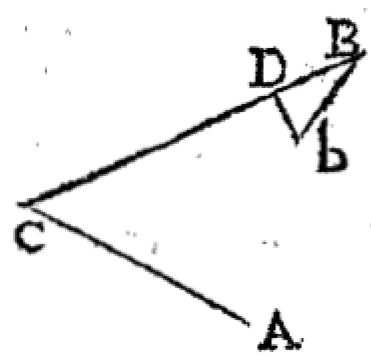
The Method of Increments.

The Second Part. [IIc]

LEMMA X. [page 94]



A particle travelling in a in given direction with a given velocity adheres to a given fixed point on a rigid length about a given axis of motion : in the figure at hand the points C, B, b, A lie on the intersection of the given plane perpendicular to the axis of rotation, with the length of the line bB proportional to the velocity, in the direction of the velocity of the particle that strikes the rigid length CB, of which the projection bB is formed in the plane of the diagram. [Thus, a particle is incident at some angle to the plane of the diagram at the point B, and the resolved component of the velocity in the plane of the diagram is *b*B, of which the component perpendicular to the fixed line CB is *b*D; the axis of rotation passes through C and is perpendicular to the plane of the diagram]*;*
and through any other given point [D]*of the same rigid length ; the normal bD is drawn to CB: then the motion produced in this rigid length by the impulse of this particle will always be the same, if the magnitude [mass] of the second particle is to the magnitude [mass]of the first particle as* CB^2 *to* CA^2*, and the velocity of the second particle is Db* $\times \frac{CA}{CB}$*, and striking that other given point A in this projection, in a direction perpendicular to the axis of rotation CA.*

[We note that the magnitude of the angular momentum of a mass m travelling with a velocity *v* in a plane at a distance *r* from a perpendicular axis is given by $L = mvr$, and that the instantaneous angular velocity is given by v = ωr, giving $L = m\omega r^2$.

The sum of the contributions of all the masses δm of the particles in a body located at distances *r* from the axis of rotation, of the form $\delta m.r^2$ can be replaced by a mass located at a certain point B at a distance CB from the axis of rotation. Thus, according to elementary statics, the centre of mass CG of mass M of such a one dimensional body is given by $CG \times M = \sum r\delta m \to \int rdm$; while the centre of oscillation BC defined here is given by $BC^2 \times M = \sum r^2\delta m \to \int r^2 dm.$]

For all the motion is taken up [absorbed] by the rigid length, except for [the component in] planes normal to the axis. Whereby for all the points, by which in the aforesaid manner, the motion can be reduced to that in such a plane, so that the velocity of the striking particle and the direction of the same in this plane is represented by the line B*b*. But the part BD of this velocity BD is taken up by the resistance of the point of rotation C, and the motion of the rigid length is produced by the remaining normal component of velocity D*b*.

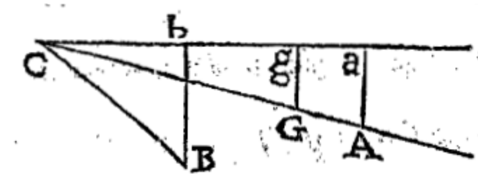
Moreover, the velocity of the point A is to the velocity of the point B, as the distance from the centre CA is to the distance CB [For constant angular velocity ω.]: whereby the corresponding velocity of the point A produced by the motion of the particle striking at B is $Db \times \frac{CA}{CB}$. Also, the [angular] momentum of the same particle [p.95] is to the [angular] momentum that is given to the point A, as CA to CB : whereby if the magnitude [mass] of the particle itself is *p*, by considering the [component of the linear] momentum of this at B to be $p \times Db$, then the [like linear] momentum contributed to the point A will be $p \times Db \times \frac{\mathrm{CA}}{\mathrm{CB}}$. But the same velocity and the same [angular] momentum is produced at the point A by a particle of mass $p \times \frac{\mathrm{CB}^2}{\mathrm{CA}^2}$ striking at that point, in the direction perpendicular to CA with the velocity $Db \times \frac{\mathrm{CA}}{\mathrm{CB}}$. Whereby with regard to the motion produced in the rigid length, in all cases, either the particle *p* strikes at B, or a particle [of size] $p \times \frac{\mathrm{CB}^2}{\mathrm{CA}^2}$ strikes at A, now by definition. *Q.E.D.*

[Thus, particles in a rotating body have their masses adjusted by this inverse square method and these contributions are then added together at the centre of oscillation to give a single mass with the same total angular momentum as the distributed mass. A particle has a certain amount of angular momentum $p \times Db \times \mathrm{CB}$ in the plane of the diagram, given as above in modern terms by $L = m\omega R^2 = mvR$. If a particle of the same mass but with velocity v' perpendicular to the arm strikes the body at A at some other distance R' from the axis, where the velocity v' at A is related to the corresponding velocity v at B defined as above, by $v/R = v'/R'$, essentially for a constant angular velocity ω, then the corresponding velocity at A is $Db \times \frac{\mathrm{CA}}{\mathrm{CB}}$, as D*b* corresponds to *v*, CB to *R*, and CA to *R'*; giving the corresponding angular momentum contribution at A as $p \times Db \times \frac{\mathrm{CA}^2}{\mathrm{CB}}$. Now, if a particle of mass $p \times \frac{\mathrm{CB}^2}{\mathrm{CA}^2}$ strikes A in the same manner, with the same speed $Db \times \frac{\mathrm{CA}}{\mathrm{CB}}$, then it contributes the angular momentum $p \times \frac{\mathrm{CB}^2}{\mathrm{CA}^2} \times Db \times \frac{\mathrm{CA}^2}{\mathrm{CB}} = p \times Db \times \mathrm{CB}$, which is the same as the initial particle at B. The same result follows if the rigid line is allowed to rotate at a uniform rate and pick up particles as it goes according to some rule of the density; thus, the rotational effect of the added mass is found as a geometrical entity, independent of the velocity, which corresponds to the later idea of the moment of inertia, due to Euler.]

