To the Reader.

In this Method of Increments, I consider quantities as increased by Increments or diminished by Decrements, and between the given relations of the Integral [whole] *quantities, I seek the relations between the Increments, and from given relations between Increments, I seek the Integral quantities themselves in turn. The uses of these methods in mathematical matters is widely known ; and moreover it is shown to the greatest effect in these circumstances where all the properties of Fluxions may hence be easily derived. The most illustrious* Newton*, by considering the Mathematical quantities written down for continuous motion, finds the ratio of the velocities by which the magnitudes are described, by means of the Method of Fluxions arising from the first ratios of the increments, and in turn from these velocities (which he calls the Fluxions of the quantities) he seeks the magnitudes of the described quantities themselves* [fluents].
[Note : To put ourselves in the correct frame of mind of the time for Newton's early calculus, we are to regard a plane curve as generated by the steady motion of a point along a principal axis, marked off in regular steps measured by the abscissa *z* from an origin, while the ordinate axis is a line in the plane passing through the origin at right angles to the principal axis, and the ordinate or *x* undergoes some motion. Quantities such as *z* and *x* that can change are called 'fluents' , while the rate of change of a fluent is called it 'fluxion' , which is the derivative of the variable. Fluents such as *x* and *z* depend on the time *t* that has elapsed since the varying point passed through the origin, and *z* commonly has a fluxion $\dot{z}$ set equal to one, and to which the other fluxions can be compared : thus, for example, for the circle $x^2 + z^2 = 1$, using Newton's rules, to be stated later, we can form the fluxional equation $2x\dot{x} + 2z\dot{z} = 0$, from which $\dot{x}/\dot{z} = -z/x$. On setting $\dot{z} = 1$, the rate of change of *x* with respect to *z* , or simply $\dot{x}$, the modern or Leibniz *dz/dy*, is equal to - *z/x*, where the unit rate of flow of *z* makes this possible. See Howard Eves *An Introduction to the History of Mathematics,* p. 339, for more of this elementary description. The reader should note that the function notation *f*(*x*) had not been invented at this time.]

Likewise, this [inverse process for finding fluents or integrals] *has been done by other mathematicians, but less generally, (as the Ancients showed in the method of exhaustion, and the methods of summation used by Cavalieri & Wallis.) The Ancients, by finding the sizes of figures inscribed and circumscribed in a circle, from finite parts and known constants, and by increasing the number and diminished the size of these parts, had made the difference between the sums of these and the size of the figure sought less than whatever was required to be given.*
[This brings to mind Archimedes' attempt to find the ratio of the circumference to the diameter of the circle; a contemporary topic pursued in evaluating π, and presented by Briggs in one of the final chapters of his *Arithmetica,* to be found in this series of translations.]

Cavalieri & the more recent workers have considered these parts as diminishing in size indefinitely. But all these, by contemplating the origins of the quantity by the addition of extra parts, have not satisfied the strict perfection [$\alpha\chi\rho\beta\varepsilon\iota\alpha$] *of the ancient*

Geometers. For the parts, in order that the Method shall be accurate, must be present from the beginning; since there are no [extra fractional] *parts of this kind in Nature, rather they are ratios arising from the original fractional parts.* [the writer means that a curve is present from the beginning, and must be worked with, rather than an approximation to the curve, such as a circle considered as an approximation from a regular n-gon inscribed or circumscribed]. *Therefore Newton, by putting the magnitudes of the small residual parts in place, and from these and their sum, has introduced the final ratios of the vanishing residual parts* [*we will call these nascent*], *arising from the first of these ratios, and from these first ratios he has laid the foundations of his Analysis. Thus by taking the first ratios of the Increments arising, or from the vanishing of the highest ratios, all the conclusions of the method of Increments, with the vanishing nascent* [i. e. higher order terms] *increments, have now been applied to the method of Fluxions, and conversely to find the whole or integral parts for fluents. And with this agreed upon, all consideration of indefinitely small quantities (or, as some would like to say, undefined quantities) is avoided. For in the method of Fluxions, in order that the conclusions are true and entirely accurate, the fractional parts or increments have to be taken together, not as very small, or as indefinitely small, but as actually zero: Indeed the ratios are not zero initially except in the very instant when they begin to arise; for when once they have arisen, they are no longer in the first ratio* [*i. e.* they are never undefined as 0/0]. *And similarly they are not the final ratios, until the quantities have now vanished and made zero. Nevertheless, for the sake of making the concept easier, these augmented ratios arising are to be taken for the Fluxions, which Newton calls moments, and designates by the letter o, which are put in place for the Fluxions.*

[The moment of a fluent such as x is the amount by which it changes in an indefinitely small length of time o, given by $\dot{x}o$, and all products in an equation that involve orders of o greater than one can be ignored. Thus, in the above example, on replacing z by $z+\dot{z}o$ and x by $x+\dot{x}o$, we have $(x+\dot{x}o)^2+(z+\dot{z}o)^2=1$; on expanding and simplifying by setting $x^2+z^2=1$, we have $x^2+2x\dot{x}o+(\dot{x}o)^2+z^2+2z\dot{z}o+(\dot{z}o)^2=1$, which gives $2x\dot{x}o+2z\dot{z}o=0$, from which dividing by o, the above result follows.]

And in this way, the relation between the method of Increments and the method of Fluxions is made easier to understand. On account of which, it is observed in several propositions that we give as examples in the use of Fluxions, that to whatever the increments taken in their generation, which are either of finite size or of the indefinitely small, the o moments are to be included.

[This last remark shows that Taylor has generalized the use of Newton's moments to produce his method of increments. One should not necessarily try to translate Newton's methods, or those of Taylor, into terms of modern calculus on going through a proof, as what is easy in one method may be cumbersome in the other; there is of course the need to establish that the two methods, and the assertions of Taylor, are in fact all equivalent. Rather, one should look on these methods as parallel developments worthy of being examined in their own right ; in this way what was 'obvious' at the time can again be understood. The same results are of course obtained by either method, Newtonian or Leibnizian, and in this translation an attempt is made to justify formulas which seem obscure. And one last word: if one was or still is, dealing only with first and second derivatives in problems, then the dot method involved far less writing and led to quicker solutions, and Newton was always a man in a hurry. On the other hand, odd drops of ink on the page might be interpreted as fluxions or increments of variables ...... not always a matter of mirth.]

The Method of Increments.

The First Part.

Where Instructions are set out about the generation of the Method of Increments and the Method of Fluxions **.**

INTRODUCTION.

I consider indeterminate quantities, [*i. e.* variables] with the Increments in these either increasing always or Decrements that are decreasing always. The indeterminate quantities are themselves the Integral or whole amounts that I designate by the letters *z, x, v*, &c. and the Increments of these, or the small parts that can be added to these, I designate by the same letters, but with points or dots below, such as $\underset{\cdot}{z}, \underset{\cdot}{x}, \underset{\cdot}{v}$, &c. The Increments of these Increments, or the second Increments I designate by the same letters with two points $\underset{\cdot\cdot}{z}, \underset{\cdot\cdot}{x}, \underset{\cdot\cdot}{v}$, &c. The Increments of these Increments, or the Increments of the second Increments of the variables, I designate by the same letters with the points $\underset{\because}{z}, \underset{\because}{x}, \underset{\because}{v}$, and so on.
[or, as here in this translation, by $\underset{\cdots}{z}, \underset{\cdots}{x}, \underset{\cdots}{v}$, as it is more convenient to produce these ; and in explanations we revert to the modern usage , and call these finite differences Δx, Δv, and Δz ; along with $\Delta^2 x$, Δx^3, etc. for the higher orders in *x*, and likewise for the other variables. It is understood that these increments are small positive or negative quantities relative to the variable quantity *x*, etc, of an arbitrary nature, but each is an order of magnitude less in absolute value than the previous one, and they form a nested set for

positive increments of the form $0 < \Delta^n < \Delta^{n-1} < .. < \Delta^i < .. < \Delta^2 < \Delta^1 = \Delta$, where i is the order of the increment Δ^i, and i is not to be considered as a power of Δ.]
But also, for higher increments for the sake of generality, and at other times, I sometimes write characters designating the number of points instead : Thus if n is 3, $\underset{\therefore}{x}$ is designated by $\underset{n}{x}$, or $\underset{3}{x}$; if n is 0, $\underset{0}{x}$ designates the Integral amount x itself ; if n is -1, the quantity designated by $\underset{-1}{x}$ is that for which the first Increment is x [page 2]; & thus for the others.
Indeed, often in this tract I designate some successive values of quantities of the same variable by the same letter described by a small line ; obviously the present value is to be designated by the simple letter, the preceding by superscript grave accents, and the subsequent by small subscript lines. Thus for example : $\overset{\backslash}{x}, \overset{\backslash\backslash}{x}, x, \underset{\backslash}{x}, \underset{\backslash\backslash}{x}$ are five successive values of the same variable quantity, of which x is the present value, $\overset{\backslash}{x}$ & $\overset{\backslash\backslash}{x}$ are the preceding values, and $\underset{\backslash}{x}$ & $\underset{\backslash\backslash}{x}$ are the subsequent values.

[Corresponding to the modern use of subscripts or superscripts, and also used for successive integrations; while Newton's dots are used for derivatives. There is unfortunately a certain vagueness about the use of some of these symbols, and occasionally Taylor does not supply an example of their usage.]

II. Fluxions, which arise in the first ratio of the increments, or in the final vanishing part, are described by points for the orders of the Increments, transposed to the upper parts of the letters : Thus $\dot{x}$ is the first Fluxion of x itself; $\ddot{x}$ is the second Fluxion of the same; $\dddot{x}$ the third Fluxion; & so on. Fluents [the integrated quantity] also are sometimes designated by small lines written above the letters (like acute accents) : Thus
$\overset{/}{x}$ designates the fluent of x itself, or the quantity of which the first Fluxion is x ;
$\overset{//}{x}$ designates the second fluent of x itself, or the quantity of which the second Fluxion is x; & thus henceforth. And these small lines in the signs of the fluents have the force of negative points (as thus I may say) in the signs of Fluxions : Thus if $n = 2$, & $\overset{n}{x}$ designates $\ddot{x}$, with the sign changed $\overset{//}{x}$ is designated by $\overset{-n}{x}$. Henceforth fluents of quantities composed of several parts are sometimes designated by the quantities themselves enclosed in parallelograms; thus $\boxed{xz^2}$ designates the fluent of xz^2.

[The reader can no doubt see that Taylor's notation lays itself open to misunderstanding and confusion, mainly by its similarity to Newton's dot notation for differentiation, to which it is distinct, and by the fact that Taylor uses Newton's notation when discussing fluxions. Taylor had little choice in his notation : he was Secretary of the Royal Society at the time, Newton was the President; it would have been a brave and rather foolish man to try to invent a completely new notation, or, heaven forbid, to adopt the continental usage of Leibniz ! Thus, he fashioned his notation around that of Newton, and if one perseveres a little, it becomes rather obvious, and it is very concise. In this translation, I have occasionally looked at the work of L. Feigenbaum : *Brook Taylor and the Method of Increments ;* published in the *Archives of the Exact Sciences*, (1981), which is good for background information on Propositions 2, 3, 4, 7, 17, and 24, which are examined there in some detail, and this is a helpful source with many references for anyone with a deep desire to learn more about the early calculus of Newton, apart from reading Newton's own work.]

[page 3]

PROP. 1. PROB. I.

To find the increments for a given equation involving variable quantities.

In the proposed equation, in place of these variable quantities, write the same quantities increased by their own increments as you see fit, and a new equation will be the result; then, by taking away the first equation, there will be left an equation from which the relations of the increments are given.

For example, let the equation be $x^3 - xv^2 + a^2 z - b^3 = 0$, where *a* & *b* are determined and unchanging quantities. And thus for the variables *x, v*, and *z* by writing $x + \underset{\cdot}{x}, v + \underset{\cdot}{v}, \& \; z + \underset{\cdot}{z}$ in their place, a new equation is produced :

$$x^3 + 3\underset{\cdot}{x}x^2 + 3\underset{\cdot}{x}^2 x + \underset{\cdot}{x}^3 - xv^2 - \underset{\cdot}{x}v^2 - 2x\underset{\cdot}{v} - x\underset{\cdot}{v}^2 - 2\underset{\cdot}{x}v\underset{\cdot}{v} - \underset{\cdot}{x}\underset{\cdot}{v}^2 + a^2 z + a^2 \underset{\cdot}{z} - b^3 = 0;$$

then with the first equation subtracted, the remainder is the equation $3\underset{\cdot}{x}x^2 + 3\underset{\cdot}{x}^2 x + \underset{\cdot}{x}^3 - \underset{\cdot}{x}v^2 - 2x\underset{\cdot}{v} - x\underset{\cdot}{v}^2 - 2\underset{\cdot}{x}v\underset{\cdot}{v} - \underset{\cdot}{x}\underset{\cdot}{v}^2 + a^2 \underset{\cdot}{z} = 0;$ with the help of which the relations between the increments are given.

[Thus, in modern terms, if we write *Δx, Δv*, and *Δz* in place of $\underset{\cdot}{x}, \underset{\cdot}{v}, \& \; \underset{\cdot}{z}$, then we have on expansion :

$$x^3 + 3x^2 \Delta x + 3x(\Delta x)^2 x + (\Delta x)^3 - xv^2 - v^2(\Delta x) - 2x(\Delta v) - x(\Delta v)^2 - 2(\Delta x)v(\Delta x)$$
$$- (\Delta x)(\Delta v)^2 + a^2 z + a^2(\Delta z) - b^3 = 0;$$

from which on subtraction of the original expression, we obtain :

$$3x^2 \Delta x + 3x(\Delta x)^2 x + (\Delta x)^3 - v^2(\Delta x) - 2xv(\Delta v) - x(\Delta v)^2 - 2(\Delta x)v(\Delta v)$$
$$- (\Delta x)(\Delta v)^2 + a^2(\Delta z) = 0;$$

This is the same as the above equation, but written in terms of *Δx, Δv*, and *Δz* in place of $\underset{\cdot}{x}, \underset{\cdot}{v}, \& \; \underset{\cdot}{z}$]

If in this solution, zero is written for the nascent increments arising [*i. e.* those of order greater than one], and for the first of these ratios the ratios associated with the fluxions are substituted, then all the operation can be performed at once simultaneously, with these terms ignored. For a proposed equation, on account of the vanishing nascent increments, can then be stated according to *Newton's Rule*, which is as follows:

> "Every term of the equation is multiplied by the index of the corresponding "quantity and by the fluent of this quantity involved, and in the individual "multiplications the power of the quantity is changed into its own Fluxion; "and the new equation is the sum of the factors with the appropriate signs, by "which the Fluxion related to the equation is defined.

EXAMPLES.

