§9.1

Chapter Nine

With these powers of the chords, and altogether everything prepared in this way, we can test any of the sections demonstrated or talked about. And from any given chord, we can find the chord for any multiples of the arc, or fractions of the given arc .

With multiples all is easily expounded; and with fractions the task is not less certain, but it is done with a lot more working. We saw the method for trisection and quinquisection for these parts of chords. Now we apply ourselves to squares for bisection and the rest.

With squares [quadratics] the governing equations can be reduced to more suitable terms: for the constant term [1] to the most distant power approach each other in even intervals [i.e. there are only even powers, going up in steps of 2], which is always the case for the powers serving these equations. As for bisection, the number of the powers is equal to **4**② — **1** ④, where there are two intervals, between unity and ②, and the same total between ② and ④: thus with the remainders from which squares are found. But for the equations for the chords the intervals are always odd. As with trisection, the powers is equal to **3**① — **1**③. Between the 0 power and ① there is a single interval; But between ① and ③ there are two intervals. Because of this, for bisection for ⓪ ② ④ we can put ⓪ ① ②; and after completing the operation we understand to have found the root by examining that square which was sought: of which the root will be the

. . . .		263191642669		
35320888862379560704786		1315959713348662533911 8		
1	**1**② *added*	2132887	1718510443603	125351659
453		20	16	12
4	**4**① *taken*	13288737	11851044360386	5351659
53		23687181	21055355413504	7895758
69	*Gnomon added*	36975918	32906399773890	13247417
122		36	32	12
12	4× *root taken*	97591895	906399773890	1247417
220		184234309	1579151656018	2368727
261	*Gnomon added*	281826204	2485551429908	3616144
481		28	24	36
4	4× *root taken*	182620460	85551429908	16144
8188 **†**		263191941	157915165602	26319
13125	*Gnomon added*	445812401	243466595510	42463
21313		4	24	4
20	4× *root taken*	4581240170	3466595510	2463
131388		7895758269	5263838853	2632
236781	*Gnomon added*	12476998439	8730434363	5095
368169		12	8	4
36	**4**① *taken*	47699843940	730434363	1096
816962		78957582789	1315959713	2105
1315925		126657426729	2046394076	2981
2132887		12	20	28
		665742672947	46394076	139
†Div. given 400		1052767770656	78957583	
Div. added 262		1718510443603	125351659	
Div. corr. 138				

[Table 9-1]

chord. E.g. the square [of the chord] subtending 90 degrees is 2 The square subtending 45 degrees is sought. I assert 2 = 4② — 1 ④, and by making the reduction 2 = 4① — 1②. [Thus, the equation is solved for x^2 initially, and then the square root is taken]

We can find the value of the root sought by two methods: With one general method, which agrees for all equations, (which before we made use of in finding these chords themselves, for trisection and quinquisection): namely by dividing the given number of the units by the number of the roots [for the first approximation], and between dividing the quotient, the square found should be added on [the gnomon]. With the other special [method], which agrees with bisection to such an extent, where as many as three powers are compared between themselves: which is the common method expounded by writers of Arithmetic [*i.e.* the usual method of solving a quadratic by completing the square].

In particular, we should make use of the general method. Let the given square of the chord of 140 degrees be $3\underline{532088886237956070404}$ The square of the chord of 70:0': degrees is sought. The equation of the chord is agreed upon 4② — 1 ④, which is reduced to 4① — 1②.

Therefore by this method we have found the value of the side [we shall call this the root henceforth] sought 13159597133486625339118. But this root is not the chord of 70:0' degrees but the square of the same chord, as warned before

3. The same value of the root is found by the usual method in most of the works of the writers of Arithmetic. If the roots are equal to the units and the square[1]: If from the square of half the number of the root is taken away the units: the root of the remainder either taken or added to the same half shall give the value of the root sought. [Thus, if $bx = x^2 + A$, $x = b/2 \pm\sqrt{(b/2)^2 - A}$].

Given the equation 4① equal to 1② + $3\underline{532088886237956070404}$. The square of half the number of the roots 4: from which if being taken the units there will remain $0\underline{467911113762043929595214}$, of which the root $0\underline{6840402866513374660882}$ which being subtracted from half the number of the roots 2, there will remain $1\underline{3159597133486625339118}$, the square of the chord of 70:0'. For if to the root found $0\underline{6840402866513374660882}$ is added to half of the number of the roots, the sum is $2\underline{6840402866513374660882}$ the square [of the chord] subtending 110:0' to the complementary arc of the semicircle.

And both of these equations satisfy the given equations.