PROP. XXIV. PROB. XIX.

To find the centre of oscillation of a hanging body from a given axis parallel to the horizontal.

By the centre of oscillation I understand the point, the velocity of which is the same as if the remainder of the body has been removed, and that point alone oscillates around the same axis [and with the whole mass concentrated at this point. This quantity had been investigated initially by Huygens in his *Horologium* regarding the oscillations of physical pendulums, and had been proposed to him in his youth by Mersenne, at which time he had not been able to solve the problem. See the text in these translations].



Therefore let the present figure be in a plane perpendicular to the horizontal and to the axis of oscillation, and which passes through the centre of gravity G, and let C be the intersection of this plane with the axis of oscillation. [Thus, take the vertical plane of the figure to be that of the page, the axis of oscillation passes through C out of the plane of the page, and the horizontal plane is also out of the page.] Draw Cg parallel to the horizontal, and A is the Centre of Oscillation sought in the line CG. Let p be [the mass of] a small prismatic shape placed parallel to the axis of the motion [i. e. coming out of the page, and at some perpendicular distance CB from the axis of rotation, but not shown on the diagram], & meeting the plane of the figure in the point B. Draw CB, and draw the perpendiculars Bb, Gg, Aa to the line Cg crossing that line in b, g, a. [p.96]

If the acceleration of gravity is 1 [down the page], then the tangential acceleration of the element p at some point on the figure at a distance CB from the axis moving around the point C is $\frac{Cb}{CB}$ Whereby the [change in the] momentum at the point A arising from the motion of the particle p is ($p \times \frac{Cb}{CB} \times \frac{CB}{CA} =$) $p \times \frac{Cb}{CA}$. But (from *Lem.* 10) the [change in the angular] momentum of the point A produced by the particle p at the point B is always the same as if produced by a particle of weight $p \times \frac{CB^2}{CA^2}$ striking the same point A. Whereby in place of the particle p in position B, by adding new particles of weight $p \times \frac{CB^2}{CA^2}$ at point A, the acceleration of the point A is given, arising from all the forces of the given body taken together, by the known rule of collisions; truly by applying the sum of all the moments $p \times \frac{Cb}{CA}$ to the sum of all the small particles by substituting $p \times \frac{CB^2}{CA^2}$, that is (on account of the given CA) by applying the sum of all $p \times Cb \times CA$ to the sum of all $p \times CB^2$: But from the noted property of the centre of gravity, if the weight of the whole given body is P, the sum of all $p \times Cb$ is equal to P × Cg. Whereby if Q is written for the sum of all $p \times \mathrm{CB}^2$, the acceleration of the point A is equal to $\frac{\mathrm{P} \times \mathrm{C}g \times \mathrm{CA}}{\mathrm{Q}}$. But from the hypothesis, this acceleration is equal to the acceleration of the point A proposed $\frac{\mathrm{C}a}{\mathrm{CA}}$ or $\frac{\mathrm{C}g}{\mathrm{CG}}$. Hence $CA = \frac{Q}{\mathrm{P} \times \mathrm{C}G}$. [Thus, the whole mass located at A will fall initially with unit acceleration, giving this result.] Thus by finding the sum of all the particles of the proposed body multiplied by the squares of their shortest distances from the axis of oscillation, by the method of inverse fluxions, the distance of the centre of oscillation from the axis will be given, by applying this sum to the product of the given body taken by the distance of the centre of gravity from the axis of oscillation. *Q.E.I.* [p.97]

[Thus, when the element is placed horizontally, it can be assumed to fall with the acceleration of gravity, taken as 1, and later when the element is moving horizontally, the downwards acceleration is zero, while for some intermediate position, the tangential acceleration is proportional to the cosine of the angle BCb, or $\frac{\mathrm{C}b}{\mathrm{CB}}$. Thus, $p \times \frac{\mathrm{C}b}{\mathrm{CB}}$ is the force acting along the tangent at B, which by the principle of moments is equivalent to a force ($p \times \frac{\mathrm{C}b}{\mathrm{CB}} \times \frac{CB}{\mathrm{CA}} =$) $p \times \frac{\mathrm{C}b}{\mathrm{CA}}$ acting at A perpendicular to the axis of rotation. This moment gives rise to a contribution to the change of angular moment at A of $p \times \frac{\mathrm{C}b}{\mathrm{CA}} \times$ CA, or simply p × C*b*. Now, according to the lemma, we can regard this moment as acting on a mass of $p \times \frac{\mathrm{CB}^2}{\mathrm{CA}^2}$ at A, in which case the sum of all the moments acting on the sum of all the masses at the same location gives a linear acceleration

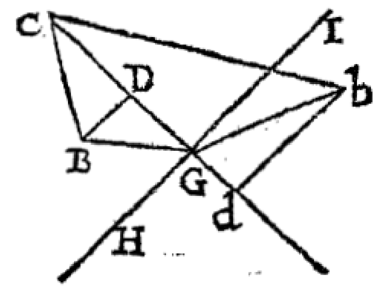
$$a = \alpha.CA = \frac{(\sum p \times Cb)}{(\sum p.CB^2)/CA^2}; \text{or } a = \frac{P \times Cg \times CA}{Q}.$$

Thus, we have the contribution to the moment of inertia I given by $\delta I = \delta m r^2 = p.CB^2; I = \sum p.CB^2 = Q;$ also, the centre of mass satisfies $P \times Cg = \sum p.Cb$. Hence, taking moments,

$P \times Cg = I\alpha = Qa/CA;$ hence the linear acceleration $a = \frac{P \times Cg \times CA}{Q}$. Finally,

$1 = \frac{P \times Cg \times CA}{Q}$, giving $\mathrm{CA} = \frac{Q}{P \times Cg}$. We note that there is no reference to angular velocity or acceleration made by Taylor, which was obviously not in vogue at the time, and he has steadily worked with the linear quantities, while we have sneaked in some angular quantities in the explanations to ease the analysis. Finally a word of caution, Taylor has not worked out what is called 'the moment of inertia' of a figure, which is normally done about a principle axis of symmetry passing through G, such as CG : he has worked out the moment of inertia about an axis of rotation passing through the point of suspension C, at right angles to CG, which amounts to the length of the equivalent simple pendulum.]