[The quote marks indicate that Newton's method of fluxions is being quoted.]
"Let *a, b, c, d*, &c. be determined and unchangeable quantities, and an equation is "proposed with several fluent [derivatives or speeds] quantities involved *z, y, x*, &c., such "as $x^3 - xy^2 + a^2 z - b^3 = 0$.
"First the terms [involving *x*] are to be multiplied by the indices corresponding to *x*, & in "the multiplications of individual terms by the appropriate power [the word *latus* or side is used, synonymous with the power or index (the original powers encountered were the squares of sides and cubes, etc), of one power less than the original],

"or in the case of *x* of one dimension, $\dot{x}$ alone is written, & the sum of the factors will be

"$3\dot{x}x^2 - \dot{x}y^2$. The same can happen in *y*, & will produce $-2x\dot{y}y$. And in z, and

"produces $a^2\dot{z}$. The sum of the factors is put equal to zero, and the equation is

"obtained : $3\dot{x}x^2 - \dot{x}y^2 - 2x\dot{y}y + a^2\dot{z} = 0$.
" In the same manner, if the equation should be :

"$x^3 - xy^2 + a^2\sqrt{ax - y^2} - b^3 = 0$ then "$3\dot{x}x^2 - \dot{x}y^2 - 2x\dot{y}y + a^2\dot{\overline{\sqrt{ax - y^2}}} = 0$ is

"produced. Where, if you wish to find the fluxion $\dot{\overline{\sqrt{ax-y^2}}}$, you put $\sqrt{ax-y^2} = z$, and

"then $ax - y^2 = z^2$, & (from this Prop.) $a\dot{x} - 2\dot{y}y = 2\dot{z}z$, or $\frac{a\dot{x}-2\dot{y}y}{2z} = \dot{z}$, that is

" $\frac{a\dot{x}-2\dot{y}y}{2\sqrt{ax-y^2}} = \dot{\overline{\sqrt{ax - y^2}}}$. And hence $3\dot{x}x^2 - \dot{x}y^2 - 2x\dot{y}y + \frac{a^2\dot{x}-2a^2\dot{y}y}{2\sqrt{ax-y^2}} = 0$.

[These examples on finding what corresponds to finding the total derivative of a function of several variables corresponding geometrically to a surface in three dimensions, are taken from, or are similar to, those in Newton's notes, which had not been published at this late date, though some 50 years had passed since Newton had first set out his ideas on the calculus and circulated to friends, but which were to be published soon afterwards. See, e. g. *The Mathematical Works of Isaac Newton* Vol. I in the Sources of Science Series, p. 31; or the original *A Treatise on the Methods of Fluxions*……. ;Newton's dot notation is not to be confused with the one now employed by Taylor. However, Taylor now shows how the total derivative or fluxion can be determined from the evaluation of finite differences, as follows.]

By the repeated operation gone through above, applied to the increments [from the first order difference], the Fluxions of the second, third, and following orders can be found. Let the equation be : $xz-av=0$. Then by the first operation [*i. e.*, of Taylor's finite difference method set out above] the equation becomes : $\underset{\cdot}{x}z+x\underset{\cdot}{z}+\underset{\cdot}{x}\underset{\cdot}{z}-a\underset{\cdot}{v}=0$. In this equation in place of $x,\underset{\cdot}{x},z,\underset{\cdot}{z},v,\underset{\cdot}{v}$ by writing $x+\underset{\cdot}{x},\underset{\cdot}{x}+\underset{\cdot\cdot}{x},z+\underset{\cdot}{z},\underset{\cdot}{z}+\underset{\cdot\cdot}{z},v+\underset{\cdot}{v},\underset{\cdot}{v}+\underset{\cdot\cdot}{v}$, and with the original equation subtracted, then from which the second operation is formed :
$2\underset{\cdot}{x}\underset{\cdot}{z}+\underset{\cdot\cdot}{x}z+2\underset{\cdot\cdot}{x}\underset{\cdot}{z}+x\underset{\cdot\cdot}{z}+2\underset{\cdot}{x}\underset{\cdot\cdot}{z}+\underset{\cdot\cdot}{x}\underset{\cdot\cdot}{z}-a\underset{\cdot\cdot}{v}=0$.

[In modern notation, we set $x+\underset{\cdot}{x},\underset{\cdot}{x}+\underset{\cdot\cdot}{x},z+\underset{\cdot}{z},\underset{\cdot}{z}+\underset{\cdot\cdot}{z},v+\underset{\cdot}{v},\underset{\cdot}{v}+\underset{\cdot\cdot}{v}$ as

$\mathrm{x}+\Delta\mathrm{x},\Delta\mathrm{x}+\Delta^2\mathrm{x},\mathrm{z}+\Delta\mathrm{z},\Delta\mathrm{z}+\Delta^2\mathrm{z},\mathrm{v}+\Delta\mathrm{v},\Delta\mathrm{v}+\Delta^2\mathrm{v}$, and the second order operation becomes :

$$(\Delta x+\Delta^2 x)(z+\Delta z)+(x+\Delta x)(\Delta z+\Delta z^2)+(\Delta x+\Delta^2 x)(\Delta z+\Delta z^2)-a(\Delta v+\Delta^2 v)\text{ -}$$
$$(\Delta x.z+x.\Delta z+\Delta x.\Delta z-a\Delta v)=0$$

Giving : $2\Delta x.\Delta z+z.\Delta^2 x+2\Delta z.\Delta^2 x+x.\Delta^2 z+2\Delta x.\Delta z^2+\Delta x^2.\Delta z^2-a.\Delta^2 v=0.$
This corresponds to the above result; corresponding to Taylor's notation for increments.]

Thus with [Newton's] fluxions for the same proposed equation, the above expression for the first fluent equation becomes $\dot{x}z+x\dot{z}-a\dot{v}=0$, and for the second $2\dot{x}\dot{z}+\ddot{x}z+x\ddot{z}-a\ddot{v}=0$. And thus you can go on as you wish by means of the increments, to the third, fourth, and fifth order fluxions, and beyond.

[Note initially that the equation with the superscript dots is a fluxional equation, corresponding to a differential equation in Newton's notation, as explained previously. Here we may note that differential equations result when the higher order increments in a Taylor incremental equation are set to zero, and here we have a first order differential equation : $z.dx/dt+x.dz/dt-a.dv/dt=0$, where all the variables are considered as functions of some parameter t from our point of view, or one variable z is considered as the free one that has a constant rate of increase $\underset{\cdot}{z}$ or $\dot{\mathrm{z}}$ set equal to one, for the incremental and fluxional cases respectively; or the equivalent equation $z.dx+x.dz-a.dv=0$, with similar expression for the second order quantities, as Taylor now demonstrates.]

But when the increments or the fluxions have gone in this way through the second, third, fourth, and those orders that follow, it is agreed to consider some quantity as uniformly increasing, and to write zero for the Increments or for the Fluxions for the

second, third, and subsequent orders of these. Thus in the equation proposed just now : $xz - av = 0$,
[p. 5] with z uniformly increasing, by going through the second operation, the equation [for the increments] becomes : $2\underset{\cdot}{x}\underset{\cdot}{z} + \underset{\cdot\cdot}{x}z + 2\underset{\cdot\cdot}{x}\underset{\cdot}{z} - a\underset{\cdot\cdot}{v} = 0$. And for the fluxions proposed for the same equation, by the second fluxion operation, the fluxion equation becomes :

$2\dot{x}\dot{z} + \ddot{x}z - a\ddot{v} = 0$, and by the third $3\ddot{x}\dot{z} + \dddot{x}z - a\dddot{v} = 0$.

And in this case it is convenient for the given Fluxion $\dot{z}$ to be written as 1. With this agreed upon, the aforementioned equations are

$\dot{x}z + x - a\dot{v} = 0$, $2\dot{x} + \ddot{x}z - a\ddot{v} = 0$, and $3\ddot{x} + z\dddot{x} - a\dddot{v} = 0$.

[Thus, Taylor shows that his method of finite differences can be used to generate fluxions on disregarding higher order differences.]

PROP. II. PROB. II.

In an incremental equation with several variables involved, every one of these variables can have a new variable substituted in its place, with the same size of increment in place, but increasing in the opposite direction.
[This theorem asserts that for an incremental equation, the principal or independent variable can be allowed to travel from left to right, or from right to left. The theorem establishes how a variable can be defined in terms of one or the other.]

Let one of the original variables in a proposed equation be *x*, and *v* one of the new variables, to be substituted in place of *x* ; thus, in such a manner that as *x* is increased by an increment, *v* is made less by the same increment. If *n* is the index of the smallest increment in the proposed equation, then the conditions of the problem for $x, \underset{\cdot}{x}, \underset{\cdot\cdot}{x}$, &c. will be satisfied [as derived in Prop. 1 above.], and a sequence of values can be written for these; where *d* is a fixed quantity to be taken as you please.

$x = d - v - \frac{n}{1}\underset{\cdot}{v} - \frac{n}{1}.\frac{n-1}{2}.\underset{\cdot\cdot}{v} - \frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}.\underset{\cdot\cdot\cdot}{v} - \&c.$, then

$\underset{\cdot}{x} = \underset{\cdot}{v} \quad + \frac{\overline{n-1}}{1}\underset{\cdot\cdot}{v} \quad + \frac{\overline{n-1}}{1}.\frac{\overline{n-2}}{2}.\underset{\cdot\cdot\cdot}{v} \quad + \frac{\overline{n-1}}{1}.\frac{\overline{n-2}}{2}.\frac{\overline{n-3}}{3}.\underset{\cdot\cdot\cdot\cdot}{v} \quad + \&c.$, and

$\underset{\cdot\cdot}{x} = -\underset{\cdot\cdot}{v} \quad - \frac{\overline{n-2}}{1}\underset{\cdot\cdot\cdot}{v} \quad - \frac{\overline{n-2}}{1}.\frac{\overline{n-3}}{2}.\underset{\cdot\cdot\cdot\cdot}{v} \quad - \&c.$, while

$\underset{\cdot\cdot\cdot}{x} = \underset{\cdot\cdot\cdot}{v} \quad + \frac{\overline{n-3}}{1}\underset{\cdot\cdot\cdot\cdot}{v} \quad + \&c.$ & thus henceforth. [p. 6]

[Note : In the scheme adopted, *x* is the value of a right-moving integral or whole variable defined at some point *z* = *a*, which is set equal to some arbitrary constant *d* from which is taken the sum of the increments of the left-moving variable *v,* with all its orders up to some finite order *n*, and with special coefficients chosen equal to the binomial coefficients of a binomial of degree *n*. It becomes clear that Taylor had invented the shift operator E defined to-day by $Ef(a) = f(a+d)$; and the above relation is postulated between the right moving increments and the left moving increments, which are shown to be equivalent. The various successive orders of the differences of *x* then follow by the application of the method of Prop. 1, in an inductive manner. An illustration of the method is then provided. Taylor does not appear to have used this Proposition in the rest of the book, and as with the next Proposition, which does find application, it has been inserted for completeness. The idea of right-handed axes is just our convention, and the idea of using backwards moving differences is avoided in this way.

In modern notation, we can write Prop. II as a set of difference equations based on the given form : $x = d-(1+\Delta_L)^n v$, where we set $(1+\Delta_R).(1+\Delta_L) = \mathrm{I}$; or $1+\Delta_R = (1+\Delta_L)^{-1}$. Thus, the action of the step or shift operator to the right E, or $(1+\Delta_R)$, increases the variable *x* in a right-going difference equation from x to $x+\underset{\cdot}{x}$, or by $\underset{\cdot}{x}$, and when acting on a set of differences defined for a variable *v* for a left going difference equation, it is equivalent to the left-going operator $(1+\Delta_L)^{-1}$; thus, these operators are formally the inverses of each other. Initially, the difference operators acts at some point *a* on the *z* axis for both *x* and *v*.

Thus, $x = d-(1+\Delta_L)^n v$ is the given difference equation, with *v* defined at *n* steps to the left of the starting point; and by acting on this equation with the operator 1 + Δ_R, for $(1+\Delta_R)x = (1+\Delta_R)[d-(1+\Delta_L)^n v] = (1+\Delta_L)^{-1}[d-(1+\Delta_L)^n v] = [d-(1+\Delta_L)^{n-1} v]$, and so on in an inductive manner for *r* = 2, ..., *n* : $(1+\Delta_R)^r x = [d-(1+\Delta_L)^{n-r} v]$ as in the text, then the variable *v* is defined at the steps *n* – 1, *n* – 2, etc, in turn, while the variable *x* is moved forwards by the corresponding steps. On taking the difference of the first two equations, we have $\Delta_R x = \underset{\cdot}{x} = -(1+\Delta_L)^{n-1}v + (1+\Delta_L)^n v = (1+\Delta_L)^{n-1}.\Delta_L v = (1+\Delta_L)^{n-1}\underset{\cdot}{v}$; again,

$\underset{\cdot}{x} + \underset{\cdot\cdot}{x} = -(1+\Delta_L)^{n-2}v + (1+\Delta_L)^{n-1}v = (1+\Delta_L)^{n-2}\underset{\cdot}{v} = (1+\Delta_L)^{n-2}\underset{\cdot}{v}$; and hence

$\underset{\cdot\cdot}{x} = (1+\Delta_L)^{n-2}\underset{\cdot}{v} - (1+\Delta_L)^{n-1}\underset{\cdot}{v} = -(1+\Delta_L)^{n-2}\underset{\cdot\cdot}{v}$, etc., as required. Thus, the whole machinery for moving a variable along an axis in steps has been established.]

DEMONSTRATION.

	A.	*B.*
1.	x...........................	$v + 3\underset{\cdot}{v} + 3\underset{\cdot\cdot}{v} + \underset{\cdot\cdot\cdot}{v}$
2.	$x + \underset{\cdot}{x}$.....................	$v + 2\underset{\cdot}{v} + \underset{\cdot\cdot}{v}$
3.	$x + 2\underset{\cdot}{x} + \underset{\cdot\cdot}{x}$...........	$v + \underset{\cdot}{v}$
4.	$x + 3\underset{\cdot}{x} + 3\underset{\cdot\cdot}{x} + \underset{\cdot\cdot\cdot}{x}$........	v

If for argument's sake $n = 3$, and in the tables A & B, four corresponding values of x & v are shown with orders increasing in opposite directions ; which are easily brought together by adding the increments. Then since from the Hypothesis the corresponding increments in both tables are always equal [and opposite], the sum of the corresponding values of these as you please ought to be x & v in these tables. Since if the sum that is given is d, then

$$x \ldots\ldots\ldots\ldots = d - v - 3\underset{\cdot}{v} - 3\underset{\cdot\cdot}{v} - \underset{\cdot\cdot\cdot}{v}$$

$$x + \underset{\cdot}{x} \ldots\ldots\ldots = d - v - 2\underset{\cdot}{v} - \underset{\cdot\cdot}{v}$$

$$x + 2\underset{\cdot}{x} + \underset{\cdot\cdot}{x} \ldots\ldots = d - v - \underset{\cdot}{v}$$

$$x + 3\underset{\cdot}{x} + 3\underset{\cdot\cdot}{x} + \underset{\cdot\cdot\cdot}{x} = d - v$$

Then by taking the differences of these :

$$\underset{\cdot}{x} \ldots\ldots\ldots\ldots = \underset{\cdot}{v} + 2\underset{\cdot\cdot}{v} + \underset{\cdot\cdot\cdot}{v}$$

$$\underset{\cdot}{x} + \underset{\cdot\cdot}{x} \ldots\ldots\ldots = \underset{\cdot\cdot}{v} + \underset{\cdot\cdot\cdot}{v}$$

$$\underset{\cdot}{x} + 2\underset{\cdot\cdot}{x} + \underset{\cdot\cdot\cdot}{x} \ldots = \underset{\cdot\cdot\cdot}{v}$$

And then by taking the differences of these equations :

$$\underset{\cdot\cdot}{x} \ldots\ldots = -\underset{\cdot\cdot}{v} - \underset{\cdot\cdot\cdot}{v}$$

$$\underset{\cdot\cdot}{x} + \underset{\cdot\cdot\cdot}{x} = -\underset{\cdot\cdot\cdot}{v}$$

[p. 7] Finally, by taking the differences of these equations : $\underset{\cdot\cdot\cdot}{x} = \underset{\cdot\cdot\cdot}{v}$.

Moreover, the values of $\underset{\cdot}{x}, \underset{\cdot\cdot}{x}, \underset{\cdot\cdot\cdot}{x},$ are themselves the same and ordered in the solution ; & and the argument is the same when n has some other value. Whereby for a certain x, by substituting the values of the increments in this way, the problem is correctly solved. Q.E.D.

