§8.1. *Synopsis: Chapter Eight.*

A difference method for finding the continued means of any required number *a* is introduced. This iterative scheme removes some of the labour associated with taking the square root repeatedly. It is successful if the first mean has three or more zeros present initially: Briggs relies on the position in the table to indicate the level [see Table 8-1], given by the left-hand column, while the orders of differences are denoted by A, B, C, etc. If the n^{th} level geometric mean of 1 and *a* has the value $1 + A_n$, evaluated in the traditional way by finding the square root of $1 + A_{n-1}$, while the next mean similarly evaluated has the value $1 + A_{n+1}$, then the second order difference is defined by: $B_{n+1} = \frac{1}{2}A_n - A_{n+1}$. Higher order differences are defined in a similar manner from subsequent means: $C_{n+2} = \frac{1}{2^2}B_{n+1} - B_{n+2}$, etc. (See Table 8-2 in the chapter notes). These differences eventually lie outside the range of the significant figures of the successive means, and the algorithm is then ran backwards to generate new means. Briggs also shows how the various differences can be expressed as powers of the initial difference. Note: Briggs enumerates the level of the mean from the smallest value in his table.

§8.2. *Chapter Eight* [p.15.]

We can find the logarithm of any proposed number by continued means according to this method which supplies our need laboriously enough by finding the square root: but the vexation of this enormous labour is most lessened through differences. I shall show the method from these means themselves, which I have written with the letter *A*, and with the help of which we can find the logarithm of six. Of these numbers, the first place on the left is assigned to one; the remaining places express the fractions to be added to one, as we advised above at the end of Chapter 6. These fractions added on to one, I call the first of the differences[1] [Table 8-1]: evidently the difference of the given number above unity. The nearest numbers placed below *A* are the halves of the nearest preceding differences, and written as $\frac{1}{2}A$. From these halves the first differences written above are taken away, and the numbers left are written below, which are written with the letter B, which is called the second differences. But the numbers placed below B are the quarters of the nearest preceding second differences: and they are written as $\frac{1}{4}B$. From which the numbers *B* taken leaves the third differences *C*, and these third differences taken from an eighth of the preceding third, denoted *C*, leaves the fourth difference *D*. By the same method, the fifth difference *E* is found, the sixth *F*, the seventh *G*, etc. By taking away the fourth *D* from a sixteenth of the preceding fourth *D*: with the fifth E, from a thirty second of the fifth: and with the sixth F, from a sixty fourth of the sixth, etc.

	10077,696		
46	10038,77283,33696,24566,38465,51		
45	10019,36766,13694,66167,58702,29		
44	10009,67914,63909,90172,88907,20		
43	10004,83840,26884,66298,54925,34	A	
42	10002,41890,87882,46856,38087,26	A	
	2,41920,13442,33149,27462,67	$^1/_2$A	
	29,25559,86292,89375,41	B	
41	10001,20938,12639,71345,94391,94	A	
	1,20945,43941,23428,19043,63	$^1/_2$A	
	7,31301,52082,24651,69	B	
	7,31389,96573,22343,85	$^1/_4$B	
	88,44490,97692,16	C	
40	1000,060467,23505,53096,80160,05	A	
	60469,06319,85672,97195,92	$^1/_2$A	
	1,82814,32576,17035,87	B	
	1,82825,38020,56162,92	$^1/_4$B	
	11,05444,39127,05	C	
	11,05561,37211,52	$^1/_8$C	
	116,98084,47	D	
39	10000,30233,16050,56577,59647,94	A	
	30233,61752,76548,40080,02	$^1/_2$A	
	45702,19970,80432,08	B	
	45703,58144,04258,97	$^1/_4$B	
	1,38173,23826,89	C	
	1,38180,54890,88	$^1/_8$C	
	7,31063,97	D	
	7,31130,28	$^1/_{16}$D	
	66,31	E	
38	10000,15116,46599,90567,29504,88	A	
	15116,58025,28288,79823,97	$^1/_2$A	
	11425,37721,50319,09	B	
	¶ 11425,54992,70108,02	$^1/_4$B	
	17271,19788,93	C	
	17271,65478,36	$^1/_8$C	
	45689,43	D	These small differences
	45691,50	$^1/_{16}$D	are found by taking
	2,07	E	away the smaller from
	2,07	$^1/_{32}$E	the larger quantity.
	65	$^1/_{32}$E	Here, the larger
	2855,589	$^1/_{16}$D	differences are
	2855,524	D	left by subtraction
	2158,89973,616	$^1/_8$C	of the smaller
	2158,87118,092	C	from the fraction
	6,34430,37579,772	$^1/_4$B	of the nearest
	6,32271,50461,680	B	above same kind.
	7558,23299,95283,64752,440	$^1/_2$A	
37	10000,07558,20443,63012,14290,706	A	
	2	$^1/_{32}$E	
	178,470	$^1/_{16}$D	
	178,468	D	
	269,85889,762	$^1/_8$C	
	269,85711,294	C	
	714,08067,87615,420	$^1/_4$B	
	714,07798,01904,126	B	
	3779,10221,81506,07145,380	$^1/_2$A	
36	10000,03779,09507,73708,05241,254	A	