COROLLARY I.



With the points C, G, B considered to be in the same plane, and in addition draw the normals HGI and BD to CG [CG is the line containing the point of suspension C and the centre of gravity G, perpendicular to the axis of oscillation; B is a point on the body, while HGI is a normal line through the centre of mass G]. Then it follows that $\mathrm{CB}^2 = \mathrm{BG}^2 + \mathrm{CG}^2 - 2\mathrm{CG} \times \mathrm{GD};$ truly by dropping a line from the point B to the same side of the line HI and the point C. But when the point *b* falls on opposite side of the line HI, then $Cb^2 = b\mathrm{G}^2 + \mathrm{CG}^2 + 2\mathrm{CG} \times \mathrm{G}d.$ [Forms of the cosine rule for triangles BCG and Co*b* when the angle BGC is acute or *b*GC obtuse.] Hence the sum of all the elements taken with their own CB^2 for the whole figure, equals the sum of all the small parts taken with their own $\overline{\mathrm{BG}^2 + CG^2}$ for the whole figure, less the sum of all $p \times 2\mathrm{CG} \times \mathrm{GD}$ from the one side of the line HI, and plus the sum of all $p \times 2\mathrm{CG} \times \mathrm{G}d$ from the other side . But from the nature of the centre of gravity, the sum of all $p \times 2\mathrm{CG} \times \mathrm{G}d$ is equal to the sum of all $p \times 2\mathrm{CG} \times \mathrm{GD}$. Whereby the sum of all $p \times \mathrm{CB}^2$, or Q, is equal to the sum of all $p \times \mathrm{GB}^2 + p \times \mathrm{CG}^2$, that is, (on account of the given CG^2) equal to $p \times \mathrm{CG}^2$ plus the sum of all $p \times \mathrm{GB}^2$. If hence D is written for the sum of all $p \times \mathrm{GB}^2$, then $\mathrm{Q} = p \times \mathrm{CG}^2 + \mathrm{D}$; and hence $\mathrm{CA}\ (= \frac{\mathrm{Q}}{\mathrm{P} \times \mathrm{CG}}) = \mathrm{CG} + \frac{\mathrm{D}}{\mathrm{P} \times \mathrm{CG}}$.

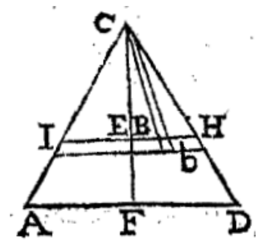
[A form of the parallel axis theorem for this kind of moment of inertia, where D is the moment of inertia about the centre of mass, and Q is the moment of inertial about the point C at a distance CG from G.]

COROLLARY II.

Hence $\mathrm{GA} = [CA - CG] = \frac{\mathrm{D}}{\mathrm{P} \times \mathrm{CG}}$. Thus for a given direction of the axis of oscillation with respect to the figure of the proposed body, the product $\mathrm{CG} \times \mathrm{GA}\ (= \frac{\mathrm{D}}{\mathrm{P}})$ is given [the square of the radius of gyration]. Hence for a given centre of oscillation arising from a single instance of the direction of the axis, the same is given in all others cases, from the last calculation. [p. 98] [See, e.g., *Analytical Experimental Physics*, Ference et al, p. 107.]

In any proposed case the calculation can be established, either by finding Q from a convenient assumption of the point C, or by finding D itself, as appears more convenient.

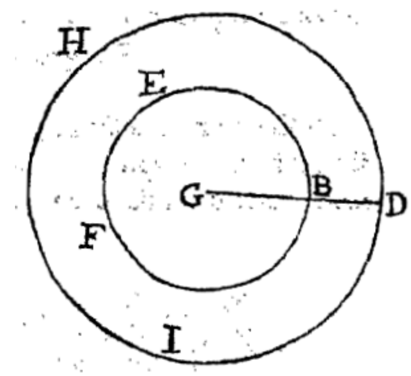
EXAMPLE I.



The centre of oscillation of the cone, generated by the rotation of the isosceles triangle ACD about the perpendicular CF, is found, hanging from the point C. Draw IEH parallel to the base crossing CA, CF, CD, in I, E, H; and let CF = a. FD = c, CE = z, and hence EH = $\frac{cz}{a}$, and take some point B on the line EH, such that EB = x. Then if for an elemental prism p for the point B, with the base consisting of $\dot{z}\dot{x}$ (= parallelogram Bb) with the height of the same considered to be $2\sqrt{\mathrm{EH}^2 - x^2}$, the volume element will be $p = 2\dot{z}\dot{x}\sqrt{\mathrm{EH}^2 - x^2}$, and $p \times \mathrm{CB}^2 = 2\mathrm{CE}^2 \times \dot{z}\dot{x}\sqrt{\mathrm{EH}^2 - x^2} + 2x^2\dot{z}\dot{x} \times \sqrt{\mathrm{EH}^2 - x^2}$. This is the fluxion of the sum of all $p \times \mathrm{CB}^2$ in the line IH. But if A is written for the area of the circle of this diameter IH, the whole fluent of this adjacent to the line IH is equal to $\frac{4a^2 + c^2}{4a^2}\dot{z}z^2A$ (by *Quad. Curvarum* : for in this case CE, EH, and $\dot{z}$ are taken as given and CE : EH :: $a : c$.) And the whole fluent of this is the sum of all $p \times \mathrm{CB}^2$ in the whole figure. But the area A is as z^2. If hence $\mathrm{A} = nz^2$, and by summing the fluent becomes $\mathrm{Q} = \frac{4a^2 + c^2}{20}n^3a^3$. Moreover the distance of the centre of gravity from the vertex C is $\frac{3}{4}a\,[= \mathrm{CG}]$, and the volume of the cone is $\frac{na^3}{3}a\,[= \mathrm{P}]$; thus it is the case that $\mathrm{CG} \times \mathrm{P} = \frac{na^4}{4}$, and hence the distance of the centre of oscillation from the point of suspension C is $\frac{4a^2 + c^2}{5a}\,[= \mathrm{CA} = \mathrm{CG} + \frac{\mathrm{D}}{\mathrm{CG} \times \mathrm{P}} = \frac{3}{4}a + \frac{4\mathrm{D}}{3a\mathrm{P}}.]$ [p. 99.]

