§12.1

Chapter Twelve

With the sines of any arcs of equal difference found by this method, the intervals can be filled out with sines in agreement, with the given intervals divided into five equal parts, in the following way.

Let the sines be given		*Of which the differences of the sines are taken which we call the first difference*
00000000	0 *Deg.*	
54514502	$3\frac{1}{8}$	54514502
108866875	$6\frac{2}{8}$	54352373
162895473	$9\frac{3}{8}$	54028598
216439614	$12\frac{4}{8}$	53544141
269340054	$15\frac{5}{8}$	52900440
321439465	$18\frac{6}{8}$	52099411

[Table 12 - 1]

The differences of these primary differences are called the secondary differences, and the difference of the secondary differences the third, and thus henceforth.

Second Diff.	*3rd Diff.*	*4th Diff.*	*5th Diff.*
000000	162129	000	483
162129	161646	483	481
323775	160682	964	474
484457	159244	1438	478
643701	157328	1916	
801029			

[Table 12 - 2]

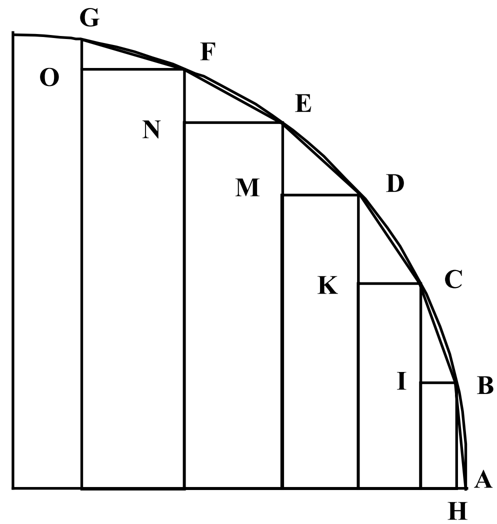
Let the given sines, and the first, second, third, etc, differences of these be set out in distinct order in this manner.

5th Diff.	*4th Diff*	*3rd Diff*	*2nd Diff*	*1st Diff*	*Sine*	*Deg.*
	000		000000			0
483		162129		54514502		
	483		162129		54514502	$3\frac{1}{8}$
481		161646		54352373		
	964		323775		108866875	$6\frac{2}{8}$
474		160682		54028598		
	1438		484457		162895473	$9\frac{3}{8}$
478		159244		53544141		
	1916		643701		216439614	$12\frac{4}{8}$
		157328		52900440		
	2382		801029		269340054	$15\frac{5}{8}$
				52099411		
	2843				321439465	$18\frac{6}{8}$

[Table 12 - 3]

2. If some allowed set of arcs are equidistant [on the quadrant of a circle]; then the differences of the sines of these arcs are proportional to the sines of the complements of the mean arcs[1].

For let the arcs of 14, 28, 42, 56, 70, 84 degrees be AB, AC, AD, AE, AF, and AG. The differences of the sines are HB, IC, KD, ME, NF, OG; If AB, the chord of 14 degrees, is the whole sine; HB is the sine of the angle BAH, 83, and CI the sine of the angle CBI, 69; DK, 55; EM, 41, FN, 27; GO, 13. Also HA is the sine of 7 degrees; IB, 21; KC, 35; MD, 49; NE, 63; OF, 77.



[Figure 12 - 1]

And because the first differences are proportional to the sines of the equidistant arcs; the second differences by the same reason are proportional to the sines of the given arcs which are the complements of the mean arcs [*i.e.* the arc formed by the complement of the half-angle]. By the same method of differences, also the third and fourth and the rest follow suite.

The second order differences, as well as the fourth, the sixth, eighth, etc, are therefore proportional to the given sines themselves. And the first difference, the third, the fifth, the seventh are proportional among themselves and to the sines of the complements of the mean arcs.

And from all these, so with the sines as with the differences, which have been set out in a like manner, are in proportion; the first and the third order differences, and the second and the fourth, correspond in this manner with the degrees.

Over and above, all of these numbers situated on the same line are in continued proportion.

COROLLARY.

Therefore, we can use this method to find, with a little more accuracy by the rule of proportionality, the final and smallest differences for a given sine. As the sines themselves are irrational, in general they are not in a proportional ratio overall.

Continued Proportionals	*Sine of* $6\frac{2}{8}$ *Degrees*	. . . 108866875
	Second Difference	 323775
	Fourth Difference	 962921
	Sixth Difference	285458
	Eighth Difference	 00848964
	Tenth Difference	 0000254
Continued Proportionals	*First Difference*	. . . 54028598
	Third Difference	. . . 160682
	Fifth Difference	447871
	Seventh Difference	 142119
	Ninth Difference	 00422664

[Table 12 - 4]

And these situated further up on the same line in continued proportion. Truly these following as they are are similarly placed are proportional , but [they are] not in continued proportion.