[Table 8-1]

[p.16.]

These differences also can be found for any given number, although none of the continued means are given. For if the difference of a given number above unity is multiplied a number of times, both itself and its products, in order that some powers of the same difference can be computed, indeed the square, cube, biquadratic, etc. The second order difference is [then] half the square, and also the rest as you see here[2]:

Differences

Second $\frac{1}{2}$ (2)

Third $\frac{1}{2}(3) + \frac{1}{8}(4)$

Fourth $\frac{7}{8}(4) + \frac{7}{8}(5) + \frac{7}{16}(6) + \frac{1}{8}(7) + \frac{1}{64}(8)$

Fifth $2\frac{5}{8}(5) + 7(6) + 10\frac{15}{16}(7) + 12\frac{69}{128}(8) + 11\frac{11}{64}(9) + 7\frac{105}{28}(10)$

Sixth $13\frac{9}{16}(6) + 81\frac{3}{8}(7) + 296\frac{87}{128}(8) + 834\frac{43}{128}(9) + 1953\frac{285}{512}(10)$

Seventh $122\frac{1}{16}(7) + 1510\frac{67}{128}(8) + 11475\frac{72}{128}(9) + 68372\frac{79}{2048}(10)$

Eighth $1937\frac{95}{128}(8) + 47151\frac{93}{128}(9) + 706845\frac{1493}{8192}(10)$

Ninth $54902\frac{89}{128}(9) + 2558465\frac{23587}{32768}(10)$

Tenth 2805527 (10)

An example of all of this with the numbers placed here is shown. But within those limits beware, coming closer or receding further from the place of unity as required. Indeed, when unity, the root, the square, the cube, etc. are in continued proportion; if the root shall be hundredth part of unity, occupying the second place from that: the square will be hundredth part of the latus, placed in the fourth position from unity, and where the further we progress, the subsequent powers have been put in more removed places, as we see: ∵

10000,00000,00000 Unity.
1st (A) 15116,46599,90567,29504,88 (1)
2nd 22850,75443,00638,16726 (2)
3rd 34542,26523,94854,62 (3)
4th 52215,69780,2288 (4)
5th 78931,68205 (5)
∵ 1,19316,81 (6)
18036 (7)