Thus also $\frac{\mathrm{D}}{\mathrm{P}} = \frac{3a^2 + 12c^2}{80} = (\frac{3\mathrm{CF}^2 + 3\mathrm{AD}^2}{80}).]$

EXAMPLE II.



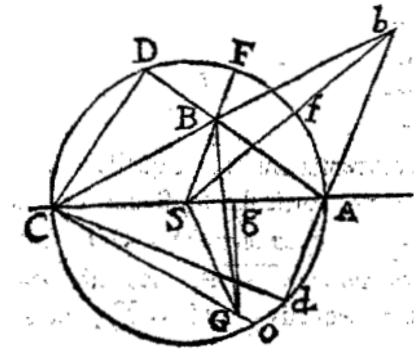
Let the figure proposed be a sphere. In this figure the calculation can proceed most conveniently by finding D, or the sum of all $p \times \mathrm{GB}^2$. Therefore let G be the centre of the sphere, DHI a great circle, and the radius of the sphere is GD = a. From the centre G and with some radius GB describe a circle BEF, and let GB = z, and the circumference BEF = nz. Then the sum of all the $p \times \mathrm{GB}^2$ in the circumference BEF is equal to the same circumference taken with the height of the small portion in B multiplied by GB^2, that is $2nz^3\sqrt{a^2 - z^2}$. Hence the sum of all $p \times \mathrm{GB}^2$ in the whole sphere is equal to the fluent of $2n\dot{z}z^3\sqrt{a^2 - z^2}$, that is, $\frac{2}{5}na^3$ [= P]; Hence $\frac{\mathrm{D}}{\mathrm{P}} = \frac{2}{5}a^2$.

SCHOLIUM.

By the line of the argument in this Proposition, the motion of the oscillating body, so with respect to the forces, as with respect to the velocities, is the same, and is equal to the motion of the particles equal to the sum of all $p \times \frac{\mathrm{GB}^2}{\mathrm{CA}^2}$ (that is of the particle $\frac{\mathrm{GG}^2 \times \mathrm{P}^2}{\mathrm{Q}}$,) oscillating at the distance CA. Hence in place of some body oscillating it is possible to substitute a particle of this kind situated at the same centre of oscillation. Therefore when the common centre of oscillation of a number of bodies is sought, it is convenient to find the centres of oscillation arising from these themselves, [p. 100.], and to substitute these themselves in place of the bodies, and hence to find the common centre of oscillation by the principles of this calculation.

PROP. XXV. PROB. XX.

To find the centre of percussion on a given line for a body of any kind rotating about a given axis.



The centre of percussion is the fixed point A in a body that is free to rotate about an axis, by which on striking an obstacle the whole rotational motion of the rotating body can be brought to rest, thus so that it does not lean in one way or the other.

It is agreed that the location of a point of this kind lies in the plane of motion of the centre of gravity; and thus prismatic elements of any kind are normal to this plane; and thus parallel to the axis of rotation; the momenta of the elements from each side of this plane are always equal to each other since the motion of each is always parallel to the other; and thus the whole motion of the rotating body can be made to stop by a resisting force acting in that plane.

If therefore C lies at the intersection of the axis of rotation with the plane of motion of the centre of gravity G, and the centre of percussion is sought on the line CS cut by the point C, and that point sought shall be taken as A. With Diameter CA describe a circle CDA*d*, the centre of which is S; and from two points taken, B within the periphery of the circle, and *b* outside the same, and from these two points particular elemental prisms of magnitude *p* are set up parallel to the axis of rotation. Draw AB, SB, A*b*, S*b*, crossing the circle in D, F, *d, f*, and draw CB, C*b*, CD, C*d*. [p. 101.] [Note that since CA is a diameter, D and *d* are right angles.]

On account of the rotational motion, the absolute velocities of the points chosen, in the direction perpendicular to CB and C*b*, are as the distances CB and C*b*. [which are in turn as the lengths BD to *db* from similar triangles.] Whereby upon striking the body in the point A, the opposing velocities on the elements taken in the interval around A are as the distances AB, A*b* to the lines BD, *bd*. Therefore the absolute forces acting on the elements at the ends of the lines AB, A*b*, are in the ratio $p\times \mathrm{DB}$ to $p\times \mathrm{d}b$; and the effectiveness of these forces in stopping the body on the opposite sides of A are as $p\times \mathrm{DB}\times \mathrm{B}A,$ and $p\times \mathrm{d}b\times \mathrm{b}A.$ Hence from the conditions of the problem, the sum of all $p\times \mathrm{DB}\times \mathrm{BA}$ within the circle is equal to the sum of all $p\times \mathrm{d}b\times b\mathrm{A}$ outside the circle.

