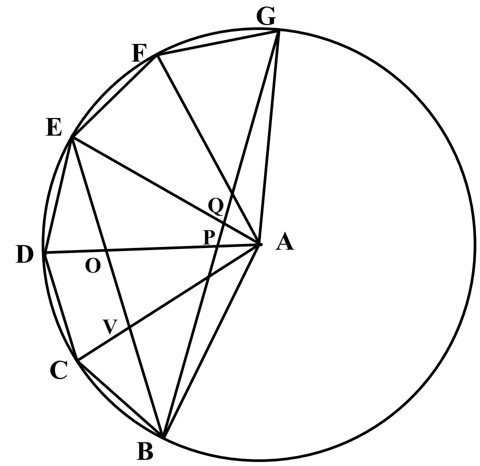
§5.1

Chapter Five

Concerning Quintuplication.

And as by trisection we found the subtended chord of the triple as well as of the third part of a subtended chord, so by quintuplication, and quinquisection we may find the subtended chord both of five times an arc and of the fifth part of an arc.

For Quinquisection.



[Figure 5-1]

1. If any right lines are in continued proportion, of which the first AB is the radius, with the second any subtended chord BC; of which five are subtended in the same way together with the sixth proportional, set equal to the subtended chord of the five-fold arc, and four of the same five proportionals are accepted.

Let BC, CD, DE, EF, FG, be equal to the subtended chords; and BE subtending the triple; BG, but the arc BC five-fold; they will be[1] AB, 1; BC, 1①; CV, 1②, because of continued proportion, and AV, 1 - 1②; and by the rule of proportion VO, will be 1① - 1③, because this is evident also through that which was said for trisection.

Proport-ionals		
	AC	1
	CD	1①
	AV	1 - 1②
	VO	1① - 1③

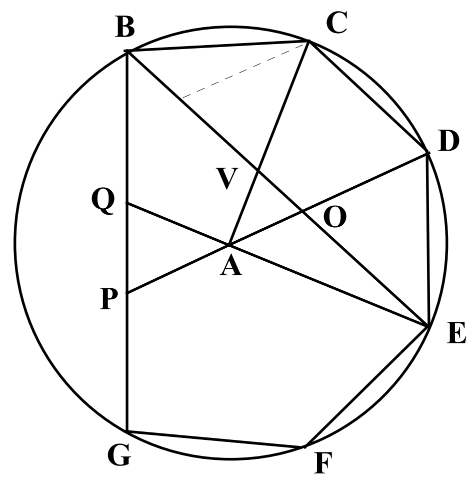
[Table 5-1]

But BO, BP are equal because PBO, OED are similar triangles: therefore BP, not less than BO, will be 2① - 1③, and PO will be 2② - 1④; And therefore OD, 1②, the total PD will be 3② - 1④ which taken from the Radius AD, 1, leaves AP, 1-3② + 1④, and by proportionality PQ will be 1①-3③ + 1⑤. But BP is 2① - 1③, and with this equal to GQ: the total BPQG is 5①-5③ + 1⑤.

Proport-ionals		
	AB	1
	BC	1①
	BP	2① - 1③
	PO	2② - 1④
	AP	1-3② + 1④
	PQ	1① -3③ + 1⑤

[Table 5-2]

2. With the total Arc BCDEFG, if it is larger than a semicircle[2], nevertheless BG is equal to 5 ① - 5 ③ + 1 ⑤. For DO is 1 ② as before; and BO, 2① - 1③, and therefore OP, 2② - 1④; to which is added DO, 1 ②, the total DP is 3② - 1④: from which if AD, 1 is taken away, AP is 3 ② -1④ -1 ; and PQ ijs 3 ③ - 1 ⑤ - 1 ①. As either BO, BP, or GQ is 2① - 1③: therefore BP and GQ have the value 4① - 2③; from which if PQ, 3 ③ - 1 ⑤ - 1 ①, is taken away there remains 5 ① -5③ + 1⑤ equal to the line BG.



[Figure 5 - 2]

Proport-ionals	AD	1	DP	3② - 1④	BO, BP	2① - 1③
	DE	1①	AP	3② -1④ -1	GQ	2① - 1③
	DE	1①	*PQ	3③ -1⑤ -1①	BP + GQ	4① - 2③
	DO	1②			*PQ	3③ -1⑤ -1①
	BO	2① - 1③			BG	5 ① -5③ + 1⑤
	OP	2② - 1④				

[Table 5 - 3]

CONCLUSIONS.

3. Given a subtended chord of any arc, we can find subtending chord of the quintuple arc; and conversely.

Let the given subtended chord of 10 Degrees be 17431148549.

This can be found by multiplication of the given subtending chord into itself, and in products by itself with itself, the square of the same, together with the cube, the biquadratic [Fourth power], and the whole [Fifth power]. If from five times the subtended chord and with one of the whole, should be taken away five of the cube; what has been left should be the subtended chord of 50 degrees.

087155742745	5①	*Continued Proport-ionals*	① 017431148549	*Sub. Ch.* 10^0:0'
000016092741	1⑤		② 00303844939793	
087171835486	*Sum*		③ 00052963662804	
002648183140	5③		④ 00009232174740	
084523652346	*Sub. Ch.* 50^0:0'		⑤ 00001609274093	
			⑥ 00000280514957	
			⑦ 00000048896979	

[Table 5-4]

Let the given subtended chord of 50 degrees be 845236523481.