$\frac{1}{2}$ (2) - - - 11425,37721,50319,08363 *B*

¶ $\frac{1}{2}$ (3) - - - 17271,13261,97427,3

$\frac{1}{8}$ (4)-- -- -- -- 6526,96222,5

$\frac{1}{2}(3) + \frac{1}{8}(4)$ ---- 17271,19788,93649,8 *C*

$\frac{7}{8}$ (4) - -- ----- - - - 45688,73557,7

$\frac{7}{8}$ (5) ------------ 69065,2

$\frac{7}{16}$ (6) ------------------522

$\frac{7}{8}(4) + \frac{7}{8}(5) + \frac{7}{16}(6)$ 45689,42623,4 *D*

$2\frac{5}{8}$ (5) ------2,07195,66

7 (6) ---------8,351

$2\frac{5}{8}(5) + 7(6)$ ---2,07204,01 *E*

All these differences are most easily computed from an uninterrupted series of numbers of continued means; but the use of these differences is not of much service with numbers of which the first difference has a value of a hundredth or thousandth part of unity. Indeed there are both too many of these and they are exceedingly labourious. Yet with unity put in the first place, and with three or more ciphers following next, these differences will be able to reduce a large part of the
[p.17.] work being undertaken, that we see in the adscribed example: in which the continued means numbers supply those differences for us by division and subtraction, until † shall have been reached; Certainly afterwards, these differences themselves produce continued means: for, 65, the thirty second part of the number *E* 207, taken from the number 2855589, with the sixteenth part of the number *D* 4568943, leaves the number 2855524 *D*, or the fourth difference. Likewise this 4th difference *D*, taken from the eighth part of the previous *C*, leaves the third difference *C* itself. Which difference, taken from the fourth of the above *B*, leaves *B* itself; and the same *B* taken from half the previous A, leaves A itself; which added to unity, is the mean itself sought 1000,07558,20443,63012,14290,760. We will find all the remaining means from differences of this kind by the same method; progressing until finally all the differences beyond the first shall gradually come to an end, which when the rest have come to an end, will be entirely equal to half the first preceding difference. The rest are to be carried out according to the proportional rule, as above.

§8.3. *Notes On Chapter 8.*

[1] Briggs' square root algorithm can be set out in modern notation (Table 8 - 2). From the 4th to the 9th square root extraction of $6^9/10^7 = 1.0077696$, presented at the end of the previous chapter, and now used as an example, a special set of differences is successively established in the third column, finally to the 28th decimal place in Briggs' Table 8 - 1; this is to be used in conjunction with Briggs' *Golden Rule*: a prerequisite of which being an approximately equal number of ciphers and digits, with the final residual root lying in the region of proportionality with the logarithm.

The algorithm so developed is then ran backwards to generate the 10^{th} root, and the roots of succeeding levels. Briggs found the method after much frustration with the long and tedious square root method of completing the square, presumably during or after the work involved in making up the successive means table, to which he alluded in the preface to the introduction to his tables. Here we will stay close to Briggs original notation, though we will replace the clumsy contemporary 'numbers in brackets' notation for powers of variables.

Mean &1st Diff.	2nd Difference	3rd Difference	4th Difference	5th Difference	6th Difference
$1 + A_n$	-	-	-	-	-
$1 + A_{n+1}$	$B_{n+1} = \frac{1}{2} A_n - A_{n+1}$	-	-	-	-
$1 + A_{n+2}$	$B_{n+2} = \frac{1}{2} A_{n+1} - A_{n+2}$	$C_{n+2} = \frac{1}{2^2} B_{n+1} - B_{n+2}$	-	-	-
$1 + A_{n+3}$	$B_{n+3} = \frac{1}{2} A_{n+2} - A_{n+3}$	$C_{n+3} = \frac{1}{2^2} B_{n+2} - B_{n+3}$	$D_{n+3} = \frac{1}{2^3} C_{n+2} - C_{n+3}$	-	-
$1 + A_{n+4}$	$B_{n+4} = \frac{1}{2} A_{n+3} - A_{n+4}$	$C_{n+4} = \frac{1}{2^2} B_{n+3} - B_{n+4}$	$D_{n+4} = \frac{1}{2^3} C_{n+3} - C_{n+4}$	$E_{n+4} = \frac{1}{2^4} D_{n+3} - D_{n+4}$	-
$1 + A_{n+5}$	$B_{n+5} = \frac{1}{2} A_{n+4} - A_{n+5}$	$C_{n+5} = \frac{1}{2^2} B_{n+4} - B_{n+5}$	$D_{n+5} = \frac{1}{2^3} C_{n+4} - C_{n+5}$	$E_{n+5} = \frac{1}{2^4} D_{n+4} - D_{n+5}$	$F_{n+5} = \frac{1}{2^5} E_{n+4} - E_{n+5}$
$1 + A_{n+6}$	$A_{n+6} = \frac{1}{2} A_{n+5} - B_{n+6}$	$B_{n+6} = \frac{1}{2^2} B_{n+5} - C_{n+6}$	$C_{n+6} = \frac{1}{2^3} C_{n+5} - D_{n+6}$	$D_{n+6} = \frac{1}{2^4} D_{n+5} - E_{n+6}$	$E_{n+6} = \frac{1}{2^5} E_{n+5}$
$1 + A_{n+7}$	$A_{n+7} = \frac{1}{2} A_{n+6} - B_{n+7}$	$B_{n+7} = \frac{1}{2^2} B_{n+6} - C_{n+7}$	$C_{n+7} = \frac{1}{2^3} C_{n+6} - D_{n+7}$	$D_{n+7} = \frac{1}{2^4} D_{n+6} - E_{n+7}$	$E_{n+7} = \frac{1}{2^5} E_{n+6}$
etc.					