But from the nature of the circle it follows that :
$\mathrm{DB}\times \mathrm{BA}=\mathrm{SF}^2-\mathrm{SB}^2=\mathrm{SA}^2-\mathrm{SB}^2,$ and $db\times b\mathrm{A}=\mathrm{S}b^2-\mathrm{SA}^2$. Thus the sum of all $p\times \mathrm{SA}^2-p\times \mathrm{SB}^2$ within the circle is equal to the sum of all $p\times \mathrm{S}b^2-p\times \mathrm{SA}^2$ outside the circle. Whereby by transferring all the terms $p\times \mathrm{S}A^2$ to one side of the equation, and the terms $p\times \mathrm{SB}^2, p\times \mathrm{S}b^2$ to the other side, the sum of all the terms is $p\times \mathrm{SA}^2$, so within as without for the whole body, equal to the sum of $p\times \mathrm{SB}^2$, also for the whole body. But by drawing SG and GB, and for the size of the body I write P, then the sum of all the

terms is $p \times \mathrm{SB}^2$ (by *Cor*. 1. *Prop*. 24). Likewise on account of the given SA^2, the sum of all $p \times \mathrm{SA}^2 = \mathrm{P} \times \mathrm{SA}^2$. Hence if for the given sum of all $p \times \mathrm{G}B^2$ I write D, then $\mathrm{P} \times \mathrm{SA}^2 = p \times \mathrm{SG}^2 + \mathrm{D}$, that is $\mathrm{SA}^2 - \mathrm{SG}^2 = \frac{\mathrm{D}}{\mathrm{P}}$.

To CA draw the normal G*g*, and draw CG; and thus $\mathrm{SG}^2 = \mathrm{C}G^2 - \mathrm{CS} \times \mathrm{CG} + \mathrm{SA}^2$, and $\mathrm{SA}^2 - \mathrm{SG}^2 = \mathrm{CA} \times \mathrm{C}g - \mathrm{C}G^2$, that is $\mathrm{CA} \times \mathrm{C}g - \mathrm{C}G^2 = \frac{\mathrm{D}}{\mathrm{P}}$. Moreover, all the terms CG, C*g*, P & D are given; whereby the point A is also given. *Q.E.I.* [p.102.]

COROLLARY.

Thus CA is to $\mathrm{CG} + \frac{\mathrm{D}}{\mathrm{P} \times \mathrm{CG}}$ as CG to C*g*. Whereby if the line CG crosses the circle at O, by considering the angle at O as right, and thus the with the triangles CAO and CG*g* similar, then $\mathrm{CO} = \mathrm{CG} + \frac{\mathrm{D}}{\mathrm{P} \times \mathrm{CG}}$. Thus (by *Cor*. 1. *Prop*. 24) O is the centre of oscillation. Hence with a line drawn from the centre of rotation to the centre of oscillation O, the perpendicular to that is the centre of percussion. And thus the centre of percussion is found from the centre of oscillation.

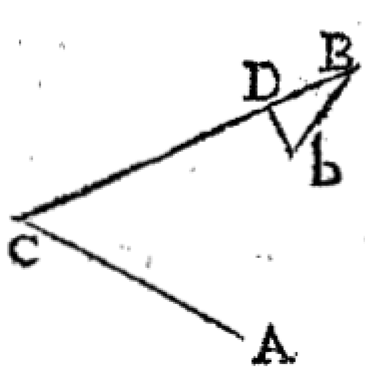
[Taylor is incorrect in his analysis, as the centre of percussion and the centre of oscillation are in fact identical; both satisfy the relation $ll' = k^2$ or $\mathrm{CG} \times \mathrm{GA}\ (= \frac{\mathrm{D}}{\mathrm{P}})$ derived above. Hence, the points O and A coalesce, and A lies on the line CG extended.]

METHODUS INCREMENTORUM.

Pars Secunda IIc.

[p. 94]

LEMMA X.



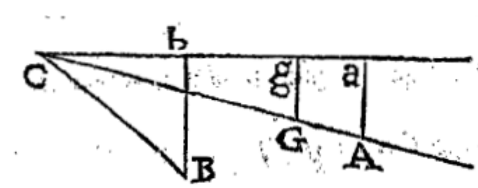
Si in punctum datum spatii rigidi circa datum axem mobilis, impingat data particula in data directione, data cum velocitate; atq; in figura praesenti sint puncta C, B, b, A intersectiones plani ad axem datum rotationis perpendicularis, cum ipso axe & cum rectis ei parallelis, transeuntibus per locum puncti in quod impingit particula data, per extremitatem rectae in directione motus, & proportionalis velocitati istius particulae impingentis; & per spatii *ejusdem aliud punctum datum : atq; ad CB ducatur normalis bD : tum erit motus istius spatii productis per impulsum istius particuale omnino idem, ac si produceretur per aliam particulam, cujus magnitudo est ad magnitudinem particulae prioris, ut* CB^2 *ad* CA^2*, & velocitatas est ut* $Db \times \frac{CA}{CB}$*, impingentem in istud aliud punctum datum cujus projectio est A, cum directione tum recta CA tum Axi rotationis perpendiculari.*

Nam per rigorem spatii omnis motus tollitur, nisi quatenus sit in planis ad axem normalibus. Quare punctis omnibus de quo agitur modo praedicto ad tale planum reductis, particulae impingentis velocitas & ejusdem directio in hoc plano repraesentabitur per rectam Bb. Sed hujus velocitatis pars BD tollitur per resistentiam puncti C, & per partem reliquam Db producitur motus spatii. Velocitas autem puncti A est ad velocitatem puncti B, ut distantia a centro CA ad distantiam CB : quare puncti A velocitas producta per motum particulae impingentis in B erit $Db \times \frac{CA}{CB}$. Sed & momentum ejusdem [p.95] particuale est ad momentum quod tribuit puncto A, ut CA ad CB : quare si particulae istius magnitudo sit *p*, ejus momento in B existente $p \times Db \times \frac{\mathrm{CA}}{\mathrm{CB}}$. Sed eadem velocitas & idem momentum producuntur in puncto A per vim particulae $p \times \frac{\mathrm{CA}^2}{\mathrm{CB}^2}$ impingentis in illud punctum, in directione ipsi CA perpendiculari cum velocitate $Db \times \frac{\mathrm{CA}}{\mathrm{CB}}$. Quare respectu motus producti in spatio rigido, perinde omnino est sive particula p impingat in B, sive particula $p \times \frac{\mathrm{CA}^2}{\mathrm{CB}^2}$ impingat in A jam definito. *Q.E.D.*

PROP. XXIV. PROB. XIX.