4226182617405	5①		1000000000000
0431411060500	1⑤		① 0845236523481
4657593677905		*Continued Proport-ionals*	② 07144247806262
3019289589325	5③		③ 06038579178650
1638304088580	*Sub. Ch.* 250^0:0'		④ 05104027671726
77	110^0:0'		⑤ 04314110605000

[Table 5-5]

4. For if the five-fold of the given arc should exceed the whole circle: the signs of the whole equation are changed, and 5① -5③ + 1⑤ is replaced with 5③ -5① - 1⑤, which is equal to the subtending chord of five times the given arc: or rather the excess of five times the subtending chord over the whole circle.

Let the arc with the given subtending chord be 140 degrees: 0': if 5① + 1⑤ is taken from 5③, the subtended chord of 20:0' or 340 degrees remains.

5① 9396926207859083840			10000000000000000000000
1⑤ 32446556060384306982	*Sub. Ch.* 140^0 *or* 220^0	①	187938524157181676810822
32843482268243390822	*Sum taken away*	②	3532088886237959070404788
5③ 33190778623577251520		③	663815572471545030432466
0347296355333860698	*Sub. Ch.* 340^0	④	1247565190028568497932258
	20^0	⑤	2344655606038430698337877
		A ⑥	4406511142557256524229240
		B ⑦	8281532008143860592945789
		C ⑧	15564189033710182271025597
		D ⑨	29251107166988838303066607

Continued proportions

[Table 5-6]

With these continued proportionals, if from the three nearest the fourth is to be found ; with three times the second added to the first; the total is the fourth. The same subtended chord of 20 degrees can be found in the same way, if the subtended chord of 76 degrees has been given: the quintuple arc of which exceeds the circle by 20 degrees.

B	*Tripled*	8281532008143860592945789
A	.. ..	4406511142557256524229240
D	.. ..	.29251107166988838303066607

[Table 5-7]

5. For if the quintuple of the given arc exceeds two whole circles; the equation resumes that which was demonstrated from the first situation, and will be 5① -5③ + 1⑤ the equal of the subtending chord over two circles. For:

Let the given subtended chord of 157:2':40" be 19600039119128

98000195595640	5 ①		
289257541566577	1 ⑤		
289267541566577		*Sum taken away*	
376479054205137	5 ③		
10778682957080			
		Subtended degrees	65:13':20"
		Five times given arc	785:13':20"
		Two circles removed	720:0':0"
		Remaining arc	65:13':20"

Continued Proportions

10000000	
19600039119128	①
38416153347635	②
752958108410275	③
1475800837990604	④
289257541566577	⑤
566945913020770	⑥
1111216207363663	⑦

[Table 5-7]

And this method will be called 'Quintuplication ' [Latin: *Quintuplatio, times five*], as with any subtended chord, the subtended chord of the quintuple sought.

§5.2 Notes on Chapter Five.

[1] This beautiful piece of elementary geometry is based on the set of similar isosceles that Briggs has considered previously; see Chapter Three, Note 1. Thus, for the first set of proportionalities, [Table 5-1] becomes:

AC	1
CD	p
AV	$1 - p^2$
VO	$p - p^3$

[Table 5 - 1A]

AB	1
BC	p
BP	$2p - p^3$
PO	$2p^2 - p^4$
AP	$1-3p^2 + p^4$
PQ	$p -3p^3 + p^5$

[Table 5 - 2A]

Hence, the total length of the subtending chord BG = 2.BP + PQ = $5p - 5p^3 + p^5$.

This derivation is almost the same as Note 1. We will, however, set out the working for this case. Again, it is based on a number of similar isosceles triangles:
OE = BV = p; AO = $1 - p^2$; AO/OV = $1/p$ gives OV = $p - p^3$. Then
BO = PV + VO = $2p - p^3$ = BP; PO = $2p^2 - p^4$, and AP = AO - PO = $1 - 3p^2 + p^4$. Again, AP/1 = PQ/p *gives* PQ = $p - 3p^3 + p^5$, and finally: BG = 2BP + PQ = $5p -5p^3 + p^5$ as before.

This may be an appropriate place to comment on Briggs' 5th power equation for finding the subtended chord of the 5-fold arc given the single arc, and the 5th part of a given subtended chord. The first is dealt with by Briggs in a straightforward manner: however, the second requires a few remarks. We note initially the customary relation:

$Sin\ 5\theta = 5sin\ \theta - 20\ sin^3\theta + 16\ Sin^5\theta$. Now for unit radius, the subtended chord has length $A = 2sin5\theta/2$; hence, $A = 2\ sin5\theta/2 = 10sin\theta/2 - 40\ sin^3\theta/2 + 32\ sin^5\theta/2 = 5(2sin\theta/2) - 5(2sin\theta/2)^3 + (2sin\theta/2)^5$. This is the equation solved by Briggs in Chapter 6, where the variable is (2sinθ/2), leading to very accurate values of the sine of the 5th part of the original angle. Four of the 5 real roots are determined by Briggs, by adding on multiples of 360, taking supplements, etc. He does not seem to know that the sum of the roots is zero, or in fact that the equation has 5 roots: his attention is focused on finding the sines of useful angles for his tables.