[Table 8 - 2]

In Table 8 - 2, in the column furthest to the left, 6 successive square roots are first evaluated using the painstaking 'completing the square' algorithm. These are labeled $1 + A_n$, $1 + A_{n+1}$, ..., $1 + A_{n+5}$, from which a set of finite differences are established column by column, until a zero value is encountered, as shown in the last row, where $F_{n+5} \sim 0$, and the scheme need proceed no further. The algorithm is then made to run backwards, to generate the differences of the different orders, starting from E_{n+6}, until finally the next root $1 + A_{n+6}$ is found; the process can then be repeated for the ensuing square roots, a welcome relief from the traditional approach. As the convergence is quite rapid, fewer orders of difference are required as this is continued, until the region of proportionality is reached.

Applying this algorithm to Briggs' Table 8 - 1, level 38 shows the algorithm running forwards to generate the required differences, where the last two differences are equal to the precision required, corresponding to n = 4 in Table 8 - 2:

38	10000,15116,46599,90567,29504,88	A_9
	15116,58025,28288,79823,97	$^1/_2 A_8$
	11425,37721,50319,09	B_9
	† 11425,54992,70108,02	$^1/_4 B_8$
	17271,19788,93	C_9
	17271,65478,36	$^1/_8 C_8$
	45689,43	D_9
	45691,50	$^1/_{16} D_8$
	2,07	E_9
	2,07	$^1/_{32} E_8$
37	65	$^1/_{32} E_{10}$
	2855,589	$^1/_{16} D_9$
	2855,524	D_{10}
	2158,89973,616	$^1/_8 C_9$
	2158,87118,092	C_{10}
	2856,34430,37579,775	$^1/_4 B_9$
	2856,32271,50461,683	B_{10}
	7558,23299,95283,64752,440	$^1/_2 A_9$
	7558,20443,63012,14290,767	A_{10}
	10000,07558,20443,63012,14290,767	$1 + A_{10}$

The 0^{th} level root is taken as :
$(1+h)^{1/2^4} = 1 + A_4$, where h = 0.0077696,
correspond ing to the index 42, see Table 8 - 1.
Subsequent ly, the 1^{st} level has
$(1+h)^{1/2^5} = 1 + A_5$, where
$\frac{1}{2}A_4 - A_5 = B_5$, etc.
At the 5^{th} level, shown opposite as index 38 :
$(1+h)^{1/2^9} = 1 + A_9$, and
$\frac{1}{2}A_8 - A_9 = B_9; \frac{1}{4}B_8 - B_9 = C_9; \frac{1}{8}C_8 - C_9 = D_9;$
$\frac{1}{16}D_8 - D_9 = E_9; \frac{1}{32}E_8 - E_9 = F_9 = 0$, to the reqiured degree of precision.

[Table 8-3]

Thus, $E_9 = 2.07(20401) \times 10^{-24}$. Note that in the adjoining working on the right, Briggs shows another 5 significant figures in the calculation: all the differences B_9, C_9, D_9, and E_9 having been worked out either from the 'forwards' method just described, or by the alternative method of using the sums of powers for the different orders, to be considered a little later. Now, by running the algorithm backwards, the next square root A_{10} can be found: thus,

$$E_{10} = \frac{1}{32}E_9 = 6.475 \times 10^{-26} \cong 6.5 \times 10^{-26}; \frac{1}{16}D_9 - D_{10} = E_{10}, \text{ giving } D_{10} = \frac{1}{16}D_9 - E_{10} = 2855524 \times 10^{-21}.$$

$$\text{Subsequently}: C_{10} = \frac{1}{8}C_9 - D_{10} = 2.2158.. \times 10^{-16};$$

$$B_{10} = \frac{1}{4}B_9 - C_{10} = 2.8563... \times 10^{-11}; \text{ and } A_{10} = \frac{1}{2}A_9 - B_{10} = 7.558... \times 10^{-6}, \text{ the next root being } 1 + A_{10}.$$

This procedure can then be repeated as often as required.