Corporis dati e dato Axe Horizonti parallelo dependentis invenire Centrum Oscillationis.

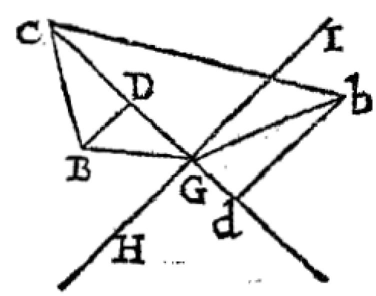
Per Centrum Oscillationis intelligo punctum, cujus velocitas idem semper est ac si corpore reliquo amoto illud solum circa eundem axem oscilletur.



Sit ergo praesens figura in plano ad horizontalem & ad Axem oscillationis perpendiculari, transeunte per corporis centrum gravitatis G, atque sit C intersectio hujus plani cum Axe oscillationis. Duc horizontali parallelam C*g*, atque in recta CG sit A Centrum oscillationis. Duc Horizontali parallelam C*g*, atque in recta CG sit A Centrum oscillationis quaesitam. Sit *p* particula prismatica Axi motus parallela, & plano figurae insistens in puncto B. Duc CB, & ad rectam C*g* duc perpendiculares B*b*, G*g*, A*a*, ei occurentes in *b, g, a*. [p.96]

Si gravitatis acceleratio datae sit 1, tum particulae *p* acceleratio, ad spatium figurae movendum circa punctum C, erit $\frac{Cb}{CB}$. Quare puncti A momentum a vi particulae *p* oriundum erit ($p \times \frac{Cb}{CB} \times \frac{CB}{CA} =$) $p \times \frac{Cb}{CA}$. Sed (per *Lem.* 10) motus puncti A per particulam *p* in puncto B productus perinde omnino est, ac si produceretur a particula $p \times \frac{CB^2}{CA^2}$ impingente in ipsum punctum A. Quare vice particulatum p in locus B, substitutis novis particulis $p \times \frac{CB^2}{CA^2}$ in puncto A, dabitur acceleratio puncti A, oriunda ex conjunctis viribus totius Corporis dati, per notissimam regulam Collisionum; nempe applicando summam omnium momentorum $p \times \frac{Cb}{CA}$ ad summam omnium particularum substitutaram $p \times \frac{CB^2}{CA^2}$, hoc est (ob datum CA) applicando summam omnium $p \times Cb \times CA$ ad summam omnium $p \times CB^2$: Sed ex notissima propietate centri gravitatis, si magnitudo corporis totius dati sit P, erit summa omnium $p \times Cb$ aequalis $P \times Cg$. Quare si pro summa omnium $p \times CB^2$ scribatur Q, erit acceleratio puncti A aequalis $\frac{P \times Cg \times CA}{Q}$. Sed, ex hypothesi, est haec acceleratio aequalis ipsius puncti A accelerationi propositae $\frac{Ca}{CA}$ vel $\frac{Cg}{CG}$. Est ergo $CA = \frac{Q}{P \times CG}$. Inventa itaque summa omnium particularum corporis propositis ductarum in quaddrata suarum distantiarum minimarum ab Axe oscillationis, per Fluxionum Methodum inversam, dabitur distantia centri oscillationis ab Axe, applicando hanc summam ad productum corporis dati ducti in distantiam centri gravitatis ab Axe oscillationis. *Q.E.I.* [p.97]

COROLL. I.



In eodem plano existentibus punctis C, G, B iisdem ac supra, ad CG duc normales HGI, BD. Tum erit $CB^2 = BG^2 + CG^2 - 2CG \times GD$; nempe cadente puncto B ad easdem partes rectae HI atque punctum C. Sed ubi cadit punctum b ad contarias partes rectae HI, erit $CB^2 = bG^2 + CG^2 + 2CG \times Gd$. Est ergo summa omnium particularum ductarum in propria sua CB^2 per totam figuram, aequalis summa particularum ductarum in propria $\overline{BG^2 + CG^2}$ per totam figuram, minus summa omnium $p \times 2CG \times GD$ ex una parte recte HI, plus summa omnium $p \times 2CG \times Gd$ ex altera parte

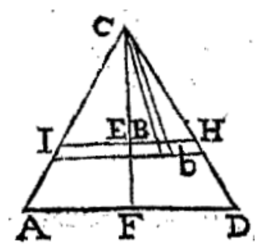
HI. Sed ex natura centri gravitatis est summa omnium $p\times 2\mathrm{CG}\times \mathrm{G}d$ aequalis summae omnium $p\times 2\mathrm{CG}\times \mathrm{GD}$ ex ultera parte HI. Quare est summa omnium $p\times 2\mathrm{CB}^2$, seu Q, aequalis summae omnium $p\times \mathrm{GB}^2+p\times \mathrm{CG}^2$, hoc est, (ob datum CG^2) aequalis $p\times \mathrm{CG}^2$ plus summa omnium $p\times \mathrm{GB}^2$. Si ergo pro summa omnium $p\times \mathrm{GB}^2$ scribatur D, erit $\mathrm{Q}=p\times \mathrm{GB}^2+\mathrm{D}$; adeoque; $\mathrm{CA}\ (=\frac{\mathrm{Q}}{\mathrm{P}\times \mathrm{CG}})=\mathrm{CG}+\frac{\mathrm{D}}{\mathrm{P}\times \mathrm{CG}}$.

