§8.1

Chapter Eight

[1.] These equations for any section can be taken from the following Table, which on account of its benefits that are both many and remarkable, I am accustomed to call the *Παγχρητος* [Abacus Panchrestos: the 'good for all-things' table].

The table is in fact in the form of distinct columns of perpendicular lines designated by the letters A B C D, with an associated power [and sign].

Between these numbers in the columns there are other nearby diagonal number, but not in the same column.

All of these numbers are composed by the addition of other diagonal numbers, of which the sum is always put in the next lower place in the same column as the first diagonal number, and furthest from the right margin. [see the bold examples in the table] Any number whatever is in the same proportion to the number in its own diagonal moving up to the left, as the beginning number in the vertical column [of the first number] is to the marginal number [*i.e.* the leftmost number in same row as the second number].

15 *in Column B*	*Proportionals*	84 *in* C
20 *in Column C*		126 *in* D
3 *From the Vertical Column B*		4
4 *From the margin of the number 20.*		6

[Table 8-1]

The numbers in Column A, are to their own diagonals in B ascending, as 2 is to the marginal [left-most number in same row] of the second. [Thus, 5: 10 as 2: 4, or 10 = 5×4/2; 15:105 as 2: 14, or 105 = 15×14/2; etc]

Proportionals	2	*Proportionals*	2
	11		9
	12		10
	66		45

[Table 8-1, cont'd]

Hence it follows that the numbers adjacent to the right margin and for the rest in succession, the nearest can be found and continued to the end of the table. It is not necessary to compute the whole table down from the head [in order to find a particular set of numbers]. For, beginning with the number 23 in the margin, there are the proportionals 2.22.23.253. The fourth [number] is formed from the diagonal with 23, which gives 276 adjacent to the diagonal, written below in the same column. Then, 3.21.253.1771 are proportionals. [In an inductive way, 253 = 23×22/2, 1771 = 253×21/3 = 23×22×21/3×2×1] To this the fourth proportional put next to 253 is 2024 written below in the same Column C. [In this case, 2024 = 276×22/3 = 24×23×22/3×2×1. Thus, Briggs has set out the two main generating properties for the binomial coefficients of Pascal's Triangle long before the arrival of Pascal: ${}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$; and ${}^{n}C_{2} = n(n-1)/2$, ${}^{n}C_{3} = n(n-1)(n-2)/3.2.1$, etc].

2. The usefulness of these numbers is manifold: the first application to be considered is the finding of the customary numbers which we use either in the *generation* or in the *analysis* of the lengths of chords in geometric figures [from first principles to date]. However the *generation* is for a given length of side, by the use of a gnomon or formula to add sums of powers to find a new power [cube etc]; on the other hand, with the *analysis* for a power given, by reckoning with the same gnomon, we may wish to extract the same root.

ABACUS ΠΑΓΧΡΗΣΟΣ.											
M	L	K	I	H	G	F	E	D	C	B	A
– (12)	– (11)	(10)	(9)	– (8)	– (7)	(6)	(5)	– (4)	– (3)	+ (2)	(1)
1	1	1	1	1	1	1	1	1	1	1	1
13	12	11	10	9	8	7	6	5	**4**	3	2
91	78	66	55	**45**	36	28	21	15	**10**	**6**	3
455	364	286	220	**165**	**120**	84	56	35	20	10	4
1820	1365	1001	715	495	330	210	126	70	35	15	5
6188	4368	3003	2002	1287	792	462	252	126	56	21	6
18564	12376	8008	5005	3003	1716	924	462	210	84	28	7
50388	31824	19448	11440	6435	3432	1716	792	330	120	36	8
125970	75582	43758	24310	12870	6435	3003	1287	495	165	45	9
293930	167960	92378	48620	24310	11440	5005	2002	715	220	55	10
646646	352716	184756	92378	43758	19448	8008	3003	1001	286	66	11
									364	78	12
									455	91	13
									560	105	14
									680	120	15
1738360									816	136	16
30421755	13037895	5311735							969	153	17
51895935	21474180	8436285	3124550						1140	171	18
			4686825	1562475	480700				1330	190	19
				2220075	657800	177100	42504			210	20
						230230	53130	10626	1771	231	21
								12650	2024	253	22
										276	23
										300	24
										325	25
										351	26
										378	27
										406	28
										415	29
											30
											1
											2
91											3
455	364	286									4
		1001	715	495							5
				1287	792	462					6
						924	462	210			7
								330	120	36	8
										45	9
17383860											
30421755	13037895										
	21474180	8436285	3124550								18
			4686825	1562275	480700						19
					657800	177100	42504				20
	13037895						53130	10626	1771		21
	21474180	8436285	3124550						2024	253	22
			4686825	1562275	480700						23

These nearby numbers [in Table 8-2] have been placed ascending order towards the left, of which the first and the last show the distance from unity of that power [by this Briggs means the magnitude of the power], which they serve. As 2 for the square; 3, 3 the cube; 4, 6, 4 the biquadratic; 5, 10, 10, 5 for the complete or fifth power, etc.

The other use, which should not to be valued less, has been for the finding of equations, with the help of which the chords are acquired for all sorts of are sections. And in particular, for any section for which the equal parts of a section add up to an off number; from which the chords themselves are found in a single operation .

Also from the same table, the equations are obtained if the equal parts of an arc add up to an even number. These equations truly do not present their own subtended chords by a single operation, but instead produce the subtended squares of chords: then the subtended chords are found from the given squares.

The powers for equations are placed one by one at the head of these columns, each with a sign of addition or subtraction.

The numbers to be added for the powers are had by the addition of two nearby [numbers] in the same column.