Proportionals	Sine $12\frac{1}{2}$	216439614
	First Difference	. 54352373
	Second Difference	. . 643701
	Third Difference	. . 161646635

[Table 12 - 5]

And these given differences, if the work were corrected by the rule of proportion, will be in accord with the above method.

Now by means of these other given differences that have been found, that serve for the insertion of the rest of the sines. These are to be found either by division or multiplication of the given [ones].

If five divides the first; 25 divides the second; 125 the third; 625 the fourth, and so on with the divisors increased by a ratio of five; The first quotients are the first means, of the second, the second, etc.

Or if we should multiply the first differences given by 2. The seconds by 4. The thirds by 8, etc., with the products increasing in the duplicate ratio, and we cut off a single place (and this the final) of the product from the first, two places for the seconds, three for the thirds, etc; these products altogether are equal to the quotients by the division first found; And these mean differences are placed in this manner.

Mean Differences

	5th Diff.	*4th Diff.*	*3rd Diff*	*2nd Diff.*	*1st Diff*	*Sines*	*Deg.*
			0	0000	0	00000000	0
		15	1297	0	10902900	4	
			77	6485	2	54514502	$3\frac{1}{8}$
		15	1293	2	10870474	6	
†		1	54	12951	0	108866875	$6\frac{2}{8}$
		15	1285	5	10805719	6	
		2	30	19378	3	162895473	$9\frac{3}{8}$
		15	1274	0	10708828	2	
		3	06	25748	0	216439614	$12\frac{4}{8}$
		15	1258	6	10580088	0	
		3	81	32041	2	269340054	$15\frac{5}{8}$
		15			10419882	2	
						321439465	$18\frac{6}{8}$

[Table 12 - 6]

And these are the mean differences which are to be corrected in order that we can use them to help find the remaining sines.

The method of correcting has been set out by me in Chapter 13 of the Arithmetica Logarithmica in the London Edition: but that chapter together with the following, without consulting me and so being unaware of it, have been omitted in the Batavian

[Dutch]Edition: but not in all things, does the Author of this other Edition, in other respects a learned and industrious man, seem to have understood my intentions : Therefore lest anyone should fall short in any of this, who would wish to complete the whole tables, for somethings I have judged of the greatest necessity ought to be transferred from here to there[2]. These mean differences are to be corrected by the following numbers placed in the following table: The first column A of the same shows the mean differences from the first as far as the twentieth. These differences are augmented with the correct differences placed in column B added and taken away and these that are placed in the following nearby column D. Hence by the same way shown: by adding B, D, F, H and taking away C, E, G, I, if the number of different differences increases.

A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
20								
19								
18	18 (20)							
17	17 (19)							
16	16 (18)	123 $\underline{1}$ (20)						
15	15 (17)	108 $\underline{0}$ (19)						
14	14 (16)	93 $\underline{9}$ (18)	400 $\underline{4}$ (20)					
13	13 (14)	80 $\underline{6}$ (17)	317 $\underline{2}$ (19)					
12	12 (14)	68 $\underline{4}$ (16)	246 $\underline{4}$ (18)	629 $\underline{64}$ (20)				
11	11 (13)	57 $\underline{2}$ (15)	187 $\underline{0}$ (17)	431 $\underline{20}$ (19)				
10	10(12)	47 $\underline{0}$ (14)	138 $\underline{0}$ (16)	283 $\underline{80}$ (18)	434 $\underline{40}$ (20)			
9	9 (11)	37 $\underline{8}$ (13)	98 $\underline{4}$ (15)	177 $\underline{84}$ (17)	236 $\underline{88}$ (19)			
8	8 (10)	29 $\underline{6}$ (12)	67 $\underline{2}$ (14)	104 $\underline{72}$ (16)	118 $\underline{72}$ (18)	111 $\underline{248}$ (20)		
7	7 (9)	22 $\underline{4}$ (11)	43 $\underline{4}$ (13)	56 $\underline{84}$ (15)	53 $\underline{20}$ (17)	36 $\underline{680}$ (19)		
6	6 (8)	16 $\underline{2}$ (10)	26 $\underline{0}$ (12)	27 $\underline{60}$ (14)	20 $\underline{40}$ (16)	10 $\underline{760}$ (18)	4 $\underline{080}$ (20)	
5	5 (7)	11 $\underline{0}$ (9)	14 $\underline{0}$ (11)	11 $\underline{40}$ (13)	6 $\underline{20}$ (15)	2 $\underline{280}$ (17)	$\underline{500}$ (19)	
4	4 (6)	6 $\underline{8}$ (8)	6 $\underline{4}$ (10)	3 $\underline{64}$ (12)	1 $\underline{28}$ (14)	$\underline{272}$(16)	$\underline{032}$ (18)	$\underline{0016}$ (20)
3	3 (5)	3 $\underline{6}$ (9)	2 $\underline{2}$ (9)	7 $\underline{2}$ (11)	1 $\underline{2}$(13)	$\underline{008}$ (15)		
2	2 (4)	1 $\underline{4}$ (6)	$\underline{4}$ (8)	$\underline{04}$ (10)				
1	1 (3)	$\underline{2}$ (5)						