131595971	*The smaller root found,* 1. *Square of the chord* 70:0':
4	
526383885	*Four of the root* 1. *Four of square of chord* 70:0':
35320888862	*Given chord of* 140:0':
17317499671	*Biquadratic of chord of* 70:0':
52638388533	*Sum equal to 4 [times] square of chord* 70:0':
26840402866	*The larger root found,* 1. *square of the chord* 110:0':
4	
107361611464	*Four of the root* 1. *Four of square of chord* 110:0':
35320888862	*Given chord of* 140:0':
72040722601	*Biquadratic of chord of* 110:0':
107361611463	*Sum equal to 4 [times] square of chord* 110:0':

[Table 9-2]

§9.2 Notes on Chapter Nine

[1] Briggs designates the terms of a polynomial $a + bx + cx^2 + dx^3 + ex^4 + \dots$ according to the following scheme: the constant term *a* he calls 'the units', from *unitas*; the linear term *x* is the root or side, from *latus,* while *b* is the number of the root, etc; x^2 is the square or quadratic term; x^3 the cubic term, x^4 the biquadratic, x^5 the quintic term, and so on.

[2.] The working of Table 9-1 is presented here for the first 10 approximations.

	f(x1)	f(x2)	Briggs' Method for Solving Quadratic Equations		f(x5)
	5.32088886E-01	5.32088886E-01	2.20888862E-02	8.18888624E-03	1.31388624E-03
A	1stx	2ndx	3rdx	4thx	5thx
3.53208889	1.00000000E+00	1.00000000E+00	1.30000000E+00	1.31000000E+00	1.31500000E+00
	del1	del2	del3	del4	del5
	0.00000000E+00	3.00000000E-01	1.00000000E-02	5.00000000E-03	9.00000000E-04
gnomonnumer	0.00000000E+00	-5.10000000E-01	-1.39000000E-02	-6.87500000E-03	-1.23219000E-03
gnomondenom	0.00000000E+00	6.00000000E-01	2.00000000E-02	1.00000000E-02	1.80000000E-03
numerator	5.32088886E-01	2.20888862E-02	8.18888624E-03	1.31388624E-03	8.16962380E-05
denominator	2.00000000E+00	1.40000000E+00	1.38000000E+00	1.37000000E+00	1.36820000E+00
correction	2.66044443E-01	1.57777759E-02	5.93397553E-03	9.59041050E-04	5.97107425E-05

f(x6)	f(x7)	f(x8)	f(x9)	f(x10)
8.16962379E-05	1.32887380E-05	9.75918950E-07	1.82620399E-08	4.58123406E-09
6thx	7thx	8thx	9thx	10thx
1.31590000E+00	1.31595000E+00	1.31595900E+00	1.31595970E+00	1.31595971E+00
del6	del7	del8	del9	del10
5.00000000E-05	9.00000000E-06	7.00000000E-07	1.00000000E-08	3.00000000E-09

-6.84075000E-05	-1.23128190E-05	-9.57656910E-07	-1.36808059E-08	-4.10424173E-09
1.00000000E-04	1.80000000E-05	1.40000000E-06	2.00000000E-08	6.00000000E-09
1.32887379E-05	9.75918950E-07	1.82620401E-08	4.58123398E-09	4.76992328E-10
1.36810000E+00	1.36808200E+00	1.36808060E+00	1.36808058E+00	1.36808057E+00
9.71327969E-06	7.13348286E-07	1.33486580E-08	3.34865800E-09	3.48658067E-10

[Table 9-2A]

As the denominator is approximately constant, and as the leading term only in the division was required, a lot of the working was mental, and so does not appear in Briggs' table.

ANOTHER EXAMPLE OF BISECTION.

Let the square of the chord of 36:0': be given. The square of the chord of 18:0': is sought from the equation 4① — 1② = 0381966011250105l5176.

. . . .		1957739348		
0381966011250105l5176		097886967409692855		
81 . . .	1②	3465056	28188146751	35324429
3900		32 . . . . 4①	1370417489	1761965
36	4①	26505650	29558564240	37086394
30066		1174596	28	36
1309		27680246	155856424076	1086394
31375		24	7830957376	39155
28.	4①	368024610	163687381452	1125549
337501		17619561	16	8
15584	1②	385644171	3687381452	325549
353085		36	176196541	15662
32	4①	2564417151	3863577993	341211
3308512		117464316	36	32
156544	1②	2681881467	263577993	21211
3465056		24	11746436	979
		28188146751	275324429	22190
			24	20
			35324429	2190

[Table 9-3]

Therefore the square sought is 0097886967409692855.
It comes out the same by the other method[1].

4① = 1② + 0381966011250105l5176. *Half the number of the root is 2.*
4 . . . . *The square of half the number of the root.*
3618033988749894848 24 *The difference: of which the root is*
1902113032590307l4 *which if taken and added to the two, the half the number of the root, comes to* 0097886967409692855 *the square of the chord of* 18 *degrees: and*

Given square of chord of 36 degrees	0381966011250105l5176
Biquadratic of the chord of 18 Degrees	0009581858388666
Sum	0391547869638771
Square of chord of 18 degrees	0097886967409692855
Four Square of Chord of 18 degrees	039154786963877144
Given square of chord of 36 degrees	0381966011250105
Biquadratic of chord of 162 degrees	15226486119111123
Sum	15608452130361228
Four square of chord of 162 degrees	1560845213036122856

[Table 9-4]

390211303259030714 *the square of the chord of* 162 *degrees, the compliment of course to the semicircle: the other [root]which will satisfy the equation given.*

Therefore with bisection we can make use of either the general method, which agrees for all equations, or use the common method, which agrees only for these equations for which three different kinds of figures equidistant are equal among themselves. [i.e. the relation can be cast as a quadratic; the other method obviously works for all degrees of polynomials.]