And for these equations the first number for the chords, is that of the section number itself in Column A. The second in C. The third in E. And the rest by the same method, with other columns placed between.

All these numbers obliquely ascend towards the left. Thus, when the number of the section is odd, we have: for trisection 3 ① — 1 ③; for quinquisection, 5 ① — 5 ③ + 1 ⑤; for septisection 7 ① — 14 ③ + 7 ⑤ — 1 ⑦; which have been shown by me before. For the remainder of the sections with odd parts the method is the same as shown, with that usefulness as before: and with the numbers found from the Table. Thus, if the arc of the periphery is cut in 45 equal parts, the equation will be

45 ① — 3795 ③ + 95634 ⑤ — 1138500 ⑦ + 7811375 ⑨ — 34512075 ⑪ + etc.

This whole equation is taken from *Adranus Romanus* [Adriaan van Roomen: see H. Goldstine, *A History of Numerical Analysis….*, Springer-Verlag, page 33, for a discussion of this famous equation proposed by Roomen, and solved by Vieta in a devastating manner] .

If the parts shall be 11, the equation is

11 ① — 55 ③ + 77 ⑤ — 44 ⑦ + 11 ⑨ — 1 ⑪. But for the equations of squares of chords, the first number is the square of the number of the section sought in column B. The second in column D. The third in column F. The fourth if H, etc. As for the bisection:

4 ② — 1 ④. For the trisection 9 ② — 6 ④ + 1 ⑥. For the quadrisection

16 ② — 20 ④ + 8 ⑥ — 1 ⑧. For quinquisection

25 ② — 50 ④ + 35 ⑥ — 10 ⑧ + 1 ⑩

3. It is also possible to prepare this second Table, in which the single numbers are made as before by the addition of the diagonals; and so it is useful for all kinds of chords with equations of squares. But to generate any numbers in particular which the table describes, without the continuation of the numbers from the beginning, it is much more difficult than with the previous table. There are still other uses of the first Table, which I am unable to explain here. For these reasons, I rate the first table more useful than the one that follows. The position of the numbers for any equation you wish is the same as in the

M	L	K	I	H	G	F	E	D	C	B	A
−(12)	−(11)	(10)	(9)	−(8)	−(7)	(6)	(5)	−(4)	−(3)	+(2)	(1)
1	1	1	1	1	1	1	1	1	1	1	1
14	13	12	11	10	9	8	7	6	5	4	3
104	90	77	65	54	44	35	27	20	14	9	5
546	442	352	275	210	156	112	77	50	30	16	7
2275	1729	1287	935	660	450	294	182	105	55	25	9
8008	5733	4004	2717	1782	1122	672	378	196	91	36	11
24752	16744	11011	7007	4290	2508	1368	714	336	140	49	13
68952	44200	27456	16445	9438	5148	2640	1254	540	204	64	15
176358	107406	63206	35750	19305	9867	4719	2079	825	285	81	17
419900	243542	136136	72930	37180	17875	8008	3289	1210	385	100	19
940576	520676	277134	140998	68068	30888	13013	5005	1716	506	121	21
					51272	20384	7371	2366	650	144	23
						30940	10556	3185	819	169	25
							14756	4200	1015	196	27
								5440	1240	225	29
									1496	256	31
										286	33
	34512075									324	35
			7811375							361	37
					1138500					400	39
										441	41
									3795	484	43
							95634			527	45

[Table 8-3]

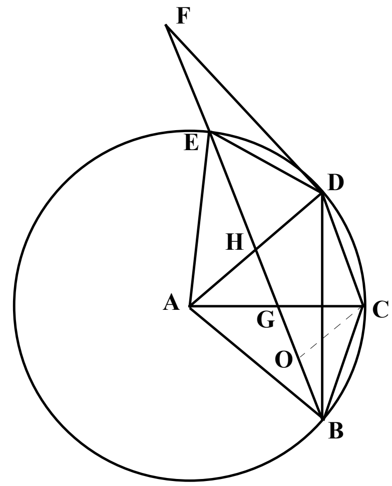
other Table, and which with two of the numbers expressed from before are here propounded singly.

Therefore for any given subtended arc, if the subtended arc of a multiple angle is sought, in the first place all the required powers of the given chord are found by continued multiplication of the same chord by itself and by its products. Secondly, the equation of any appropriate section can then be found by addition or subtraction of the powers of the chords of the multiples of the arcs themselves, or of their subtended squares [*i. e*. according to the table coefficients].

For these chords themselves where the number of the section is odd; we had the examples used for illustration above: but if the section is even, the squares are found, as we now examine.

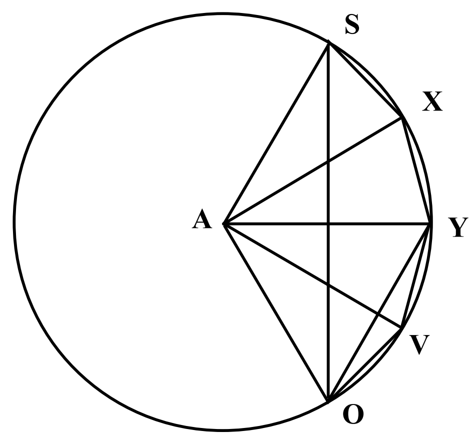
4. And first for bisection, where four square chords for any given arc you please, is equal to the square of the subtended arc doubled, and to the biquadratic of the given chords. Which can be shown thus:

Let BC, CD, DE be equal inscribed chords in a circle, and BE is continued in F, in order that EF and ED are equal: and BD, DF are draw. The triangles BCD, BDF have equal angles. For EFD, EDF are equal by the construction, and by *Prop.* 5, *Book* 1 . And the angle DEB is double the angle EFD, by *Prop. 32, Book 1* ; and double of the angle EBD, or DBC, by *Prop. 33, Book 6*. The triangles BCD, BDF are therefore equal angled, And, by *Prop. 4, book 6,* similar. And the sides BC, BD, BF are continued proportionals. And the square BD, is equal to the mean of the extremes to the rectangle BC, BF by *Prop. 17, Book 6*. [as BC/BD = BD/BF]



[Figure 8-1]

But BC is **1** ①; and BE, **3** ① — **1** ③, as is evident from the trisection [For BC = BG = p, and as ΔBCG is similar to ΔABC, GC = p^2 = HD; hence, AH = HG = 1 - p^2 ; from AG/AC = HG/DC, it follows that HG = (1 - p^2)p = p - p^3. Hence, BE = 3p - p^3]; and EF, from the construction, certainly equals the line BC. The total therefore BF, is **4** ① — **1** ③, which taken with BC, gives the rectangle **4** ② — **1** ④. [As noted above], equal to the square of the line BD of the mean of the three proportionals. Therefore the square of the line BD is **4** ② — **1** ④. Which was to be shown.



[Figure 8-2]

If an arc of the periphery is cut in four equal parts, as OV, VY, YX, XS; then the square of the line OS is thus computed.

The square of the line OY (as we have shown above), is equal to four times the square of the line OV, less the biquadratic of the same OV. For the same reason, the square of the line OS, is equal to four times the square of OY, less the biquadratic of the same line OY. The square of the line OY is **4** ② — **1** ④, Which going into itself makes the biquadratic of the line OY; **16** ④ — **8** ⑥ + **1** ⑧

Square OY,	$4\textcircled{2} - 1\textcircled{4}$
	$16\textcircled{4} - 4\textcircled{6}$
	$4\textcircled{6} + 1\textcircled{8}$
Biquadratic OY	$16\textcircled{4} - 8\textcircled{6} + 1\textcircled{8}$
Four Squares of OY	$16\textcircled{2} - 4\textcircled{4}$
Biquadratic OY being taken	$+16\textcircled{4} - 8\textcircled{6} + 1\textcircled{8}$
Square of line OS left	$16\textcircled{2} - 20\textcircled{4} + 8\textcircled{6} - 1\textcircled{8}$

[Table 8 - 4]

[$OY^2 = 4p^2 - p^4$; $OS^2 = 4OY^2 - OY^4 = 16p^2 - 4p^4 - 16p^4 + 8p^6 - p^8 = 16p^2 - 20p^4 + 8p^6 - p^8$]

If the Periphery is cut in eight equal parts, and the square of the line subtending four parts is $16\textcircled{2} - 20\textcircled{4} + 8\textcircled{6} - 1\textcircled{8}$, then this square multiplied into itself gives the biquadratic:

$256\textcircled{4} - 640\textcircled{6} + 656\textcircled{8} - 352\textcircled{10} + 104\textcircled{12} - 16\textcircled{14} + 1\textcircled{16}$.

Four squares are $64\textcircled{2} - 80\textcircled{4} + 32\textcircled{6} - 4\textcircled{8}$.

By taking the biquadratic the square of the line OS is left.

$64\textcircled{2} - 336\textcircled{4} + 672\textcircled{6} - 660\textcircled{8} + 352\textcircled{10} - 104\textcircled{12} + 16\textcircled{14} - 1\textcircled{16}$.

By the same method the square of the line subtending 16 equal parts is computed.

If the number of equal parts is odd, the chords (as we found above) multiplied by themselves will give the same square of the chord.

As the chord of the triple arc is equal to three of the roots less the cube

$3\textcircled{1} - 1\textcircled{3}$

$9\textcircled{2} - 3\textcircled{4}$

$- 3\textcircled{4} + 1\textcircled{6}$

$9\textcircled{2} - 6\textcircled{4} + 1\textcircled{6}$ *The square subtending three equal arcs.*

This square is multiplied by itself gives the biquadratic:

$81\textcircled{4} - 108\textcircled{6} + 54\textcircled{8} - 12\textcircled{10} + 1\textcircled{12}$.

Now four of the square are $36\textcircled{2} - 24\textcircled{4} + 4\textcircled{6}$

Therefore by taking the biquadratic there remains

$36\textcircled{2} - 105\textcircled{4} + 112\textcircled{6} - 54\textcircled{8} + 12\textcircled{10} - 1\textcircled{12}$.

So this is the square subtending six equal parts.

5. By the same method the squares of the lines are found subtending the arcs which are any multiple of the given arc. And if the chord is given of any arc you please for the parts [i.e. fractions] of which the radius is 100000, the square of the arc for any multiples of the chord can be found from the same parts.

For as many as are necessary powers of chords should be prepared , and by the multiplication, addition, and subtraction of these; according as are required by some section equation, it is then easy to calculate the chord itself or the square of any multiple arc. As an example, the chord of 36 degrees is taken, which is equal to the larger segment of the radius proportionally cut. Of which any powers you wish are calculated by subtraction alone, as you see here.