[2] Let us now consider Briggs' other idea for finding the differences associated with a given level, without having to actually form the differences. Inspection of Table 8 - 2 shows that all the differences originate from those of the preceding level and above. We note first that some entry with 3 or 4 zeros is selected as a starting point, let this occur at the n^{th} row. Suppose we want to

find the differences associated with $1 + A_n = 1 + \alpha$ as sums of powers of α; The previous known root is $1 + A_{n-1} = (1 + \alpha)^2$, while one more level up we have : $1 + A_{n-2} = (1 + \alpha)^4$, etc. The initial choice of n must insure that sufficient higher powers of $1 + \alpha$ are available to satisfy the required degree of precision. In general, the higher powers of $1 + \alpha$ are sums of powers of α, and also each of these small additions to 1 is approximately half that preceding. Briggs 'factors out' this dependence on 2 by making up a special set of difference equations, which we illustrate as far as the 4th power as follows:

First Order: The 'latus' or linear term is simply $A_n = (1 + \alpha) - 1$, for the index n selected initially. Briggs has chosen A_9 in the above example.

Second Order: $B_n = \frac{1}{2} A_{n-1} - A_n = \frac{1}{2}((1 + \alpha)^2 - 1) - ((1 + \alpha)^1 - 1) = \frac{1}{2}\alpha^2$, called ½ (2) by Briggs, with a similar notation (3), (4), etc for the cube, biquadratic, and higher powers.

Third Order: $C_n: = \frac{1}{2}^2 B_{n-1} - B_n = \frac{1}{2}^2 [\frac{1}{2}((1 + \alpha)^4 - 1) - ((1 + \alpha)^2 - 1)] - \frac{1}{2}\alpha^2$

$$= \tfrac{1}{2}\alpha^3 + \tfrac{1}{8}\alpha^4 = \tfrac{1}{2}(3) + \tfrac{1}{8}(4)$$

Fourth Order: $D_n = \frac{1}{2}^3 C_{n-1} - C_n = \frac{1}{2}^3[\frac{1}{2}^2 B_{n-2} - B_{n-1}] - [\frac{1}{2}^2 B_{n-1} - B_n]$

$$= \tfrac{1}{2}^5[\tfrac{1}{2}A_{n-3} - A_{n-2}] - \tfrac{1}{2}^3[\tfrac{1}{2}A_{n-2} - A_{n-1}] - [\tfrac{1}{2}\alpha^3 + 1/8\alpha^4]$$

$$= \tfrac{1}{2}^5[\tfrac{1}{2}((1 + \alpha)^8 - 1) - ((1 + \alpha)^4 - 1)] - \tfrac{1}{2}^3[\tfrac{1}{2}((1 + \alpha)^4 - 1) - ((1 + \alpha)^2 - 1)] - [\tfrac{1}{2}\alpha^3 + \tfrac{1}{8}\alpha^4]$$

$$= \tfrac{7}{8}\alpha^4 + \tfrac{7}{8}\alpha^5 + \tfrac{7}{16}\alpha^6 + \tfrac{1}{8}\alpha^7 + \tfrac{1}{64}\alpha^8 = \tfrac{7}{8}(4) + \tfrac{7}{8}(5) + \tfrac{7}{16}(6) + \tfrac{1}{8}(7) + \tfrac{1}{64}(8) .$$

This line of enquiry has been continued as far as the tenth order difference by Briggs, but only retaining powers of α up to the 10th power as needed. The right-hand side of Table 8 - 1 shows how the various powers of α generate the required differences, as found by the first method.