COROLL. II.

Hinc est $\mathrm{GA}=+\frac{\mathrm{D}}{\mathrm{P}\times \mathrm{CG}}$. Unde data directione Axis Oscillationis respectu figurae corporis propositi, dabitur productum $\mathrm{CG}\times \mathrm{GA}\ (=\frac{\mathrm{D}}{\mathrm{P}})$. Adeoque dato centro oscillationis in uno casu ejusdem directionis Axis, dabitur idem in omnibus aliis, absque ulteriori calculo. [p. 98]

In casu aliquo proposito calculus institui potest, vel per inventionem ipsius Q ex commoda assumptione puncti C, vel per inventionem ipsius D, prout commodius videbitur.

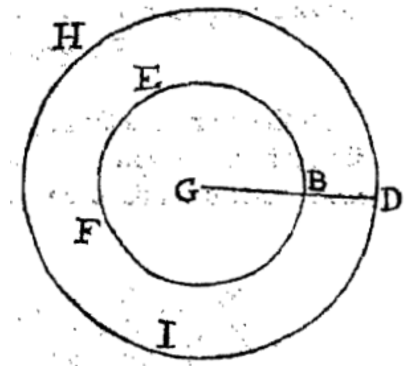
EXEMPL. I.



Inventium sit centrum oscillationis Coni recti geniti per rotationem per rotationem trianguli isoscelis ACD circa perpendiculum CF, & dependentis ab ibsius vertice C. Duc basi parallelam IEH occurrentem ipsis CA, CF, CD, in I, E, H; & sint CF = a. FD = c, CE = z, adeoque; $\mathrm{EH}=\frac{cz}{a}$, & in recta EH sumpto quovis puncto B, sit EB = x. Tum si particulae prismaticae p puncto B insistentis basis sit $\dot{z}\dot{x}$ (= parallelogrammo Bb) ejusdem altitudine existente $2\sqrt{\mathrm{EH}q-xx}$, erit particula ipsa $p=2\dot{z}\dot{x}\sqrt{\mathrm{EH}^2-xx}$, atque; $p\times \mathrm{CB}^2=2\mathrm{CE}^2\times \dot{z}\dot{x}\sqrt{\mathrm{EH}^2-xx}+2xx\dot{z}\dot{x}\times\sqrt{\mathrm{EH}^2-xx}$. Et haec est fluxio omnium $p\times \mathrm{CB}^2$ in recta IH. Sed si pro Area circuli cujus diameter est IH scribatur A, erit hujus fluens totalis adjacens rectae IH aequalis $\frac{4aa+cc}{4aa}\dot{z}z^2A$ (per *Quad. Curvarum* : nam in hoc casu sumuntur CE, EH, & $\dot{z}$ pro datis & est CE : EH :: $a : c$.) Et hujus fluens totalis est summa omnium $p\times \mathrm{CB}^2$ in tota figura. Sed est area A ut z^2. Si ergo $\mathrm{A}=nz^2$, & sumendo fluentem fiet $\mathrm{Q}=\frac{4a^2+c^2}{20}n^3a^3$. Distantia autem centri gravitatis a vertice C est $\frac{3}{4}a\ [=\mathrm{CG}]$, atque magnitudo Coni est $\frac{na^3}{3}a\ [=\mathrm{P}]$; unde sit $\mathrm{CG}\times \mathrm{P}=\frac{na^4}{4}$, adeoque; distantia centri Oscillationis a puncto suspensionis C est

$\frac{4a^2+c^2}{5a}$ [$= \mathrm{CA} = \mathrm{CG} + \frac{\mathrm{D}}{\mathrm{CG}\times\mathrm{P}} = \frac{3}{4}a + \frac{4\mathrm{D}}{3a\mathrm{P}}$.] [p. 99.] Unde etiam sit
$\frac{\mathrm{D}}{\mathrm{P}} = \frac{3a^2+12c^2}{80} = (\frac{3\mathrm{CF}^2+3\mathrm{AD}^2}{80})$.]

EXEMPL. II.



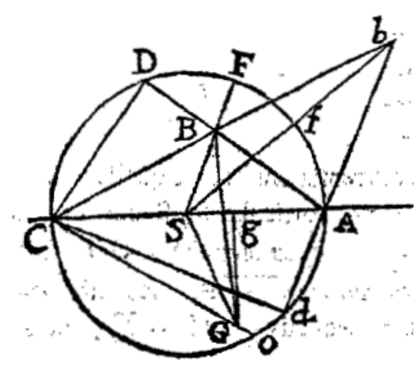
Sit figura proposa Sphaera. In hac figura calculus commodissime procedit per inventionem ipsius D, seu summae omnium $p\times\mathrm{GB}^2$. Sit ergo G centrum Sphaerae, DHI circulus maximus, & sit Sphaerae radius GD = a. Centro G & radio quovis GB describe circulum BEF, & sit GB = z, & circumferentia BEF = nz. Tum summa omnium $p\times\mathrm{GB}^2$ incircumferentia BEF erit aequalis eidem circumferentiae ductae in altitudinem particulae in B ductae in GB^2, hoc est $2nz^3\sqrt{a^2-z^2}$. Est ergo summa omnium $p\times\mathrm{GB}^2$ in tota sphaera aequalis fluenti ipsius $2n\dot{z}z^3\sqrt{a^2-z^2}$, hoc est, $\frac{2}{5}na^3$ [= P] ; Unde sit $\frac{\mathrm{D}}{\mathrm{P}} = \frac{2}{5}a^2$.

SCHOLIUM.