100000	*Chord 36:0:'*
61803398874989484824	(1)
38196601125010515176	(2)
23606797749978969648	(3)
14589803375031545528	(4)
9016994374947424120	(5)
5572809000084121408	(6)
3444185374863302712	(7)
2128623625220818696	(8)
1315561749642484016	(9)
813061875578334680	**(10)**
502499874064149336	**(11)**
310562001514185344	**(12)**
191937872549963992	**(13)**
118624128964221352	**(14)**
73313743585742640	**(15)**
45310385378478712	**(16)**
28003358207263928	**(17)**
17307027171214784	**(18)**
10696331036049144	**(19)**
6610696135165640	**(20)**

[Table 8 - 5]

100000	n
.618033988749894848 2050	1
.381966011250105151 7954	2
.236067977499789696 4092	3
.145898033750315455 3863	4
.090169943749474241 02294	5
.055728090000841214 36332	6
.034441853748633026 65964	7
.021286236252208187 70368	8
.013155617496424838 95596	9
.008130618755783348 747726	10
.005024998740641490 208229	11
.003105620015141858 539497	12
.001919378725499631 668733	13
.001186241289642226 870764	14
.000733137435857404 7979690	15
.000453103853784822 0727946	16
.000280033582072582 7251746	17
.000173070271712239 3476201	18
.000106963310360343 3775545	19
.000066106961351895 97006554	20

[Corrected Table 8 - 5 for comparison]

6. If the Square of the nine times the arc of the chord is sought, or of 324 degrees,

81 (2) — 540 (4) + 1386 (6) — 1782 (8) + 1287 (10) — 546 (12) + 135 (14) — 18 (16) + 1 (18)

For which duly by adding and subtracting there will remain the square of the chord 324:0'; or 36:0': 038196601125010515176.

38196601125010515176	**81 (2)**
3055728090000841214 08	
33436854000504728448	**1386 (6)**
44582472000672971264.	
16718427000252364224. .	
5572809000084121408. . .	
5691433129048342760	**1287 (10)**
6504495004626677440.	
1626123751156669360. .	
813061875578334680. . .	
593120644821106760	**135 (14)**
355872386892664056.	
118624128964221352. .	
17307027171214784	**1 (18)**
† 118802801635491018312 08	*Sum*

[Table 8 - 6]

583592135001261821120	**540 (4)**
72949016875157727640 . .	
4257247250441637392	
17028989001766549568 .	**1782 (8)**
14900365376545730872 . .	
2128623625220818696 . . .	
186337200908511206 4	
12422480060567413 76 .	**546 (12)**
15528100075709267 20 . .	
453103853784787120	**18 (16)**
362483083027829696	
118420835624240913 16032	**540 (4)** + **1782 (8)** + **546 (12)** + **18 (16)**
†118802801635491018 31208	**81 (2)** + **1386 (6)** + **1287 (10)** + **135 (14)** + **1 (18)**
038196601125010515176	The square of the chord of 324 or sq. of the chord of 36.

[Table 8 - 6, cont'd]

If the square of double the chord is sought, indeed of 72:0':

4 (2)	1527864045000420607 04	
1 (4)	145898033750315455 28	
	138196601125010515176	Square of the Chord of 72:0':

[Table 8 - 7]

If the square of the quintuple of the chord is sought, indeed of 180:0':

25 (2)	954915028125262879400
35 (6)	1950483150029442492 80
1 (10)	813061875578334680
	11507764050037854633 60
50 (4)	729490168751577276400
10 (8)	2128623625220818696 0
	75077640500378546336 0
	4000 *Square of Chord of* 180:0':

[Table 8 - 8]

7. The chord of 12:0': is taken and several powers of this .

100000 *Radius*	
20905692653530694279966 8	*Chord of 12:0':*
4370479853238872414286 6	(2)
913679085602598019441 8	(3)
191010941475668757572 1	(4)
39932160359518697374 9	(5)
8348094714675996394 4	(6) A
1745227023475803952 1	(7) B
364851797634213553 0	(8) C
76274795454290458 2	(9) D
15945774309783167 9	**(10) E**
3333574568428924 9	**(11) F**
696906853652023 0	**(12) G**

[Table 8 - 9]

With these continued proportionals, if five are given together, and from the largest and five times the forth is taken five times the second, the remainder is equal to the sixth. Thus this series may be continued, as the use of which will have been seen. For given ABCDE, F is sought.

A the first	83480947146759963944
Five times the forth D	3813739772714522910
Sum	87294686919474486854
Five times the second B	87261351173790197605
Required sixth F	33335745684289249
B the first	17452270234758039521
Five times the forth E	797288715489158359
Sum	18249558950247197880
Five times the second C	18242589881710
Required sixth G	6969068536520230

[Table 8 - 10]

[This amounts to saying that if $p^6=2^6\sin^6(\pi/60)$ is the first term, then $p^6 + 5p^9 - 5p^7 = p^{11}$, or $1 + 5p^3 = 5p + p^5$: see Note 1 of Chapter 5, where A = 1 is the length of the quintuple of the chord].

If the square of the chord of the sextuple, namely of 72:0': degrees is required, the equation is **36**(2) — **105**(4) + **112**(6) —**54**(8) + **12**(10) — **1**(12).
This equation accordingly solved gives square of the chord of 72 degrees 1381966011250105151795497; [Compare with the true value: 1.381966011250105151795413165 63…] Which clearly agrees with the square of the radius and the sides of the inscribed hexagon, as you see here.

2622287911943323448571 96	36 (2)
1311143955971661724285980	
1669618942935199278 88	112 (6)
918290418614359603384 0	
318915486195663358	12 (10)
1594577443097831679 0	
1582722452673934835908505 2	36 (2) + 112 (6) + 12 (10)
9550547073783437878605	105 (4)
191010941475668757572 1 . .	
1459407190536854212 0	54 (8)
182425898817106776 50	
6969068536520230	1 (12)
2007585154892432072895 55	105 (4) + 54 (8) + 1 (12)
158272452673934835908505 2	36 (2) + 112 (6) + 12 (10)
1381966011250105151795497	*Square of the chord of* 72:0':

[Table 8 - 11]

8. The chords of 20. 100. 156. 84 follow together with several powers of the same series, the use of which can be seen to be continued without much difficulty : as is evident for single examples.