[Table 12 - 7]

The numbers placed in Column A are the mean differences, of the first, second, third, etc, which have been corrected by the following numbers placed in the following columns in this way[3].

To the sixth mean, have been added 6 (8), namely by six of the eighth mean corrected: and besides being corrected by 26 (12) + 20 $\underline{4}$ (16) + 4 $\underline{08}$ (20). From the sum of these should be taken away 16 $\underline{2}$ (10) + 27 $\underline{6}$ (14) + 10 $\underline{76}$ (18). With these correction differences added or taken in this way, what has been left gives the correct sixth difference. Which everything in the following example will try to show.

4. With sines the mean differences should be increased and decreased, as we have shown. But with tangents, secants, logarithms, and powers or figures with equally spaced roots [i.e. powers of integers and the like], the mean differences are corrected by subtraction alone of all the correction differences placed in the same line with the means.

For sines, tangents, secants and logarithms, and in everything with the ratio surely increasing or decreasing: If there shall be irrational [numbers]; it will be most suitable [to have] two places in the furthest the differences; but with the First and following at least one place beyond the limits to be conceded; as safe and sure (with one place always being added or taken, if required) we are able to find the correct differences and these numbers sought. As with the example of the sine above placed on page 37 † [here the previous page] and following next ✠. Where the fifth and fourth differences have two places beyond the line which is the boundary of the given sines and of the genuine differences: But the 3rd, 2nd, and 1st differences have as much as a single place beyond the same limit place.

The fifth and fourth differences in this example are not corrected; because not having seventh and sixth differences: but the third, second, and first are increased in this way.

Third mean		1	297	0
Three of the 5th added			. . .	45
Third corrected		1	297	4
Second mean		6	485	2
Two of the 4th added. . .			. .1	54
Second corrected . . .		6	486	7
First mean	10	902	900	4
Third added on		. .1	297	4
	10	904	197	8
$^{2}/_{10}$ of the 5th taken away		. . .	. .	030
First Corrected	10	904	197	8

[Table 12 - 8]

5th	*4th*		*3rd Diff.*		*2nd Diff.*		*1st Diff.*		*Sine*	*Deg.*
	=	00			=000	0			= 00000000	
			1297	9			10908091	5		
		15			1298	0			10908092	$^{5}/_{8}$
			1297	7			10906793	5		
		31			2595	7			21814885	1 $^{2}/_{8}$
15			=1297	4			10904197	8		
		46			3893	2			32719083	1 $^{7}/_{8}$
			1297	0			10900304	6		
		61			5190	2			43619388	2 $^{4}/_{8}$
			1296	4			10895114	3		
	=	77			= 6486	7			= 54514502	3 $^{1}/_{8}$
			1295	6			10888627	6		
		92			7782	3			65403130	3 $^{6}/_{8}$
			1294	7			10880845	2		
	1	08			9077	0			76283965	4 $^{3}/_{8}$
15			= 1293	6			= 10871768	2		
	1	23			10370	7			87155743	5
			1292	4			10861397	5		
	1	39			11663	1			98017141	5 $^{5}/_{8}$
			1291	0			10849734	4		
	= 1	54			= 12954	1			= 108866875	6 $^{2}/_{8}$
			1289	5			10836780	4		
	1	69			14243	6			119703655	6 $^{7}/_{8}$
			1287	8			10822536	8		
	1	85			15531	4			130526192	7 $^{4}/_{8}$
15			= 1285	9			= 10807005	5		

[Table continues until….].

	3	66			30796	9			258819045	15
			1251	8			10521008	6		
	= 3	81			= 32048	8			= 269340054	15 $^{5}/_{8}$

[Table 12 - 9]

[See the Notes, where a similar table has been calculated, 12-9A.]

5. Also with rational numbers, as with homogeneous powers of equidistant sides [i.e. powers of numbers such as the integers]; it will not be necessary to continue the corrected

differences beyond the given limits: because for all of these, as the powers sought shall be rational, so the differences [whole numbers].
For let the cubes by the squares be given, or the sixth powers of the sides [numbers] 50:55:60:65:70: I desire to find the powers inserted between for the sides 51.52, etc.