COROLLORY.

On account of the vanishing of the increments in the solution of Fluxions, the task is easier by proving to be:

$x = d - v,\ \dot{x} = \dot{v},\ \ddot{x} = -\ddot{v},\ \dddot{x} = \dddot{v},\ \ddddot{x} = -\ddddot{v}$, & thus henceforth [as the higher increments vanish.]

SCHOLIUM.

Truly, incremental equations can be transformed with the help of any assumed equations that pleases you. Thus if you should make $x = vv$, by taking the increments (according to Prop. I) the equations become :

$\underset{\cdot}{x} = 2\underset{\cdot}{v}v + \underset{\cdot}{v}\underset{\cdot}{v}$; $\underset{\cdot\cdot}{x} = 2\underset{\cdot\cdot}{v}v + 4\underset{\cdot\cdot}{v}\underset{\cdot}{v} + \underset{\cdot\cdot}{v}\underset{\cdot\cdot}{v} + 2\underset{\cdot}{v}^2$; & thus henceforth ; hence the equation for $\underset{\cdot}{x}, \underset{\cdot\cdot}{x}, \underset{\cdot\cdot\cdot}{x},$ &c. will be transformed by substituting these values [into the incremental equation for x and its increments]. The same happens if $x = d - v$. But in this case v is a negative quantity, since $\dot{x} = -\dot{v}$; whereby the substituted quantity v is not in fact an increasing quantity in the transformed equation, but decreasing, with the nearby decrement to be taken away from the value of v present. Hence if you wish the equation to transform thus, in order that the quantities v are decreasing with increasing x, & yet in the transformed equation the increments are truly those of $\dot{v}$, as the former increments are indeed those of $\dot{x}$, then the procedure is according to this Proposition.

PROP. III. PROB. III.

A fluxional equation, in which there are two fluents z and x, of which z is the uniform fluent, can be transformed thus in order that x is the uniform fluent.
[A differential equation can be changed from one involving differentials of x with respect to z to one involving differentials of z with respect to x.]

The problem is solved for $\ddot{x}$, $\dddot{x}$, $\ddddot{x}$, etc., and by substituting the sequences of their values, truly $\ddot{x} = -\frac{\ddot{z}\dot{x}}{\dot{z}}$, $\dddot{x} = \frac{-\dot{z}^{z}\dot{x}+3\dot{z}^{2}\dot{x}}{\dot{z}^{2}}$, $\ddddot{x} = \frac{-\ddddot{z}\dot{z}^{3}\dot{x}+10\dddot{z}\ddot{z}\dot{z}\dot{x}-15\ddot{z}^{3}\dot{x}}{\dot{z}^{3}}$, $\&c.$ [by means of which the inversion of the equation is effected.]

DEMONSTRATION.

Let $A, B, C, D, E,$ &c. be quantities derived from x, for a given composition, and let the equations be $\dot{A} = B\dot{x}$, $\dot{B} = C\dot{x}$, $\dot{C} = D\dot{x}$, $\dot{D} = E\dot{x}$, and thus henceforth. Then if we put $z = A$, $\dot{z} = \dot{A}$, & (with z the uniform fluent) :

$0 = (B\ddot{x} + \dot{B}\dot{x} =)\quad B\ddot{x} + C\dot{x}^{2}$, $0 = (B\dddot{x} + \dot{B}\ddot{x} + 2C\ddot{x}\dot{x} + \dot{C}\dot{x}^{2} =)\quad B\dddot{x} + 3C\ddot{x}\dot{x} + D\dot{x}^{3}$,

$0 = B\ddddot{x} + 4C\dddot{x}\dot{x} + 3C\ddot{x}^{2} + 6D\dot{x}^{2}\ddot{x} + E\dot{x}^{4}$, &c. again (by Prop. I).

But when x is the uniform fluent, and z is the variable fluent, then the equations are :

$B = \frac{\dot{z}}{\dot{x}}$, $C = \frac{\ddot{z}}{\dot{x}^{2}}$, $D = \frac{\dddot{z}}{\dot{x}^{3}}$, $E = \frac{\ddddot{z}}{\dot{x}^{4}}$, &c. Thus with these values written for $B, C, D, E,$ &c., the values of $\ddot{x}$, $\dddot{x}$, $\ddddot{x}$, etc, are themselves found, as are shown above. From which hence in on substituting in the equation, then x is the uniform fluent rather than z. Q.E.D.

[Notes : In the initial setting, where z is the independent variable, according to the scheme proposed, we have :
$1 = Bdx/dz$; while $dB/dz = Cdx/dz$; $dC/dz = Ddx/dz$; $dD/dz = Edx$; etc.

Thus, according to Taylor, but written in modern notation, these relations give :
$0 = (dB/dz)(dx/dz) + B(d^2x/dz^2) = B(d^2x/dz^2) + C(dx/dz)^2$; while similarly,
$0 = B(d^3x/dz^3) + 3C(d^2x/dz^2)(dx/dz) + D(dx/dz)^3$;and
$0 = B(d^4x/dz^4) + 4C(d^3x/dz^3)(dx/dz) + 3C(d^2x/dz^2)^2 + 6D(d^2x/dz^2)(dx/dz)^2 + E(dx/dz)^4$; etc, for higher orders. The change from *dx/dz* to *dz/dx* is effected by extracting this factor from each differentiation.

In the standard notation of the calculus, there is a symbolic distinction in the notation of the numerator and denominator of the ratio; the upper line d, d^2, d^3, etc., refers to the order of the difference from which the limiting value of the ratio is derived, while the lower line refers to the number of times the difference has been divided by the constant difference before the limit is taken: hence the order number is placed in a different location on the numerator and denominator lines. This distinction is more apparent in Taylor's or Newton's notation.

Later, on p. 25, Taylor considers the fluxional or differential equation $\ddot{x}z + n\ddot{x}x - \dot{x} - \dot{x}^2 = 0.$ It is useful of us to consider this equation here as an example of his method of transforming fluxional or differential equations, as he makes use of the formulae for the derivatives to solve a fluxional equation involving an infinite series in a finite form. Following the original, we write $\dot{z} = 1$ or (with y written for $z + nx$), $\ddot{x} = \frac{\dot{x}+\dot{x}^2}{y}$ (*). Thus, it is an easy matter to find the second derivative. Then by continued differentiation :

$$\dddot{x} = \frac{(\ddot{x}+2\dot{x}\ddot{x})(z+nx)-(1+n\dot{x})(\dot{x}+\dot{x}^2)}{y^2} = \frac{(\dot{x}+\dot{x}^2)(1+2\dot{x})-(\dot{x}+\dot{x}^2)(1+n\dot{x})}{y^2} = (2-n)\frac{(\dot{x}^2+\dot{x}^3)}{y^2} = (2-n)\frac{\dot{x}}{y}\ddot{x},$$

on substitution of *. Thus, the third derivative is found $\dddot{x} = (2-n)\frac{\dot{x}}{y}\ddot{x}$, (**). Again,

$$\overset{4}{x} = (2-n)\left[\frac{\ddot{x}^2+\dot{x}\dddot{x}}{y} - \frac{\dot{x}\ddot{x}(1+n\dot{x})}{y^2}\right] = (2-n)\left(\frac{\ddot{x}^2+\dot{x}\dddot{x}}{y}\right) - \frac{\dddot{x}(1+n\dot{x})}{y} = \frac{\dot{x}}{y}\dddot{x}(2-2n) + \frac{(2-n)\ddot{x}^2-\dddot{x}}{y}]$$

To proceed further, we must express $\ddot{x}^2$ in terms of $\dot{x}$ and $\dddot{x}$: Thus, from *
$\ddot{x}^2 = \left(\frac{\dot{x}+\dot{x}^2}{y}\right)^2 = \frac{\dot{x}^2+\dot{x}^3+\dot{x}(\dot{x}^2+\dot{x}^3)}{y^2} = \frac{(1+\dot{x})\dddot{x}}{2-n}$. Hence, $\frac{(2-n)\ddot{x}^2}{y} = \frac{(1+\dot{x})\dddot{x}}{y}$, and the fourth derivative is found from :

$\overset{4}{x} = \frac{\dot{x}}{y}\dddot{x}(2-2n) + \frac{(2-n)\ddot{x}^2-\dddot{x}}{y} = \frac{\dot{x}}{y}\dddot{x}(2-2n) + \frac{\dot{x}}{y}\dddot{x} = \frac{\dot{x}}{y}\dddot{x}(3-2n)$, (***) as required; while the fifth derivative is found from :

$\overset{5}{x} = \frac{\ddot{x}}{y}\dddot{x}(3-2n) + \frac{\dot{x}}{y}\overset{4}{x}(3-2n) - \frac{(3-2n)\dot{x}\dddot{x}(1+n\dot{x})}{y^2}$; To proceed, we need two expressions that can be derived from these above : $\dot{x}\dddot{x} = \frac{\overset{4}{x}y}{3-2n}$ and $\ddot{x}\dddot{x} = \frac{\overset{4}{x}(1+\dot{x})}{3-2n}$; On substitution into the

expression for $\overset{5}{x}$, we obtain $\overset{5}{x} = \frac{\overset{4}{x}.\dot{x}}{y}(3-2n)$,(****) as required; etc. Thus, each derivative can be written as a product of $\dot{x}$ and the previous derivative. In which case, from this theorem, we can write a series of differential equations of increasing order. Thus for the original equation chosen as an example :
$\ddot{x}z + n\ddot{x}x - \dot{x} - \dot{x}^2 = 0$, or $(d^2x/dz^2)(z+nx) - dx/dz - (dx/dz)^2 = 0$: from
$\ddot{x} = \frac{\dot{x}+\dot{x}^2}{y}$, we have on transforming, $\ddot{x} = -\frac{\ddot{z}\dot{x}}{\dot{z}} = \frac{\dot{x}+\dot{x}^2}{y}$, *or* $\ddot{z}(z+nx) + \dot{z}(1+\dot{x}) = 0$, which we can write as $(d^2z/dx^2)(z+nx) + (1+dz/dx) = 0$, or $\ddot{z}z + nx\ddot{z} + \dot{x}\dot{z} + \dot{z}\dot{z} = 0$; and likewise for equations of higher orders. Note the change in sign of the last two terms. Taylor is later to show that this equation cannot be solved in general, but the inverse equation can be solved, which he does, as noted.]

[p. 9]

SCHOLIUM.

And in the same way it is possible to transform an equation, which involves more fluents *v*, *y*, &c. besides *z* and *x*; but no fluxion [derivative] of *v, y*, &c., is involved beyond the first order in the equation. Whereby if for an equation, if you want to transform it by this proposition, there are certain fluxions of *v, y*, &c. of the second and third orders, and consequently these are first to be eliminated with the help of the given equations, and then the transformation can proceed to be done by this Proposition.

PROP. IV. THEOR. I.

For a given [finite-difference] *equation, for which all the values are given for a variable z* [the independent variable for which $\dot{z} = 1$], *there are besides some increments of another variable quantity x involving the orders* $\underset{m}{x}$, $\underset{m+1}{x}$, $\underset{m+2}{x}$ *etc* ; *the first of these is* $\underset{m}{x}$, *and the last* $\underset{m+n}{x}$, (*also it is possible that there is a region in which both z, and all the increments* $\underset{m+1}{x}$, $\underset{m+2}{x}$, &. *between* $\underset{m}{x}$ & $\underset{m+n}{x}$ *need not be defined* ;) *and in addition there are m + n* [starting values or boundary] *conditions given for the observed m + n given values z, and with as many corresponding values taken between for the* [boundary] *values of x,* $\underset{\cdot}{x}$, $\underset{\cdot\cdot}{x}$, *&c .and so on indefinitely ; thus as no more than n values of x can be taken for* $\underset{m}{x}$ *or* $\underset{m+1}{x}$, *or of any lesser increments you wish to consider* [i. e. higher in order and smaller in magnitude], *or between the several increments* $\underset{m}{x}$, $\underset{m+1}{x}$, $\underset{m+2}{x}$, *& those below*; *neither more than n + 1 values can be taken of* $\underset{m-1}{x}$; *nor more than n + 2 values can be* *taken of* $\underset{m-2}{x}$ *; and thus henceforth; all the values of x itself are given from the given values of z.*

[This is a general theorem, and care has to be taken to understand what is being said. Essentially, there is a difference equation involving the fluents *z* and *x*, corresponding to the independent and dependent variables, that starts at some difference order *m* and ends at order *m* + *n,* where the starting order *m* can of course be zero or some small number, in which case the integral quantity *z* itself is involved, perhaps the most common circumstance. The extra conditions that Taylor considers are added increments that insure the correct starting or boundary conditions on the difference equations of the various orders. Thus, there will be a difference of the same order as a starting value for each order of *z*, and ending with a difference of order *m* + *n*. An example done a little later by Taylor helps to clarify matters. An obvious example for the time being, is the initial displacement and speed of a body undergoing some kind of accelerated motion, in which each boundary condition can be set to be the difference of the two initial values of the preceding order.]

[p. 10.]

DEMONSTRATION.

Throughout a given difference equation, $\underset{m+n}{x}$ is expressed in terms of *z* and by the increments $\underset{m}{x}$, $\underset{m+1}{x}$, $\underset{m+2}{x}$, &c. with the upper value $\underset{m+n}{x}$. Thus, (by Prop. 1.) the next closest increment $\underset{m+n+1}{x}$ is expressed by the same quantities ; then (by the same Proposition) the next increment $\underset{m+n+2}{x}$ can be expressed by the same quantities ; and with the operations continued indefinitely all the increments below $\underset{m+n}{x}$ [in a table of values where each row has the order increased by one] can be expressed in terms of the same quantities *z*, and by the increments $\underset{m}{x}$, $\underset{m+1}{x}$, $\underset{m+2}{x}$, &c., as far as $\underset{m+n}{x}$. Thus if *a*, *c,* and $\underset{\cdot}{c}$, $\underset{\cdot\cdot}{c}$ [are starting value of] *z* and *x*, $\underset{\cdot}{x}$, $\underset{\cdot\cdot}{x}$, &c. are indeed corresponding values, then all the values $\underset{m+n}{c}$, $\underset{m+n+1}{c}$, $\underset{m+n+2}{c}$, &c. are given in the indefinitely continued expressions through *a*, & $\underset{m}{c}$, $\underset{m+1}{c}$, $\underset{m+2}{c}$, &c. themselves above $\underset{m+n}{c}$. Thus through addition all the values of z itself $\underset{m}{x}$ are given in the table of values before being continued. And for $\overset{o}{z}$ & *x* with new values substituted (Per Prop. 2) in the same way all the values $\underset{m}{x}$ can be continued backwards in the table. Hence by continued addition and subtraction all the nearby increment above $\underset{m-1}{x}$, can be expressed by the same quantities, and through $\underset{m-1}{c}$: and then by the same method all the values of the increments still above $\underset{m-2}{x}$ can be expressed by the same quantities and by $\underset{m-2}{c}$. And thus by proceeding finally all the values of x are expressed by [p. 11.] *a* & by the terms $c, \underset{\cdot}{c}, \underset{\cdot\cdot}{c}$, &c. and by $\underset{m+2}{c}$ themselves above. But the number of terms before the term $\underset{m+2}{c}$ is $m + n$. Whereby all the terms $c, \underset{\cdot}{c}, \underset{\cdot\cdot}{c}$, &c. are determined by the number $m + n$ according to the conditions, and thus all x are given. Q.E.D.

