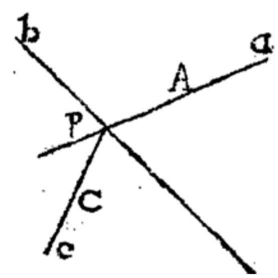
The Method of Increments.

The Second Part. [IIb]

[page 74]
LEMMA V.

If a rigid solid is held in equilibrium by three forces, the lines in the directions of the forces pass through the same point, and lie in the same plane.



The forces can be applied at the points A, B, C, and act in the directions A*a*, B*b*, C*c*. Since any point A is held in equilibrium, the forces B*b* and C*c* taken together add to give a force equal and opposite to the force A*a*. But (by the Principles of Statics) the forces B*b* and C*c* cannot be added together in this way, unless each line B*b* and C*c* passes through some point *p* on the line A*a*, and all the lines A*a*, B*b*, and C*c* lie together in the same plane. Thus the situation is resolved in this manner. *Q. E. D.*

LEMMA VI.

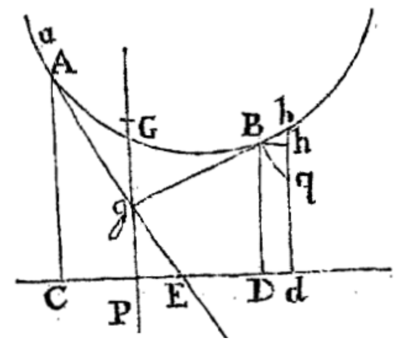
If the volume is laden with heavy material and is held in place by two strings, the relative sizes of the forces in the strings depends on how the matter has been arranged within the volume; if the centre of gravity always acts in this manner normal to the horizontal.

In accordance with statics.

N. B. The following four propositions are concerned with the figures formed by thin ropes or chains, sails filled with water, and arches supporting given loads. All these figures, as they are made from physical materials, have a density [or weight per unit length], and are subject to bending, and give a little with the forces, either by being extended or compressed [now accounted for by Young's modulus]. Hence it is necessary for anyone who wishes to describe the figures with accuracy to take these things into account. But since the solutions to these problems are found with difficulty, the calculation itself being of great enough complexity, these will impede our progress exceedingly, and in short we neglect the effect of these, and agree to work out the proposed figures from materials that can be simply extended or compressed with complete flexibility [meaning that a segment of the material does not respond to the external forces, as a spring would do, by generating internal forces, and is completely limp]; and thus by being made thin, so that the painful [inclusion of] density disappears with respect to the length given [in the equation]. Nevertheless, regarding this, it is not always possible to make the density completely zero, since with thin ropes, and with arches, on account of so much weight being supported, the weight associated with the structure is part of the solution of the problem.

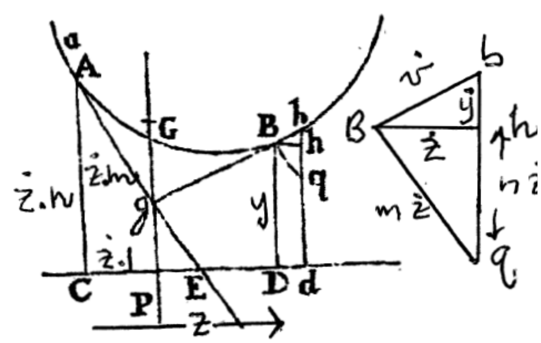
PROP. XVIII. PROB. XIII.

For a given law of the density of a thin rope [elsewhere Taylor refers to chains] *suspended from two points; to find the relations between the abscissae, the ordinates, and the length of the length of the curve; and to define the conditions under which the figure to be described can be subjected.*



AB is part of a certain thin rope hanging from the points A and B; the normals AC and BD are drawn to the line with the given position CD parallel to the horizontal, and the tangents A*g* and B*g* are drawn at the points A and B, of which A*g* crosses CD in E, and the points *a* and *b* are themselves close to A and B, and by drawing the new ordinate *bd*, the line B*h* parallel to the horizontal crosses *bd* in *h*, and B*q* is drawn parallel to the tangent A*g* crossing *bd* at *q*; and AC is the ordinate with the given fixed position, and BD the ordinate for the moving point; and let CD = *z*, DB = *y*, with the length of the curve AB = *v*, with the density at B = *x*, the weight of the piece of rope AB = *p*, CE : CA : AE :: 1 : *n : m*, and the given weight *a* is equal to the tension of a thread at A.

Since the whole weight of the length of rope AB is considered to be held in place by threads of the shortest length *a*A and *b*B, in the directions of the tangents A*g*, B*g*, the whole rope is put in a plane normal to the horizontal; and if the centre of gravity of the part AB is G, by what has been said, GP is normal to the horizontal, the same GP meets the mutual tangents that cross at *g*. For (by *Lemma* 5) with respect to the threads A*a* and B*b*, likewise , and as if all the material of the rope AB is suspended by a thread GP : thus the volume of the figure is set up in equilibrium from the tensions of the three threads A*a*, B*b*, GP, that cross each other at [p. 76] *g*, and placed in the same plane, (by *Lem.* 4) that on account of the normal GP is perpendicular to the horizontal. Therefore the tensions of these threads are in proportion to the sides of the triangle B*bq* , which are parallel to the directions of these lines. For if we make B*h* = $\dot{z}$, then *bh* = $\dot{y}$, and B*b* = $\dot{v}$, and (on account of the similar triangles B*hq* and ECA,) $hq = n\dot{z}$, and $bq = \dot{y} + n\dot{z}$, and $Bq = m\dot{z}$. Whereby



$a : p :: m\dot{z} : \dot{y} + n\dot{z}$, that is $a\dot{y} + na\dot{z} - mp\dot{z} = 0.$ Moreover $\dot{p} = \dot{v}x.$ Hence by eliminating p from this equation here (as indeed with the uniform fluent $\dot{z}$ also) , $a\ddot{y} - mx\dot{v}\dot{z} = 0.$

Thus by resolving the equation $a\ddot{y} - mx\dot{v}\dot{z} = 0,$ and $\dot{v}\dot{v} = \dot{y}\dot{y} + \dot{z}\dot{z},$ (by *Prop*. 6) the relations between *z, y, v* themselves are given; for by hypothesis [the density] *x* is given, either by the first power, or by the second power, or some other powers of *z, y,* and *v*.

[The modern intrinsic standard equation for the catenary is $s = c\tan\psi = cdy/dx$,where *s* is the length of the curve and *c* is some unknown length of the curve. Hence $1 = cd^2y/dsdx;$ or $d^2y/dsdx = 1/c.$ Hence, in the above equation for constant line density, $c = a/mx$; Thus, Taylor has taken as the boundary condition the angle and tension at A, rather than the horizontal tension at the lowest point, as is the case in modern analysis, as the latter is more convenient mathematically, but certainly not so in a practical sense, where Taylor's approach must prevail. Now, in modern terms, *c* is the catenary constant related to the minimum tension T_0 at the lowest point, and the line density *w,* by $T_0 = wc$, which is taken to be the constant horizontal tension ; i. e. $T_0 = T\cos\psi = wc$ everywhere; while $T\sin\psi = ws$, is equal to the weight of the chain above the lowest point, and this leads to $c\tan\psi = s$ as above. In the present case, T= *a* is given at the boundary point A, and $\sin\psi = n/m;$ while $T\cos\psi = wc$, or $a/m = wc$; hence $c = a/mx$ as required. The whole analysis can of course be performed starting from Taylor's basic equations $a\ddot{y} - mx\dot{v}\dot{z} = 0,$ and $\dot{v}\dot{v} = \dot{y}\dot{y} + \dot{z}\dot{z}.$ We should also note the clever means of establishing the 'force triangle', where the top triangle relates to conditions at any point on the curve, and the lower triangle relates to the starting conditions.]

CASE I.

If the value of *x* depends on both *y* and *v*, the curve is described by the [boundary] conditions, which can be applied as you please to the values of *y* and *v*, and their fluxions (by *Prop*. 5).

CASE II.

If the value of *x* does not depend on *y,* there are three [boundary] conditions in all, of which one at the least is to be applied to the value of *y*, and the two remaining can be applied as you wish to the values of *v* and *y* and their fluxions.

CASE III.

If the value of *x* depends on *y* only, there are also three [boundary] conditions that can be applied as you wish to the values of *v* and *y* and their fluxions, as there is only one applied to the value of *v* itself.

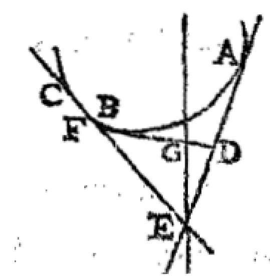
CASE IV.

Thus if *x* does not depend on *v* or *y*, there are three [boundary] conditions, only one of which is applied to the the value of *v*, the second [p. 77] to the value of *y*, and the third to be applied as you wish to the values *v* and *y* and their fluxions.

COROLL. I.

For this [last] solution, the tension of the string at B to the given tension at A, is as Bb to Bq, that is as $\dot{v}$ to $m\dot{z}$. Therefore the tension in B is as $\frac{\dot{v}}{m\dot{z}}$, or on account of the given fixed m, as $\frac{\dot{v}}{\dot{z}}$: But by the equation $a\dot{y}+na\dot{z}-mp\dot{z}=0$, that is $\dot{v}(=\sqrt{\dot{y}\dot{y}+\dot{z}\dot{z}})=\frac{\dot{z}}{a}\times\sqrt{n^2a^2+1-2mnap+m^2p^2}$; and the tension at A is equal to a. Thus the tension at B is $[\frac{a\dot{v}}{m\dot{z}}=]\sqrt{\frac{n^2a^2+1}{m^2}a^2-\frac{2n}{m}ap+p^2}$. And because this tension is proportional to $\frac{\dot{v}}{\dot{z}}$, this will be a minimum when $\dot{v}=\dot{z}$, that is at the lowest point of the curve at which the tangent is parallel to the horizontal, by considering this to be the case, the tension here is equal to $\frac{a}{m}$. And hence the law governing the density of the rope is of such a kind, that for a given tension at the one point A, by drawing the tangent, the tension is given at some other point B.

COROLL. II.



Indeed the rope can be divided into parts in which the weights are in a given ratio by drawing the tangents. For let ABC be the rope, and the tangents ADE, DEF, and EFC are drawn at the three points A, B, and C, crossing in turn at D, E, and F. Then (by this proposition) the centres of gravity of the ropes AB,AC and BC, are [p.78] on the perpendiculars passing through the points of mutual concurrence of the respective tangents at D, E, and F. Hence if the perpendicular EG is drawn through the point of concurrence E of the tangents AD and CF crossing the third tangent DBF in G, the weights of the parts AB and BC are to each other in the reciprocal ratio of the distances of their own centres of gravity from the centre of gravity of the whole rope ABC, this is, the weight of AB will be to the weight of BC in the reciprocal ratio of DG to GF [just by taking moments]. Therefore, for a given ratio of the weights, the ratio DG to GF is given. Hence, from the positions for the given lines DE, GE and FE, the direction is given of the third tangent DBF, by which the weight of the rope ABC is divided in the given ratio.

COROLL. III.

If in the expression for the density x, only one of z, y, v is present, then the problem be solved by the quadrature of the curve.

CASE I.

For in the first place if only v is present in the expression for the density x, then at first I calculate using the equations $a\dot{y}+na\dot{z}-mp\dot{z}=0$, and $\dot{v}\dot{v}=\dot{y}\dot{y}+\dot{z}\dot{z}$, and for the sake of brevity I write R for $\sqrt{n^2a^2+1-2mnap+m^2p^2}$. It is found that $\dot{z}=\frac{a\dot{v}}{\mathrm{R}}$, and $\dot{y}=\frac{mp-na}{a}\dot{z}=\frac{\overline{mp-na}}{\mathrm{R}}\dot{v}$. Thus for a given p by the quadrature of this curve , the abscissa is v, and the ordinate x; then z and y are given, by quadrature, the common abscissa is v, and the ordinates are $\frac{a}{\mathrm{R}}$ and $\frac{mp-na}{\mathrm{R}}$.