The chord of 20:0' is taken with a few powers of the same as you wish.

	100000 *Radius*	
	3472963553338606977034 34	*Chord of 20:0':*
	1206147584281832318917 80	(2)
	418890660015820931102 97	(3)
	145479199506889979719 08	(4)
	505243957656304743911 1	(5)
	175469385048490080542 7	(6)
	609398779000144345425	(7)
	211641974891654977170	(8)
	735024 86515532230848	(9)
A	. .2552714567482058608 5	(10)
B	. . .8865484654941715374	(11)
C	. . .3078950508929527407	(12)
D	. . .1069308290004560037	(13)

[Table 8 - 12]

.3472963553338606977034 34	1
.1206147584281832318917 82	2
.0418890660015820931103000	3
.0145479199506889979719112	4
.0050524395765630474391178 9	5
.0017546938504849008054335 1	6
.0006093987790001443454424 36	7
.0002116419748916549771826 28	8
.0000735024865155322308937953	9
.0000255271456748205861054464	10
.0000088654846549417154987579 8	11
.0000030789505089295274225435 8	12
.0000010693082900045603908274 8	13

[Table 8 - 12A, true values for comparison]

If the square of the chord of six times the arc, surely of 120:0' is sought: the equation is

36 (2) — 105 (4) + 112 (6) — 54 (8) + 12 (10) — 1 (12).

7236885505690993913506 80	**36** (2)
36184427528454969567534 . .	
35093877009698016108 54	**112** (6)
193016323553339088596 97 .	
510542913496411721 70	**12** (10)
2552714567482058608 5 .	
†4538963340417003085344924	**36** (2) + **112** (6) + **12** (10)
7273959975344498985954 0	**105** (4)
14547919950688997971908 . .	
84656789956661990868 0	**54** (8)
105820987445827488585 0..	
3078950508929527407	**1** (12)
1538963340417003085344927	= **105** (4) + **54** (8) + **1** (12)
†4538963340417003085344924	= **36** (2) + **112** (6) + **12** (10)
30000	*Square of the chord of* 120:0':

[Table 8 - 13]

Let the given chord be 100:0':

1000 *Radius*

	1532088886237956070404786	
	2347296355333860697703430	(2)
	3596266658713868211214358	(3)
	5509800179763626022705504	(4)
	8441503620807743935939644	(5)
	12933133880577009856902154	(6)
A . .	19814710682659605785113428	(7)
B . . .	30357898020923285634766818	(8)
C . . .	46510998167401807498438130	(9)
D . . .	71258983380110251119187026	**(10)**

[Table 8 - 14]

1.532088886237956070404785	1
2.347296355333860697703433	2
3.596266658713868211214355	3
5.509800179763626022705514	4
8.441503620807743935939632	5
12.933133880577009856902188	6
19.814710682659605785113390	7
30.357898020923285634766930	8
46.510998167401807498437984	9
71.258983380110251119187404	10

[Table 8 - 14A, for comparison]

For these continuing proportionals with three given together it is possible to find the fourth: For as before the first A taken from three of the second leaves the fourth D.

Let the given chord be140:0': and powers of the same.

100000 *Radius Chord of 140:0':*

	1879385241571816768108218	
	3532088886237956070404786	(2)
	6638155724715450304324654	(3)
	12475651900285684979322576	(4)
	23446556060384306983378748	(5)
	44065111425572505242292382	(6)
A . . .	82.815320081438605929458820	(7)
B . . .	155641890337101822710255994	(8)
C . .	.292511071669888323030668842	(9)

[Table 8 - 15]

1.879385241571816768108218 5	1
3.532088886237956070404785 2	2
6.638155724715450304324655 3	3
12.475651900285684979322574	4
23.446556060384306983378750	5
44.065111425572505242292374	6
82.815320081438605929458821	7
155.641890337101822710255890	8
292.511071669888323030668875	9

[Table 8 - 15A, for comparison]

If the first is added to three of the second, the total is equal to the fourth

Let the given chord be156:0': and powers of the same.

100000 *Radius. Chord of 156:0':*

1956295201467611276	
3827090915285201791	2
7486919593152728676	3
14646624873858523591	4
28653121958425587001	5
56053964994334220276	6
109658102741649493416	7
214523620195531212101	8
419671528789978055666	9
821001397964410546126	10

[Table 8 - 17]

1.9562952014676112758	1
3.82709091528520179101	2
7.48691959315272867578	3
14.6466248738585235917	4
28.6531219584255869996	5
56.0539649943342202794	6
109.658102741649493410	7
214.523620195531212114	8
419.671528789978055646	9
821.001397964410546174	10

[Table 8 - 17A, for comparison]

With five closest neighbours given, if 5 of the second is taken away from five of the fourth and the first, the remainder will be the equal of the sixth.

Let the given chord be 84: and powers of the same.

100000 *Radius. Chord of 84:0':*

133826121271771640	
179094307346469306	2
239674964940325439	3
320747709239116095	4
429244218342768993	5
574440888191462305	6
768751959665748464	7
1028790929821405611	8
1376790997375781260	9
1842505989806947537	10

[Table 8 - 18]

1.338261212717716427	1
1.790943073464693057	2
2.396749649403254434	3
3.207477092391160952	4
4.292442183427690035	5
5.744408881914623050	6
7.687519596657484937	7
10.28790929821405604	8
13.76790997375781358	9
18.42505989806947495	10

[Table 8 - 18A, for comparison

Given 5 neighbouring [terms], if the first and five of the second are taken from five of the fourth, the remainder will be the sixth.