But since the values of $\underset{m}{x}, \underset{m+1}{x}, \underset{m+2}{x}$, &c. themselves only include the terms $\underset{m}{c}, \underset{m+1}{c}, \underset{m+2}{c}$, &c., the number of which is just n ; hence there are no more than n conditions that can be applied to the values of the increments $\underset{m}{x}$ & of those below. Likewise since the values of $\underset{m-1}{x}$ include just as many terms $\underset{m-1}{c}, \underset{m}{c}, \underset{m+1}{c}$, &c. , the number of which is only $n + 1$, there cannot be more than $n + 1$ values applied to the increments $\underset{m-1}{x}$: And the argument is similar for the rest. Thus by this given theorem are correctly determined, and the conditions of these, for as many as two whole quantities, and the increments involved with these.

[p. 12.]

PROP. V. THEOR. II.

For two given difference equations, for which all the values of [the independent variable] *z are given, there are a number of increments involving the variables v & x as well as z. For these, the largest increments* [*i. e*. the finite differences of the smallest order] *taken from both the difference equations are* $\underset{p}{v}$ *and* $\underset{\pi}{x}$*, while the smallest increments* [*i. e*. those differences of the highest order] *in the one difference equation are* $\underset{p+a}{v}$ *and* $\underset{\pi+\alpha}{x}$*, and in the other equation they are* $\underset{p+b}{v}$ *and* $\underset{\pi+\beta}{x}$*; if m is made the maximum of the numbers a + β and α + b, then all the differences for v are given by the number of conditions m + p applied to the m + p values of the differences of z itself, and to just as many corresponding values of* $v, \underset{\cdot}{v}, \underset{\cdot\cdot}{v}$*, etc, and* $\underset{\pi}{x}, \underset{\pi+1}{x}, \underset{\pi+2}{x}$*, etc. ; also, all the differences for x are given by the number m + π conditions with regard to the m + π* [orders] *of z itself, and to the same corresponding values of* $x, \underset{\cdot}{x}, \underset{\cdot\cdot}{x}$*, etc. and of* $\underset{p}{v}, \underset{p+1}{v}, \underset{p+2}{v}$*, etc., themselves. Thus it is insured that it will not be possible to apply more than m* [boundary or starting] *conditions to the values of the initial increments of the smallest order* $\underset{p}{v}$ *and* $\underset{\pi}{x}$*, and for the rest of the conditions to be applied to the intermediate values of those smaller differences* $x, \underset{\cdot}{x}, \underset{\cdot\cdot}{x}$*, etc, and* $v, \underset{\cdot}{v}, \underset{\cdot\cdot}{v}$ *etc. themselves, provided that* $\underset{\pi}{x}$ *and* $\underset{p}{v}$ *follow the above rules for the values of the increments set out above for the* $\underset{m}{x}$ *in Proposition four; that is, in order that for the one smallest value given* *to x itself, there are two values required for x and* $\underset{\cdot}{x}$*, three values of* x, $\underset{\cdot}{x}, \underset{\cdot\cdot}{x}$, *etc*; *and thus similarly regarding* v, $\underset{\cdot}{v}, \underset{\cdot\cdot}{v}$, *etc. , and thus henceforth.*

[p. 13.]

[Note: The lowest orders are found from both difference equations by inspection, which must be reducible to similar forms on taking repeated differences: $\underset{p}{v}$ *and* $\underset{\pi}{x}$. Additional finite differences are formed from both the first and the second finite difference equations, taking each maximum order of difference of *x* and *v* up to the level of other equation acted on in the same way. Thus, $\underset{p+a}{v}$ and $\underset{\pi+\alpha}{x}$ are the given maximum orders in I, and $\underset{p+b}{v}$ *and* $\underset{\pi+\beta}{x}$ in II.

The differences are augmented in I to $\underset{p+a+\beta}{v}$ and $\underset{\pi+\alpha+\beta}{x}$, while those in II are increased to orders $\underset{p+b+\alpha}{v}$ and $\underset{\pi+\beta+\alpha}{x}$; the maximum of these two operations $a+\beta$ and $\alpha+b$ for the final order of difference of *v* is taken as *m,* while the *x* value can be substituted into the other equation in terms of other *v* increments. Thus, there are $m+p$ conditions applied to *v*, and $m+p+\pi$ conditions applied to *x*.]

DEMONSTRATION.

For according to the given equations, and thence from new equations derived according to Prop. 1, by eliminating *v* with its increments, a difference equation is given besides *z* involving just as many increments of *x* itself, the greatest of which is $\underset{\pi}{x}$, and the smallest is $\underset{\pi+m}{x}$ [thus, a difference of lowest order in one variable can be taken from one difference equation and inserted into the other]. And an equation is given as found before above in terms of *z* and the increments of *x* itself and from the values $\underset{\pi}{x}$ and for higher differences, involving terms such as $\underset{p}{v}$. Hence, $\underset{p}{v}$ is given by an expression through *z* and the increments of $\underset{\pi}{x}$ and higher orders. But if *a*, & $c, \underset{\cdot}{c}, \underset{\cdot\cdot}{c}$, *etc*. are the values of these, and *z*, & $x, \underset{\cdot}{x}, \underset{\cdot\cdot}{x}$, *etc*., are certain corresponding values, all the increments of $\underset{\pi}{x}$ and for higher values are given expressed by a number *m* of these difference equations : *a*, and $\underset{\pi}{c}$, $\underset{\pi+1}{c}$, $\underset{\pi+2}{c}$, *etc*., according to Prop. 4. Whereby also all the $\underset{p}{v}$ are expressed by the same quantities. Thus if $d, \underset{\cdot}{d}, \underset{\cdot\cdot}{d}$, etc. are values of $v, \underset{\cdot}{v}, \underset{\cdot\cdot}{v}$, *etc*. corresponding to the value *a* of *z*, and the series $d, \underset{\cdot}{d}, \underset{\cdot\cdot}{d}$, etc. is continued as far as to include $\underset{p-1}{d}$, in order that the number of terms is *p*, all *v* are given expressed by the quantities $d, \underset{\cdot}{d}, \underset{\cdot\cdot}{d}$, *etc*. and $\underset{\pi}{c}$, $\underset{\pi+1}{c}$, $\underset{\pi+2}{c}$; *etc*. of which the number of values is $m+p$. Moreover all the values of *x* can be expressed by the quantities $c, \underset{\cdot}{c}, \underset{\cdot\cdot}{c}$, *etc.* , the number of which is $m + \pi$. Whereby with all the values $c, \underset{\cdot}{c}, \underset{\cdot\cdot}{c}$, *etc*. determined by the number of conditions $m + \pi$ with regard to the values of x and of its increments, then all of $d, \underset{\cdot}{d}, \underset{\cdot\cdot}{d}$, *etc*., can be determined by other conditions with regard to the number p for the values $v, \underset{\cdot}{v}, \underset{\cdot\cdot}{v}$, *etc*., [p. 13.] or by the determination of all $d, \underset{\cdot}{d}, \underset{\cdot\cdot}{d}$, *etc*, and from some terms $\underset{\pi}{c}, \underset{\pi+1}{c}, \underset{\pi+2}{c}$, *etc.,* by conditions with respect to the values $v, \underset{\cdot}{v}, \underset{\cdot\cdot}{v}$, *etc*. of the rest of the terms $c, \underset{\cdot}{c}, \underset{\cdot\cdot}{c}$, *etc*. that are determined by conditions regarding the values $x, \underset{\cdot}{x}, \underset{\cdot\cdot}{x}$, *etc*. Hence by the $m + p$ conditions all the v terms are given, and by the $m + \pi$ conditions, all the x terms are given, (Q.E.D.) and by the entire $m + p + \pi$ set of all conditions is given, as for v, and so for x. Moreover in this manner the conditions are to be applied to the values of v and x, and to the higher increments of $\underset{p}{v}$ and $\underset{\pi}{x}$, in agreement with Proposition four.

SCHOLIUM.

In the same manner through the elimination of variables it is also possible to precede to find the condition, for which it is possible to form the boundary for three or four or more incremental equations involving three or four or more variables besides z, all the values of which are given.

Let $z^2 \underset{\cdot\cdot}{x} - z \underset{4}{x} + b^3 = 0$ be the equation. In this equation $\underset{\cdot\cdot}{x}$ is the same as $\underset{m}{x}$ of Proposition Four, and likewise $\underset{4}{x}$ is the same as $\underset{m+\pi}{x}$; hence $m = 2$, and $n = 2$. Whereby the four conditions give all the values of x, for all the given values of z. And since $\underset{\cdot\cdot}{x}$ is the first of all the differences $x, \underset{\cdot}{x}, \underset{\cdot\cdot}{x}$; *etc.* which occur in the equation, at least two conditions are applied to the values of x and $\underset{\cdot}{x}$ themselves; thus , as either one condition is applied to the value x, and the other to the value of $\underset{\cdot}{x}$*,* or each is applied to the value at π. Moreover the rest of the conditions can be applied as you please to the values of all $x, \underset{\cdot}{x}, \underset{\cdot\cdot}{x}$, *etc*. indefinitely; thus, as either one condition is applied to one of these terms, and the other to the other term, or also all the conditions can be applied to the different values of the same term.

[p. 15.]

There are two equations $\underset{\cdot\cdot}{x} - \underset{\cdot}{v} + \underset{\cdot\cdot\cdot}{v} = 0$, and $\underset{\cdot\cdot\cdot}{x} z - \underset{\cdot\cdot}{v} = 0$. For these equations, the terms of the proposition are $p = 1$ [from $\underset{\cdot}{v}$ in I], $\pi = 2$ [from $\underset{\cdot\cdot}{x}$ in I], $a = 2$ [from $\underset{\cdot}{v} \rightarrow \underset{\cdot\cdot\cdot}{v}$ in I], $\alpha = 0$ [from $\underset{\cdot\cdot}{x}$ in I, as there is higher difference], $b = 1$ [from $\underset{\cdot}{v} \rightarrow \underset{\cdot\cdot}{v}$ in II], $\beta = 1$ [from $\underset{\cdot\cdot\cdot}{x}$ in the second equation]. Thus $a + \beta = 3$ and $\alpha + b = 1$, hence :
$m = 3$ $(= a + \beta)$ and $m + p = 4, m + \pi = 5$, and $m + p + \pi = 6$. Hence all the values of v are given by four conditions, all the values of x are given by five conditions, and all the

values both of x and v are given by six conditions, with the two values of x and $\underset{\cdot}{x}$, with the remaining three values applied in some manner to the values of the increments $\underset{\cdot}{v}, \underset{\cdot\cdot}{x}$, and of those smaller increments.
[One can of course adopt the method of the proof, and take successive differences until a common difference is found, at which time the common value is substituted in the other equation, etc.]

There are two fluxional equations $\dot{v}^2 - \dot{x}^2 - \dot{z}^2 = 0$, and $\dot{z}\,\dddot{v} - z\,\dot{x}^2\,\ddot{x} - \overset{4}{x}x = 0$. Then according to method of this Proposition:

$[\text{common}:]\; \underset{p}{v} = \dot{v},\quad \underset{\pi}{x} = x; [1st:]\; \underset{p+a}{v} = \dot{v},\quad \underset{\pi+\alpha}{x} = \dot{x}; [2nd:]\; \underset{p+b}{v} = \dddot{v},\quad \&\quad \underset{\pi+\beta}{x} = \overset{4}{x};$

thus $p = 1, \pi = 0, a = 0, \alpha = 1, b = 2, \beta = 4, a + \beta = 4, \alpha + b = 3$; and hence $m = 4, m + p = 5, m + \pi = 4$, and $m + p + \pi = 5$. Hence all the terms v and x are given entirely for all z by five conditions ; of which the smallest one relates to a value of v, and the rest can be applied to the values of v, x, and of their fluxions. Hence if two curves are to be described, the ordinates of which are v and x, and with the common abscissa z, the curve described with this ordinate v is given by five points; or by four points and with the fifth point cutting the ordinate axis at a given angle; or is to generated by a curve that passes through one given point, and four ordinates with the position given cut either in points or in given angles, or in their extremities have a given curvature, or with a certain indication of the curvature depending on the values of the fluxions $\ddot{v}, \dddot{v}$, and lower orders; all the points of each kind of curve are given. Or also if the other curve is described, of which the ordinate is x; thus, in order that it cuts four given ordinates, or in other cases it cuts in points, others in

[p. 16.]

given angles; or in given curvatures it may have of these for the boundaries, or it may have some other kind of curvature depending on the third, fourth, and higher order differences; in addition, all the points of each curve are given from the one value of v. And in the same manner one condition can be given regarding the initial value of v, and the rest of the conditions can be chosen freely between the values of the ordinates x and v, and of their fluxions [*i. e.* derivatives].

Again in these cases there can be two or more values equal to each other for a given z, or (which is the same in geometry) two or more values taken from the positions of the given ordinates can coincide. [*i. e.* the curves can cross each other] But this depends on certain conditions arising from considerations of the nature of any proposed equations.

Thus for the two proposed equations : $xv = z$ and $\dot{x}z + \dddot{x} - \ddot{v}\overset{4}{v} = 0$, from the present proposition there are four conditions which can be applied to the values of x and v and their fluxions as you wish. But on account of the first equation $xv = z$, it is not possible for the ordinates to coincide, when the conditions in the first equation are applied regarding each of v and x; neither on account of the second equation can all four ordinates coincide, with regard to the conditions applied to all the values $\dot{x}, \dddot{x}, \ddot{v}, \overset{4}{v}$; for each x and v

given together, and z determined by the first equation; likewise for all $\dot{x}, \dddot{x}, \ddot{v}, \overset{4}{v}$ given at the same time, and z determined by the second equation; each contrary to the hypothesis. In the same way the fluxion of the first equation can be taken (where for $\dot{z}$ I write 1, as previously) and you will find that $\dot{x}v + \dot{v}x = 1$; hence also it is agreed that it is not possible that all the ordinates coincide, when the conditions are applied regarding the values of all $x, \dot{x}, v, \dot{v}$. And again by taking the fluxions of the proposed equations perhaps all the boundaries of the conditions are given in this way.

For these, just as in equations involving variables, so in incremental equations, certain variable quantities are liable to be defined within certain limits. Let $x^2 - a\ddot{x} + z = 0$ be an example of a fluxional equation.

[p. 17.]

Hence $x = \sqrt{a\ddot{x} - z}$; thus where it is impossible for $a\ddot{x} < z$, (and thus for all its fluxions). The second fluxion of the same equation is $2\dot{x}^2 + 2x\ddot{x} - a\overset{4}{x} = 0$; thus $x = \sqrt{\tfrac{1}{2}a\overset{4}{x} - x\ddot{x}}$;

hence always $x\ddot{x} < \tfrac{1}{2}a\overset{4}{x}$ [The original has a '>' sign]. And in the same manner from the further fluxional equations proposed, perhaps you will find elsewhere the limits of the variables.

LEMMA 1.

In a fluxional equation, many increments of the same variable quantity do not involve a general rule that can be defined, by which the extent of the increase can given, in the variable equation defining the relation to the other variable quantities.

In the fluxional equation $\ddot{x}x + \dot{x}\dot{x} + n\ddot{x}z + \dot{x}\dot{z} = 0$; the quantities are only affected once, and when $n = 2$, x is found from a given z by a quadratic equation; but however if $n = 3$, x is only given by a cubic equation; if $n = 4$, x is given by a bi-quadratic [fourth power] equation; if $n = \frac{1}{32}$, x is given by an equation of the fifth dimension : then if the coefficients of the rest of the terms are made general, so that the equation becomes $\ddot{x}x + m\dot{x}^2 + n\ddot{x}z + p\dot{x}\dot{z} = 0$; I doubt whether it is possible to find the dimensions of the equation sought by any known rule, if indeed it is possible to find x in terms of z by an equation with a finite number of terms.