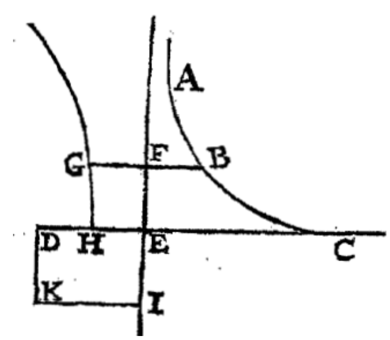
CASE II.

If in the expression for x only z is present, by multiplying each part of the equation by x, $\dot{z}=\frac{a\dot{v}}{R}$ becomes $\dot{z}x=\frac{a\dot{p}}{R}$. Hence by quadrature, the curves [p. 79] of which the abscissae are z and p, and of which the ordinates are x and $\frac{a}{R}$, give the relation between z and p. Hence on account of $\dot{v}=\frac{\dot{p}}{x}$, and $\dot{y}=\frac{mp-na}{a}\dot{z}$, by quadrature the curves of which the abscissae are p and z, and the ordinates are $\frac{1}{x}$ and $\frac{mp-na}{a}$, give v and y.

CASE III.

Hence if the value x depends only on y is, by multiplying the equation $\dot{y}=\frac{mp-na}{R}\dot{v}$ by x, it becomes $\dot{y}x=\frac{\overline{mp-na}}{R}\dot{p}(=\frac{R}{m})$. Whereby by making $\frac{R}{m}$ equal to the area of the curve of which the abscissa is y, and with the ordinate x, the relation between p and y is given. Hence, then by quadrature, the curve with this abscissa y and with the ordinate $\frac{a}{mp-na}$, gives z. Moreover this gives v by quadrature, as in the second case the abscissa of this curve is p, and the ordinate $\frac{1}{x}$.

COROLL. IV.



By *Cas*. 3 *Cor*. 3., $\frac{\mathrm{R}}{m}$ is equal to the fluent $\dot{y}x$, that likewise is equal to the tension in the thread, (by *Cor*. 1). Hence ABC is a thin rope, at the lowest point of which C the tangent CEHD is drawn parallel to the horizontal, and the normal EF is drawn to that; and draw BFG [p. 80] parallel to the horizontal, in that always FG = x = density of the rope at B. Then if GH is the curve, that always touches the point G, and the area GFEH of the end to the tangent CE is added to the area EDKI = $\frac{a}{m}$, the total area IKDHGFEI is always equal to the tension in the thread at B. For EF = y, and thus the fluxion of the area is $\dot{y}x$ = fluxion of the tension which is equal to $\frac{a}{m}$ at the point C.

COROLL.V.

By *Ex*.4 *Prop*. 15, the radius of curvature is equal to $\frac{\dot{v}^3}{\dot{z}\ddot{y}}$; whereby (by equation $a\ddot{y} - mx\dot{v}\dot{z} = 0$.) the radius of the rope is equal to $\frac{a\dot{v}^2}{mx\dot{z}^2}$, that is, equal to the area DI (fig. *Cor*. 4) to be applies to FG, and then by multiplying by $\frac{\dot{v}^2}{\dot{z}^2}$, that is by the square of the secant of the angle, that the curve makes with the horizontal.

COROLL. VI.

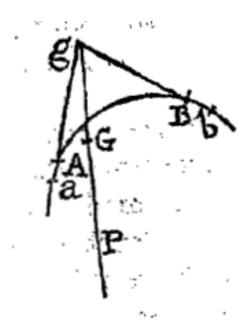
For a given shape of the rope, it is easy to find the ratio of its density. For given relations of the fluxions from the form of the figure : hence through the equation found in this problem $a\ddot{y} - mx\dot{v}\dot{z} = 0$, for a given density : $x = \frac{a\ddot{y}}{m\dot{v}\dot{z}} = \frac{a\dot{v}^2}{m\dot{z}^2} \times \frac{\dot{z}\ddot{y}}{\dot{v}^2} = \frac{a\dot{v}^2}{m\dot{z}^2}$ from the application of the radius of curvature.

[p. 81]

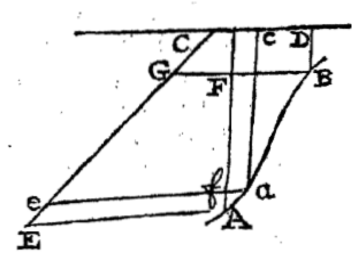
PROP. XIX. PROB. XIV.

For a given ratio of the density, to find the figure of the arch supported by its own weight.

Let AB be a certain part of the arch [formed by the flexing of a plane], and *a* and *b* points close to A and B, and the tangents A*g* and B*g* are drawn. Then if the centre of gravity of the portion AB is at G, through this point is drawn the perpendicular to the horizontal GP that cuts the point of concurrence of the tangents *g*, (by *Lem*. 5 & 6) since the weight of the arch AB is supported by the short lines *a*A and *b*B. Thus with these forces interpreted in the same way as in the preceding proposition, it is agreed that the figure of the arch is the same as that of the rope yet inverted in position.



LEMMA VII.



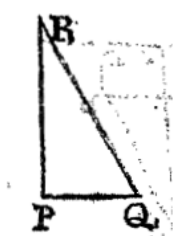
Let AB be some curved line in a plane perpendicular to the horizontal, and a surface is described by a motion of the line normal to the same plane, by which a fluid is supported, the surface of which runs to meet the same plane in the line CD parallel to the horizontal.

Then I say,

Since if the perpendiculars CA and DB are drawn to the horizontal, meeting the curve in A & B, and the line CD in C & D, and the parallels AE and BFG are drawn parallel to the horizontal, of which BF crosses [p. 82] *the line CA in F, and AE is made equal to CA, and CE is drawn crossing BF in G, then the total lateral pressure of the fluid, by which the surface AB has a force acting on it horizontally, is to the weight of the fluid enclosed in the space CABD, as the area AEGF is to the area CABD.*

Draw *ca* perpendicular to the horizontal, near to CA itself , and crossing the curve AB and the line CD at *a* and *c*; through *a* draw *ae* parallel to the horizontal, crossing CA and CE in *f* and *e*. Then with the points A and *a* coinciding, the volume of fluid enclosed in CA*ac* by being considered to weigh the same as the volume, that is, as CA × *fa*, [*fa* is a differential quantity, and so the pressure acting on the differential A*a* due to the head of fluid CA is the same as on *fa*; the difference being a second order quantity which is ignored; note that C has been displaced erroneously to the left on the diagram; and it is common knowledge that pressure in a fluid has no fixed direction, and acts the same in all directions.] the absolute pressure of this in the short line A*a* produced perpendicularly is as CA × A*a*; and thus the lateral part of the same pressure parallel to the horizontal in the direction *fa* is as CA × *f*A, that is as EA × *f*A. Therefore by considering CA*ac* as the fluxion of the weight, EA*fe* is the fluxion of the lateral pressure; hence [on summing over all increments of weight and pressure] when the weight is equal to the area CABD, the lateral pressure is made equal to the area AEGF. *Q.E.D.*

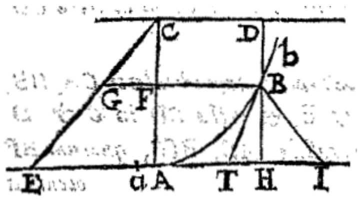
COROLLARY.



Hence if a right-angled triangle PQR is set up, the base of which PQ, parallel to the horizontal, is to the perpendicular RP as the area AEGF to the area ACDB, the resultant force of the fluid on the surface AB is in the direction of the hypotenuse RQ, and by that it is represented, if the weight is represented by the perpendicular RP.

[Thus, the single horizontal force representing the pressure averaged over the vertical distance FA exerted on AB is PQ, while the single vertical force representing the pressure exerted on AB averaged over FB is RP, and these are balanced by a single force RQ produced by the sail.]

PROP. XX. PROB. XV.



To find the figure of a sail filled with water.

Some part of the sail is represented by the curve AB, and [p. 83] the horizontal surface of the water by the line CD. The two lines EATHI and BT are tangents to the sail at A and B, themselves mutually crossing at T, of which AT is horizontal, and through the points A and B from CD and AT normals are drawn, with these crossing at C, D, and H; and on AT, AE = AC, and draw CE, and BG crosses that parallel to the horizontal at G; and with CA in F.

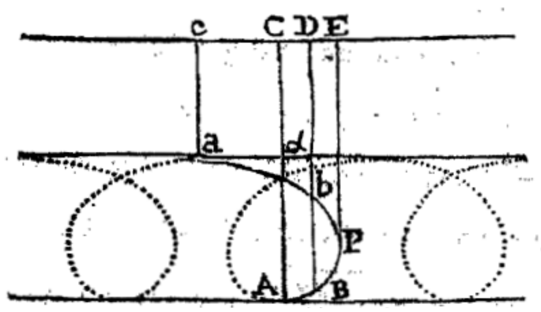
Now if on AT the ratio HI to HB is taken, as the area AEGF to the area ACDB, then the hypotenuse BI is parallel to the direction of the total absolute force of the pressure on the sail AB (by *Lem.* 7). But there is resistance to this force offered by the tensions of the threads A*a* and B*b* in the directions of the tangents AT and BT : whereby (by *Lem.* 5) the absolute force of the fluid pressure on the sail AB is equal in strength to the force applied to the point T in the direction of the line BI [for the point T is where the mechanical forces are applied, along the direction IB]. Therefore the tensions in the threads A*a* and B*b*, and the force of the pressure on the sail AB, are between themselves as the lines parallel to the directions TI, TB, BI; likewise with the weight of fluid in the volume CABD to the lateral force on the surface AB proving to be as BH to HI. But on account of the fluid nature of the liquid, the sail will be freely moved by small parts of this, just as by pulleys, until the tension in the string B*b* is equal to the tension in the string A*a*, and hence TB = TI [The magnitude of the tension in the sail is taken as constant]. Hence if the tension in the given string is designated by the given length *a*, and the weight of the fluid held by the volume ACDB, that is also called A, and the lateral pressure by the volume AEGF proportional to that, called B, the ratio
[of the three sides of the triangle BHI are in proportion to the forces, where
$\vec{\mathrm{IT}}+\vec{\mathrm{TB}}=\vec{\mathrm{IB}}$ is the resultant force exerted by the sail] is $a : \mathrm{A} : \mathrm{B} :: \mathrm{TB} : \mathrm{BH} : \mathrm{HI}$.
Moreover, $\mathrm{BH}=\sqrt{2\mathrm{TB}\times\mathrm{HI}-\mathrm{HI}^2}$; (as TI = TB) whereby also by this analogy
$\mathrm{A}=\sqrt{2a\mathrm{B}-\mathrm{BB}}$.

[For : $a/\mathrm{TB} = \mathrm{A}/\mathrm{BH} = \mathrm{B}/\mathrm{HI}$; and $\mathrm{HI} = a - \mathrm{B}$, hence $\mathrm{BH}^2 = a^2 - (a - \mathrm{B})^2$ giving the vertical pressure force $\mathrm{A}=\sqrt{2a\mathrm{B}-\mathrm{B}^2}$.]

Now CA = c, AH = z, DB = y (= CF = FG.) Then [the area EAFG (recall the 45^0 triangles!) is $\frac{c^2-y^2}{2}$ or] $B=\frac{c^2-y^2}{2}$, and [the inverse gradient of the curve at the point B is]TH : HB :: $\dot{z}:\dot{y}$, that is, (by the analogy found above, for the value of A)

$a-B:\sqrt{2a\mathrm{B}-\mathrm{B}^2}::\dot{z}:\dot{y}$. Hence $\dot{z}=\frac{\overline{B-a}\times\dot{y}}{\sqrt{2a\mathrm{B}-\mathrm{B}^2}}$ naturally with the ratio having the signs of $\dot{z}$ and $\dot{y}$. For when the curve is convex, with increasing z, and decreasing y, and we have $B=\frac{c^2-y^2}{2}$: But when the curve is concave, the fluxions of z and y have the same sign, with it being understood that $B=\frac{y^2-c^2}{2}$. [p. 84] Moreover the fluxion of z is of the same form in each case, with only the sign change. In the case of the present figure the fluid is indeed contained in the sail : in another case it lies at the lower part of the sail, and it is acted upon by a force (consider the action of a siphon) at some point proportional to the perpendicular height of the fluid below the highest horizontal surface.