Finally, let us consider the connection between Briggs' algorithm and the $n = \frac{1}{2}$ case of the Binomial Theorem, established by Whiteside. First, however, on the practical side, Briggs' approach is easy to use, and highly effective, as the convergence is very rapid - some 4 or 5 decimal places per iteration. By going as high as α^{10} in order, square roots up to around 50 decimal places will be accommodated: however, this would be incredibly tedious to evaluate manually with the binomial theorem. From the work in Chapter Thirteen on subtabulation, we may surmise that Briggs had evaluated logarithms to this degree of precision. It is possible to show that the two methods are equivalent, up to some degree of precision, such as that in the table. According to the Binomial Theorem, for $|\alpha| \leq 1$:

$$(1+\alpha)^{1/2} = 1 + \tfrac{1}{2}\alpha - \tfrac{1}{8}\alpha^2 + \tfrac{1}{16}\alpha^3 - \tfrac{5}{128}\alpha^4 + \tfrac{7}{256}\alpha^5 ..$$

According to the Briggs algorithm, setting $\alpha = A_N < 1$:

$$(1+\alpha)^{1/2} = 1 + A_{N+1} = 1 + \tfrac{1}{2}A_N - B_{N+1} = 1 + \tfrac{1}{2}A_N - \tfrac{1}{4}B_N + C_{N+1} = ...(etc.)$$

$$= 1 + \tfrac{1}{2}A_N - \tfrac{1}{4}B_N + \tfrac{1}{8}C_N - \tfrac{1}{16}D_N + \tfrac{1}{32}E_N - \tfrac{1}{64}F_N + ...(finally).$$

$$= 1 + \tfrac{1}{2}\alpha - \tfrac{1}{8}\alpha^2 + \tfrac{1}{8}(\tfrac{1}{2}\alpha^3 + \tfrac{1}{8}\alpha^4) - \tfrac{1}{16}(\tfrac{7}{8}\alpha^4 + \tfrac{7}{8}\alpha^5 + \tfrac{7}{16}\alpha^6 + \tfrac{1}{8}\alpha^7 + \tfrac{1}{64}\alpha^8) + \tfrac{1}{32}(\tfrac{21}{8}\alpha^5 + 7\alpha^6 + \tfrac{175}{16}\alpha^7 + \tfrac{21}{8}\alpha^8 + ..) - ..$$

$$= 1 + \tfrac{1}{2}\alpha - \tfrac{1}{8}\alpha^2 + \tfrac{1}{16}\alpha^3 - \tfrac{5}{128}\alpha^4 + \tfrac{7}{256}\alpha^5 ..$$

By collecting terms of the same powers together, we see that the terms in the binomial expansion are reproduced, to whatever precision we have in mind: thus demonstrating the equivalence of the two methods. See D. T. Whitesides's note: '*Henry Briggs: The Binomial Theorem Anticipated*', *The Mathematical Gazette*, 1964, p. 9. Also of considerable interest in this context is the book by Herman H. Goldstine mentioned above: '*A History of Numerical Analysis from the 16th Through the 19th Century*', (Springer-Verlag, 1977,N.Y.), pp.13 - 20.

§8.1. Caput VIII. [p.15.]

Atque ad hunc modum, cuiuscunque numeri propositi Logarithmum, per continue Medios invenire poterimus. quos nobis lateris quadrati inventio suppeditat satis laboriose. huius autem tanti laboris molestia minuetur plurimum per Differentias. modum in his ipsis Mediis ostendam, quos A litera notavi, & quorum ope Senarii Logarithmum invenimus. Horum numerorum prima nota versus sinistrum, designat Unitatem; relique exprimunt partes Unitati adijciendas: ut supra ad finum sexti Capitis monuimus. Has partes Unitati adiectas, appelo Differentium primam: Differentiam scilicet dati numeri supra unitatem. Proximi numeri infra A positi, sunt semisses Differentiarum primarum proxime praecedentium : & notantur $\tfrac{1}{2}$A . ex istis semissibus auferendae sunt Differentiare primae suprascriptae, scribanturque numeri reliqui inferius, qui notari litera B, appellentur Differentiae secundae. Numeri autem infra B positi sunt quadrantes Differentiarum secundarum proxime praecedentium: & notantur $\tfrac{1}{4}$B . e quibus numeri B ablati relinquunt Differentias tertias C. & hae tertiase ablata ex octavis praecedentium tertiarum notatis C relinquunt Differentias quartas D. Eodem modo inveniuntur quintae Differentiae E, sextae F, septimae G, &c. ablatis quartis D e decimus sextis quartarum praecedentium: quintis E, e tricesimis secundis quintarum: & sextis F, e sexagesimis quartis sextarum, &c.