Differences Given

Sixth Powers - The Given
given Roots.

Sixth	*Fifth*	*Fourth*	*Third*	*Second*	*First*	*Square of cube*	
		564375000		4734406250		15625000000	50
	118125000		2185312500		12055640625		
11250000		682500000		6919718750		27680640625	55
	129375000		2867812500		18975359375		
11250000		811875000		9787531250		46656000000	60
	140625000		3679687500		28762890625		
11250000		952500000		13467218750		75418890625	65
	151875000				42230109375		
		1104375000				117649000000	70

[Table 12 - 10]

Mean Differences

Sixth	*Fifth*	*Fourth*	*Third*	*Second*	*First*	*Square of cube*	*side*
720		903000		189376250		15625000000	50
	37800		17482500		2411128125		
720		1092000		276788750		27680640625	55
	41400		29437500		3995071875		
720		1299000		391501250		46656000000	60
	45000		37057500		5752578125		
720		1524000		538688750		75418890625	65
	48600				8446021875		
						117649000000	70

[Table 12 - 11]

Example of the Corrected Differences.

4^{th} mean ….. ……. 903000	3^{rd} mean 17482500	2^{nd} mean 189376250
4 of the 6^{th} taken 2880	3 of 5^{th} taken 113400	2 of 4^{th} corrected 1800240
4^{th} corrected 900120	3^{rd} corrected 17369100	14 of 6^{th} 1008
		2^{nd} corrected 187575002
4^{th} mean 1092000	3^{rd} mean 22942500	1^{st} mean 2411128125
4 of the 6^{th} taken 2880	3 of 5^{th} taken 124200	3^{rd} corr. taken: 17369100
4^{th} corrected 1089120	3^{rd} corrected 22818300	$^{1}/_{5}$ of 5^{th} taken: 7560
		1^{st} corr. 2393751465

The sixth and fifth are not corrected.

[Table 12 - 11]

The cubes by the squares sought, together with these differences actually found by the corrected mean differences.

6^{th}	5^{th}	4^{th} Diff.	3^{rd} Diff.	2^{nd} Diff.	1^{st} Diff.	Sq. of Cubes	Side
720	36360	_900120	15459060	_187575002	1971287801	_15625000000	50
720	37080	936480	16395540	203034062	2174321863	17596287801	51
720	_37800	973560	_17369100	219429602	_2393751465	19770609664	52
720	38520	1011360	18380460	236798702	2630550167	22164361129	53
720	39240	1049880	19430340	255179162	2885729329	24794911296	54
720	39960	_1089120	20519460	_274609502	3160338831	_27680640625	55
720	40680	1129080	21648540	295128962	3455467793	30840979456	56
720	41400	_1169760	22818300	_316777502	3772245295	34296447249	57
720	42120	1211160	24029460	339595802	4111841097	38068692544	58
720	42840	1253280	25282740	363625262	4475466359	42180533641	59
720	43560	_1296120	26578860	_388908002	4864374361	_46656000000	60
720	44280	1339680	27918540	415486862	5279861223	51520374361	61
720	_45000	1383960	_29302500	443405402	_5723266625	56800235584	62
720	45720	1428960	30731460	472707902	6195974527	62523502209	63
720	46440	1474680	32206140	503439362	6699413889	68719476736	64
720	47160	_1521120	33727260	_535645502	7202853251	_75418890625	65
720	47880	1568280	35295540	569372762	7738498753	82653950016	66
720	_48600	1616160	_36911700	604668302	_8409100455	90458382169	67
720	49320	1664760	38576460	641580002	9050680457	98867482624	68
720	50040	1714080	40290540	680156462	9730836919	107918163081	69
720		_1767000		_723976250		_117649000000	70

[Table 12-13]

§12.2 Notes On Chapter Twelve.

[1] These follow from the identity $\sin(n + 1)\theta - \sin(n\theta) = 2\sin(\theta/2).\ \cos((2n + 1)\theta/2) = 2\sin(\theta/2).\ \sin(\pi/2 - (n + {}^{1}/_{2})\theta)$. Thus, e.g. if $\theta = 3.125^0$ and $n = 3$, then $\sin(12.5) - \sin(9.375) = 2\sin(1.5625)\ .\sin(79.0625) = 0.053544141$. As $2\sin(1.5625)$ is constant for the set of differences, it follows that the differences are proportional to the sines of the complements: recall that the cosine had not been defined as such at this time. As Briggs indicates, the next set of differences reverts to sines related to the original angles.