[This is the equation, modified by making the last two terms negative, that we have mentioned in Prop. III, to which Taylor returns in the Scholium of Prop. VIII on page 25, to effect a solution, by interchanging x and z according to the rules of Prop. III, as we have shown there.]

Let another equation be : $4x^3\dot{z}^2 - 4x^2\dot{z}^2 = (1+x^2)^2\dot{x}^2$. In this equation x rises to the third degree, and $\dot{x}$ rises to the second degree; but yet x is given from z by an equation of the second dimension, the root of which is $x = \frac{1+z^2}{\overline{a+\sqrt{1-a^2z}}|^2}$. Truly with the coefficients changed I hardly know whether it is possible to get x from z by a finite equation. [Thus Taylor bemoans the sad truth that there are no analytical solutions for many differential equations. However, he now proposes possible ways of solving differential equations.]

[p. 18.]

PROP. VI. PROB. IV.

For many given [difference] *equations, commonly involving some integral and incremental quantities, and for which there are several variables x, v, y, &c., which are to be referred to the* [principal] *variable z, all the values of which are given; these equations are to be solved, to find equations between the variables that are free from increments, which are to be found with undetermined coefficients, and these can then be adapted for the conditions of the problem.*

SOLUTION:

In attempting to solve some proposed difference equation, either by multiplying or dividing it somehow, some other known increment of some quantity will be recognised. If this is the case, then instead of the proposed equation, substitute that known quantity that is equal to the first increment of this quantity present, or the second, or even some other that is in fact equal to zero ; on account of which the proposed equation, or some multiple or fraction of it shall be the first increment, or the second, or some other known quantity of this known variable. Likewise in this way for all the proposed equations, if only equations involving integral quantities are found, then the solution is given by a finite number of terms; which can be accommodated to the conditions of the problem by adjusting the indeterminate coefficients in the assumed quantities. Q.E.F.

But if by this arrangement the problem cannot be solved, then by the elimination of variables (with the help of the given equations, and of new equations thus derived by Prop. 1, if it is necessary) in which case all the new equations are in terms of the same number of variables *x, v, y* etc., as well as *z*. One of these equations only involves *x* with its increments (truly besides *z*,) and the rest involve only two variables, *x* and *v*, *x* and *y*, etc. (one of which is always *x*) with their increments (always with *z* understood in the equations). From the equation involving only *x* and its increments, the value of *x* itself can be found, expressed as a power of *z*, with the help of some of the following propositions. Then with the help of these equations, *x* and its increments can be

eliminated [p. 19.] from the equations involving only x and v, x and y, etc., with their increments; and through the equations in this manner the resulting values of v, y, etc, are sought, to be expressed in terms of the powers of z, in the same way that the value of x itself was found. And by this arrangement all the values of x, v, y, etc. are to be expressed by the powers of z. Which values are to be adapted to the conditions of the problem with the help of the undetermined coefficients. And if of these values some are produced in a finite number of terms, or in series that can be reduced to a finite expressions, truly the solution for this part is given by the mathematics in a finite number of terms. But where it is not possible for the series to be reduced to a finite number of terms, a mechanical solution is required, and a series is used for finding the approximate roots of x, or v, or y, etc.
[Note : In all of these, the word 'integral' indicates that the variable does not involve increments or differences, but is whole ; it does not mean integral in the modern sense of being an integral, at least not directly.]

SCHOLIUM.

The reduction of many proposed equations to integral equations in this kind of solution depends on the skill of analyst in finding increments (by Prop. 1.) to be acted on. Whereby to this end, there may be a useful quantity found from various ways of composition that are possible, and with the help of the other increments the integral can be found. The following table is of this kind.

Increments	Integrals.
1. $a\underset{\bullet}{z}+b\underset{\bullet}{z}z+c\underset{\bullet}{z}z\overset{\backslash}{z}+d\underset{\bullet}{z}z\overset{\backslash}{z}\overset{\backslash\backslash}{z}+\&c.$	$A+\frac{1}{1}az+\frac{1}{2}bz\overset{\backslash}{z}+\frac{1}{3}cz\overset{\backslash}{z}\overset{\backslash\backslash}{z}+\frac{1}{4}dz\overset{\backslash}{z}\overset{\backslash\backslash}{z}\overset{\backslash\backslash\backslash}{z}+\&c.$
2. $\frac{a\dot{z}}{z\underset{\backslash}{z}}+\frac{b\dot{z}}{z\underset{\backslash}{z}\underset{\backslash\backslash}{z}}+\frac{c\dot{z}}{z\underset{\backslash}{z}\underset{\backslash\backslash}{z}\underset{\backslash\backslash\backslash}{z}}+\frac{d\dot{z}}{z\underset{\backslash}{z}\underset{\backslash\backslash}{z}\underset{\backslash\backslash\backslash}{z}\underset{\backslash\backslash\backslash\backslash}{z}}+\&c.$	$A-\frac{a}{1z}-\frac{b}{2z\underset{\backslash}{z}}-\frac{c}{3z\underset{\backslash}{z}\underset{\backslash\backslash}{z}}-\frac{d}{4z\underset{\backslash}{z}\underset{\backslash\backslash}{z}\underset{\backslash\backslash\backslash}{z}}-\&c.$

[p. 20.]

Fluxions.	Fluents.
1. $\dot{z}\,z^{\pm n}$	$\frac{1}{\pm n+1}z^{\pm n+1}+\text{A}.$
2. $(\vartheta\dot{z}x+\lambda\dot{x}z)\times z^{\vartheta-1}x^{\lambda-1}$	$\text{A}+z^{\vartheta}x^{\lambda}$
3. $(\vartheta\dot{z}xv+\lambda\dot{x}zv+\mu\dot{v}zx)\times z^{\vartheta-1}x^{\lambda-1}v^{\mu-1}$	$\text{A}+z^{\vartheta}x^{\lambda}v^{\mu}$
4. $(\vartheta\dot{z}xvy+\lambda\dot{x}zvy+\mu\dot{v}zxy+\pi\dot{y}zxv)\times z^{\vartheta-1}x^{\lambda-1}v^{\mu-1}y^{\pi-1}$	$\text{A}+z^{\vartheta}x^{\lambda}v^{\mu}y^{\pi}.$

[Note : In the first table, the results 1. for the increments and the integral quantities follow in an inductive formal manner. We may illustrate the notation in the finite difference case by considering the difference of the quantities $\frac{1}{2}bz\overset{\backslash}{z}$: $\frac{1}{2}bz(z+\underset{\cdot}{z})-\frac{1}{2}b(z-\underset{\cdot}{z})z=b\underset{\cdot}{z}z$;and $\frac{1}{3}cz\overset{\backslash}{z}\overset{\backslash\backslash}{z}$: $\frac{1}{3}cz\overset{\backslash}{z}\overset{\backslash\backslash}{z}=\frac{1}{3}cz(z+\underset{\cdot}{z})(z+2\underset{\cdot}{z})-\frac{1}{3}c(z-\underset{\cdot}{z})z(z+\underset{\cdot}{z})=c\underset{\cdot}{z}z(z+\underset{\cdot}{z})=c\underset{\cdot}{z}z\overset{\backslash}{z}$;etc. The results 2. follow in the same manner. The results 3. and 4. are of course the familiar ones for powers and their derivatives.]

By comparing expressions with example of this kind certain problems can be solved. If the fluxional equation is given: $\dddot{x}\dot{x}x-\ddot{x}^2x-2\dot{x}^2\ddot{x}=0.$ By comparing this equation with the fluxions of the third order constructed, $\vartheta\dot{z}xv+\lambda\dot{x}zv+\mu\dot{v}zx$, then it is found that $\vartheta=1,\ \lambda=-1,\ \mu=-2,$ with $\ddot{x},\dot{x},x$ entered in this equation in place of *z, x, v* in that fluxion. Then the fluent $\ddot{x}\dot{x}^{-1}.x^{-2}$ is equal to the given quantity, (since the fluxion of this is equal to zero.) And, for a given quantity, $\ddot{x}\dot{x}^{-1}.x^{-2}=a\dot{z}$. (truly for the complete orders of the fluxions by the degrees of the given $\dot{z}$.) Thus $\ddot{x}=a\dot{z}\dot{x}x^2$; then again by returning to the table of the fluents (to the example $\dot{z}z^{\pm n}$ from the fluxions) it follows that $\dot{x}=\frac{a\dot{z}x^3}{3}+b\dot{z}$. From which the equation of the fluxes of the third order are reduced to a fluxional equation of only the first order.

In the same manner it can be shown that the equation $\ddot{x}^2=x^3\dot{z}^4$, or (where I write 1 for $\dot{z}$), $\ddot{x}^2=x^3$ can be returned to the higher order. [p. 21] For by extracting the root it becomes $\ddot{x}=x^{\frac{3}{2}}$. Multiply the equation by $\dot{x}$, and it is $\ddot{x}\dot{x}=\dot{x}x^{\frac{3}{2}}$, then by taking the fluents it is $\frac{\dot{x}}{2}=\frac{2}{5}x^{\frac{5}{2}}+a.$

Another equation is $\ddot{x}^2=\dot{x}x$: then by extracting the root it is $\ddot{x}=\dot{x}^{\frac{1}{2}}x^{\frac{1}{2}}$, that is $\ddot{x}\dot{x}^{\frac{1}{2}}=\dot{x}x^{\frac{1}{2}}$, then by taking the fluents it is $\frac{2}{3}\dot{x}^{\frac{3}{2}}=\frac{2}{3}x^{\frac{3}{2}}+a,$ or if it pleases, $\dot{x}^{\frac{3}{2}}=x^{\frac{3}{2}}+a.$ And in this manner by multiplications, divisions, and by the extractions of roots, the expressions are to be reduced to known forms of fluxes, or the fluents themselves are to be found, or the equations are reduced to fluxions of lower order.

LECTORI.

In Methodo hac Incrementorum quantitates considero ut Incrementis auctas vel Decrementis imminutae, & ex datis relationis Integralium quaero relationes Incrementorum, atque vicissim ex datis relationibus Incrementorum quaero quantitates ipsas Integrales. Horum usus in rebus Mathematicis satis late patet; sed in eo maxime elucet, quod hinc facile deriventur omnes proprietates Fluxionum. Clarissimus Dominus Newtonus quantitates Mathematicas considerando ut motu perpetuo descriptas, per Methodum Fluxionum ex rationibus primis Incrementorum nascentium quaerit rationes velocitatum quibus magnitudines describuntur, & vicissim ex velocitatibus hisce (quas Fluxiones quantitatum vocat) quaerit magnitudines quantitatum ipsarum descriptarum. Hoc idem, sed minus generaliter, secere alii (ut Veteres in Methodo exhaustionum, Cavellerius & Wallisius in Methodis summatoriis.) Veteres investigando magnitudines figurarum inscripserunt & circumscripserunt figuras ex partibus finitis & cognitis constantes, & partium istarum numerum auxerunt & magnitudinem minuerunt, donec differentia inter earum summam & figuram quaesitum esset minor quavis data. Caballeius & Recentiores contemplarunt partes istas ut in infinitum diminutas. Sed hi omnes, contemplando eneses quantitatum per additiones partium, non satis consuluerunt severae isti αχρβεια Geometrarum. Partes enim, ut Methodus sit accurata, deberent esse primae nascentes; at nullae sunt ejusmodi partes in rerum Natura, sunt tantum rationes primae partium nascentium.; Ergo Newtonus missis partium magnitudinibus, missis & earum summis, rationes ultimas partium evanescentium, & primas nascentium introduxit, & in his rationibus Analysin suam fundavit. Sumptis itaque rationibus primis Incrementorum nascentium, vel ultimis evanescentium, accommodantur omnes Conclusiones Methodi Incrementorum ad Methodum Fluxionum, Incrementis jam evanescentibus, & Integralibus in fluentes conversis. Et hoc pacto vitatur omnis consideratio quantitatum infinite (seu, ut aliqui loqui ament, indefinite) parvarum. Nam in Methodo Fluxionum, ut Conclusiones sint verae & omnino accuratae, partes seu incrementa concipienda sunt, non ut perexigua, seu infinite parva, sed ut revera nulla: Rationes enim primae non sunt nisi in ipso momento ubi quantitates nasci incipiunt; ubi semel nascuntur jam desinunt esse primae. Similiter & rationes ultimae non sunt, nisi ubi quantitates jam evanescunt & fiunt nullae. Facilioris tamen conceptus gratia possunt pro Fluxionibus sumi augmenta illa nascentia, quae Newtonus momenta vocat, atque designat litera o Fluxionibus apposita. Et in hac modo concipiendi facilius cernitur relatio inter Methodum hanc Incrementorum & Methodum Fluxionum. Quapropter etiam in Propositionibus nonnullis generalibus spectantibus ad Incrementa quavis in genere, vel finitae magnitudenis, vel infinite parva, exempla damus in Fluxionibus, vice tamen Fluxionum sumendo momenta.

Methodus Incrementorum.

Pars Prima.

Ubi traduntur Praecepta, cum Methodi Incrementorum in genere, tum Methodi Fluxionum.

INTRODUCTIO.

Quantitates indeterminatas in his considero ut Incrementis perpetuo auctas, vel Decrentis diminutas. Indeterminatas ipsas Integrales designo literis *z, x, v*, &c. earumque Incrementa, seu partes mox addendas designo iisdem literis a parte inferiori punctatis $\underset{\cdot}{z}, \underset{\cdot}{x}, \underset{\cdot}{v}$, &c. Quorum Incrementorum Incrementa, seu Integralium ipsarum Incrementa designo iisdem literis bis punctis $\underset{\cdot\cdot}{z}, \underset{\cdot\cdot}{x}, \underset{\cdot\cdot}{v}$, &c. Quorum Incrementorum Incrementa, seu Integralium Incrementa secunda designo iisdem literis punctis $\underset{\because}{z}, \underset{\because}{x}, \underset{\because}{v}$, & sic porro. Quin etiam majoris generalitatis gratia, vice punctorum nonnunquam scribo characteres punctorum numeros designantes : Sic si n sit 3, per $\underset{n}{x}$, vel $\underset{3}{x}$, designatur $\underset{\because}{x}$; si n sit *o,* per $\underset{n}{x}$, vel $\underset{o}{x}$ designatur ipsa Integralis x ; si n sit -1, per $\underset{n}{x}$, seu $\underset{-1}{x}$ designatur quantitas cujus Incrementorum primum est x ; & sic de caeteris. Saepe etiam in hoc Tractatu quantitatis ejusdem variabilis valores aliquot successivos designo per eandem literam lineolis insignitam; nempe praesentem valorem designando per literam simplicem, praecedentes per accentus graves suprascriptos, & subsequentes per lineolas subscriptas. Sic exempli gratia sunt $\overset{\backslash}{x}, \overset{\backslash\backslash}{x}, x, \underset{\backslash}{x}, \underset{\backslash\backslash}{x}$, ejusdem quantitas valores quinque successivi, quorum est x valor praesents, sunt $\overset{\backslash}{x}$ & $\overset{\backslash\backslash}{x}$ valores praecedens, atque $\underset{\backslash}{x}$ & $\underset{\backslash\backslash}{x}$ valores subsequentes.

