb = P. Per *Lem*. 3 eadem proprietas convenit curvae particulae cuivis datae. Sit ergo (vid. fig. *Lem*. 4) punctum A extremitas ordinatae BE in praesenti figura, atque sit AEFG particula curvae quaesita, & existente P ordinata bE pertinente ad punctum A, sit $\overset{/}{P}$ similis ordinata pertinens ad punctum E, atque $\overset{//}{P}$ tertia ordinata bE pertinens ad punctum F. Tum areae DabE particula respondends curvae particulae AEFG erit $zP + \dot{z}\overset{/}{P} + \ddot{z}\dot{\overset{/}{P}}$, quae cum debeat esse extrema, ejus fluxio erit nihil (per method. maximorum & minorum.) Fluxiones autem sunt aestimandae per motus punctorum tantum E & F sursum & deorsum. Unde existentibus $\ddot{z}$ & $\dot{\text{P}}$ aequalibus nihilo, ob defectum motus punctorum A, B, C, D, erit $\dot{z}\dot{\overset{/}{P}} + \dot{z}\dot{\overset{//}{P}} = 0$.

Sit in genere $\dot{P} = Q\dot{z} + R\dot{x} + S\dot{v}$. Tum pro Q, R, & S pertinentibus ad puncta E & F scriptis Q, R, S, & $\overset{/}{Q}, \overset{/}{R}, \overset{/}{S}$; atque pro $\dot{z}, \dot{x}, \dot{v}$, in valoribus ipsorum $\dot{\overset{/}{P}}$ & $\dot{\overset{//}{P}}$ scriptis motibus punctum E, B, F, & C, prout designantur in *Lemmate* 4 erit $\dot{\overset{//}{P}} = -\overset{/}{R}\dot{c} - \overset{/}{S}\dot{f}$, vel pro $\dot{d}$ & $\dot{f}$ scriptis ipsorum valoribus $\frac{a\dot{a}}{d}$ atque $\frac{c\dot{c}}{f}$; $\dot{\overset{/}{P}} = R\dot{a} + S\frac{a\dot{a}}{d}$, atque [p. 70.] $\dot{\overset{//}{P}} = -\overset{/}{R}\dot{c} - \overset{/}{S}\frac{c\dot{c}}{f}$. Unde sit $\dot{c} : \dot{a} :: R + S\frac{a}{d} : \overset{/}{R} + \overset{/}{S}\frac{c}{f}$. Sed (per *Lem*. 4) $\dot{c} : \dot{a} :: \dot{y} : \dot{y}+\ddot{y}$. Quare (ut in Lemmate isto, pro $\frac{a}{d}$ vel $\frac{\dot{x}}{\dot{v}}$ scripto y, ut sit etiam $\frac{c}{f} = y + 2\dot{y} + \ddot{y}$.) erit

$R + Sy : \overset{/}{R} + \overset{/}{S} \times \overline{y + 2\dot{y} + \ddot{y}} :: \dot{y} : \dot{y} + \ddot{y}$. Pro $\overset{/}{R} + \overset{/}{S}$ scribe $\text{R} + \dot{R}$ atque $\text{S} + \dot{\text{S}}$, atque fiet

$R+Sy : R+\dot R+Sy+2S\dot y+S\ddot y+\dot S y+2\dot S\dot y+\dot S\ddot y :: \dot y : \dot y+\ddot y$, atque dividendo

$R+Sy : \dot R+2S\dot y+S\ddot y+\dot S y+2\dot S\dot y+\dot S\ddot y :: \dot y : \ddot y$, vel (in primo consequente rejectis

evanescentibus $S\ddot y, 2\dot S\dot y, \& \dot S\ddot y$) $R+Sy : \dot R+2S\dot y+\dot S y :: \dot y : \ddot y$, hoc est

$\dot R\dot y-R\ddot y+\dot S y\dot y+2S\dot y\dot y-Sy\ddot y=0$, Est haec aequatio fluxionalis irreducibilis; quare

pro $y, \dot y, \ddot y$,scriptis eorum valoribus per $\dot x, \ddot x, \dddot x$ & $\dot v, \ddot v, \dddot v$ expressis fiet

$\dot R\dddot v\dot v^2-R\dddot x\dot v^2+3R\ddot x\ddot v\dot v+\dot S\ddot x\dot x\dot v+2S\ddot x^2\dot v-S\dddot x\dot x\dot v+S\ddot x\dot x\ddot v=0$. Et ope hujus aequationis, una cum

aequatione $\dot v\dot v=\dot x\dot x+\dot z\dot z$, (nempe pro $\mathrm{R}, \dot{\mathrm{R}}, \mathrm{S}, \dot{\mathrm{S}}$,scriptis eorum valoribus per *z, x, v*, & eorum fluxiones expressis) dabuntur fluentes *x*, & *v* (per *Prop*. 6). In resolutione autem harum aequationum erunt quatuor coefficients indeterminati, (per *Prop*. 5) quorum duo determinantur faciendo *v* = 0, & *x* = AD ad punctum D, atque reliqui duo determinantur faciendo *v* = datae longitudini ABC, & *x* = FC ad punctum C.

COROLL. I.