Moreover z is given from a given y by integrating the curve of this fluxion, for which the abscissa is y; and the ordinate $\frac{B-a}{\sqrt{2a\mathrm{B}-\mathrm{BB}}}$. [In modern terms, this becomes $z=\int\frac{(\mathrm{B}-a)dy}{\sqrt{2a\mathrm{B}-\mathrm{BB}}}$, where $B=\frac{y^2-c^2}{2}$ and a and c are constants. This expression reappears again for the arch and the vibrating string, and is finally integrated in Prop. XXIII.] And from the undetermined coefficient in the value of the fluent z, and from the

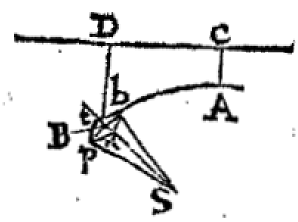


two coefficients a and c the solution can accommodate three [boundary]conditions, of which at least one is in respect to the value of z; by which indeed the position of the curve is determined. From the equation: $\dot{z}=\frac{\overline{B-a}\times\dot{y}}{\sqrt{2a\mathrm{B}-\mathrm{BB}}}(=\frac{B-a}{A}\dot{y})$ est $\frac{\dot{z}}{\dot{y}}=0,$ [*i.e.* where the curve is vertical] that is, the ordinate y is a tangent to the curve, when [the pressure force] B = a, *i.e.* $\frac{c^2-y^2}{2}=a$, [p. 85] or $y=\sqrt{c^2-2a}$. And in this case, the area [corresponding to this force] is A = a. Likewise when A = 0, $\dot{z}$ is infinite with respect to $\dot{y}$, that is the curve is a tangent to a line parallel to the horizontal. And this shall be when $2a\mathrm{B}-\mathrm{B}^2=0$, that is , either when [the lateral force] B = 0, and thus $y = c$, or when B = $2a$, and thus $y=\sqrt{c^2-4a}$. Hence if CD is the horizontal surface of the fluid, and the normal CA is drawn to the horizontal, on which are taken CA = c, and $C\alpha=\sqrt{c^2-4a}$, and through the point A and α AI and αa are drawn parallel to the horizontal, with the curve described tangent to each, and between these the whole curve is situated. And it is made under this condition, that as the ordinate first by going from CA to DB arrives at EP, where EP = $\sqrt{c^2-2a}$, and the area CABPE = a; then by going backwards from EP by Db it arrives at ca, where $ca=\sqrt{c^2-4a}$, and the area EPbac = a. For by the ordinate progressing, the area increases : whereby if the area is zero, this is the evidence, either the ordinate has not yet moved from its place, or since the decrement of the area by the regression of the ordinate is equal to the increment of the same first made by the progression of the ordinate. And hence by the infinite repetitions of the figure APB*ba* made on both sides the curve will constantly crawl between the parallel lines AI and *αa* to the horizontal in the [epicycloid form of] image of a cycloid, where the point describing the curve is taken beyond the circumference of the wheel.

The radius of curvature is equal to $\frac{a}{y}$ for this curve, that we will soon show in the following proposition.

PROP. XXI. PROB. XVI.

To find the shape of the arch supporting the weight of a liquid from above.



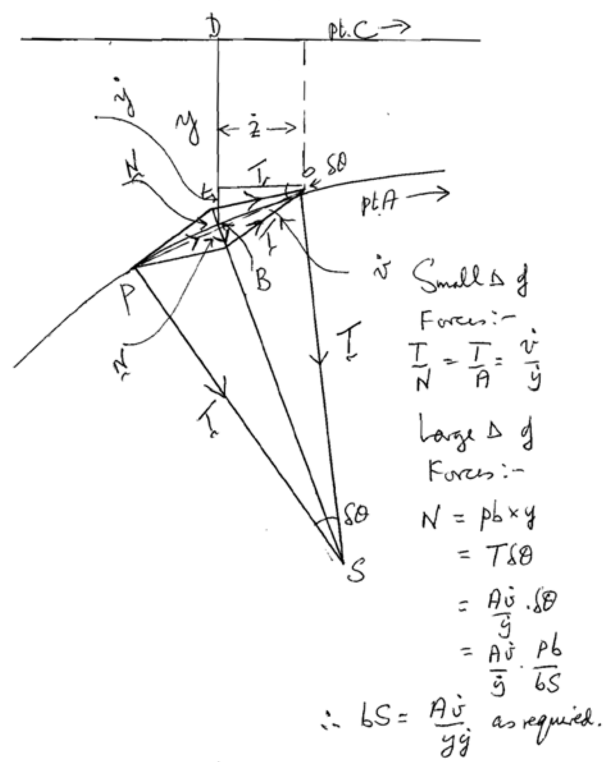
In this figure the forces are understood entirely as in Prop. 20. Since the shape of this arch and the figure of the sail is the same, only with the weight applied to different parts of the figures; for in this case the fluid is applied to convex parts of the curve, while in the other case to it is applied to concave parts of the surface. [p.86]

But in order that the situation can somehow be made clearer for the different solutions, some part of the arch is represented by the curve AB; I take A for the vertex by considering where the tangent is parallel to the horizontal. Also, the whole surface of the fluid is represented by the line CD parallel to the horizontal, and with the normals AC and BD drawn to the horizontal, these are AC = c, CD = z, DB = y, and the length of the curve AB = v, and A is written for the area CABD. Take each point *b* and *p* at equal small distances from the point B, and the normals *b*S and *p*S are drawn to the curve concurrent in S, and the tangents *bt* and *pt* are drawn meeting in *t*, and the parallelogram *btpr* is completed.

It is understood that the points *b* and *p* are each acted upon by forces trying to bring them closer together in the directions of the tangents *bt* and *pt* [and we note that the internal tension T is of constant magnitude along the curve]; for moreover these forces are to be resisted by the weight of the fluid pressing upon the base *bp*, and in the direction perpendicular to the base *bp*. Hence these forces are as the sides of the triangle *btr* parallel to these directions, or as the sides of the similar triangle S*bp* [as triangle *t*B*b* is similar to triangle SBb and likewise for triangles *t*P*b* and SB*p*; hence *tb*/*b*S = *pt*/*p*S = *tr*/BS; note however that the forces in the larger triangle have all been rotated through a right angle]. But the point *b* is acted on by the weight of the whole fluid in the area CABD, the component of the weight in the direction *bt* is to the normal component of the same weight A as the secant of *b*BD to the radius (*b*S), that is as $\dot{v}/\dot{y}$.

[Perhaps some explanation is needed for these assertions. The usual way of going about the creation of a fluxional equation is to consider the statics of some finite object from a geometrical point of view, on to which is tagged an incremental part, showing how the quantities change when a small or incremental adjustment is made to the values. In the present case, a finite weight of fluid ACDB is held up by a tension force acting along the surface of the arch. Thus, a finite arch length *v* results a tension *T* acting along the curve at some small angle θ; the other right-hand end of the finite triangle of forces should be horizontal, and the weight A is balanced by the upward tension force N = Tθ. Meanwhile the curve has the ordinate y, while the distance along the curve is v; and an incremental triangle with sides $\dot{z}$, $\dot{y}$, and $\dot{v}$ is considered, for which sinθ is equal to $\frac{\dot{v}}{\dot{y}}$. The incremental triangle has the same curvature as the original triangle of forces. This can be seen from the added drawn figure, for the small triangle of forces : the weight of the fluid A acting vertically is the normal force N, while the tangential force at the point *b* on the curve is T, then the ratio of T/N = $\frac{\dot{v}}{\dot{y}}$; and likewise the remarks concerning the larger diagram. Note that the curvature is small, and squares and higher powers of derivatives are ignored in the analysis, while the small angle *θ* is the complement of the angle *b*BD, and the normal forces are taken as vertical, even though the rather small diagram given in the text does not follow this edict.]

Likewise the weight on the [incremental] base *bp* is to the weight of the fluid in the area ACDB as $bp \times y$ is to A. Whereby with these ratios taken together, $bp \times y\dot{y}$ is to $A\dot{v}$ as *bp* to *b*S; and thus $bS = \frac{A\dot{v}}{y\dot{y}}$. But for the given $\dot{v}$, $bS = \frac{\dot{v}\dot{z}}{\ddot{y}}$ (by *Ex*. 4. *Prop*. 15).[In this situation, $\tan\theta = \frac{\dot{y}}{\dot{z}}$, and $\sec^2\theta.\delta\theta \sim \frac{\ddot{y}}{\dot{z}}$, where for small θ this becomes $\delta\theta \sim \frac{\ddot{y}}{\dot{z}}$.



Hence, $b\mathrm{S} = \frac{\dot{v}}{\delta\theta} = \frac{\dot{v}\dot{z}}{\ddot{y}}$.]Thus $\frac{\dot{v}\dot{z}}{\ddot{y}} = \frac{\mathrm{A}\dot{v}}{\dot{y}y}$, that is $\dot{z}\,y\,\dot{y} = \mathrm{A}\,\ddot{y}$, or $\dot{\mathrm{A}}\,\dot{y} - \mathrm{A}\,\ddot{y} = 0$. Hence by taking fluents [*i. e.* integrating, where a constant *a* is introduced for A and $\dot{y} = \dot{v}$ initially when z = 0], $\frac{\mathrm{A}}{\dot{y}} = \frac{a}{\dot{v}}$, that is $\mathrm{A}\,\dot{v} = a\,\dot{y}$. Moreover $\dot{v}\dot{v} = \dot{z}\dot{z} + \dot{y}\dot{y}$, [p.87] hence $\dot{y} = \frac{\mathrm{A}\dot{z}}{\sqrt{a^2 - \mathrm{A}^2}}$, [on eliminating $\dot{v}$;] that is $\dot{y}\,y = \frac{\mathrm{A}\dot{\mathrm{A}}}{\sqrt{a^2 - \mathrm{A}^2}}$, [as $\dot{z}\,y = \dot{\mathrm{A}}$.]Thus (by taking the fluents, and on adding $a + \frac{c^2}{2}$, for on taking $y = c$ at the vertex of the curve, where A = 0,) $\frac{y^2}{2} = a + \frac{c^2}{2} - \sqrt{a^2 - \mathrm{A}^2}$. For $\frac{y^2 - c^2}{2}$ write B, and with the calculation carried out it is found that $\mathrm{A} = \sqrt{2a\mathrm{B} - \mathrm{B}^2}$; and thus $\dot{z} = \frac{\overline{a - \mathrm{B}} \times \dot{y}}{\sqrt{2a\mathrm{B} - \mathrm{B}^2}}$, just as in Prop. 20.

COROLLARY.

In this solution it was found that $\frac{\mathrm{A}}{\dot{y}} = \frac{a}{\dot{v}}$, and that $b\mathrm{S} = \frac{\mathrm{A}\dot{v}}{\dot{y}y}$. Hence the pressure $\frac{\mathrm{A}\dot{v}}{y}$ multiplied by the given line increment *b*B is equal to the given [force] *a*, and the radius of curvature is $b\mathrm{S} = \frac{a}{\dot{y}}$.

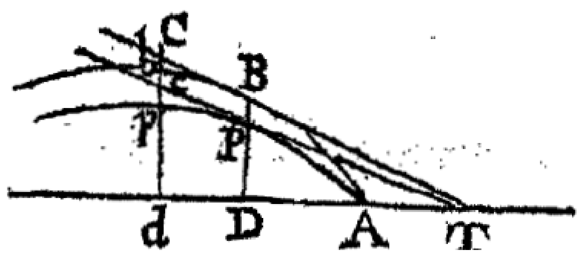
SCHOLIUM.