II. Fluxiones, quae sunt in ratione prima Incrementorum nascentium, vel ultima evanescentium, designantur punctis indicibus Incrementorum ad literarum partes superiores transpositis : Sic est $\dot{x}$ Fluxio prima ipsius x ; $\ddot{x}$ est ejusdem Fluxio secundo secunda; $\overset{\therefore}{x}$ Fluxio tertia; & sic porro. Fluentes etiam nonnumquam designantur per lineolas (similes accentus acuti) literis suprascriptas : Sic $\overset{/}{x}$ designat fluentem ipsius x, seu quantitatem cujus Fluxio prima est x ; $\overset{//}{x}$ designat fluentem ipsius x, seu quantitatem cujus Fluxio secunda est x ; & sic porro. Et hae lineolae in indicibus fluentum vim habent punctorum (ut ita dicam) negativorum in indicibus Fluxionum : Sic si sit $n = 2$, &

$\overset{n}{x}$ designet $\ddot{x}$, mutato signo designantur $\overset{//}{x}$ per $\overset{-n}{x}$. Porro fluentes quantitatum compositarum designantur nonnumquam per quantitates ipsas parallelogrammis inclusas; sic designat xz^2 fluentem ipsius xz^2.

PROP. 1. PROB. I.

Data Aequatione quantitates variabiles involvente invenire Incrementa.

In Aequatione proposita vice quantitatis cujusvis variabilis scribe eandem quantitatem proprio Incremento auctam, & resultabit Aequatio nova; unde ablata Aequatione priori, residuum erit Aequatio, per quam dabitur relatio Incrementorium.

Exempli gratia sit Aequatio $x^3 - xv^2 + a^2z - b^3 = 0$, ubi *a* & *b* sunt quantitates determinatae & immutabiles. Itaque pro *x, v,* & *z* scriptis $x + \underset{\cdot}{x}, v + \underset{\cdot}{v}, \& \; z + \underset{\cdot}{z}$, prodit

Aequatio nova

$x^3 + 3\underset{\cdot}{x}x^2 + 3\underset{\cdot}{x}^2x + \underset{\cdot}{x}^3 - xv^2 - \underset{\cdot}{x}v^2 - 2x\underset{\cdot}{v} - x\underset{\cdot}{v}^2 - 2\underset{\cdot}{x}v\underset{\cdot}{v} - \underset{\cdot}{x}\underset{\cdot}{v}^2 + a^2z + a^2\underset{\cdot}{z} - b^3 = 0$; unde subducta Aequatione priori, residuum $3\underset{\cdot}{x}x^2 + 3\underset{\cdot}{x}^2x + \underset{\cdot}{x}^3 - \underset{\cdot}{x}v^2 - 2x\underset{\cdot}{v} - x\underset{\cdot}{v}^2 - 2\underset{\cdot}{x}v\underset{\cdot}{v} - \underset{\cdot}{x}\underset{\cdot}{v}^2 + a^2\underset{\cdot}{z} = 0$; sit Aequatio, cujus ope dantur relationes Incrementorum.

In hac Solutione, si pro Incrementis nascentibus scribatur nihil, & pro earum rationibus primis substituantur rationes Fluxionum, eo pacto dabuntur relationes Fluxionum. Et potest operatio simul & semel perfici, ab initio neglectis terminis, cum Aequationis propositae, tum & ob Incrementa nascentia evanescentibus, omnino ut docetur in Regula *Newtoniana*, quae haec est ;

"Multiplicetur omnis Aequationis terminus per indicem
"dignitatis quantitatis cujusque fluentis quam involvit, &
"in singulis multiplicationibus mutetur dignitatis latus
"in Fluxionem suam; & aggregatum factorum sub pro-
"priis signis erit Aequatio nova, per quam definitur
"relatio Fluxionum.

EXPLICATIO.

"Sunto *a, b, c, d*, &c. quantitates determinatae & immutabiles, & proponatur Aequatio
"quaevis quantitates fluentes *z, y, x*, &c. involvens, uti $x^3 - xy^2 + a^2z - b^3 = 0$.
"Multiplicentur termini primo per indices dignitatum *x*, & in singulis multiplicationibus
"pro dignitatis latere, seu x unius dimensionis, scribatur $\dot{x}$, & summa factorum
"erit $3\dot{x}x^2 - \dot{x}y^2$. Idem fiat in *y*, & prodibit $-2x\dot{y}y$. Idem fiat in z, & prodibit $a2\dot{z}$.
"Ponatur summa factorum aequalis nihilo, & habebitur Aequatio
"$3\dot{x}x^2 - \dot{x}y^2 - 2x\dot{y}y + a^2z = 0$.

" Ad eundem modum si Aequatio esset $x^3 - xy^2 + a^2\sqrt{ax-y^2} - b^3 = 0$ produceret

" $3\dot{x}x^2 - \dot{x}y^2 - 2x\dot{y}y + a^2\dot{\overline{\sqrt{ax-y^2}}} = 0$. Ubi si Fluxion $\sqrt{ax-y^2}$ tollere velis, pone $\sqrt{ax-y^2} = z,$ & erit $ax - y^2 = z^2$, & (per hanc Prop.) $a\dot{x} - 2\dot{y}y = 2\dot{z}z$, seu $\frac{a\dot{x}-2\dot{y}y}{2z} = \dot{z}$, hoc est $\frac{a\dot{x}-2\dot{y}y}{2\sqrt{ax-y^2}} = \sqrt{ax-y^2}$. Et unde $3\dot{x}x^2 - \dot{x}y^2 - 2x\dot{y}y + \frac{a^2x-2a^2\dot{y}y}{2\sqrt{ax-y^2}} = 0.$

Per operationem repetitam pergitur ad Incrementa, ut & ad Fluxions secundas, tertias, & sequentes. Sit Aequatio $xz - av = 0$. Tum per operationem primam erit $\underset{\cdot}{x}z + x\underset{\cdot}{z} + \underset{\cdot}{x}\underset{\cdot}{z} - a\underset{\cdot}{v} = 0$. In hac Aequatione pro $x, \underset{\cdot}{x}, z, \underset{\cdot}{z}, v, \underset{\cdot}{v}$ scriptis $x+\underset{\cdot}{x}, \underset{\cdot}{x}+\underset{\cdot\cdot}{x}, z+\underset{\cdot}{z}, \underset{\cdot}{z}+\underset{\cdot\cdot}{z}, v+\underset{\cdot}{v}, \underset{\cdot}{v}+\underset{\cdot\cdot}{v}$, & subducta Aequatione, per operationem secundam fiet $2\underset{\cdot}{x}\underset{\cdot}{z} + \underset{\cdot\cdot}{x}z + 2\underset{\cdot\cdot}{x}\underset{\cdot}{z} + x\underset{\cdot\cdot}{z} + 2\underset{\cdot}{x}\underset{\cdot\cdot}{z} + \underset{\cdot\cdot}{x}\underset{\cdot\cdot}{z} - a\underset{\cdot\cdot}{v} = 0$. Sic in Fluxionibus proposita eadem Aequatone, fiet per operationem primam $\dot{x}z + x\dot{z} - a\dot{v} = 0$, per secundam $\dot{x}z + x\dot{z} - a\dot{v} = 0$, per secundam $2\dot{x}\dot{z} + \ddot{x}z + x\ddot{z} - a\ddot{v} = 0$. Et sic pergere licet ad Incrementa, & ad Fluxiones tertias, quartas, & sequentes.

Sed ubi hoc modo pergitur ad Incrementa, vel ad Fluxiones secundas, tertias, & sequentes, convenit quantitatem aliquam considerare ut uniformiter crescentem, & pro ejus Increments, vel Fluxionibus, secunda, tertia, & sequentibus scribere nihil. Sic in Aequatione modo proposita $xz - av = 0$, [p. 5] uniformiter crescente z, erit per operationem secundam $2\underset{\cdot}{x}\underset{\cdot}{z} + \underset{\cdot\cdot}{x}z + 2\underset{\cdot\cdot}{x}\underset{\cdot}{z} - a\underset{\cdot\cdot}{v} = 0$. Et in Fluxionibus proposita eadem Aequatione, erit per operationem secundam $2\dot{x}\dot{z} + \ddot{x}z - a\ddot{v} = 0$, per tertiam $3\ddot{x}\dot{z} + \dddot{x}z - a\dddot{v} = 0$. Et in hoc casu potest commode pro Fluxione data $\dot{z}$ scribi 1. Hoc pacto Aequationes praedictae sunt $\dot{x}z + x - a\dot{v} = 0, 2\dot{x} + \ddot{x}z - a\ddot{v} = 0, 3\ddot{x} + \dddot{x}z - a\dddot{v} = 0$.

PROP. II. PROB. II.

In Aequatione incrementali variabiles quotvis involvente, vice omnium istarum variabilium substituere totidem novas per eadem Incrementa in ordine inverso crescentes.

Sit x quaevis variabilium in Aequatione proposita, & v nova variabilis in ejus locum substituenda; ita ut dum augetur x, minuatur v per eadem Incrementa. Tum si sit n index infimi Incrementi in Aequatione proposita, satisfiet Problemati pro $x, \underset{\cdot}{x}, \underset{\cdot\cdot}{x}$, &c. scribendo sequentes ipsorum valores; ubi est d quantitas determinata ad libitum sumpta,

$x = d - v - n\underset{\cdot}{v} - n.\frac{n-1}{2}.\underset{\cdot\cdot}{v} - n.\frac{n-1}{2}.\frac{n-2}{3}.\underset{\cdot\cdot\cdot}{v} - \&c.$

$$\underset{\cdot}{x} = \underset{\cdot}{v} \quad + \overline{n-1}\,\underset{\cdot\cdot}{v} \quad + \overline{n-1}.\tfrac{n-2}{2}.\,\underset{\cdot\cdot\cdot}{v} \quad + \overline{n-1}.\tfrac{n-2}{2}.\tfrac{n-3}{3}.\,\underset{\cdot\cdot\cdot\cdot}{v} \quad + \&c.$$

$$\underset{\cdot\cdot}{x} = -\underset{\cdot\cdot}{v} \quad - \overline{\tfrac{n-2}{1}}\,\underset{\cdot\cdot\cdot}{v} \quad - \overline{\tfrac{n-2}{1}}.\tfrac{n-3}{2}.\,\underset{\cdot\cdot\cdot\cdot}{v} \quad - \&c.$$

$$\underset{\cdot\cdot\cdot}{x} = -\underset{\cdot\cdot\cdot}{v} \quad - \overline{n-3}\,\underset{\cdot\cdot\cdot}{v} \quad + \&c.$$ & sic porro. [p. 6]

DEMONSTRATIO.

$$\begin{array}{r|l|l}
 & A. & B. \\
1. & x - - - - - - - - - & v + 3\underset{\cdot}{v} + 3\underset{\cdot\cdot}{v} + \underset{\cdot\cdot\cdot}{v} \\
2. & x + \underset{\cdot}{x} - - - - - - & v + 2\underset{\cdot}{v} + 2\underset{\cdot\cdot}{v} \\
3. & x + 2\underset{\cdot}{x} + 2\underset{\cdot\cdot}{x} - - - & v + \underset{\cdot}{v} \\
4. & x + 3\underset{\cdot}{x} + 3\underset{\cdot\cdot}{x} + \underset{\cdot\cdot\cdot}{x} & v
\end{array}$$

Sit verbi gratia $n = 3$, & in tabulis A & B exhibeantur quatuor valores correspondentes ipsorum x & v in contrario ordine crescentium; qui peradditionem Incrementorum facile colliguntur. Tum quoniam ex Hypothesi Incrementa correspondentia in utraque tabula sunt semper aequalia, debitur summa suorum quorumvis valorum correspondentium x & v in his tabulis. Quare si summa ista data sit d, erit

$$\begin{aligned}
&x - - - - - - - - - = d - v - 3\underset{\cdot}{v} - 3\underset{\cdot\cdot}{v} - \underset{\cdot\cdot\cdot}{v} \\
&x + \underset{\cdot}{x} + - - - - - - = d - v - 2\underset{\cdot}{v} - 2\underset{\cdot\cdot}{v} \\
&x + 2\underset{\cdot}{x} + 2\underset{\cdot\cdot}{x} - - - = d - v - \underset{\cdot}{v} \\
&x + 3\underset{\cdot}{x} + 3\underset{\cdot\cdot}{x} + \underset{\cdot\cdot\cdot}{x} = d - v
\end{aligned}$$

Tum sumendo Differentias harum Aequationum sit

$$\begin{aligned}
&\underset{\cdot}{x} - - - - - - = \underset{\cdot}{v} + 2\underset{\cdot\cdot}{v} + \underset{\cdot\cdot\cdot}{v} \\
&\underset{\cdot}{x} + \underset{\cdot\cdot}{x} - - - = \underset{\cdot}{v} + \underset{\cdot\cdot}{v} \\
&\underset{\cdot}{x} + 2\underset{\cdot\cdot}{x} + \underset{\cdot\cdot\cdot}{x} - = \underset{\cdot}{v}
\end{aligned}$$

Et tum sumendo Differentias harum Aequationes sit

$$\begin{aligned}
&\underset{\cdot\cdot}{x} - - - = -\underset{\cdot\cdot}{v} - \underset{\cdot\cdot\cdot}{v} \\
&\underset{\cdot\cdot}{x} + \underset{\cdot\cdot\cdot}{x} = -\underset{\cdot\cdot}{v}
\end{aligned}$$

[p. 7] Denique sumendo Differentias harum Aequationum sit $\underset{\cdot\cdot\cdot}{x} = \underset{\cdot\cdot\cdot}{v}$.
Sed hi valores ipsorum $\underset{\cdot}{x}, \underset{\cdot\cdot}{x}, \underset{\cdot\cdot\cdot}{x}$, iidem sunt, ac in solutione jubentur; & eadem est argumentatio ubi est *n* alterius cujusvis valoris. Quare pro singulis x, & suis Incrementis substituendo hujusmodi valores, recte solvitur Problema. Q.E.D.

COROL.

Ob Incrementa evanescentia in Fluxionibus Solutio, sit simplicior, existente

$x = d - v,\ \dot{x} = \dot{v},\ \ddot{x} = -\ddot{v},\ \dddot{x} = \dddot{v},\ \ddddot{x} = -\ddddot{v}$, & sic porro.

SCHOLIUM.

Possunt equidem Aequationes incrementales pro lubitu transformari ope Aequationum assumptarum. Sic se feceris $x = vv$, capiendo Incrementa (per Prop. I) erit $\underset{\cdot}{x} = 2\underset{\cdot}{v}v + \underset{\cdot}{v}\underset{\cdot}{v}$; $\underset{\cdot\cdot}{x} = 2\underset{\cdot\cdot}{v}v + 4\underset{\cdot\cdot}{v}\underset{\cdot}{v} + \underset{\cdot\cdot}{v}\underset{\cdot}{v}$; & sic porro ; unde transformabitur Aequatio pro $\underset{\cdot}{x}, \underset{\cdot\cdot}{x}, \underset{\cdot\cdot\cdot}{x}$, &c. substituendo hos ipsorum valores. Idem fiet si fit $x = d - v$. Sed in hoc casu quoniam est $\underset{\cdot}{x} = -\underset{\cdot}{v}$, erit *v* quantitatas negativa; quare quantitas substituta v non erit quantitas revera increscens in Aequatione transforma, sed decrescens, existente v ipsius Decdremento proxime auferendo. Proinde si cupis Aequationem ita transforma sint v descrescant crescentibus x, & tamen in Aequatione transforma sint v ipsarum vera Incrementa, ut sunt x vera Incrementa ipsius x, procedendum erit per hanc Propositionem. [p. 8]

PROP. III. PROB. III.

Aequationem fluxionalem, in qua sunt fluents tantum duae z & x, quarum z fluit uniformiter, ita transformare ut fluat x uniformiter.