Si curva *v* non ingreditur valorme ordinatae P, existente S = 0, erit $\dot{\mathrm{R}}\dot{\mathrm{y}}-\mathrm{R}\ddot{\mathrm{y}}=0$. Qau aequatione comparata cum fluxione *n* : 2 . *Schol*. *Prop*. 6, [p. 71.] invenetur $\frac{R}{\dot{\mathrm{y}}}=\frac{a}{\dot{\mathrm{z}}}$. Ubi

pro $\overset{/}{\dot{\mathrm{y}}}$ scripto ipsius valore $\frac{\dot z^2\ddot x}{\dot v^3}$, fiet $\mathrm{R}.\frac{\dot v^3}{\dot z\ddot x}=a$. Sed est $\frac{\dot v^3}{\dot z\ddot x}$ aequale radio curvaturae (per

Prop. 15, *Ex*.4) quare in hoc casu est radius curvaturae aequalis $\frac{a}{\mathrm{R}}$.

COROLL. II.

Iisdem positis, si in expressione ordinatae P desit etiam *z*, erit $\mathrm{R}x=\dot{\mathrm{P}}$, quo pacto

fiet $\dot{\mathrm{P}}.\frac{\dot v^3}{\dot z\dot x\ddot x}=a$. Vice $\dot x\ddot x$ scribe ipsius valorem $\dot v\ddot v$, atque hinc fiet $\dot{\mathrm{P}}=\dot z a\ddot v\dot v^{-2}$. Unde

capiendo fluentes erit $P=b-\frac{a\dot z}{\dot v}$. Quo pacto in hoc casu revocatur Problema ad

fluxiones primas.

Solvi etiam potest per quadraturam curvarum. Nam in valore ipsius P nulla involvitur

variabilis nisi *x*. Est ergo $a\dot z=b\dot v-\mathrm{P}\dot v$, adeoque $a^2\dot z^2=\overline{b-\mathrm{P}}|^2\dot v^2=\overline{b-\mathrm{P}}|^2\times\overline{\dot z^2+\dot x^2}$, hoc est

$\dot z\sqrt{a^2+2b\mathrm{P}-\mathrm{P}^2-b^2}=\overline{b-\mathrm{P}}\times\dot x$,or $\dot z=\frac{\overline{b-\mathrm{P}}\times\dot x}{\sqrt{a^2+2b\mathrm{P}-\mathrm{P}^2-b^2}}$. Etiam $\dot v=\frac{a\dot x}{\sqrt{a^2+2b\mathrm{P}-\mathrm{P}^2-b^2}}$.

Ergo quadrando curvas quarum [p. 72.] abscissa communis est *x*, & ordinatae sunt

$\frac{(b-\mathrm{P})x}{\sqrt{a^2+2b\mathrm{P}-\mathrm{P}^2-b^2}}$ & $\frac{ax}{\sqrt{a^2+2b\mathrm{P}-\mathrm{P}^2-b^2}}$, dabuntur *z* & *v*.

COROLL. III.

Si in expressione ordinatae P defit x, existente R = 0, erit $\dot{S}\,y\,\dot{y} + 2\dot{S}\,\dot{y}\,\dot{y} - S\,y\,\ddot{y} = 0$, Qua aequatione collata cum fluxione n : 3 . *Schol*. *Prop*. 6. Invenietur $S\frac{y^2}{\dot{y}} = \frac{a}{\dot{z}}$, hoc est (pro y & $\dot{y}$ scriptis suis valoribus $\frac{\dot{x}}{\dot{v}}$ atque $\frac{\dot{z}^2\ddot{x}}{\dot{v}^3}$) $S\frac{\dot{x}^2}{\dot{v}^2}\times\frac{\dot{v}^3}{\dot{z}\ddot{x}} = a$. Unde in casu erit radius curvature ($=\frac{\dot{v}^3}{\dot{z}^2\ddot{x}}$) aequalis $\frac{a\dot{v}^2}{S\dot{x}^2}$.

COROLL. IV.

Iisdem positis, si praeterea desit z in expressione ipsius P ut sit $\dot{P} = \dot{S}\,v$, $\frac{P'\dot{z}^2}{\dot{z}\ddot{x}} = a$, hoc est $P = a\dot{z}\ddot{x}\dot{x}^{-2}$. Unde regrediendo ad fluentes sit $P = b - \frac{a\dot{z}}{\dot{x}}$. Adeoque etiam in hoc casu revocatur Problema fluxiones primas.

Solve etiam potest per quadraturam curvarum. Nam in hoc casu valorem ipsius P nulla ingreditur variabilis nisi v. Ergo est [p. 72.] $a\dot{z} = b\dot{x} - P\dot{x}$, adeoque $a^2\dot{z}^2 = \overline{b-P}|^2\,\dot{x}^2 = \overline{b-P}|^2 \times \overline{\dot{v}^2 - \dot{z}^2}$, hoc est $\dot{z} = \frac{\overline{b-P}\times\dot{v}}{\sqrt{a^2+b^2-2bP+P^2}}$. Etiam $\dot{x} = \frac{a\dot{v}}{\sqrt{a^2+b^2-2bP+P^2}}$. Itaque quadrando curvas, quarum abscissa communis est b, & ordinatae sunt $\frac{(b-P)v}{\sqrt{a^2+b^2-2bP+P^2}}$, and $\frac{av}{\sqrt{a^2+b^2-2bP+P^2}}$ dabuntur z & x.

COROLL. V.

Et hinc vice versa, data curva ABC, innotescit cujusmodi factum sit extremum in hac curva. Nam si quaeritur ordinata bE quae componatur ex dignitatibus ordinatae BE, dabitur per aequationem $P = b - \frac{a\dot{z}}{\dot{x}}$, (*Cor*. 2). Et si quaeritur ordinata bE quae componatur ex dignitatibus v, invenietur per aequationem $P = b - \frac{a\dot{z}}{\dot{x}}$. (*Cor*. 4).