From this expression for the radius of curvature it is agreed that this curve is also the figure of the flexed lamina for the given force. For this will be bent in the reciprocal ratio of the radius of curvature, and thus in this curve in the direct ratio of the height *y*. But the force of the given weight to the curvature of the plane is as the same minimum distance of the same from the point of curvature. Whereby if the weight is applied to the plane on the line CD, the figure generated is that we have described here.

Indeed for the remaining points, in which this curve meets the surface of the liquid to the horizontal CD, if the maximum height *c* is lessened by an infinitesimal amount, [p. 88],this will be the curve of the same figure a vibrating musical string puts in place, in some part of its motion. Which we now hurry to demonstrate.

[This aspect of the curve being the same as that of a vibrating string, or of a sine curve, is explained in Vol. II of Feynmann's *Lectures On Physics*, p. 38-11. Essentially the curvature measures the deflection *y* of the lamina which is given approximately by $\frac{d^2y}{dx^2}$, and as these are in proportion to each other, the usual simple harmonic equation $\frac{d^2y}{dx^2} = -ky$ arises with negative curvature. One might presume that Taylor was the first person to investigate surface forces in this systematic manner, and for which the equation for the tangent to the curve of the same form keeps reappearing.]

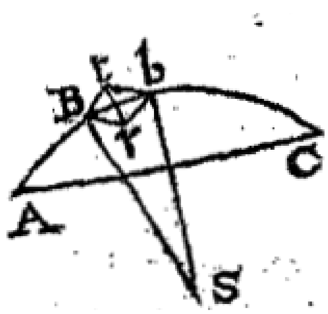
LEMMA VIII.



If for two curves AB and AP having the common abscissa AD, the ordinates DB and DP are reciprocally in a given ratio, with these diminishing indefinitely, in order that finally they coincide with the axis AD [at A]*, the final ratio of the flections* [curvatures] *is the same as that of the ordinates.*

Draw the new ordinate *dp* crossing the curves in *p* and *b*, and draw the tangents at the points B and P, crossing *dp* in C and *c*. Then on account of the given ratio of the ordinates, the tangents produced meet at some point T on the axes AD. Hence on account of the parallel lines *db* and DB, it follows that *d*C : *dc* :: DB : DP :: *db : dp* :: *d*C – *db* : *dc* – *dp*, that is *b*C : *pc* :: DB : DP.
[Or : *d*C / *dc* = DB / DP; DB / DP = *db/ dp*; and *db/ dp* = (*d*C – *db*) /(*dc* – *dp*). The last ratio following from *d*C/*dc* = *db/ dp* or *d*C / *db* – 1 = *dc / dp* – 1, giving (*d*C – *db*)/*db* = (*dc* - *dp*)/ *dp,* or *b*C /*db* = *pc*/ *dp*]
Now the ordinates *db* and DB coincide, and the vanishing increments *b*C and *pc* are as the subtangents of the angles of contact *b*BC and *p*P*c*; and they are proportional to the indefinitely diminishing ordinates of the angles. But the amount of flexion is estimated by these angles. Whereby with the curves AB and AP coinciding with the axis AD, the flexion at B to the flexion at P is in the ratio of the ordinate DB to the ordinate DP. *Q.E.D.*

LEMMA IX.



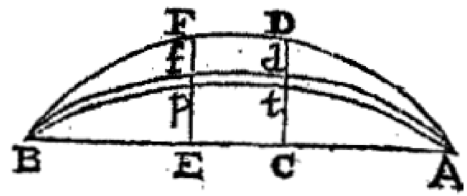
For a given density of a stretched string, the accelerating force at any point is as the curvature at that point.

The string is in the position of the curve ABC. Take a point *b* close to B, and draw the tangents B*t* and *bt* meeting in *t*, [p. 89], and the parallelogram B*tbr* is completed, and the normals B*s* and *b*S are drawn to the curve meeting in S. Then (by the principals of Statics) the resultant force acting in the movement of the element Bb in the direction *tr* to the tension in the string at B or *b*, by which that force is generated, is as *tr* ad *tB*, that is as Bb to BS, and thus the force is as $\frac{\text{Bb}}{\text{BS}}$; since the tension of the string is given. But the acceleration of the force is in the direct ratio of the absolute force and in the inverse ratio of the matter to be moved; and in this case the matter to be moved is in proportion to Bb. Whereby the acceleration of the force is as $\frac{1}{BS}$, that is as the curvature at B; for the curve is in the inverse ratio of the radius. [The unbalanced force is Tδθ, and the acceleration is as Tδθ/B*b*, which varies as 1/radius of curvature.]

PROP. XXII. PROB. XVII.

To define the motion of a stretched string.

With these I consider the string to be constructed from the thinnest material of uniform thickness; and the maximum elongation of this from the axis of the motion AB is to be infinitely small; thus in order that the tension is not changed by the increase in length of the string at its greatest distances from the axis AB, and so the inclination of the radius of curvature to the axis is always negligible.



SOLUTION.

The curve ADFB is drawn through the points A and B, an inbuilt characteristic of which is, that for any ordinates CD and EF drawn as you wish perpendicular to the axis, the curvature at D to the curvature at F shall be as DC to FE. [which is tantamount to saying in modern terms that the curve is sinusoidal.] I say that is the figure put in place in any part of its motion; likewise since all the points D and F arrive at the axis at the same time, and likewise their returning vibrations are carried out in the same period of time, the counterpart of the oscillating pendulum in the cycloid. [See Huygens' *Pendulum Clock* in these translations.] *Q.E.F.* [p. 90].

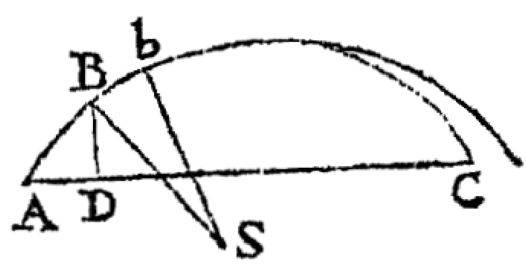
DEMONSTRATION.

For let the maximum distance of the string from the axis AB be ADFB , with all the points now at rest. Then since the curvature at D is to the curvature at F as the distance CD to the distance EF (from the hypothesis), and hence the acceleration at D to the acceleration at F is in the same ratio of the distances (by *Lem*.9); and thus in the initial motion the distances traversed D*d* and F*f* are in the same ratio: and the separate intervals traversed C*d* and E*f* are in the same ratio : and hence also the new accelerations for the points *d* and *f* are in the same ratio (by *Lem*.8 & 9) ; the initial accelerations at D and F are as the distances *d*C and *f*E to the distances DC and FE (from the same *Lemm*.) Hence the acceleration of some point D, either in the same curve ADFB, or seen to be in the different curves ADFB and A*df*B, is always as the same distance from the axis of the motion AB. Whereby (by *Prop. 51. Book.1. Phil.Nat.Principia Mathematica*) all the points of the string reach the axis at the same time, and return at the same time and the individual vibrations are performed in the given period of time, the image of a body oscillating in a cycloid. *Q.E.D.*

Again if [we suppose that] the string struck by means of a plectrum had not yet attained the form of the curve just described, then the form ADFB of this shall be that with the curvature at F to the curvature at D considered to be in a greater ratio than the distance FE to the distance DC. In this case the velocity at F is to the velocity at D, is either in a greater or smaller ratio than the distance FE to the distance DC. If the velocity at F to the velocity at D is in a greater ratio than FE to DC, then the interval F*f* described in the least time to the interval D*d* described in the same time is in a ratio greater than EF to CD; and thus the part *f*R is less with respect to FE, than EF is to CD; and thus the part *f*E is less with respect to FE, than dC is with respect to DC, and thus (by the preceding *Lemma*) the acceleration in *f* is less with respect to the acceleration in F, than the acceleration in *d* is with respect to the acceleration in D. Thus with the acceleration of the greater velocity always decreasing, and from the contrary reasoning, with the acceleration of the lesser velocity always increasing (with respect to the distances from the axis AB) the motions can be combined together between themselves finally, in order that with the points F and D arriving at some points *p* and *t*, then the velocities are to the accelerations as the distances *p*er to *t*ic; and thus with the curve A*pt* now considered to be the same as that which we have described, hence all the motions [p. 91] are in agreement. And the same eventuates if the velocity at F to the velocity at D is in a smaller ratio than for the distance FE to the distance DC. Whereby in whatever manner the string may be struck, the form of the curve here described is quickly adapted, and goes on to be moved in the manner now described. [The mode of exciting the string does explain the presence of harmonics, however.] *Q.E.D.*

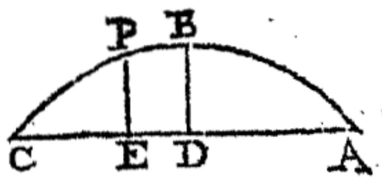
PROP. XXIII. PROB. XVIII.

With the length of the string, and also the weight, and the stretching force given, to find the periodic time of the vibrations..



Let a length L of string be extended between the points A and C, N is the weight of the same; the stretching weight is P, and the string is put in place in the position ABC ; with the points B and *b* taken close together, the normals BS and *b*S are drawn to the curve, meeting in S, and the ordinate BD is drawn normal to the axis.

By *Lem*. 9 the force of the tension is to the absolute force in moving the element of length B*b* is as BS to B*b*. But the acceleration of the force is in the ratio composed directly from the absolute force and inversely as the matter to be moved. Whereby if the weight of the increment to be moved Bb is called *p* [we would call this mass rather than weight], the acceleration of the element B*b* to the acceleration of the weight P arising from its own gravity, that is, the acceleration of the force of the string in moving the point B to the acceleration due to the force of gravity, is thus $\frac{\mathrm{B}b}{p}$ to $\frac{\mathrm{BS}}{\mathrm{P}}$; thus if the acceleration of gravity given is called 1, the acceleration of the point B is $\frac{\mathrm{B}b\times \mathrm{P}}{\mathrm{BS}\times p}$. [Thus, triangle bBS is similar to the force triangle acting on the element of length Bb; where if the tension in the string is called T, then BS is proportional to T, and Bb is proportional to T$\delta\theta$, where $\delta\theta$ is the angle BSb.



Hence $\frac{\mathrm{B}b}{p}$ is in proportion to the acceleration of the element a, while the tension T is equal to a mass Mg = P, giving T or BS/P = g, the acceleration of gravity. Hence, $a/g = \frac{\mathrm{B}b}{p} / \frac{\mathrm{BS}}{\mathrm{P}}$ as required.] But P to p is in the ratio composed from P to N, and N to p or L to Bb, hence $\frac{\mathrm{P}}{p} = \frac{\mathrm{P}\times \mathrm{L}}{\mathrm{N}\times \mathrm{B}b}$. And thus, the acceleration of the point B is $\frac{\mathrm{P}\times \mathrm{L}}{\mathrm{N}\times \mathrm{BS}}$. But since [p. 92] the curvature is proportional to the distance BD (*per Prop*. 22.) which is the same as $\frac{1}{\mathrm{BS}}$, the given quantity is $\mathrm{BS}\times \mathrm{BD}$. That shall be called a. [Thus, BS = a/BD, where a is the constant of proportionality.] Then by substituting $\frac{a}{\mathrm{BD}}$ for BS, the acceleration of the point B becomes equal to $\frac{\mathrm{P}\times \mathrm{L}\times \mathrm{BD}}{\mathrm{N}\times a}$. But for pendulums, the periodic times vary directly as the square roots of the lengths, and inversely as the force of gravity, (by *Prop. 52. Book1, Newton's Phil.Nat.Principia Mathematica.*) Whereby if a pendulum is set up, the length of which is D, the periodic time of the string to the period of this pendulum is in the root ratio (that is, for BD is applied to the acceleration $\frac{\mathrm{P}\times \mathrm{L}\times \mathrm{BD}}{\mathrm{N}\times a}$, or) $\frac{\mathrm{N}\times a}{\mathrm{P}\times \mathrm{L}}$ to D. [The cycloidal pendulum of Huygens was the archetypal simple harmonic motion, for which the time for a swing from one side to the other $T_p = \frac{1}{\pi}\sqrt{D/g}$; Taylor oscillating system behaves in the same manner, and he considers the equivalent pendulum to have a length BD and an acceleration $\frac{\mathrm{P}\times \mathrm{L}\times \mathrm{BD}}{\mathrm{N}\times a}$, hence the period $T_s/T_p = \sqrt{\frac{\mathrm{N}\times a\times \mathrm{BD}}{\mathrm{P}\times \mathrm{L}\times \mathrm{BD}}/D}$. However , the period is hence independent of BD, as it should be, presumably the reason for assuming this form.] And the number of vibrations of the string in the time of one vibration of the pendulum is $\frac{\mathrm{D}^{\frac{1}{2}}\mathrm{P}^{\frac{1}{2}}\mathrm{L}^{\frac{1}{2}}}{\mathrm{N}^{\frac{1}{2}}a^{\frac{1}{2}}}$.