Solvitur Problema pro $\ddot{x}$, $\dddot{x}$, $\ddddot{x}$, &c. substituendo sequentes ipsorum valores, nempe

$$\ddot{x} = -\frac{\ddot{z}\dot{x}}{\dot{z}},\quad \dddot{x} = \frac{-\dot{z}^{z}\dot{x} + 3\ddot{z}^{2}\dot{x}}{\dot{z}^{2}},\quad \ddddot{x} = \frac{-\ddddot{z}\dot{z}^{3}\dot{x} + 10\dddot{z}\ddot{z}\dot{z}\dot{x} - 15\ddot{z}^{3}\dot{x}}{\dot{z}^{3}},\ \&c.$$

DEMONSTRATIO.

Sunto *A, B, C, D, E,* &c. quantitates ex *x* datis compositae, & sint $\dot{A} = B\dot{x},\ \dot{B} = C\dot{x},\ \dot{C} = D\dot{x},\ \dot{D} = E\dot{x}$, et sic porro. Tum si ponatur $z = A$, erit $\dot{z} = \dot{A}$, & (fluente uniformiter z) $0 = (B\ddot{x} + \dot{B}\dot{x} =)\quad B\ddot{x} + C\dot{x}^{2}$, $0 = (B\dddot{x} + \dot{B}\ddot{x} + 2C\ddot{x}\dot{x} + \dot{C}\dot{x}^{2} =)\quad B\dddot{x} + 3C\ddot{x}\dot{x} + D\dot{x}^{3}$,

$0 = B\ddddot{x} + 4C\dddot{x}\dot{x} + 3C\ddot{x}^2 + 6D\dot{x}^2\ddot{x} + E\dot{x}^4$, &c. porro (per Prop. I.) Sed ubi fluit uniformiter *x,* & fluit inequaliter *z*, sunt $B = \frac{\dot{z}}{\dot{x}}$, $C = \frac{\ddot{z}}{\dot{x}^2}$, $D = \frac{\dddot{z}}{\dot{x}^3}$, $E = \frac{\ddddot{z}}{\dot{x}^4}$, &c. Itaque pro *B, C, D, E,* &c. scriptis his ipsorum valoribus, invenientur ipsorum $\ddot{x}$, $\dddot{x}$, $\ddddot{x}$, etc, valores, ut supra exhibentur. Quibus proinde in Aequatione substitutis, deinde fluet *x* uniformiter, atque *z* inequaliter. Q.E.D.

[p. 9]

SCHOLIUM.

Et eodem modo transformari potest Aequatio, quae plures fluentes involvit *v, y*, &c. praeter *z* & *x*; modo nullius *v, y*, &c. Fluxio ultra primam in Aequatione involatur. Quare si in Aequatione, quam per hanc Propositionem transformare velis, sint quaedam ipsorum *v, y,* &c. Fluxiones secundae, tertiae, & sequentes, primum eliminandae sunt Fluxiones istae ope Aequationum datarum, & deinde precedet transformatio per hanc Propositionem facta.

PROP. IV. THEOR. I.

Data Aequatione praeter variabilem z, cujus valores omnes dantur, involvente alterius variabilis x Incrementa aliquot, $\underset{m}{x}$, $\underset{m+1}{x}$, $\underset{m+2}{x}$ &c. *quarum prima sit* $\underset{m}{x}$, & *ultima* $\underset{m+n}{x}$, (*ubi etiam desse possunt z, & omnia Incrementa* $\underset{m+1}{x}$, $\underset{m+2}{x}$, *&c. inter* $\underset{m}{x}$ *&* $\underset{m+n}{x}$ *media* ;) *atque datis praeterea m + n conditionibus spectantibus ad m + n datos valores z, & ad totidem valores correspondentes quovis modo sumptos inter valores ipsorum x,* $\underset{\cdot}{x}$, $\underset{\cdot\cdot}{x}$, *&c. in infinitum ; ita tamen ut non plures quam n valores sumantur ipsius* $\underset{m}{x}$, *vel* $\underset{m+1}{x}$, *vel cuiusvis Incrementi inferiora; vel inter plura Incrementa* $\underset{m}{x}$, $\underset{m+1}{x}$, $\underset{m+2}{x}$, & inferiora; *nec plures quam n + 1 valores sumantur ipsius* $\underset{m-1}{x}$; *nec plures quam n + 2 valores sumantur ipsius* $\underset{m-2}{x}$ *; atque sic deinceps; dabuntur omnes valores ipsius x ex datis omnibus valoribus z.*

[p. 10.]

DEMONSTRATIO.

Per Aequationem datam datur $\underset{m+n}{x}$ expressa per z & per Incrementa $\underset{m}{x}, \underset{m+1}{x}, \underset{m+2}{x}$, &c. ipso $\underset{m+n}{x}$ superiora. Unde (per Prop. 1.) dabitur proximum Incrementum $\underset{m+n+1}{x}$ per easdem quantitaties expressum; deinde (per eandem Propositionem) dabitur proximum Incrementum $\underset{m+n+2}{x}$ per easdem quantitates expressum; & operationibus in infinitum continuatis dabuntur omnia Incrementa ipso $\underset{m+n}{x}$ inferia expressa per eadem quantitates z, & per Incrementa $\underset{m}{x}, \underset{m+1}{x}, \underset{m+2}{x}$, &c. ipso $\underset{m+n}{x}$ superiora. Si itaque sint a, & $c, \underset{\cdot}{c}, \underset{\cdot\cdot}{c}$, &c. ipsorum z, & $x, \underset{\cdot}{x}, \underset{\cdot\cdot}{x}$, &c. certi quidam valores correspondentes, dabuntur omnia $\underset{m+n}{c}, \underset{m+n+1}{c}, \underset{m+n+2}{c}$, &c. in infinitum expressa per a, & $\underset{m}{c}, \underset{m+1}{c}, \underset{m+2}{c}$, &c. ipso $\underset{m+n}{c}$ superiora. Unde per additionem continuam dabuntur omnes valores ipsius $\underset{m}{x}$ in tabula ante continuata. Et pro $\overset{o}{z}$ & x substitutis novis variabilis (Per Prop. 2) eodem modo dabuntur omnes valores $\underset{m}{x}$ in tabula retro continuata. Deinde per additionem & subductionem continuam dabuntur omnes valores Incrementi proxime superioris $\underset{m-1}{x}$, expressi per easdem quantitates, & per $\underset{m-1}{c}$: Et deinde eodem modo dabuntur omnes valores Incrementi adhuc superioris $\underset{m-2}{x}$ expressi per easdem quantitaties & per $\underset{m-2}{c}$. Et sic pergendo dabuntur tandem omnes valores [p. 11.] ipsius x expressi per a & per terminos $c, \underset{\cdot}{c}, \underset{\cdot\cdot}{c}$, &c. ipso $\underset{m+2}{c}$ superiores. Sed terminorum $c, \underset{\cdot}{c}, \underset{\cdot\cdot}{c}$, &c. ante $\underset{m+2}{c}$ numerus est $m + n$. Quare per conditiones numero $m + n$ determinabuntur omnes $c, \underset{\cdot}{c}, \underset{\cdot\cdot}{c}$, &c. adeoque dabuntur omnes x. Q.E.D. Sed quoniam valores ipsorum $\underset{m}{x}, \underset{m+1}{x}, \underset{m+2}{x}$, &c. includant terminos tantum $\underset{m}{c}, \underset{m+1}{c}, \underset{m+2}{c}$, &c., quorum numerus est tantum n ; ergo nequeunt plures quam n conditiones applicari ad valores Incrementi $\underset{m}{x}$ & inferiorum. Item quoniam valores ipsius $\underset{m-1}{x}$ includantur terminos tantum $\underset{m-1}{c}, \underset{m}{c}, \underset{m+1}{c}$, &c. quorum numerus est tantum $n + 1$, nequeunt plures conditiones quam n + 1 applicari ad valores Incrementi $\underset{m-1}{x}$: Et similia est argumentio in caeteris. Unde per hoc Theorema recte determinantur data, & eorum conditiones, in Aequationibus duas tantum Integrales & eorum Incrementa involventibus.

[p. 12.]

PROP. V. THEOR. II.

Datis duabus Aequationes, praeter z, cujus valores omnes dantur, involventibus ipsorum v & x Incrementa aliquot, quarum suprema in utraque Aequatione sint $\underset{p}{v}$ *&* $\underset{\pi}{x}$*, & infima* *in una Aequatione sint* v_{p+a} *&* $x_{\pi+\alpha}$, *& infima in altera Aequatione sint* v_{p+b} *&* $x_{\pi+\beta}$ *; si sit m numerorum a + β & α + b maximus, dabuntur omnes v per conditiones numero m + p spectantes ad m + p valores ipsius z, & ad totidem valores respondentes ipsorum* $v, \underset{\cdot}{v}, \underset{\cdot\cdot}{v}$ & $x_{\pi}, x_{\pi+1}, x_{\pi+2}$, *&c. atque dabantur omnes x per conditiones numero m + π spectantes ad m + π valores ipsius z, & ad totidem valores respondentes isporum* $x, \underset{\cdot}{x}, \underset{\cdot\cdot}{x}$ & v_{p}, v_{p+1}, v_{p+2}, *&c. Ita quidem ut conditionum numerus non amplius m applicari possit ad valores Incrementorum* v_{p} *&* x_{π} *,& inferiorum, reliquis conditionibus applicandis ad valores* $x, \underset{\cdot}{x}, \underset{\cdot\cdot}{x}$, &c. $v, \underset{\cdot}{v}, \underset{\cdot\cdot}{v}$ *&c. ipsis* x_{π} *&* v_{p} *superiorum juxta leges valorum Incrementorum ipsa* x_{m} *superiorum in Propositione quarta, hoc est, ut ad minimum unus valor sit ipsius x, duo valores sint ipsorum x &* $\underset{\cdot}{x}$, *tres valores sint ipsorum* $x, \underset{\cdot}{x}, \underset{\cdot\cdot}{x}$ *; & sic de* $v, \underset{\cdot}{v}, \underset{\cdot\cdot}{v}$, *&c. atque deinceps.*

[p. 13.]

DEMONSTRATIO.

Nam per Aequationes datas, & per Aequationes novas inde derivatas (per Prop. 1) eliminato *v* cum suis Incrementis, dabitur Aequatio praeter *z* involvens tantum Incrementa ipsius *x*, quorum supremum est x_{π}, & infinum est $x_{\pi+m}$. Et proxime ante inventam Aequationem istam dabitur Aequatio praeter z & ipsius x Incrementa x_{π} & inferiora, involvens tantum v_{p}. Unde dabitur v_{p} expressum per *z*, & per ipsius *x* Incrementa x_{π} & inferiora. Sed si sint *a*, & $c, \underset{\cdot}{c}, \underset{\cdot\cdot}{c}$, &c. ipsorum *z*, & $x, \underset{\cdot}{x}, \underset{\cdot\cdot}{x}$, &c., valores quidam correspondentes, dabantur omnia Incrementa x_{π} & inferiora impressa per a, & pr ipsorum $c_{\pi}, c_{\pi+1}, c_{\pi+2}$, &c. numerum *m* (per Prop. 4). Quare etiam dabantur omnia v_{p} per eadem quantitaties expressa. Unde si sint $d, \underset{\cdot}{d}, \underset{\cdot\cdot}{d}$, &c. ipsorum $v, \underset{\cdot}{v}, \underset{\cdot\cdot}{v}$, &c.valores ipsius z valori *a* respondes, & continetur series $d, \underset{\cdot}{d}, \underset{\cdot\cdot}{d}$, &c. usque d_{p-1} inclusive, ut sit terminorum numerus *p*, dabantur omnes *v* expressa per quantitaties $d, \underset{\cdot}{d}, \underset{\cdot\cdot}{d}$, &c & $c_{\pi}, c_{\pi+1}, c_{\pi+2}$; &c. quorum omnium numeros est *m* + *p*. Exprimentur autem omnes valores ipsius *x* per quantitaties $c, \underset{\cdot}{c}, \underset{\cdot\cdot}{c}$, &c. quarum numerus est *m* + *π*. Quare determinatis omnibus $c, \underset{\cdot}{c}, \underset{\cdot\cdot}{c}$, &c. per conditiones numero *m* + *π* spectantes ad valores x & suorum Incrementorum, deinde determinabuntur omnes $d, \underset{\cdot}{d}, \underset{\cdot\cdot}{d}$, &c per alias conditiones numero p spectantes ad valores $v, \underset{\cdot}{v}, \underset{\cdot\cdot}{v}$, &c. vel determinatis omnibus $d, \underset{\cdot}{d}, \underset{\cdot\cdot}{d}$, &c & aliquot ex terminis $\underset{\pi}{c}, \underset{\pi+1}{c}, \underset{\pi+2}{c}$; &c. per conditiones spectantes ad valores $v, \underset{\cdot}{v}, \underset{\cdot\cdot}{v}$, &c. reliqui terminorum $c, \underset{\cdot}{c}, \underset{\cdot\cdot}{c}$, &c. determinatur per conditiones spectantes ad valores $x, \underset{\cdot}{x}, \underset{\cdot\cdot}{x}$, &c. Unde per conditiones $m + p$ dabantur omnes v, & per conditiones $m + \pi$ dabantur omnes x, (Q.E.D.) & per conditiones omnino $m + p + \pi$ dabuntur omnes, cum v, tum x. Quomodo autem conditiones applicandae sint ad valores v & x, & suorum incrementorum ipsis $\underset{p}{v}$ & $\underset{\pi}{x}$ superiorum, satis constat ex Propositione quarta.

SCHOLIUM.

Ad eundem modum per eliminationes variabilium etiam pergere licet ad inventionem conditionum, quibus astringi possunt tres, vel quatuor, vel plures Aequationes incrementales involventes tres, vel, vel plures variabiles praeter z, cujus valores omnes dantur.

Sit Aequatio $z^2 \underset{\cdot\cdot}{x} - z \underset{4}{x} + b^3 = 0.$ In hac Aequatione ad mentem Propositionis quartae est $\underset{\cdot\cdot}{x}$ idem ac $\underset{m}{x}$, atque $\underset{4}{x}$ idem ac $\underset{m+\pi}{x}$; unde sunt $m = 2$, & $n = 2$. Quare per conditiones quatuor dabuntur omnes x, ex datis omnibus z. Et quoniam est $\underset{\cdot\cdot}{x}$ primum omnium $x, \underset{\cdot}{x}, \underset{\cdot\cdot}{x}$; &c. quae in Aequatione occurrunt, ad minimum duae conditiones applicationae sunt ad valores ipsorum x & $\underset{\cdot}{x}$; ita , ut vel una conditio applicetur ad valorem x, & alia ad valorem $\underset{\cdot}{x}$, vel utraque applicetur ad valorem π. Reliquae autem conditiones possunt pro lubitu applicari ad valores omnium $x, \underset{\cdot}{x}, \underset{\cdot\cdot}{x}$; &c.in infinitum; ita, ut vel una conditio applicetur ad unum ex istis terminis, & alia ad alium, vel etiam ut omnes conditiones applicentur ad diversos valores ejusem termini.

[p. 15.]

Sint duae Aequatones $\underset{\cdot\cdot}{x} - \underset{\cdot}{v} + \underset{\cdot\cdot\cdot}{v} = 0,$ & $\underset{\cdot\cdot\cdot}{x} z - \underset{\cdot\cdot\cdot}{v} = 0.$ In his Aequationibus ad mentem hujus Propositionis est $p = 1$, $\pi = 2$, $a = 2$, $\alpha = 0$, $b = 1$, $\beta = 1$. Unde sit $\alpha + \beta = 3$ & $a + b = 1$, adeoque;
$m = 3$ $(= \alpha + \beta)$ atque $m + p = 4, m + \pi = 5,$ & $m + p + \pi = 6.$ Proinde per quatuor conditiones dabuntur omnes valores v, per quinque dabuntur omnes x, & per sex conditiones dabuntur omnes cum x, tum v, duae ad valores x & $\underset{\cdot}{x}$, reliquis tribus utcunque applicandis ad valores Incrementorum $\underset{\cdot}{v}, \underset{\cdot\cdot}{x}$, & inferiorum.