[2] I.e. from the *Arithmetica* to the envisaged work. Thus Briggs did not acknowledge that Vlacq had in fact finished his tables, at least in the manner he had intended; and still held hopes that someone would do so; Vlacq was to upset matters again by not adopting Briggs' method of using decimal fractions of degrees rather than minutes and seconds, in subsequent publications of tables, after the death of Briggs. This relic from antiquity has since stayed with us.

[3] It is worthwhile to insert here the argument presented by Herman H. Goldstine in his book: *A History of Numerical Analysis*..(Springer-Verlag, New York, 1977, pp. 27 - 30.), which formalises the numerical scheme used by Briggs for interpolation or sub-tabulation. It is evident that Briggs had discovered this process, though he would not have used an algebraic notation, but rather relied on the position of the number in his table to indicate what it was, or was the result of doing. So, following Goldstine's lead, who also provides references to the work of later mathematicians such as Lagrange and Legendre, subsequently dealing with interpolation in a way similar to Legendre.

Before doing this, let us summarise the initial state of affairs, as presented in the following Table 12-9:

1. The sines of the angles of multiples of $25/8 = 3^1/_8$ degrees are given, having been found by the methods considered in the earlier chapters, though Briggs does not seem to indicate by which particular route he went to get the sine of the above angle.
2. The uncorrected differences of the various orders up to five have been evaluated for these angles, each one obtained from the previous by subtraction and division by 5. The odd orders are placed in the second slots, while the even orders are placed in the zero slots. To the accuracy considered, the 5^{th} U. D.'s are constant. We show these in **bold type**
3. The sines of the sub-multiples of these angles on division by 5 are required to the same accuracy, formed from their corrected finite differences, by the process of sub-tabulation. Relations are found between the intervals of length 5 and those of unit length, for the subdivision, for the various orders of differences, which in addition are different for odd and even differences. This is explained below in the next note,
4. Briggs has observed that the higher order differences are zero, to this degree of accuracy: in this case the fifth order is observed from the table to be constant, and hence this order can be filled in immediately, and the fourth order thus differs by constant amounts, and can also be filled in. We have done this with *italics*. The

remaining differences are to be corrected, according to Table 12-7, where some smart thinking has been done by Briggs. See note 4.