It remains for us to find the size of the quantity a. Thus the string is set up in the position ABPC, and the normal ordinate DB is set up at the mid-point D of AC, and EP is some other ordinate; let DB = c, DE = z, EP = y. Then, since the radius of curvature is equal to $\frac{a}{y}$, it follows that (by Prop. 21.) $\dot{z} = \frac{\overline{\mathrm{B}-\mathrm{a}}\times \dot{y}}{\sqrt{2a\mathrm{B}-\mathrm{BB}}}$, truly with $\frac{cc-yy}{2}$ taken for B.

But with c and y vanishing, [so that B^2 is negligible w.r.t. $2aB$ as the maximum value is nearby, and B << a] this expression is $\dot{z} = \frac{-a \times \dot{y}}{\sqrt{ac^2 - ay^2}} = \frac{-a^{\frac{1}{2}} \times \dot{y}}{\sqrt{c^2 - y^2}}$, or $\dot{z} = \frac{-a^{\frac{1}{2}}}{c} \times \frac{c\dot{y}}{\sqrt{cc - yy}}$. But also $\frac{c\dot{y}}{\sqrt{cc-yy}}$ is the fluxion [$\dot{s}$] of the circular arc, of which the sine is y, and the radius c. [For, the arc length $s = c\theta$, and $\dot{s} = c\dot{\theta}$. If $y = c\sin\theta$ then $\dot{y} = c\cos\theta\dot{\theta}$, and $\dot{s} = c\dot{\theta} = \frac{c\dot{y}}{\sqrt{c^2 - y^2}}$.]Whereby with the arc of the quadrant [p. 93] in this circle considered as q, then $\mathrm{DC} = \frac{a^{\frac{1}{2}}}{c} \times q = \frac{1}{2}\mathrm{L}$. Thus $a^{\frac{1}{2}}$ is to $\frac{1}{2}\mathrm{L}$ as the radius of the circle to the arc of the quadrant; or thus the diameter of the circle is to the periphery of the same as $a^{\frac{1}{2}}$ to L [i. e. π]. Therefore p is the periphery of the circle of which the diameter is 1, and now by considering $a^{\frac{1}{2}} = \frac{\mathrm{L}}{p}$ the number of vibrations of the string, in the time of one vibration of the given pendulum of length D, is equal to $\frac{\mathrm{D}^{\frac{1}{2}}\mathrm{P}^{\frac{1}{2}}p}{\mathrm{L}^{\frac{1}{2}}\mathrm{N}^{\frac{1}{2}}}$. Q.E.I.

COROLLARY I.

By comparing the motions of the strings between each other, on account of the given p and D, the periodic time of the string is as $\frac{\mathrm{L}^{\frac{1}{2}}\mathrm{N}^{\frac{1}{2}}}{\mathrm{P}^{\frac{1}{2}}}$.

COROLLARY II.

When the strings are of the same composition, the weight of the string N is proportional to the length L of the same. Whereby in comparing the motions of strings of this kind, the periodic time is as $\frac{\mathrm{L}}{\mathrm{P}^{\frac{1}{2}}}$.

COROLLARY III.

With the same quantities in place, if besides the weight P is given, that is, if the lengths taken for the same string are stopped up in different ways, then the periodic time will be as the length L. But with regard to the uncovered strings, the half part will give the whole octave, the $\frac{2}{3}$ will give the musical fifth sound, the $\frac{3}{4}$ will give the musical fourth, and thus for the rest. Whereby the proportions of these tones are correctly defined by the numbers $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}$, etc. proportional to the lengths of the strings.

[We may note that Taylor performed experiments with his harpsichord and a chamber clock from which he had removed the timing mechanism, so that the clock, which normally kept going for several days would run down in a short time; a quill touching a cog wheel gave a note corresponding to a known number of contacts per second, which could be compared with a musical note set up in a string of the instrument set to a known length, tension, and linear density. The agreement of his formula, which is really a theoretical derivation of Mersenne's Law, was excellent. See *The Evolution of Dynamics. Vibration Theory from 1687 to 1742*, Cannon & Dostrovsky. Springer. (1981). This book, which looks at 18th century work from a modern viewpoint, also discusses some of the assumptions made by Taylor in his derivation.]

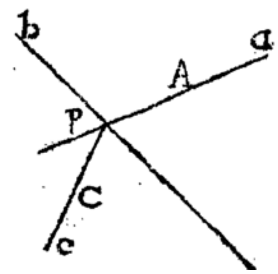
METHODUS INCREMENTORUM.

Pars Secunda IIb.

[p. 74]

LEMMA V.

Si spatium rigidum a tribus potentiis in aequilibrio tenetur, lineae directionum transibunt per idem punctum, & in eodem plano jacebunt.



Applicentur potentiae ad puncta A, B, C, atque agant in directionibus A*a*, B*b*, C*c*. Quoniam punctum quavis A tenetur in aequilibrio, vires B*b*, C*c* conjunctim sumptae componunt vim vi A*a* contrariam & aequalem. Sed (per Principia Statices) vires B*b*, C*c* hoc modo componi nequeunt, nisi transeat recta utraque B*b*, C*c* per punctum aliquod *p* in recta A*a*, atque omnes A*a*, B*b*, C*c* jaceant in plani communi. Ergo ita se res habet. *Q. E. D.*

LEMMA VI.

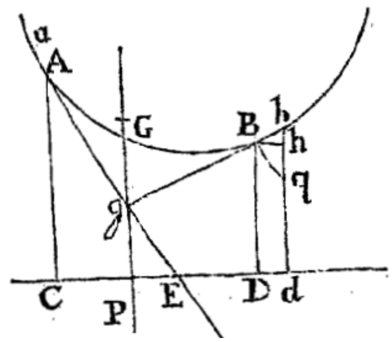
Si spatium materia gravi onustum a duobus filis sustinetur, respectu virium, quibus fila ista tenduntur, perinde est quonam modo disponatur materia in spatio isto; si modo Centrum Gravitatis semper versetur in eadem recta ad Horizontem normali.

Constat ex Staticis.

N. B. In propositionibus quatuor sequentibus sumus acturi de figuris funiculorum, linteorum aqua plenorum, atque fornicum data onera sustinentium. Omnes hae figurae, utpote ex materia physica compositae, veram habent crassitiem, sunt ad flexuram nonnihil ineptae, & cedunt aliquantulum viribus, vel extendentibus vel comprimentibus. Ergo ad haec omnia attendere oportet eum, qui velit has figuras accurate describere. Sed cum ea ad computum mathematicum difficule revocentur, & calculum, per se satis prolixum, nimis impedirent, nos, eorum effectus prorsus negligentes, fingimus figuras propositas constare ex materia perfecte flexili extentioni, atq; contractioni prorsus inepta, atque adeo tenui, ut ejus crassities poene evanescat respectu longitudinis datae. Respectu tamen sui ipsius non semper fingimus crassitiem esse absolute nullam, quoniam in funiculis, & in fornicibus propria tantum pondera sustinentibus, ea ad figuras formandas plurimum valet.

PROP. XVIII. PROB. XIII.

Data lege crassitudinis Funiculi dependentis a duobus punctis; invenire relationes fluxionum abscisae, ordinata, & curva; & definire conditiones quibus figura describenda subjici potest.



Sit AB funiculi pars quaedam dependens a punctis A & B, atque; ad rectam positione datam CD Horizonti parallelam ducantur normales AC, BD, & ad puncta A & B ducantur tangentes A*g*, B*g*, quarum A*g* occurrat ipsi CD in E, atque sint puncta *a* & *b* ipsis A & B proxima, & ducta nova ordinata *bd*, ei occurrat B*b* Horizonti parallela in *b*, atque B*q* tangenti A*g* parallela in *q*; atque sit AC ordinata positione data, & BD ordinata mobilis; & sint CD = *z*, DB = *y*, longitudo curvae AB = *v*, crassitudo in B = *x*, pondus funiculi AB = *p*, CE : CA : AE :: 1 : *n : m*, atque *a* pondus datum aequale tensioni fili in A.

Quoniam pondus totius funiculi AB sustinetur a filis brevissimis *a*A, *b*B, in directionibus tangentium A*g*, B*g*, jacet totus funiculus in plano ad Horizontalem normali; atque si partis AB centrum gravitatis sit G, per quod ducatur Horizontali normalis GP, transibit eadem GP per tangentium concursum mutuum *g*. Nam (per *Lemma* 5) respectu filorum A*a*, B*b*, perinde est, ac si omnis materia funiculi AB appendatur ad filum GP : unde spatio figurae constituto in aequilibrio per tensiones trium filorum A*a*, B*b*, GP, transibunt ea per [p. 76] *g*, atque jacebunt in eodem plano, (per *Lem.* 4) quod ob normalem GP est Horizonti perpendicularis. Sunt ergo tensiones horum filorum, ut latera trianguli Bbq eorum directionibus parallela. Sed si fiat B*b* = $\dot{z}$, erit $bb = \dot{y}$, atque B*b* = $\dot{v}$, & (ob similia triangula Bb*q*, ECA,) $bq = n\dot{z}$, atque $bq = \dot{y} + n\dot{z}, \& \ Bq = m\dot{z}$. Quare est $a : p :: m\dot{z} : \dot{y} + n\dot{z}$, hoc est $a\dot{y} + na\dot{x} - mp\dot{z} = 0.$ Est autem $\dot{p} = \dot{v}x.$ Unde eliminato *p* ab aequatione hac (nempe fluente uniformiter z) erit

$$a\ddot{y} + mx\dot{v}\dot{z} = 0.$$

Itaque resolvendo aequatione $a\ddot{y} + mx\dot{v}\dot{z} = 0, \& \ \dot{v}\dot{v} = \dot{y}\dot{y} + \dot{z}\dot{z}$, (per *Prop*. 6) dabuntur relationes ipsorum *z, y, v*; nam ex hypothesi datur *x*, vel dignatates unius, vel duorum, vel omnium *z, y, v*.

CASUS I.

Si utrumque y & v ingrediuntur valorem ipsius x, describetur curva per conditiones, quae possunt pro lubitu applicari ad valores y, & v, & fluxionum suarum (per *Prop*. 5).

CASUS II.

Si desit y in valore x erunt conditiones omnino tres, quarum una ad minimum applicanda est ad valorem ipsius y, & reliquae duae possunt pro lubitu applicari ad valores ipsorum v & y, & fluxionum suarum.

CASUS III.

Si tantum y ingreditur valorem ipsius x erunt etiam conditiones tres, applicandae pro lubitu ad valores ipsorum v, y, & fluxionum suarum modo ut una applicetur ad valorem ipsius v.

CASUS IV.

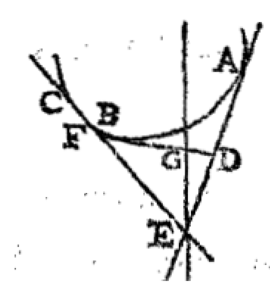
Denique si in valore ipsius x desit utrumque v & y, erunt conditiones tres, quarum una applicanda est ad valorem ipsius v, altera [p. 77] ad valorem ipsius y, & tertia pro lubitu applicati potest ad valores v, y, & fluxionum suarum.