4. Because if either there will have been more kinds, or the intervals shall be unequal, we ought to have recourse to that more general method. As with trisection if the square of the third of the chord is sought: 9(2) — 6(4) + 1(6) is equal to the square of the chord: and by the factor reduction 9(1) — 6(2) + 1(3) is equal to the given square.

Let the given square of the chord be 72:0': 1381966011250105152. Sought is the square of the chord 24:0': the equation is 9(1) — 6(2) + 1(3) = 1381966.

13819660112501051	(172909084714798		345818	300	7
6	6(2) *added*		17290908	30	49
1441			1	2100	343
9	9(1) *taken*		189	1470	
1	1(3) *taken*		684	343	
540966		† 63727761105	29584 [$=172^2$]	3913	[$g(x_1)$]
1134	6(2) *added*	18673686 6(2)	31041	86700	2
				510	4
654366		82401447105	2989441 [$= 1729^2$]	173408	8
				204	
63	9(1) *taken*	81. . . . . 6(2)	3112281	175448	[$g(x_2)$]
3913	1(3) *taken*	807191085 1(3)	29897522281 [$=172909^2$]	8875200	9
				5160	81
20453011		594256020	276654464	79876800	729
4104	6(2)	1659926784 6(2)		5160	
				4128	
24557011		7602486984		729	
18175448	9(1) + 1(3)	72 . . . . . 9(1)		80295489	[$g(x_3)$]
6381563		71754087 1(3)	89692566843	8968323 .	
				518	9
186246	6(2)	30732897 [$=f(x_6)$]	5187	80714907 .	
8244023250			717540534744	518	81
				41496	729
8180295489	— 9(1) + 1(3)		20748	807191085	[$g(x_4)$]
† 63727761105			31122		
			71754086671 [$g(x_5)$]		

[Table 9-5]

Chapter Nine Notes (Cont'd)

f(x1)	f(x2)	Briggs' Method for Solving Cubic Equations	
5.4096601125E-01	2.0453011250E-02	6.3815632501E-03	6.3727761100E-05
1stx	2ndx	3rdx	4thx
1.0000000000E-01	1.7000000000E-01	1.7200000000E-01	1.7290000000E-01
del1	del2	del3	del4
7.0000000000E-02	2.0000000000E-03	9.0000000000E-04	9.0000000000E-06
gnomon = $-3x_1^2\Delta - 3x_1\Delta^2 - \Delta^3$:			
-2.1000000000E-03	-1.7340000000E-04	-7.9876800000E-05	-8.0714907000E-07
-1.4700000000E-03	-2.0400000000E-06	-4.1796000000E-07	-4.2014700000E-11
-3.4300000000E-04	-8.0000000000E-09	-7.2900000000E-10	-7.2900000000E-16
-3.9130000000E-03	-1.7544800000E-04	-8.0295489000E-05	-8.0719108543E-07
$12x_1\Delta + 6\Delta^2 - 9\Delta$:			
1.1340000000E-01	4.1040000000E-03	1.8624600000E-03	1.8673686000E-05
-6.3000000000E-01	-1.8000000000E-02	-8.1000000000E-03	-8.1000000000E-05
num1	num2	num3	num4
2.0453011250E-02	6.3815632501E-03	6.3727761100E-05	5.9425601455E-07
f'_x1	f'_x2	f'_x3	f'_x4
7.8300000000E+00	7.0467000000E+00	7.0247520000E+00	7.0148832300E+00
-8.2236000000E-01	-2.3983920000E-02	-1.0796734080E-02	-1.0799967297E-04
7.0076400000E+00	7.0227160800E+00	7.0139552659E+00	7.0147752303E+00
divisor	divisor2	divisor3	divisor4
7.0076400000E+00	7.0227160800E+00	7.0139552659E+00	7.0147752303E+00
quot1	quot2	quot3	quot4
2.9186732267E-03	9.0870301140E-04	9.0858522309E-06	8.4714904617E-08

f(x5)	num5
5.9425601462E-07	3.3073284352E-08
5thx	
1.7290900000E-01	f'_x5
del5	7.0147845668E+00
8.0000000000E-08	-9.5999997416E-07
	7.0147836068E+00
-7.1754053474E-09	divisor5
-3.3198528000E-15	7.0147836068E+00
-5.1200000000E-22	
-7.1754086673E-09	quot5
	4.7147975198E-09
1.6599267840E-07	

-7.2000000000E-07 [Table 9-5A]

Table 9-5 contains a fragment of the calculations of the root, the reader should be able to find what the various numbers mean either from the brief notes inserted into the table itself, or from this adjoining spreadsheet calculation, Table 9-5A. The work follows from the extensive discussion in Chapter four.