	(Deg.)	Sine	1st *U.*/C.D.	2nd *U.*/C.D.	3rd *U.*/C.D.	4th	5th
0	0.000E+00	**0.00000000E+00**	*1.09067936E-02*	**0.00000E+00**	1.2975E-06	**0.00E+00**	
			1.09080913E-02		1.2977E-06		1.54E-10
1	6.250E-01	1.09080915E-02	*1.09054958E-02*	*1.29764E-06*	*1.2793E-06*	*1.54E-10*	
			1.09067934E-02	1.29795E-06	1.2977E-06		*1.54E-10*
2	1.250E+00	2.18148850E-02	**1.09029003E-02**	*2.59514E-06*	**1.2970E-06**	*3.08E-10*	
			1.09041977E-02	2.59575E-06	1.2974E-06		**1.54E-10**
3	1.875E+00	3.27190828E-02	*1.08990075E-02*	*3.89232E-06*	*1.2966E-06*	*4.62E-10*	
			1.09003045E-02	3.89324E-06	1.2970E-06		*1.54E-10*
4	2.500E+00	4.36193874E-02	*1.08938179E-02*	*5.18904E-06*	*1.2959E-06*	*6.16E-10*	
			1.08951142E-02	5.19027E-06	1.2964E-06		*1.54E-10*
5	3.125E+00	**5.45145016E-02**	*1.08873320E-02*	**6.48514E-06**	*1.2952E-06*	**7.71E-10**	
			1.08886276E-02	6.48668E-06	1.2956E-06		*1.54E-10*
6	3.750E+00	6.54031292E-02	*1.08795506E-02*	*7.78047E-06*	*1.2942E-06*	*9.21E-10*	
			1.08808453E-02	7.78232E-06	1.2947E-06		*1.54E-10*
7	4.375E+00	7.62839745E-02	**1.08704746E-02**	*9.07487E-06*	**1.2932E-06**	*1.08E-9*	
			1.08717682E-02	9.07703E-06	1.2936E-06		**1.54E-10**
8	5.000E+00	8.71557427E-02	*1.08601052E-02*	*1.03682E-05*	*1.2919E-06*	*1.23E-9*	
			1.08613976E-02	1.03707E-05	1.2924E-06		*1.54E-10*
9	5.625E+00	9.80171403E-02	*1.08484435E-02*	*1.16603E-05*	*1.2905E-06*	*1.38E-9*	
			1.08497345E-02	1.16631E-05	1.2910E-06		*1.54E-10*
10	6.250E+00	**1.08866875E-01**	*1.08354910E-02*	**1.29510E-05**	*1.2890E-06*	**1.54E-09**	
			1.08367805E-02	1.29541E-05	1.2895E-06		*1.54E-10*
11	6.875E+00	1.19703655E-01	*1.08212492E-02*	*1.42401E-05*	*1.2873E-06*	*1.69E-9*	
			1.08225369E-02	1.42435E-05	1.2878E-06		*1.54E-10*
12	7.500E+00	1.30526192E-01	**1.08057197E-02**	*1.55276E-05*	**1.2855E-06**	*1.85E-9*	
			1.08070056E-02	1.55313E-05	1.2859E-06		**1.53E-10**
13	8.125E+00	1.41333198E-01	*1.07889045E-02*	*1.68132E-05*	*1.2835E-06*	*2.00E-9*	
			1.07901884E-02	1.68172E-05	1.2839E-06		*1.53E-10*
14	8.750E+00	1.52123386E-01	*1.07708055E-02*	*1.80969E-05*	*1.2813E-06*	*2.15E-9*	
			1.07720872E-02	1.81012E-05	1.2818E-06		*1.53E-10*
15	9.375E+00	**1.62895473E-01**	*1.07514248E-02*	**1.93783E-05**	*1.2790E-06*	**2.31E-09**	
			1.07527043E-02	1.93829E-05	1.2795E-06		*1.53E-10*
16	1.000E+01	1.73648178E-01	*1.07307649E-02*	*2.06575E-05*	*1.2765E-06*	*2.46E-9*	
			1.07320419E-02	2.06624E-05	1.2770E-06		*1.53E-10*
17	1.063E+01	1.84380220E-01	**1.07088281E-02**	*2.19342E-05*	**1.2739E-06**	*2.61E-9*	
			1.07101025E-02	2.19394E-05	1.2744E-06		**1.52E-10**
18	1.125E+01	1.95090322E-01	*1.06856171E-02*	*2.32083E-05*	*1.2712E-06*	*2.76E-9*	
			1.06868887E-02	2.32138E-05	1.2716E-06		*1.52E-10*
19	1.188E+01	2.05777211E-01	*1.06611346E-02*	*2.44796E-05*	*1.2683E-06*	*2.91E-9*	
			1.06624032E-02	2.44854E-05	1.2687E-06		*1.52E-10*
20	1.250E+01	**2.16439614E-01**	*1.06353835E-02*	**2.57480E-05**	*1.2652E-06*	**3.06E-09**	
			1.06366491E-02	2.57541E-05	1.2657E-06		*1.51E-10*
21	1.313E+01	2.27076263E-01	*1.06083669E-02*	*2.70134E-05*	*1.2620E-06*	**3.21E-9**	
			1.06096293E-02	2.70198E-05	1.2624E-06		*1.51E-10*
22	1.375E+01	2.37685892E-01	**1.05800880E-02**	*2.82755E-05*	**1.2586E-06**	**3.36E-9**	
			1.05813470E-02	2.82822E-05	1.2591E-06		***1.50E-10***
23	1.438E+01	2.48267239E-01	*1.05505502E-02*	*2.95343E-05*	*1.2551E-06*	**3.51E-9**	
			1.05518057E-02	2.95413E-05	1.2556E-06		*1.50E-10*
24	1.500E+01	2.58819045E-01	*1.05197570E-02*	*3.07895E-05*	*1.2514E-06*	**3.66E-9**	
			1.05210089E-02	3.07969E-05	1.2519E-06		*1.50E-10*
25	1.563E+01	**2.69340054E-01**		**3.20411E-05**		**3.81E-09**	

Table 12-9A

[4] A unit shift operator E is defined for a function f(x), initially assumed to be increasing in the interval considered, which satisfies E f(x) = f(x + 1)
Now, in a numerical manner, Briggs defines the 1st order mean difference by the relation (f(x + 5) - f(x))/5, which corresponds to $(E^5 - I)f(x)/5$ in operator notation, where I is the zero operator. A 1st order mean difference of this form with n = 5 intermediate steps, with the result placed in the second slot, can be written symbolically:

$${}_5\Delta_2^1 f(x) = E^2\frac{(E^5 - I)f(x)}{5}, \text{ or even as: } {}_5\Delta_2^1 = E^2\frac{(E^5 - I)}{5}$$