COROLL. I.

Per hanc solutionem, est tensio fili in B ad tensionem datam in A, ut Bb ad Bq, hoc est ut $\dot{v}$ ad $m\dot{x}$. Est ergo tensio in B ut $\frac{\dot{v}}{m\dot{z}}$, vel ob datam m, ut $\frac{\dot{v}}{\dot{z}}$: Sed per aequationem $a\dot{y}+na\dot{x}-mp\dot{z}=0$, est $\dot{v}(\sqrt{\dot{y}\dot{y}+\dot{z}\dot{z}})=\frac{\dot{z}}{a}\times\sqrt{n^2a^2+1-2mnap+m^2p^2}$; atque est tensio in A aequalis a. Tensio itaque in B est $\sqrt{\frac{n^2a^2+1}{m^2}a^2-\frac{2n}{m}ap+p^2}$. Et cum haec tensio sit proportionalis ipsi $\frac{\dot{v}}{\dot{z}}$, ea erit minima quando est $\dot{v}=\dot{z}$, hoc est in curvae puncto infimo ad quod tangens est Horizonti parallala, existente tensione ista aequali $\frac{a}{m}$. Et hinc qualiscunque sit lex crassitudinis funiculi, data tensione in uno puncto A, ducendo tangentem dabitur tensio in alio quovis puncto B.

COROLL. II.

Quinetiam ducendo tangentes dividi potest funiculus in partes quarum pondera sint in data ratione. Sit enim ABC funiculus, & ad puncta tria A, B, C ducantur tangentes ADE, DEF, EFC sibi invicem occurrentes in D, E, & F. Tum (per hanc propositionem) centra gravitatis funiculorum AB,AC, BC, erunt [p.78] in perpendicularibus transeuntibus per respectivos tangentium concursus mutuos D, E, & F. Proinde si per



duarum tangentium AD, CF concursum E ducatur perpendicularis EG occurrens tangenti tertiae DBF in G, erunt pondera partium AB,BC inter se in ratione reciproca distantiarum propriorum centrorum gravitatis a centro gravitatis totius funiculi ABC, hoc est pondus ipsius AB erit ad pondus ipsius BC in ratione reciproca DG ad GF. Data ergo ratione ponderum, dabitur ratio DG ad GF. Unde datis positione rectis DE, GE, FE, dabitur directio tangentis tertiae DBF, per quam invenitur punctum B, quo dividitur funiculi ABC pondus in data ratione.

COROLL. III.

Si in expressione crassitudinis x insit tantum unum ex ipsius z, y, v, solvetur Problema per quadraturam curvarum.

CASUS I.

Primo enim insit tantum curva v in expressione crassitudinis x. Tum inito calculo per aequationes $a\dot{y} + na\dot{x} - mp\dot{z} = 0$, $\dot{v}\dot{v} = \dot{y}\dot{y} + \dot{z}\dot{z}$, & brevitatis causa pro

$\sqrt{n^2a^2 + 1 - 2mnap + m^2p^2}$ scripto R. Invenietur $\dot{z} = \frac{a\dot{v}}{R}$, atque $\dot{y} = \frac{mp-na}{a}\dot{z} = \frac{\overline{mp-na}}{R}\dot{v}$. Unde dato p per quadraturum curvae cujus abscissa est v, & ordinata x, deinde dabuntur z & y, quadrando curvas abscissa communis est v, & ordinatae sunt $\frac{a}{R}$, atque; $\frac{mp-na}{a}$.

CASUS II.

Si in expressione ipsius x insit tantum z, ducta utroque; membro aequationis $\dot{z} = \frac{a\dot{v}}{R}$ in x,

fiet $\dot{z}x = \frac{a\dot{p}}{R}$. Unde quadrando curvas [p. 79], quarum abscissae sunt z & p, & quarum

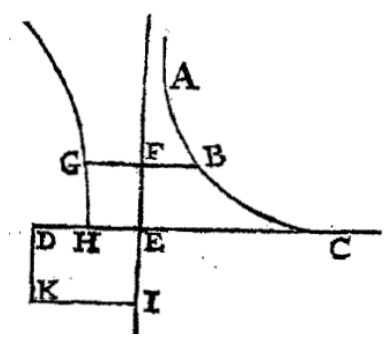
ordinatae sunt x & $\frac{a}{R}$, dabitur relatio inter z & p. Deinde ob $\dot{v} = \frac{\dot{p}}{x}$, & $\dot{y} = \frac{mp-na}{a}\dot{z}$,

quadrando curvas quarum abscissae sunt p & z, & ordinatae sunt $\frac{1}{x}$ & $\frac{mp-na}{a}$, dabuntur v & y.

CASUS III.

Denique si in valore x insit tantum y, ducendo aequationem $\dot{y} = \frac{mp-na}{R}\dot{v}$ in x, fiet

$\dot{y}x = \frac{\overline{mp-na}}{R}\dot{p}(=\frac{R}{m})$. Quare faciendo $\frac{R}{m}$ aequale areae curvae cujus abscissae est y, & ordinata x, dabitur relatio inter p & y. Unde deinde quadrando curvam cujus abscissa est y, & ordinata $\frac{a}{mp-na}$ dabitur z. Dabitur autem v, ut in casu



secundo, quadrando curvam cujus abscissa est p, & ordinata $\frac{1}{x}$.

COROLL. IV.

Per *Cas.* 3 *Cor.* 3., est $\frac{\mathrm{R}}{m}$ aequale fluenti $\dot{y}x$, quod idem aequale est tensioni fili, (per *Cor.* 1). Ergo sit ABC funiculus, ad cujus punctum infimum C ducatur tangens CEHD horizonti parallela, & ad eum ducatur normalis EF; atque ducta BFG [p. 80] horizonti parallela, in ea sit semper FG = x = crassitudini funiculi in B. Tum si sit GH curva, quam perpetuo tangit punctum G, atque areae GFEH terminatae ad tangentem CE addatur area EDKI = $\frac{\mathrm{a}}{m}$, erit area tota IKDHGFEI semper aequalis tensioni fili in B. Nam est EF = y, adeoque fluxio areae est $\dot{y}x$ = fluxioni tensioni quae in puncto C aequalis est $\frac{\mathrm{a}}{m}$.

COROLL. IV.

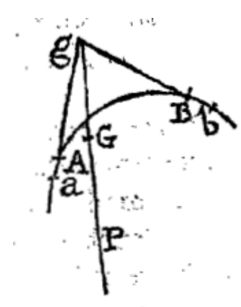
Per *Ex*.4 *Prop*. 15, est radius curvaturae aequalis $\frac{\dot{v}^3}{\dot{z}\ddot{y}}$; quare (per aequationem $a\ddot{y} - mx\dot{v}\dot{z} = 0.$) in funiculo est radius iste aequalis $\frac{a\dot{v}^2}{mx\dot{z}^2}$, hoc est, aequalis areae DI (fig. *Cor*. 4) applicatae ad FG, & deinde ductae in $\frac{\dot{v}^2}{\dot{z}^2}$, hoc est in quadratum secantis anguli, quem facit curva cum horizonte.

COROLL. IV.

Data figura funicui facile invenitur ratio crassitudinis suae. Nam dantur relationes fluxionum per speciem figurae : unde per aequationem in hoc Problemate inventam $a\ddot{y} - mx\dot{v}\dot{z} = 0$, datur crassitudo $x = \frac{a\ddot{y}}{m\dot{v}\dot{z}} - = \frac{a\dot{v}^2}{m\dot{z}^2} \times \frac{\dot{z}\ddot{y}}{\dot{v}^2} = \frac{a\dot{v}^2}{m\dot{z}^2}$ applicato ad radium curvaturae.

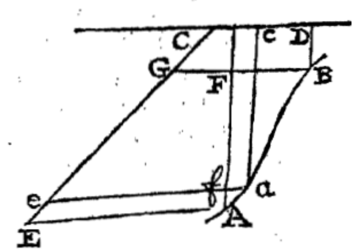
PROP. XIX. PROB. XIV. [p. 81]

Data ratione crassitudinis, invenire figuram Fornicis proprium pondus sustintentis.



Sit fornicis porito quaedam AB, & sint *a* & *b* puncta ipsis A & B proxima, & ducantur tangentes A*g*, B*g*. Tum si portionis AB centrum gravitatis sit G, per id ducta horizonti perpendicularis GP transibit per tangentium concursum *g*, (per *Lem*. 5 & 6) quoniam sustinetur pondus arcus AB pe lineolas aA, bB. Viribus itaque eodem modo interpretatis, atq; in Propositioni praecedenti, constat figuram hujusmodi fornicis, eandem esse atque; funiculi, in situ tamen inverso.

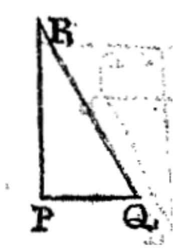
LEMMA VII.



Sit AB linea quaevis curva in plano ad horizontem perpendiculari, & per motum rectae eidem plano normalis per curvam AB describatur superficies, per quam sustineatur fluidum cujus superficies Horizontali parallela occurrat plano eidem in recta CD.
Tum dico,
Quod si ad Horizontem ducantur perpendiculares CA, DB, occurrentes curva in A & B, & recta CD in C & D, & ducantur Horizontali parallelae AE, BFG, quarum BF occurrat [p. 82] *recta CA in F, & fiat AE = CA, atque ducatur CE occurrens BF in G, erit fluidi pressio tota lateralis, per quam urget fundam AB in directione Horizonti parallela, ad pondus fluidi spatio CABD inclusi, ut area AEGF, ad aream CABD.*

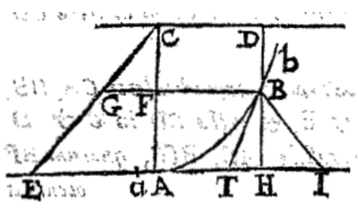
Duc *ca* horizonti perpendicularem, ipsi CA proximam, atque occurrentem curvae AB & rectae CD in *a* & *c*, atq; per a duc ae horizonti parallelam, occurrentem ipsis CA,CE in *f* & *e*. Tum coincidentibus punctis A &a, fluidi spatio CA*ac* inclusi pondere existente ut idem spatium, hoc est ut CA × *fa*, ejus pressio absoluta in lineolam A*a* perpendiculariter facta erit ut CA × A*a*; adeoque ejusdem pressionis pars lateralis horizontali parallela in directione *fa* erit ut CA × *f*A, hoc est ut EA × *f*A. Existente igitur CA*ac* fluxione ponderis, erit EA*fe* fluxio pressionis lateris; adeoque; ubi pondus sit aequale areae CABD, pressio lateralis fiet aequalis areae AEGF. *Q.E.D.*

COROLLARIUM.



Hinc si constituatur triangulum rectangulum PQR, cujus basis PQ horizontali parallela sit ad perpendicularem RP. ut spatium AEGF ad spatium ACDB, fluidi vis tota absoluta in fundum AB erit in directione hypothenusae RQ, & per eam repraesentabitur, si repraesentetur pondus per perpendicularem RP.

PROP. XX. PROB. XV.



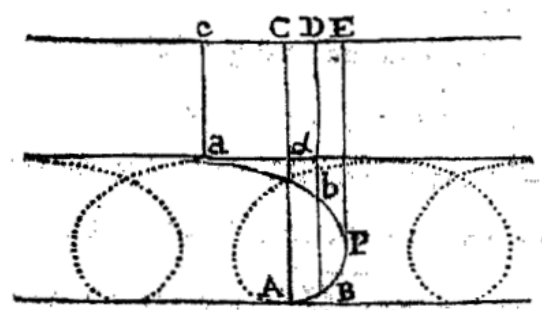
Invenire figuram Lentei aqua pleni.

Repraesentetur Lentei partio quaedam per curvam AB, & [p. 83] aquae superficies horizontali parallela per rectam horizontali parallela CD. Tangant linteum rectae duae EATHI, BT in A & B, sibi muto occurrentes in T, quarum AT sit horizontali parallela, & per puncta A & B ducantur ipsis CD, AT normales, iis occurrantes in C, D, & H, & in AT sit AE = AC, & ducta CE, ei occurrat BG horizonti parallela in G, atq; ipsi CA in F.