Per argumentationem in hac Propositione, corporis oscillantis motus, tam respectu virium, quam respectu velocitatis, idem est, ac foret motus particulae aequalis summae omnium $p\times\frac{\mathrm{GB}^2}{\mathrm{CA}^2}$ (hoc est particulae $\frac{\mathrm{GG}^2\times\mathrm{P}^2}{\mathrm{Q}}$,) oscillantis ad distantiam CA. Adeoque; vice corporis cujusvis oscillantis substitui potest hujusmodi particula sita in ejusdem centro Oscillationis. Ubi ergo quaeritur corporum plurium centrum [p. 100.] Oscillationis commune, convenit singulorum centra Oscillationis seorsim quaerere, & in iis substituere hujusmodi particulas vice corporum ipsorum, atque deinde quaerere centrum commune oscillationis per principia hujus calculi.

PROP. XXV. PROB. XX.

Corporis cujusvis circa datum Axem revolventis invenire Centrum Percussionis in recta data.



Est Centrum Percussionis punctum in corpore circa axem mobilem, sed immotum, revolvente, quo in obstaculum impingente sistitur motus totus corporis revolventis, ita ut nec huc neq; illuc inclinet.
Constat hujusmodi puncti locum esse in plano motus centri gravitatis; nam elementi cujusvis prismatici huic plano normalis, adeoque; axi revolutionis paralleli, motus, cum sit sibi semper

parallelus, momenta ex utraque; parte hujus plani erunt aequalia; adeoque per resistentiam in eo factam sisti potest motus totus corporis revolventis.

Sit ergo G interfectio axis revolutionis cum plano motus centri gravitatis G, & quaeratur centrum percussionis in recta CS transeunte per punctum C, & sit punctum illud quaesitum A. Diametro CA describe circulum CDA *d*, cujus centrum sit S; & sumptis punctis duobus, B intra circuli peripheriam, & *b* extra eandem, iis insistant particulae duae prismaticae p axi revolutionis parallelae. Duc AB, SB, A*b*, S*b*, circulo occurrentes in D, F, *d, f*, & duc CB, C*b*, CD, C*d*. [p. 101.]

Ob motum revolutionis, punctorum velocitates absolutae, in directionibus ipsis CB, C*b* normalibus, sunt ut distantiae CB,C*b*. Quare impingente corpore in punctum A, particularum velocitates ad trhhendum spatium circum A in partes contrarias per radios AB, A*b* erunt ut rectae BD, *bd*. Vires ergo absolutae particularum in extremitatibus radiorum AB, A*b*, sunt ut $p\times \mathrm{DB}$, & $p\times \mathrm{d}b$; & harum virium efficaciae ad corpus trahendum in partes contrarias circum A, sunt ut $p\times \mathrm{DB}\times \mathrm{B}A$, & $p\times \mathrm{d}b\times b\mathrm{A}$. Ergo per conditiones Problematis, debet summa omnium $p\times \mathrm{DB}\times \mathrm{BA}$ intra circulum aequari summae omnium $p\times \mathrm{d}b\times b\mathrm{A}$ extra circulum.

Sed ex natura circuli est
$\mathrm{DB}\times \mathrm{BA}=\mathrm{SF}^2-\mathrm{SB}^2=\mathrm{SA}^2-\mathrm{SB}^2$, atque $db\times b\mathrm{A}=\mathrm{S}b^2-\mathrm{SA}^2$. Unde est summa omnium $p\times \mathrm{SA}^2-p\times \mathrm{SB}^2$ intra circulum aequalis summae omnium $p\times \mathrm{S}b^2-p\times \mathrm{SA}^2$ exta circulum. Quare transferendo omnes terminos $p\times \mathrm{S}A^2$ ad unam partem aequationis, & terminos $p\times \mathrm{SB}^2, p\times \mathrm{S}b^2$, ad alteram partem, erit summa omnium $p\times \mathrm{SA}^2$, tam intra quam extra circulum in universo corpore, aequalis summae omnium $p\times \mathrm{SB}^2$, etiam in iniverso corpore. Sed ductis SG & GB, & pro Corporis magnitudine scripto P, erit summa omnium $p\times \mathrm{SB}^2$ (per Cor. 1. Prop. 24). Item ob datum SA^2, est summa omnium $p\times \mathrm{SA}^2=\mathrm{P}\times \mathrm{SA}^2$. Proinde pro data summa omnium $p\times \mathrm{G}B^2$ scripto D, erit $\mathrm{P}\times \mathrm{SA}^2=p\times \mathrm{SG}^2+\mathrm{D}$, hoc est $\mathrm{SA}^2-\mathrm{SG}^2=\frac{\mathrm{D}}{\mathrm{P}}$.

Ad CA duc normalem Gg, & duc CG; atque erit $\mathrm{SG}^2=\mathrm{C}G^2-\mathrm{CS}\times \mathrm{CG}+\mathrm{SA}^2$, adeoque; $\mathrm{SA}^2-\mathrm{SG}^2=\mathrm{CA}\times \mathrm{C}g-\mathrm{C}G^2$, hoc est $\mathrm{CA}\times \mathrm{C}g-\mathrm{C}G^2=\frac{\mathrm{D}}{\mathrm{P}}$. Dantur autem omnes CG, C*g*, P & D; quare etiam datur punctum A. *Q.E.I.* [p.102.]

COROLLARIUM.

Hinc est CA ad $\mathrm{CG}+\frac{\mathrm{D}}{\mathrm{P}\times \mathrm{CG}}$ ut CG ad C*g*. Quare si recta CG occurrat circulo in O, existente angulo ad O recto, adeoque & triangulis CAO. CG*g* similis, erit $\mathrm{CO}=\mathrm{CG}+\frac{\mathrm{D}}{\mathrm{P}\times \mathrm{CG}}$. Unde (per *Cor*. 1. *Prop*. 24) est O centrum Oscillationis. Proinde per centrum rotationis C ducta recta ad centrum Oscillationis O, ei perpendicularis OA erit locus centri Percussionis. Invenire itaque; centre Percussionis per calculum centri Oscillationis.