Now, it is useful to know that: $E + E^{-1} = E^{-1}(E - I)^2 + 2I = \Delta_{-1}^2 + 2I$;
and subsequently that: $(E + E^{-1})^2 - 2I = E^2 + E^{-2} = (\Delta_{-1}^2 + 2I)^2 = \Delta_{-2}^4 + 4\Delta_{-1}^2 + 2I$, where (Note: $\Delta^5 \equiv {}_1\Delta^5$, $\Delta_{-1}^2 \times \Delta_{-1}^2 = \Delta_{-2}^4$, etc). Hence, symbolically, the 2nd order difference from the 1st order difference that occupies the 0th point of subdivision, can be expressed as:

$${}_5\Delta_0^2 = E^3\frac{((-E^{-5} + I)}{5}.\frac{E^2(E^5 - I)}{5} = \frac{(E^5 - I)^2}{5^2};$$

$${}_5\Delta_2^3 = \frac{E^2(E^5 - I)}{5}.\frac{(E^5 - I)^2}{5^2} = \frac{E^2(E^5 - I)^3}{5^3}; {}_5\Delta_0^4 = E^3\frac{((-E^{-5} + I)}{5}.\frac{E^2(E^5 - I)^3}{5^3} = \frac{(E^5 - I)^4}{5^4};$$

$${}_5\Delta_2^5 = \frac{E^2(E^5 - I)}{5}.\frac{(E^5 - I)^4}{5^4} = \frac{E^2(E^5 - I)^5}{5^5}; {}_5\Delta_0^6 = E^3\frac{((-E^{-5} + I)}{5}.\frac{E^2(E^5 - I)^5}{5^5} = \frac{(E^5 - I)^6}{5^6}. \quad (12\text{-}1)$$

Hence, for even powers 2p, where p is an integer ≥ 1:

$${}_5\Delta_0^{2p} = \frac{(E - I)^{2p}}{5^{2p}}.E^{4p}(E^2 + E^{-2} + E + E^{-1} + I)^{2p} = \frac{\Delta_{4p}^{2p}}{5^{2p}}(\Delta_{-2}^4 + 4\Delta_{-1}^2 + 2I + \Delta_{-1}^2 + 2I + I)^{2p}$$

$$= (\frac{1}{5}\Delta_0^5 + \Delta_1^3 + \Delta_2^1)^{2p}.$$

While, for odd powers 2p + 1:

$${}_5\Delta_0^{2p+1} = \frac{(E - I)^{2p+1}}{5^{2p+1}}.E^2.E^{2(2p+1)}(E^2 + E^{-2} + E + E^{-1} + I)^{2p+1} = \frac{E^2\Delta_{2(2p+1)}^{2p+1}}{5^{2p+1}}(\Delta_{-2}^4 + 4\Delta_{-1}^2 + 2I + \Delta_{-1}^2 + 2I + I)^{2p+1}$$

$$= E^2(\frac{1}{5}\Delta_0^5 + \Delta_1^3 + \Delta_2^1)^{2p+1}.$$

(12.2)

Similar results are derived by Goldstine, who refers to forward differences only, which is rather misleading, as Briggs uses both forward and backward differences to get his central difference results: though the same coefficients are obtained as in Briggs' Table 12-7. A great simplification is obtained if the differences in the final column are considered to be equal, which is the case if the results are being calculated to a finite degree of accuracy, by setting equal the different row levels in the final results for (12.10 and (12.2). In the case considered with constant 5th order differences, to the accuracy required, it follows that the 3rd order differences are also correct for a given row, and so the 1st order - from which it follows that the 2nd and 4th orders are also correct. There is hence a great deal of sense in using differences that obey these rules, in easing the arithmetical work. Let us see how this works out in practise:

When p =1, we have the corrected 1st order mean, obtained from the original mean, written in the second slot, together with the corrected 3rd order mean, and $^1/_5$ of the 5th order mean; :

$$_5\Delta^1 = \frac{1}{5}\Delta^5 + \Delta^3 + \Delta^1 \,;\, or\ \Delta^1 = {_5\Delta^1} - \Delta^3 - \frac{1}{5}\Delta^5 \tag{12.3}$$

Now, this is the final result for the log function, or any other f(x) that increases monotonically, as we shall see with the 6th powers of integers tackled a little later in the Chapter as a sort of *tour de force* to vindicate the method; however, for the sine function, the finite differences of differing orders have signs attached - following the same rules as differentiation - as the various differences are themselves either increasing or decreasing functions in the interval considered: hence, the 3rd order differences are made negative:

$$_5\Delta^1 = \frac{1}{5}\Delta^5 - \Delta^3 + \Delta^1 \,;\, or\ \Delta^1 = {_5\Delta^1} + \Delta^3 - \frac{1}{5}\Delta^5 \tag{12.4}$$

to give agreement with Tables 12 - 7 & 8. To solve (12-1 & 2), the various levels can be expanded out to give Briggs' Table 12-7. Thus, for 2nd order:

$$\begin{aligned} &_5\Delta^2 = (\frac{1}{5}\Delta^5 - \Delta^3 + \Delta^1)^2 = \frac{1}{25}\Delta^{10} - \frac{2}{5}\Delta^8 + 1\frac{2}{5}\Delta^6 - 2\Delta^4 + \Delta^2 \,;\text{hence:} \\ &\Delta^2 = {_5\Delta^2} + 2\Delta^4 - 1\frac{2}{5}\Delta^6 + \frac{2}{5}\Delta^8 - \frac{1}{25}\Delta^{10} \end{aligned} \tag{12.5}$$