Jam si in AT sumatur HI ad HB, ut est spatium AEGF ad aream ACDB, erit hypothenusa BI parallela directioni totius vis absolutae pressionis fluidi in fundum AB (per *Lem.* 7) Sed huic vi resistitur per tensiones filorum Aa, Bb, in directionibus tangentium AT, BT : quare (per *Lem.* 5) vis absoluta pressionis fluidi in fundum AB aequipollet vi applicatae ad punctum T in directione rectae BI. Tensiones ergo filorum A*a*, B*b*, & vis pressionis fluidi in fundum AB, sunt inter se ut rectae earum directionibus parallelae TI, TB, BI; simul existentibus pondere fluidi in spatio CABD, & pressione laterali in fundum AB ut BH & HI. Sed ob liquoris fluiditatem, libere movebitur linteum per ejus particulas, tanquam per trochleas, adeoque est tensio fili Bb aequalis tensioni fili A*a*, & inde TB = TI. Ergo si tensio fili data se signetur per datam spatium *a*, atque pondus fluidi per spatium contiens ACDB, quod etiam dicetur A, & pressio lateralis per spatium ei proportionale AEGF, quod dicetur B, erit *a* : A : B :: TB : BH : HI. Est autem $\mathrm{BH}=\sqrt{2\mathrm{TB}\times\mathrm{HI}-\mathrm{HI}q};$ (ob TI = TB) quare etiam per hanc analogiam erit $\mathrm{A}=\sqrt{2a\mathrm{B}-\mathrm{BB}}$.

Sit jam CA = *c*, AH = *z*, DB = *y* (= CF = FG.) Tum erit $B=\frac{c^2-y^2}{2}$, atque TH : HB :: *z* : *y*, hoc est, (per analogiam supra inventam, per valorem ipsius A) $a-B:\sqrt{2a\mathrm{B}-\mathrm{BB}}::\dot{z}:\dot{y}$.

Unde sit $\dot{z}=\frac{\overline{B-a}\times\dot{y}}{\sqrt{2a\mathrm{B}-\mathrm{BB}}}$ quippe ratione habita signorum ipsorum $\dot{z}$ & $\dot{y}$. Nam ubi curva est versus convexa, crescente *z*, decrescit *y*, atque est $B=\frac{c^2-y^2}{2}$: Sed ubi curva est ad concava, ipsorum *z* & *y* fluxiones sunt ejusdem signi, existente etiam $B=\frac{yy\ -cc}{2}$. [p. 84] Ipsius autem *z* fluxio est ejusdem formae in utroque casu, signo tantum mutato. In casu figurae praesentis fluidum vere continetur in linteo: in casu altero jacet ad lintei partes inferiores, idque (puta mediante syphone) sursum urget cum vi in quovis puncto proportionali fluidi altitudini perpendiculari infra altissimam



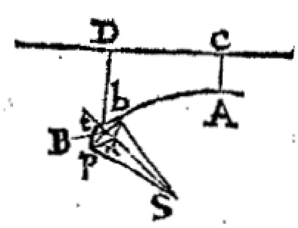
superficiem horizontalem.

Datur autem z ex dato y, quadrando curvam cujus abscissa est y, atq; ordinata $\frac{B-a}{\sqrt{2aB-BB}}$. Et per coefficentem indeterminatum in valore fluentis z, & per coefficientes duos a & c accommodari potest solutio ad tres conditiones, quarum ad minimum una respectum habebit ad valorem ipsius z; per quem nempe determinatur positio curvae. Per aequationem $\dot{z} = \frac{\overline{B-a} \times \dot{y}}{\sqrt{2aB-BB}} (= \frac{B-a}{A}\dot{y})$ est $\frac{\dot{z}}{\dot{y}} = 0$, hoc est, ordinata y tangit curvam, quando est B = a, *i.e.* $\frac{cc-yy}{2} = a$, [p. 85] vel $y = \sqrt{cc-2a}$. Et in hoc casu est area A = a. Item ubi A = 0, erit $\dot{z}$ infinita respectu $\dot{y}$, hoc est curva tanget rectam horizonti parallelam. Et hoc sit ubi est $2a$B – BB = 0, id est , vel quando B = 0, adeoque $y = c$, vel quando B = $2a$, adeoque; $y = \sqrt{cc-4a}$. Ergo si sit CD fluidi superficies horizontalis, atque ad horizontem ducatur normalis CA, & in ea sumantur CA = c, & $C\alpha = \sqrt{cc-4a}$, & per puncta A & α ducantur AI & αa horizontali parallelae, curva descripta utrasque tanget, & inter eas tota jacebit. Idque fiet sub hac conditione, ut ordinata DB primum pergendo de CA per DB perveniat in EP, ubi sit $\mathrm{EP} = \sqrt{cc-2a}$, atque area CABPE = a; deinde regrediendo de EP per Db perveniat in ca, ubi sit $ca = \sqrt{cc-4a}$, atque area EPbac = a. Nam progrediente ordinata area increscit, & regrediente ordinata decrescit: quare si area sit nihil, hoc est indicium, vel ordinatum de loco suo nondum movisse, vel quod areae decrementum per regressionem ordinatae aequale sit ejusdem incremento prius facto per progressionem ordinatae. Et hinc per infinitas repetitiones figurae APBba utrinque factas curva serpet perpetuo inter horizonti parallelas AI, αa ad instar cycloidis, ubi punctum describens sumitur peripheriam rotae.

In hac curva est radius curvaturae aequalis ipsi $\frac{a}{y}$. Quod mox demonstrabimus in propositione sequenti.

PROP. XXI. PROB. XVI.

Invenire figuram Fornicis sustinentis onus fluidis superincumbentis.



In hac figura vires omnino interpretantur ut in Prop. 20. Quare est hujus fornicis figura eadem, atque figura lintei, onere tantum ad contrarias partes figurarum applicato; in hac enim applicatur fluidum ad curvae partes convexas, in illa ad partes concavas. [p.86]

Sed ut per varias solutiones res quodammodo fiat illustrior, repraesentetur fornicis portio quaedam per curvam AB; vertice summo ubi tangens est horizontali parallela, existente A. Repraesentatur etiam fluidi superficies summa per rectam horizontali parallalem CD,

& ductis horizontali normalibus AC, BD, sint AC = c, CD = z, DB = y, atque curva AB = v, & pro area CABD scribatur A. Sume puncta b & p utrinque ad distantias minimas a puncto B, & ducatur ad curvam normales bS, pS concurrentes in S, atque ducantur tangentes bt, pt concurrentes in t, & compleatur parallelogrammum $btpr$.

Constat puncta b & p versus invicem urgeri per vires duas in directionibus tangentium bt & pt; his autem viribus resisti per pondus fluidi insistentis fundo bp, idque in directione ad fundum perpendiculari. Sunt ergo vires illae ut latera trianguli btr earum directionibus parallela, vel ut latera trianguli consimilis Sbp. Sed punctum b sustinet pondus totius fluidi in area CABD, cujus pressio in directione bt est ad ejusdem pressionem perpendicularem ut secans anguli bBD ad radium, hoc est ut $\dot{v}$ ad $\dot{y}$: item pressio in fundum bp est ad pondus fluidi in spatio ACDB ut $ad \times y$ ad A. Quare conjunctis his rationibus sit $bp \times y\dot{y}$ ad $\mathrm{A}\dot{v}$ ut bp ad bS; adeoque; $b\mathrm{S} = \frac{\mathrm{A}\dot{v}}{y\dot{y}}$. Sed data $\dot{v}$ est $b\mathrm{S} = \frac{\dot{v}\dot{z}}{\ddot{y}}$ (per *Ex*. 4. *Prop*. 15). Unde sit $\frac{\dot{v}\dot{z}}{\ddot{y}} = \frac{\mathrm{A}\dot{v}}{y\dot{y}}$, hoc est $zy\dot{y} = \mathrm{A}\ddot{y}$, vel $\dot{\mathrm{A}}\dot{y} - \mathrm{A}\ddot{y} = 0.$ Unde capiendo fluentes sit $\frac{\mathrm{A}}{\dot{y}} = \frac{a}{\dot{v}}$, hoc est $\mathrm{A}\dot{v} = a\dot{y}$. Est autem $\dot{v}\dot{v} = \dot{z}\dot{z} + \dot{y}\dot{y}$, [p.87] unde sit $\dot{y} = \frac{\mathrm{A}\dot{z}}{\sqrt{a^2 - \mathrm{A}^2}}$; hoc est $\dot{y}y = \frac{\mathrm{A}\dot{\mathrm{A}}}{\sqrt{a^2 - \mathrm{A}^2}}$. Unde (capiendo fluentes, & addendo $a + \frac{cc}{2}$, ut fiat y = c in vertice curvae, ubi est A = 0,) sit $\frac{y^2}{2} = a + \frac{c^2}{2} - \sqrt{a^2 - \mathrm{A}^2}$. Pro $\frac{y^2 - c^2}{2}$ scribe B, atque calculo peracto invenietur $\mathrm{A} = \sqrt{2a\mathrm{B} - \mathrm{B}^2}$; adeoque $\dot{z} = \frac{\overline{a - \mathrm{B}} \times \dot{y}}{\sqrt{2a\mathrm{B} - \mathrm{B}^2}}$, omnino ut in Prop. 20.

COROLLARIUM.

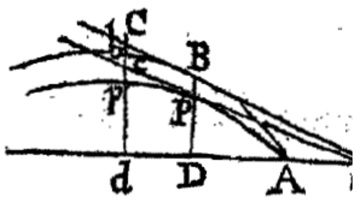
In hac solutione inveniebatur $\frac{\mathrm{A}}{\dot{y}} = \frac{a}{\dot{v}}$, atque erat $b\mathrm{S} = \frac{\mathrm{A}\dot{v}}{y\dot{y}}$. Est ergo pressio $\frac{\mathrm{A}\dot{v}}{y}$ facta in lineolam bB aequalis datae a, atque radius curvaturae bS = $\frac{a}{y}$.

SCHOLIUM.

Ex hac expressione radii curvaturae constat eandem hanc curvam etiam esse figuram laminae data vi flexae. Nam est flexura in ratione reciproca radii curvaturae, adeoque in hac curva in directa ratione altitudinis y. Sed dati ponderis vis ad laminam incurvandam est ut ejusdem distantia minima a puncto flexurae. Quare si pondus applicetur ad laminam in recta CD, figura genita ea erit quam hic descripsimus.

Quinetiam manentibus punctis, in quibus haec curva occurrit fluidi superficiei horizontali CD, si minuatur altitudo maxima c in infinitum [p. 88], erit haec eadem curva cujus figuram induit Nervus musicus vibrans, in quovis articulo motus sui. Quod jam demonstrare properamus.

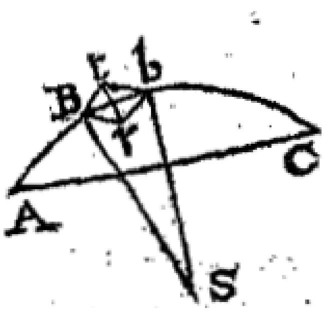
LEMMA VIII.



Si curvarum duarum AB, AP abscissam communem habentium AD, ordinatae DB, DP sint ad invicem in data ratione, imminutis iis in infinitum, ut curvae tandem coincidant cum axe AD, erit ultima ratio flexurae eadem, quae ordinatarum.

Duc novam ordinatam *dp* curvis occurrentem in *p* & *b*, & ad puncta B & P, duc tangentes occurrentem *dp* in C & *c*. Tum ob datam rationem ordinatarum, tangentes productae concurrent in aliquo puncto T in axe AD. Unde ob parallelas *db*, DB, erit *d*C : *dc* :: DB : DP :: *db : dp* :: DC – *db* : *dc* – *dp*, hoc est *b*C : *pc* :: DB : DP. Coincidant jam ordinatae *db*, DB, atque lineolae evanescentes *b*C, *pc* erunt subtensae angulorum contactus *b*BC , *p*P*c*; atque imminutis ordinis in infinitum erunt angulis ipsis proportionales. Sed per hos angulos aestimatur flexura. Quare coincidentibus curvis AB, AP cum axe AD, erit flexura in B ad flexuram in P ut ordinata DB ad ordinatam DP. *Q.E.D.*

LEMMA IX.