Sint Aequationes fluxionales duae $\dot{v}^2 - \dot{x}^2 - \dot{z}^2 = 0,$ & $\dot{z} \dddot{v} - z \dot{x}^2 \ddot{x} - \overset{4}{x} x = 0.$ Tum ad mentem hujus Propositionis erunt

$\underset{p}{v} = \dot{v}, \quad \underset{\pi}{x} = x, \quad \underset{\pi+a}{v} = \dot{v}, \quad \underset{\pi+\alpha}{x} = \dot{x}, \quad \underset{p+b}{v} = \dddot{v}, \quad \& \quad \underset{\pi+\beta}{x} = \overset{4}{x};$

adeoque $p = 1, \pi = 0, a = 0, \alpha = 1, b = 2, \beta = 4, a + \beta = 4, \alpha + b = 3;$ unde sit $m = 4, m + p = 5, m + \pi = 4,$ atque $m + p + \pi = 5.$ Dabuntur ergo omnes *v* & *x* ex datis omnibus *z* per conditiones omnino quinque; quarum ad minimum una pertinet ad valorem ipsius *v*, reliquae possunt utcunque applicari ad valorem ipsius *v*, reliquae possunt utcunque applicari ad valores ipsorum *v, x*, & Fluxionum suarum. Proinde si curvae duae sint describendae, quarum ordinatae sint v & x, & abscissa communis z, descripta curva cujus ordinata est v per quinque puncta datal vel per quatuor puncta & secante quintam ordinatam in angulo dato; vel in genere quae transeat per unum punctum datum, & quatuor ordinias positione datas fecet vel in punctis, vel in angulis datis, vel in earum extremitatibus habeat curvaturam datam, vel symptomata quadam curvaturae pendentia a valoribus Fluxionum $\ddot{v}, \dddot{v}$, & inferiorum; dabantur omnia puncta utriusque curvae. Vel etiam si describatur altera curva, cujus ordinata est *x*; ita, ut quatuor ordinatas vel secet in punctis datis, vel alias

[p. 16.]

secet in punctis, alias in angulis datis; vel in earum extremitatibus curvaturas datas habeaat, vel habeat ulla quaevis symptomata curvaturae pendentia a Fluxionibus tertiis, quartis, & inferioribus; dabuntur omnia puncta utriusque curvae ex dato praeterea uno valore *v*. Et modo detur una conditio respiciens valorem *v*, conditiones reliquae poterunt pro lubitu distribui inter valores ordinatarum *x* & *v*, & Fluxionum suarum.

Porro in his casibus possunt duo vel plures dati valores z interse aequari, vel (quod idem est in Geometria) possunt duae vel plures ex ordinatis positione datis coincidere. Sed hoc pendet a certis conditionibus petendis ex natura Aequationum quarumvis propositarum. Sic propositis duabus Aequationibus $x\mathrm{v} = \mathrm{z}$ & $\dot{\mathrm{x}}z + \dddot{x} - \ddot{v}\overset{4}{v} = 0,$ per hanc propositionem erunt conditiones quatuor, quae possunt prolubitu applicari ad valores *x* & *v*, & Fluxionum suarum. Sed ob Aequationem primam $xv = z$ non possunt coincidere ordinatae, quibus applicantur conditiones spectantes ad valores utriusque *v* & *x*; nec ob Aequationem secundam possunt coincidere omnes quatuor ordinatae, quibus applicantur conditiones spectantes ad valores omnium $\dot{\mathrm{x}}, \dddot{x}, \ddot{v}, \overset{4}{v}$; nam datis simul utrisque *x* & *v* determinatur z per Aequationem primam; & datis simul omnibus $\dot{\mathrm{x}}, \dddot{x}, \ddot{v}, \overset{4}{v}$, item determinatur *z* per Aequationem secundam; utrumque contra Hypothesin. Eodem modo capeindo Fluxionem Aeqationis primae (pro $\dot{z}$ scripto 1 ut prius) invenies $\dot{\mathrm{x}}v + \dot{v}x = 1;$ unde etiam constat non posse coincidere omnes ordinatas, quibus applicantur conditiones spectantes ad valores omnium $\mathrm{x}, \dot{x}, v, \dot{v}$. Et iterum capiendo Fluxiones Aequationum propositarum forsan dabuntur alii limites hujusmodi conditionum.

Ad haec, ut in Aequationibus integrales tantum involventibus, sic in Aequationibus incrementalibus, quantitates variables sunt certis limitibus obnoxiae. Sit exemplum in Aequatione fluxionali $x^2 - a\ddot{x} + z = 0.$

[p. 17.]

Hinc est $x = \sqrt{a\ddot{x} - z}$; unde erit $\ddot{x}$ (adeoque & omnes ipsius Fluxiones) impossibilis ubi est $\ddot{x} < z$. Ejusdem Aequationis Fluxio secunda est $2\dot{x}^2 + 2x\ddot{x} - a\overset{4}{x} = 0$; unde sit $x = \sqrt{\tfrac{1}{2}ax^4 - x\ddot{x}}$; erit ergo semper $x\ddot{x} > \tfrac{1}{2}ax^4$. Et eodem modo per ulteriores Fluxiones Aequationis propositae forsan invenies alio limites variabilium.

LEMMA 1.

In Aequatione plura ejusdem quantitatis variabilis Incrementa involva per nullam regulam generalem certo definiri potest ad quot dimensi ascendat quantitaties illa in Aequatione integrali definiente ejus relatio ad alias quantitates variabiles.

In Aequatione fluxionali $\ddot{x}x + \dot{x}\dot{x} + n\ddot{x}z + \dot{x}\dot{z} = 0$; quantitates sunt semel tantum affectae, & ubi est $n = 2$, datur x ex dato z per Aequationem quadraticam; sed tamen si fiat $n = 3$, non dabitur x, nisi per Aequationem cubicam; si $n = 4$, non dabitur x, nisi per Aequationem biquadraticam; si $n = \frac{1}{32}$, non dabitur x, nisi per Aequationem quinque dimension : Deinde si terminorum reliquorum coefficientes fiant generales, ut sit $\ddot{x}x + m\dot{x}^2 + n\ddot{x}z + p\ddot{x}\dot{z} = 0$; dubito an dimensiones Aequationis quaesitae per ullam certam legem definiri possint, si quidem omnino dari potest x ex dato z per Aequationem terminorum numero finitam. Sit alia Aequatio $4x^3\dot{z}^2 - 4x^2\dot{z}^2 = \overline{1+z^2}|^2\, x^2$. In hac Aequatione ascendit x ad tertiam dignitatem, & ascendit $\dot{x}$ ad secundam; attamen datur x ex data z per Aequationem duarum dimensionum, cujus radix est $x = \dfrac{1+z^2}{\overline{a+\sqrt{1-a^2z}}|^2}$.

Mutatis vero coefficientibus haud certo scio an dari possit x ex dato z per Aequationem finitam.

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PROP. VI. PROB. IV.

Datis tot Aequationibus, quantitates integrales & Incrementa utcunque promiscue involentibus, quot sunt variabiles x, v, y, &c. ad z referendae, cujus valores omnes dantur; invenire relationes Integralium per Aequationes ab Incrementis liberas, quae per coefficientes invariabiles indeterminatas adaptari possint ad conditiones Problematis per has Aequationes solvendi.

SOLUTIO:

Tenta num aliqua Aequatio proposita, vel ejus multiplex, vel submultiplex aliqua sit cogitum aliquod Incrementum quantitatis alicujus congnitae. Hoc si sit, vice istius Aequationis propositae substitue quantitatem illam cognitam factam aequalem quantitati cujus Incrementum primum, vel secundum vel aliud quoddam est nihil; prout Aequatio proposita,vel ejus multiplex, vel submultiplex sit Incrementum primum, vel secundum, vel aliud quoddam quantitatis istius cognitae. Hoc idem facto in omnibus Aequationibus propositis, si Aequationes inventae integrales tantum involvunt, dabitur Solutio in terminis numero finitis; quae per coefficientes indeterminatos in quantitatibus assumptis accommodabitur ad conditiones Problematis. Q.E.F.

Sed si hoc pacto Problema solvi nequit, per eliminationes variabilium (ope Aequationum datarum, & novarum Aequationum inde derivatarum per Prop. 1, si opus est) quare tot novus Aequationes quot sunt variabiles *x,v, y* &c. praeter *z*, quarum una involvat tantum *x* cum suis Incrementis (nempe praeter *z*,) reliquae involvant duas tantum variables *x* & *v*, *x* & *y*, &c. (quarum una sit semper *x*) cum suis Incrementis (in omnibus Aequationibus semper subintellecto *z*). Per Aequationem involventum tantum x cum suis Incrementis quare valorem ipsius *x*, expressum per dignitates ipsius *z*, ope Propositionis alicujus sequentis. Deinde ope hujus Aequationis eliminentur x cum suis Incrementis ab Aequationibus involventibus tantum *x* & *v, x* & *y*, &c. cum suis incrementis; & per aequationes hoc modo resultantes quaerantur valores ipsorum *v, y,* &c. expressi per dignitates ipsius *x*, eodem modo quo quaerebatur valor ipsius *x*. Atque hoc pacto dabantur omnes *x, v, y*, &c. expressi per dignitates ipsius *z*. Qui valores accommodabuntur ad conditiones Problematis ope coefficientium adhuc indeterminatorum. Et si horum valorum aliqui prodeunt in terminis numero finitis, vel in seriebus quae ad expressiones finitas reduci possunt, ex hac parte dabitur solutio ver Mathematica in terminis numero finitis. Sed ubi Series ad terminos finitos reduci nequet, solutio pro Mechanica est habenda, atque, seriei usus erit in inventione radicis quaesitae *x*, vel *v*, vel *y*, &c. per approximationes.

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SCHOLIUM.

Reductio aequationum propositarum ad aequationes integrales in hac solutione plurimum pendet a solertia Analystae in inventione incrementorum (per Prop. 1.) exercitati. Quare in hunc finem utile est quantitatum variis modiis compositarum possunt, quoties opus est alicujus incrementi integralem invenire. Hujus generis est sequens tabella.

Incrementa	Integrales.
1. $az+b\underset{\cdot}{z}z+c\underset{\cdot}{z}z\overset{\backslash}{z}+d\underset{\cdot}{z}z\overset{\backslash}{z}\overset{\backslash\backslash}{z}+\&c.$	$A+\frac{1}{1}az+\frac{1}{2}bz\overset{\backslash}{z}+\frac{1}{3}cz\overset{\backslash}{z}\overset{\backslash\backslash}{z}+\frac{1}{4}dz\overset{\backslash}{z}\overset{\backslash\backslash}{z}\overset{\backslash\backslash\backslash}{z}+\&c.$
2. $\frac{a\underset{\cdot}{z}}{z\underset{\backslash}{z}}+\frac{b\underset{\cdot}{z}}{z\underset{\backslash}{z}\underset{\backslash\backslash}{z}}+\frac{c\underset{\cdot}{z}}{z\underset{\backslash}{z}\underset{\backslash\backslash}{z}\underset{\backslash\backslash\backslash}{z}}+\frac{d\underset{\cdot}{z}}{z\underset{\backslash}{z}\underset{\backslash\backslash}{z}\underset{\backslash\backslash\backslash}{z}\underset{\backslash\backslash\backslash\backslash}{z}}+\&c.$	$A-\frac{a}{1z}-\frac{b}{2z\underset{\backslash}{z}}-\frac{c}{3z\underset{\backslash}{z}\underset{\backslash\backslash}{z}}-\frac{d}{4z\underset{\backslash}{z}\underset{\backslash\backslash}{z}\underset{\backslash\backslash\backslash}{z}}-\&c.$

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Fluxiones.	Fluentes.
1. $\dot{z}\, z^{\pm n}$	$\frac{1}{\pm n+1} z^{\pm n+1} + \mathrm{A}.$
2. $(\vartheta \dot{z} x + \lambda \dot{x} z) \times z^{\vartheta-1} x^{\lambda-1}$	$\mathrm{A} + z^{\vartheta} x^{\lambda}$
3. $(\vartheta \dot{z} xv + \lambda \dot{x} zv + \mu \dot{v} zx) \times z^{\vartheta-1} x^{\lambda-1} v^{\mu-1}$	$\mathrm{A} + z^{\vartheta} x^{\lambda} v^{\mu}$
4. $(\vartheta \dot{z} xvy + \lambda \dot{x} zvy + \mu \dot{v} zxy + \pi \dot{y} zxv) \times z^{\vartheta-1} x^{\lambda-1} v^{\mu-1} y^{\pi-1}$	$\mathrm{A} + z^{\vartheta} x^{\lambda} v^{\mu} y^{\pi}.$

Comparando expressiones cum hujusmodi exemplis solvuntur quaedam Problemata. Sit aequatio $\dddot{x}\dot{x}x - \ddot{x}^2 x - 2\dot{x}^2\ddot{x} = 0.$ Comparando hanc aequationem cum fluxionis tertiae factore $\vartheta \dot{z} xv + \lambda \dot{x} zv + \mu \dot{v} zx$, invenitur $\vartheta = 1, \lambda = -1, \mu = -2,$ ipsis $\ddot{x}, \dot{x}, x$ in hac aequatione subeuntibus vices ipsorum *z, x, v*, in fluxione ista. Unde sit fluens $\ddot{x}\dot{x}^{-1}.x^{-2}$ aequalis quantitati datae, (quoniam est ipsius fluxio aequalis nihilo.) Sit atque, *a* quantitas data, atque, erit $\ddot{x}\dot{x}^{-1}.x^{-2} = a\dot{z}$. (nempe completis ordinibus fluxionum per dignitates fluxiones datae $\dot{z}$.) Hinc sit $\ddot{x} = a\dot{z}\dot{x}x^3$; unde iterum regrediendo ad fluentes (ad exemplum fluxionis $\dot{z}\, z^{\pm n}$) sit $\dot{x} = \frac{a\dot{z}x^3}{3} + b\dot{z}$. Quo pacto jam revocatur aequatio fluxionalis ordinis tertii ad aequationem fluxionalem ordinis tantum primi.

Ad eundem modum potest aequatio $\ddot{x}^2 = x^3\dot{z}^4$, vel (pro $\dot{z}$ scribo 1) $\ddot{x}^2 = x^3$ revocari ad ordinem superorem. [p. 21] Nam extrahendo radicem sit $\ddot{x} = x^{\frac{3}{2}}$. Duc aequationem in $\dot{x}$, atque sit $\ddot{x}\dot{x} = \dot{x}x^{\frac{3}{2}}$, unde capeindo fluentes sit $\frac{\dot{x}}{2} = \frac{2}{5}x^{\frac{5}{2}} + a.$

Sit alia aequatio $\ddot{x}^2 = \dot{x}x$: tum extracta radice sit $\ddot{x} = \dot{x}^{\frac{1}{2}} x^{\frac{1}{2}}$, hoc est $\ddot{x}\dot{x}^{\frac{1}{2}} = \dot{x}\, x^{\frac{1}{2}}$, unde capiendo fluentes sit $\frac{2}{3}\dot{x}^{\frac{3}{2}} = \frac{2}{3}x^{\frac{3}{2}} + a,$ vel si placet $\dot{x}^{\frac{3}{2}} = x^{\frac{3}{2}} + a.$ Et hoc modo per multiplicationes, divisiones, & extractiones radicum reducendo expressiones ad formas fluxorum cognitarum, vel inveniuntur ipsae fluentes, vel revocantur aequationes ad fluxionum ordines superiores.