Now, the *initial* 1st order difference and subsequent odd orders occupy the 2 mod(5) slots, while the *initial* even orders always occupy the 0 mod(5) slots, in order that the differences are centred on the place in the table being interpolated. The rest of Table 12-7 now follows:

$$\begin{aligned} &_5\Delta^3 = (\frac{1}{5}\Delta^5 - \Delta^3 + \Delta^1)^3 = (\frac{1}{5}\Delta^5 - \Delta^3 + \Delta^1)(-\frac{2}{5}\Delta^8 + \frac{1}{25}\Delta^{10} + \Delta^2 - 2\Delta^4 + 1\frac{2}{5}\Delta^6) \\ &\therefore \Delta^3 = {_5\Delta^3} + 3\Delta^5 - 3\frac{3}{5}\Delta^7 + 2\frac{1}{5}\Delta^9 - \frac{18}{25}\Delta^{11} + \frac{3}{25}\Delta^{13} - \frac{1}{125}\Delta^{15}. \end{aligned} \tag{12.6}$$

$$_5\Delta^4 = (\frac{1}{5}\Delta^5 - \Delta^3 + \Delta^1)^4 = (\frac{1}{5}\Delta^5 - \Delta^3 + \Delta^1)(\Delta^3 - 3\Delta^5 + 3\frac{3}{5}\Delta^7 - 2\frac{1}{5}\Delta^9 + \frac{18}{25}\Delta^{11} - \frac{3}{25}\Delta^{13} + \frac{1}{125}\Delta^{15}).$$

$$\therefore \Delta^4 = {_5\Delta^4} + 4\Delta^6 - 6\frac{4}{5}\Delta^8 + 6\frac{2}{5}\Delta^{10} - \frac{91}{25}\Delta^{12} + 1\frac{7}{25}\Delta^{14} - \frac{34}{125}\Delta^{16} + \frac{4}{125}\Delta^{18} - \frac{1}{625}\Delta^{20}, \text{etc.}$$

[5] An attempt has been made in Table 12-13A below to show how the interpolated values can be built up from these calculated, in bold, with the corrected mean written immediately below. The two columns to the right are not corrected, the 6th mean being 6!

There are a number of typographical errors in the original to Table 12-13, which have been corrected here.

720	36360	**903000**		**189376250**		**15625000000**	50
		900120	15459060	187575002	1971287801		
720	37080					17596287801	51
		936480	16395540	203034062	2174321863		
720	**37800**		**17482500**		**2411128125**	19770609664	52
	37800	973560	17369100	219429602	2393751465		
720	38520					22164361129	53
		1011360	18380460	236798702	2630550167		
720	39240					24794911296	54
		1049880	19430340	255179162	2885729329		
720	***39960***	***1092000***		***276788750***		***27680640625***	55
	39960	*1089120*	20519460	274609502	3160338831		
720	40680					***30840979456***	56
		1129080	21648540	295128962	3455467793		
720	**41400**		**22942500**		**3795071875**	***34296447249***	57
	41400	1169760	22818300	316777502	3772245295		
720	42120					***38068692544***	58
		1211160	24029460	339595802	4111841097		
720	42840					***42180533641***	59
		1253280	25282740	363625262	4475466359		
720	**43560**	**1299000**		**391501250**		**46656000000**	60
	43560	*1296120*	26578860	388908002	4864374361		
720	44280					**51520374361**	61
		1339680	27918540	415486862	5279861223	**0**	
720	**45000**		**29437500**		**5752578125**	**56800235584**	62
	45000	1383960	29302500	443405402	5723266625		
720	45720					**62523502209**	63
		1428960	30731460	472707902	6195974527		
720	46440					**68719476736**	64
		1474680	32206140	503439362	6699413889		
720	**47160**	**1524000**		**538688750**		**75418890625**	65
720	**47160**	1521120	33727260	535645502	7202853251		
720	47880					**82653950016**	66
		1568280	35295540	569372762	7738498753		
720	**48600**		**37057500**		**8446021875**	**90458382169**	67
	48600	1616160	36911700	604668302	8409100455		
720	49320					**98867482624**	68
		1664760	38576460	641580002	9050680457		
720	50040					**107918163081**	69
		1714080	40290540	680156462	9730836919		
720		**1767000**		**723976250**		**117649000000**	70

Table 12-13A