Notes on Chapter Eight.

[1] Table 8-2 contains the Binomial coefficients, yet it is not exactly Pascal's triangle, and has great versatility. It is hence a useful exercise to locate these coefficients within the table, a part of which is reproduced here:

(9)	−(8)	−(7)	(6)	(5)	−(4)	−(3)	+(2)	(1)
$^{9}C_0 = 1$	$^{8}C_0 = 1$	$^{7}C_0 = 1$	$^{6}C_0 = 1$	$^{5}C_0 = 1$	$^{4}C_0 = 1$	$^{3}C_0 = 1$	$^{2}C_0 = 1$	$^{1}C_0 = 1$
$^{10}C_1 = 10$	$^{9}C_1 = 9$	$^{8}C_1 = 8$	$^{7}C_1 = 7$	$^{6}C_1 = 6$	$^{5}C_1 = 5$	$^{4}C_1 = 4$	$^{3}C_1 = 3$	$^{2}C_1 = 2$
$^{11}C_2 = 55$	$^{10}C_2 = 45$	$^{9}C_2 = 36$	$^{8}C_2 = 28$	$^{7}C_2 = 21$	$^{6}C_2 = 15$	$^{5}C_2 = 10$	$^{4}C_2 = 6$	$^{3}C_2 = 3$
$^{12}C_3 = 220$	$^{11}C_3 = 165$	$^{10}C_3 = 120$	$^{9}C_3 = 84$	$^{8}C_3 = 56$	$^{7}C_3 = 35$	$^{6}C_3 = 20$	$^{5}C_3 = 10$	$^{4}C_3 = 4$
$^{13}C_4 = 715$	$^{12}C_4 = 495$	$^{11}C_4 = 330$	$^{10}C_4 = 210$	$^{9}C_4 = 126$	$^{8}C_4 = 70$	$^{7}C_4 = 35$	$^{6}C_4 = 15$	$^{5}C_4 = 5$
$^{14}C_5 = 2002$	$^{13}C_5 = 1287$	$^{12}C_5 = 792$	$^{11}C_5 = 462$	$^{10}C_5 = 252$	$^{9}C_5 = 126$	$^{8}C_5 = 56$	$^{7}C_5 = 21$	$^{6}C_5 = 6$
$^{15}C_6 = 5005$	$^{14}C_6 = 3003$	$^{13}C_6 = 1716$	$^{12}C_6 = 924$	$^{11}C_7 = 462$	$^{10}C_6 = 210$	$^{9}C_6 = 84$	$^{8}C_6 = 28$	$^{7}C_6 = 7$
$^{16}C_7 = 11440$	$^{15}C_7 = 6435$	$^{14}C_7 = 3432$	$^{13}C_7 = 1716$	$^{12}C_7 = 792$	$^{11}C_7 = 330$	$^{10}C_7 = 120$	$^{9}C_7 = 36$	$^{8}C_7 = 8$
$^{17}C_8 = 24310$	$^{16}C_8 = 12870$	$^{15}C_8 = 6435$	$^{14}C_8 = 3003$	$^{13}C_8 = 1287$	$^{12}C_8 = 495$	$^{11}C_8 = 165$	$^{10}C_8 = 45$	$^{9}C_8 = 9$
$^{18}C_9 = 48620$	$^{17}C_9 = 24310$	$^{16}C_9 = 11440$	$^{15}C_9 = 5005$	$^{14}C_9 = 2002$	$^{13}C_9 = 715$	$^{12}C_9 = 220$	$^{11}C_9 = 55$	$^{10}C_9 = 10$
$^{19}C_{10} = 92378$	$^{18}C_{10} = 43758$	$^{17}C_{10} = 19448$	$^{16}C_{10} = 8008$	$^{15}C_{10} = 3003$	$^{14}C_{10} = 1001$	$^{13}C_{10} = 286$	$^{12}C_{10} = 66$	$^{11}C_{10} = 11$

The numbers in the second Table are formed from the vertical sums of two adjacent numbers in the same column: We now have an extended note concerning the connection of these

(9)	−(8)	−(7)	(6)	(5)	−(4)	−(3)	+(2)	(1)
1	1	1	1	1	1	1	1	1
11	10	9	8	7	6	5	4	3
65	54	44	35	27	20	14	9	5
275	210	156	112	77	50	30	16	7
935	660	450	294	182	105	55	25	9
2717	1782	1122	672	378	196	91	36	11
7007	4290	2508	1368	714	336	140	49	13
16445	9438	5148	2640	1254	540	204	64	15
35750	19305	9867	4719	2079	825	285	81	17
72930	37180	17875	8008	3289	1210	385	100	19
140998	68068	30888	13013	5005	1716	506	121	21

numbers with the coefficients in the section equations: Thus:

$121 = 55 + 66 = {}^{11}C_9 + {}^{12}C_{10}$, etc. Now, the odd sections have expansions such as :
$7p - 14p^3 + 7p^5 - p^7$, which become in terms of binomial coefficients:
$({}^4C_3 + {}^3C_2)p - ({}^5C_2 + {}^4C_1)p^3 + ({}^6C_1 + {}^5C_0)p^5 - p^7$; again, for the 11th power section:
$11p - 55p^3 + 77p^5 - 44p^7 + 11p^9 - p^{11}$,
$({}^6C_5 + {}^5C_4)p - ({}^7C_4 + {}^6C_3)p^3 + ({}^8C_3 + {}^7C_2)p^5 - ({}^9C_2 + {}^8C_1)p^7 + ({}^{10}C_1 + {}^9C_0)p^9 - p^{11}$. Hence, we surmise that for the odd section of order $M = 2N + 1$, the governing equation will be:

$$({}^{N+1}C_N + {}^{N}C_{N-1})p - ({}^{N+2}C_{N-1} + {}^{N+1}C_{N-2})p^3 + ({}^{N+3}C_{N-2} + {}^{N+2}C_{N-3})p^5 - \ldots..$$