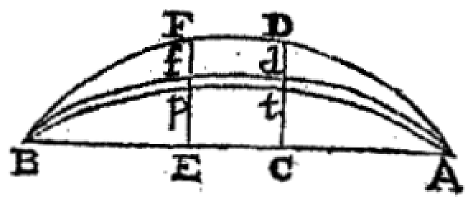


Data crassitudine nervi tensi, vis accelatrix cujusvis puncti est ut flexurain isto puncto.

Sit nervus in positione curvae ABC. Sume punctum *b* puncto B proximum, & duc tangentes B*t, bt* concurrentes in *t*, [p. 89], & compleatur parallelogrammum B*tbr*, atque ad curvam ducantur normales B*s*, *b*S concurrentes in S. Tum (per principia Statices) erit vis absoluta ad movendam particulam Bb in directione tr ad fili tensionem in B, vel in b, per quam generatur vis illa, ut tr ad tB, hoc est ut Bb ad BS, adeoque vis illa est ut $\frac{\mathrm{Bb}}{\mathrm{BS}}$; quoniam fili tensio est data. Sed est vis acceleratrix in ratione vis absolutae directe & materiae movendae inverse; & in hoc casu est materia movenda ut Bb. Quare est vis acceleratrix ut $\frac{1}{\mathrm{BS}}$, hoc est ut curvatura in B; curvam enim est in ratione reciproca radii.

PROP. XXII. PROB. XVII.

Definire motum Nervi tensi.



In his pono nervum constare ex materia tenuissima uniformiter crassa, ejusque; elongationem maximum ab axe motus AB esse infinite parvam; ita ut vis tensionis non mutetur per auctam longitudinem nervi in majoribus suis distantiis ab axe AB, & ut inclinatio radiorum curvaturae ad axem sit semper insensibilis.

SOLUTIO.

Per puncta A & B describatur curva ADFB, cujus indoles sit, ut, ductis ad libitum ordinatis ad axem normalibus CD, EF, sit curvatura in D ad curvaturam in F, ut DC ad FE. Dico quod haec sit figura, quam induit nervus in quovis articulo motus sui; item quod puncta omnia D, F simul ad axem pervenientia & simul redeuntia vibrationes suas omnes peragunt in eodem tempore periodico, ad instar penduli oscillantis in Cycloide. *Q.E.F.* [p. 90].

DEMONSTRATIO.

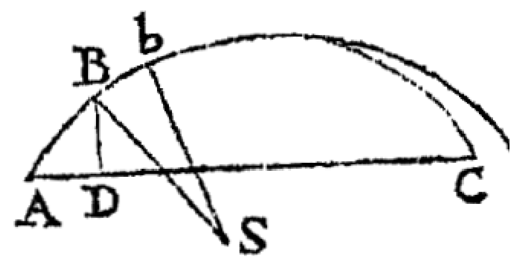
Sit enim curva ADFB nervi distantia maxima ab axe AB, puctis omnibus jam quiescentibus. Tum quoniam curvatura in D est ad curvaturam in F ut distantia CD ad distantiam EF (ex hypothesi) erit acceleratio in D ad accelerationem in F in eadem ratione distantiarum (per *Lem*.9) adeoque; in initio motus spatia simul percursa D*d*, F*f* erunt in eadem ratione: adeoque; & divisim spatia percurrenda Cd, Ef erunt in eadem ratione : unde etiam accelerationes novae in punctis d, & f erunt in eadem ratione (per *Lem*.8, & 9) atque; erunt ad accelerationes priores in D & F, ut distantiae *d*C & *f*E ad distantias DC & FE (per eadem *Lemm*.) Ergo puncti cuiusvis D, vel ut in eadem curva ADFB, vel ut in diversis ADFB & A*df*B spectati, acceleratio semper est ut ejusdem distantia ab axe motus AB. Quare (per *Prop. 51. Lib.1. Phil.Nat.Princ.Math.*) puncta omnia Nervi ad axem simul perveniunt, simul redeunt, & vibrationes singulas peragunt in dato tempore periodico, ad instar corporis in Cycloide oscillantis. *Q.E.D.*

Porro si Nervus plectro modo percussus nondum induerit formam curvae jam descriptae; sit ejus forma ADFB, curvatura in F existente ad curvaturam in D in majori ratione quam distantiae FE ad distantiam DC. In hoc casu velocitas in F est ad velocitatem in D, vel in majori, vel in minor ratione, quam distantia FE ad distantiam DC. Si sit velocitas in F ad velocitatem in D in ratione majore quam FE ad DC, erit spatium F*f* in tempore minimo descriptum ad spatium D*d* eodem tempore descriptum in ratione majore quam EF ad CD; adeoque divisim erit *f*R minor respectu FE, quam est EF ad CD; adeoque divisim erit *f*E minor respectu FE, quam est dC respectu DC, indeque (per *Lemm*. praed.) acceleratio in *f* minor erit respectu accelerationis in F, quam est acceleratio in *d* respectu acceleratione in D. Itaque majoris velocitatis acceleratione semper decrescente, & minor velocitatis acceleratione e contra semper crescente (respectu distantiarum ab axe AB) motus inter se tandem ita temperabuntur, ut perventis punctis F

& D in puncta quaedam p & t, erunt tum velocitates tum accelerationes ut distantiae *p*E, *t*C; adeoque curva A*tp*B jam existente eadem quam descripsimum, motus dehinc [p. 91] omnes conspirent. Atque idem eveniet si sit velocitas in F ad velocitatem in D in minore ratione quam distantiae FE ad distantiam DC. Quare quocunque modo percutiatur nervus, quam citissime induet formam curvae hic descriptae, atque perget moveri more jam descripto. *Q.E.D.*

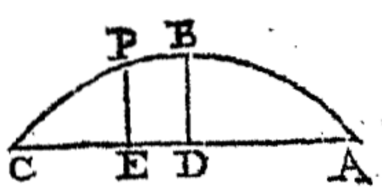
PROP. XXIII. PROB. XVIII.

Datis longitudine Nervi, ejusdem pondere, & pondere tendiente; invenire tempus periodicum.



Sit Nervi inter puncta A & C extensi longitudo L, ejusdem pondus N, atque; pondus tendens P, & constituatur nervus in positione ABC; & sumptis punctis B & b proximis, ducantur ad curvam normales BS, bS, concurrentes in S, & ducantur ordinata axi normalis BD.

Per Lem. 9 est vis tensionis Nervi ad vim absolutam ad movendam particulam B*b* ut BS ad B*b*. Sed est vis acceleratrix in ratione composita vis absolutae directe & materiae movendae inversae. Quare si particulae movendae B*b* pondus dicatur *p*, erit acceleratio particuale B*b* ad accelerationem ponderis P ab ipsius propria gravitate oriundam, hoc est, vis acceleratrix vervi ad movendum punctum B ad vim acceleratricem gravitatis, ut $\frac{Bb}{p}$ ad $\frac{BS}{P}$; unde si gravitatis acceleratio data dicatur 1, erit puncti B acceleratio $\frac{Bb\times P}{BS\times p}$. Sed est P ad *p* in ratione composita P ad N, & N ad *p*, seu L ad B*b*, unde sit $\frac{P}{p}=\frac{P\times L}{N\times Bb}$. Adeoque; acceleratio puncti B est $\frac{P\times L}{N\times BS}$. Sed quoniam [p. 92] est curvatura ut distantia BD (*per Prop*. 22.) quae eadem est ut $\frac{1}{BS}$, erit $BS\times BD$ quantitas data. Sit illa *a*. Tum pro BS substituto $\frac{a}{BD}$, fiet acceleratio puncti B aequalis $\frac{P\times L\times BD}{N\times a}$. Sed in funipendulis tempora periodica sunt in dimidiata ratione longitudinum directe, & virium acceleratricium inverse (per *Prop. 52. Lib.1. Phil.Nat.Princ.Math.*) Quare si constituatur pendulum cujusvis longitudinis D, erit tempus periodicum nervi ad tempus periodicum istius penduli in dimitiata ratione (BD applicatae ad $\frac{P\times L\times BD}{N\times a}$, hoc est) $\frac{N\times a}{P\times L}$ ad D. Atque numerus vibrationum nervi in tempore unius vibrationis penduli erit $\frac{D^{\frac{1}{2}}P^{\frac{1}{2}}L^{\frac{1}{2}}}{N^{\frac{1}{2}}a^{\frac{1}{2}}}$.



Superest ut inveniamus quantitatem a. Constituatur itaque nervus in positione ABPC, & ad axis AC punctum medium D erigatur ordinata normalis DB, & sit alia quaevis ordinata EP, atque; sint DB = c, DE = z, EP = y. Tum ob radium curvatura aequalem $\frac{a}{y}$, erit (per Prop. 21.) $\dot{z} = \frac{\overline{\mathrm{B\text{-}a}} \times \dot{y}}{\sqrt{2a\mathrm{B} - \mathrm{BB}}}$, nempe pro B sumpto $\frac{cc - yy}{2}$.

Sed evanescentibus c & y haec expressio sit $\dot{z} = \frac{-a \times \dot{y}}{\sqrt{ac^2 - \mathrm{a}y^2}} = \frac{-a^{\frac{1}{2}} \times \dot{y}}{\sqrt{c^2 - y^2}}$, vel $\dot{z} = \frac{-a^{\frac{1}{2}}}{c} \times \frac{c\dot{y}}{\sqrt{cc - yy}}$. At enim est $\frac{c\dot{y}}{\sqrt{cc - yy}}$ fluxio arcus circularis, cujus sinus est y, & radius c. Quare arcu quadrantali [p. 93] in isto circulo existente q, erit $\mathrm{DC} = \frac{-a^{\frac{1}{2}}}{c} \times q = \frac{1}{2}\mathrm{L}$. Unde est $a^{\frac{1}{2}}$ ad $\frac{1}{2}\mathrm{L}$ ut radius circuli ad arcum quadrantalem, vel $a^{\frac{1}{2}}$ ad L ut diameter circuli ad peripheriam ejusdem. Sit ergo p peripheria circuli cujus diameter est 1, atque jam existente $a^{\frac{1}{2}} = \frac{\mathrm{L}}{p}$ erit numerus vibrationum Nervi, in tempore unius vibrationis penduli datae longitudinis D, aequalis $\frac{\mathrm{D}^{\frac{1}{2}}\mathrm{P}^{\frac{1}{2}}p}{\mathrm{L}^{\frac{1}{2}}\mathrm{N}^{\frac{1}{2}}}$. Q.E.I.

COROLL. I.

Comparatis motibus Nervorum inter se, ob data p & D, erit tempus periodicum Nervi ut $\frac{\mathrm{L}^{\frac{1}{2}}\mathrm{N}^{\frac{1}{2}}}{\mathrm{P}^{\frac{1}{2}}}$.

COROLL. II.

Ubi Nervi constituuntur ex eodem filo, est Nervi pondus N ut ejusdem longitudo L. Quare in comparandis motibus hujusmodi Nervorum, est Tempus periodicum ut $\frac{\mathrm{L}}{\mathrm{P}^{\frac{1}{2}}}$.

COROLL. III.

Iisdem positis, si praeterae detur pondus P, hoc est, si longitudines sumantur in eodem Nervo diversimodo obturato, erit Tempus periodicum ut L. Sed respectu Nervo aperti, pars dimidia edit sonum Diapason, pars $\frac{2}{3}$ edit sonum Diapente, pars $\frac{3}{4}$ edit sonum Diatessaron, & sic de caeteris. Quare a Musicis recte definiuntur proportiones hujusmodi tonorum per numeros $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}$, &c. longitudinibus Nervorum proportionales.