$(-1)^r({}^{N+r+1}C_{N-r} + {}^{N+r}C_{N-r-1})p^{2r+1}$ + ….. $({}^{2N}C_1 + {}^{2N-1}C_0)p^{2N-1} - p^{2N+1}$. Thus, Adriaan van Roomen's famous equation of the 45th degree has N = 22, and the general term is of the form: $(-1)^r({}^{23+r}C_{22-r} + {}^{22+r}C_{21-r})p^{2r+1}$: e.g. setting r = 3 gives $-({}^{26}C_{19} + {}^{25}C_{18})p^7$
= - (657800 + 480700) = - 1138500.

We should be able to derive these equations starting from De Moivre's Theorem: According to which, for any given angle θ, usually taken with $0 < \theta < 2\pi$,

$e^{i\theta} = \cos\theta + i\sin\theta$, while $e^{iM\theta} = (\cos\theta + i\sin\theta)^M = \cos M\theta + i\sin M\theta$.
In the present circumstances, a chord of length A in a circle of radius R subtends an angle 2Mθ, where A = 2Rsin Mθ, and where M is odd as above, and equals 2N + 1. Hence we may write De Moivre's Theorem in the form:

$${}^{2N+1}C_0\cos^{2N+1}\theta + i\,{}^{2N+1}C_1\cos^{2N}\theta\sin\theta - {}^{2N+1}C_2\cos^{2N-1}\theta\sin^2\theta - i\,{}^{2N+1}C_3\cos^{2N-2}\theta\sin^3\theta + \ldots$$

$$+ {}^{2N+1}C_{2r-2}\cos^{2N-2r+3}\theta\sin^{2r-2}\theta + i\,{}^{2N+1}C_{2r-1}\cos^{2N-2r+2}\theta\sin^{2r-1}\theta -$$

$${}^{2N+1}C_{2r}\cos^{2N-2r+1}\theta\sin^{2r}\theta - i\,{}^{2N+1}C_{2r+1}\cos^{2N-2r}\theta\sin^{2r+1}\theta + \ldots.$$

$$+ {}^{2N+1}C_{2N-2}\cos^3\theta\sin^{2N-2}\theta + i\,{}^{2N+1}C_{2N-1}\cos^2\theta\sin^{2N-1}\theta - {}^{2N+1}C_{2N}\cos^1\theta\sin^{2N}\theta$$

$$- \quad i\,{}^{2N+1}C_{2N+1}\sin^{2N+1}\theta$$

$$= \cos(2N+1)\theta + i\sin(2N+1)\theta.$$

Hence, $\sin(2N+1)\theta = {}^{2N+1}C_1\cos^{2N}\theta\sin\theta - {}^{2N+1}C_3\cos^{2N-2}\theta\sin^3\theta + \ldots.$

$$+ {}^{2N+1}C_{2r-1}\cos^{2N-2r+2}\theta\sin^{2r-1}\theta - {}^{2N+1}C_{2r+1}\cos^{2N-2r}\theta\sin^{2r+1}\theta + \ldots$$

$$+ {}^{2N+1}C_{2N-1}\cos^2\theta\sin^{2N-1}\theta - {}^{2N+1}C_{2N+1}\sin^{2N+1}\theta$$

$$= {}^{2N+1}C_1(1-\sin^2\theta)^N\sin\theta - {}^{2N+1}C_3(1-\sin^2\theta)^{N-1}\sin^3\theta + \ldots.$$

$$+ {}^{2N+1}C_{2r-1}(1-\sin^2\theta)^{N-r+1}\theta\sin^{2r-1}\theta - {}^{2N+1}C_{2r+1}(1-\sin^2\theta)^{N-r}\sin^{2r+1}\theta +$$

$$+ {}^{2N+1}C_{2N-1}(1-\sin^2\theta)\sin^{2N-1}\theta - {}^{2N+1}C_{2N+1}\sin^{2N+1}\theta$$

$$= {}^{2N+1}C_1\sin\theta(1 - {}^NC_1\sin^2\theta + {}^NC_2\sin^4\theta - {}^NC_3\sin^6\theta + \ldots (-1)^N\,{}^NC_N\sin^{2N}\theta)$$

$$- {}^{2N+1}C_3\sin^3\theta\,(1 - {}^{N-1}C_1\sin^2\theta + {}^{N-1}C_2\sin^4\theta - {}^{N-1}C_3\sin^6\theta + \ldots (-1)^{N-1}\,{}^{N-1}C_{N-1}\sin^{2N-2}\theta) + \ldots.$$

$${}^{2N+1}C_{2r-1}\sin^{2r-1}\theta(1 - {}^{N-r+1}C_1\sin^2\theta + {}^{N-r+1}C_2\sin^4\theta - {}^{N-r+1}C_3\sin^6\theta + \ldots (-1)^{N-r+1}\,{}^{N-r+1}C_{N-r+1}\sin^{2N-2r+2}\theta)$$

$$- {}^{2N+1}C_{2r+1}\sin^{2r+1}\theta(1 - {}^{N-r}C_1\sin^2\theta + {}^{N-r}C_2\sin^4\theta - {}^{N-r}C_3\sin^6\theta + \ldots (-1)^{N-r}\,{}^{N-r}C_{N-r}\sin^{2N-2r}\theta) + \ldots$$

$+ {}^{2N+1}C_{2N-1} \sin^{2N-1}\theta(1 - {}^{1}C_{1}\sin^{2}\theta) - {}^{2N+1}C_{2N+1} \sin^{2N+1}\theta.$

The terms are now gathered as a finite expansion of powers of sinθ, taking R = 1, where θ is the half-angle subtended by the chord: then A = 2sin(2N+1)θ =

$\sin\theta.2({}^{2N+1}C_1) - \sin^3\theta.2({}^{2N+1}C_1\ {}^{N}C_1 + {}^{2N+1}C_3) + \sin^5\theta.2({}^{2N+1}C_1\ {}^{N}C_2 + {}^{2N+1}C_3\ {}^{N-1}C_1 + {}^{2N+1}C_5)$

$- \sin^7\theta.2({}^{2N+1}C_1\ {}^{N}C_3 + {}^{2N+1}C_3\ {}^{N-1}C_2 + {}^{2N+1}C_5\ {}^{N-2}C_1 + {}^{2N+1}C_7) - \ldots.$

$+ (-1)^r.\mathrm{Sin}^{2r+1}\theta.2({}^{2N+1}C_1\ {}^{N}C_r + {}^{2N+1}C_3\ {}^{N-1}C_{r-1} + {}^{2N+1}C_5\ {}^{N-2}C_{r-2} +..\ {}^{2N+1}C_{2r-1}\ {}^{N-r+1}C_1 + {}^{2N+1}C_{2r+1}) +$

$\ldots + \mathrm{Sin}^{2N-1}\theta.2({}^{2N+1}C_1\ {}^{N}C_{N-1} + {}^{2N+1}C_3\ {}^{N-1}C_{N-2} + {}^{2N+1}C_5\ {}^{N-2}C_{N-3} +..\ {}^{2N+1}C_{2N-3}\ {}^{2}C_1 + {}^{2N+1}C_{2N-1}) -$

$\mathrm{Sin}^{2N+1}\theta.2({}^{2N+1}C_1\ {}^{N}C_N + {}^{2N+1}C_3\ {}^{N-1}C_{N-1} + {}^{2N+1}C_5\ {}^{N-2}C_{N-2} +..\ {}^{2N+1}C_{2N-1}\ {}^{1}C_1 + {}^{2N+1}C_{2N+1})$

If we identify these coefficients with those present in Briggs' s second Abacus: for the linear and cubic terms, we have:

$2({}^{N+1}C_N + {}^{N}C_{N-1}) = 2({}^{N+1}C_1 + {}^{N}C_1) = 2.{}^{2N+1}C_1 = 2.(2N + 1);$
$2^3({}^{N+2}C_{N-1} + {}^{N+1}C_{N-2}) = 2^3({}^{N+2}C_3 + {}^{N+1}C_3) = 2^3(2N + 1)(N + 1)N/3!$, while
$2({}^{2N+1}C_1\ {}^{N}C_1 + {}^{2N+1}C_3) = 2(2N + 1)N + 2(2N + 1)2N(2N - 1)/3! = 2^3(2N + 1)(N + 1)N/3!$.

For the $(2r + 1)^{th}$ term, we have $2^{2r+1}({}^{N+r+1}C_{N-r} + {}^{N+r}C_{N-r-1}) =$

$2({}^{2N+1}C_1\ {}^{N}C_r + {}^{2N+1}C_3\ {}^{N-1}C_{r-1} + {}^{2N+1}C_5\ {}^{N-2}C_{r-2} +..\ {}^{2N+1}C_{2r-1}\ {}^{N-r+1}C_1 + {}^{2N+1}C_{2r+1});$

For the last term, consider $2({}^{2N+1}C_1 + {}^{2N+1}C_3 + {}^{2N+1}C_5 +.. + {}^{2N+1}C_{2N-1} + {}^{2N+1}C_{2N+1})$.

Now, $2^{2N+1} = (1 + 1)^{2N+1} =$

${}^{2N+1}C_0 + {}^{2N+1}C_1 + {}^{2N+1}C_2 + \ldots\ {}^{2N+1}C_{2N-1} + {}^{2N+1}C_{2N} + {}^{2N+1}C_{2N+1}$, while

$0^{2N+1} = (1 - 1)^{2N+1} = {}^{2N+1}C_0 - {}^{2N+1}C_1 + {}^{2N+1}C_2 - \ldots - {}^{2N+1}C_{2N-1} + {}^{2N+1}C_{2N} - {}^{2N+1}C_{2N+1};$

Hence: $2^{2N+1} = 2({}^{2N+1}C_1 + {}^{2N+1}C_3 + {}^{2N+1}C_5 +.. + {}^{2N+1}C_{2N-1} + {}^{2N+1}C_{2N+1}).$

From this investigation, we assert:

$2^{2r}({}^{N+r+1}C_{N-r} + {}^{N+r}C_{N-r-1}) =$

${}^{2N+1}C_1\ {}^{N}C_r + {}^{2N+1}C_3\ {}^{N-1}C_{r-1} + {}^{2N+1}C_5\ {}^{N-2}C_{r-2} +..\ {}^{2N+1}C_{2r-1}\ {}^{N-r+1}C_1 + {}^{2N+1}C_{2r+1}$

is a valid Binomial Identity, for 0 <= r <=N.

End of current Note