Hae omnes differentiae habentur facillime e continuata serie numerorum continue Mediorum, earum autem usus non multum conducit, in numeris quorum Differentia prima, valet centesimam aut millesimam partem Unitatis. Sunt enim & nimis multae & earum aliquae nimis operosae. Verum cum Unitatis notam primo loco positam, tres vel plures cyphrae proxime sequantur, poterunt hae Differentiae magnum suscepti laboris partem minuere. uti [p.16.]

	10077,696	
46	10038,77283,33696,24566,38465,51	
45	10019,36766,13694,66167,58702,29	
44	10009,67914,63909,90172,88907,20	
43	10004,83840,26884,66298,54925,34	A
42	10002,41890,87882,46856,38087,26	A
	2,41920,13442,33149,27462,67	$^1/_2$A
	29,25559,86292,89375,41	B
41	10001,20938,12639,71345,94391,94	A
	1,20945,43941,23428,19043,63	$^1/_2$A
	7,31301,52082,24651,69	B
	7,31389,96573,22343,85	$^1/_4$B
	88,44490,97692,16	C
40	1000,060467,23505,53096,80160,05	A
	60469,06319,85672,97195,92	$^1/_2$A
	1,82814,32576,17035,87	B
	1,82825,38020,56162,92	$^1/_4$B
	11,05444,39127,05	C
	11,05561,37211,52	$^1/_8$C
	116,98084,47	D
39	10000,30233,16050,56577,59647,94	A
	30233,61752,76548,40080,02	$^1/_2$A
	45702,19970,80432,08	B
	45703,58144,04258,97	$^1/_4$B
	1,38173,23826,89	C
	1,38180,54890,88	$^1/_8$C
	7,31063,97	D
	7,31130,28	$^1/_{16}$D
	66,31	E
38	10000,15116,46599,90567,29504,88	A
	15116,58025,28288,79823,97	$^1/_2$A
	11425,37721,50319,09	B
¶	11425,54992,70108,02	$^1/_4$B
	17271,19788,93	C
Hucusque Differentiae minores sunt inventae per subductionem maiorum e partibus homogenearum praecedentium .	7271,65478,36	$^1/_8$C
	45689,43	D
	45691,50	$^1/_{16}$D
	2,07	E
	2,07	$^1/_{32}$E
Hic, maiores Ddifferentia relinquuntur per subductionem minorem e partibus proxime superioris speciei	65	$^1/_{32}$E
	2855,589	$^1/_{16}$D
	2855,524	D
	2158,89973,616	$^1/_8$C
	2158,87118,092	C
	85 6,34430,37579,772	$^1/_4$B
	85 6,32271,50461,680	B
	3299,95283,64752,440	$^1/_2$A
37	10000,07558,20443,63012,14290,706	A
	2	$^1/_{32}$E
	178,470	$^1/_{16}$D
	178,468	D
	269,85889,762	$^1/_8$C
	269,85711,294	C
	714,08067,87615,420	$^1/_4$B
	714,07798,01904,126	B
	3779,10221,81506,07145,380	$^1/_2$A
36	10000,03779,09507,73708,05241,254	A

Poterunt etiam hae differentiae inveniri pro dato quolibet numero, licet continue Medij nulli dati fuerint. Nam si dati numeri differentia supra unitatem, seipsam suosque factos aliquoties multiplicet, ut eiusdem differentiae aliquot potestates habeantur, nempe Quadratus, Cubus, Biquadratus, &c. erit Differentia secunda semissis Quadrati, reliquae etiam ut hic vides.

Secunda $\frac{1}{2}$ (2)

Differentia

Tertia $\frac{1}{2}(3) + \frac{1}{8}(4)$

Quarta $\frac{7}{8}(4) + \frac{7}{8}(5) + \frac{7}{16}(6) + \frac{1}{8}(7) + \frac{1}{64}(8)$

Quinta $2\frac{5}{8}(5) + 7(6) + 10\frac{15}{16}(7) + 12\frac{69}{128}(8) + 11\frac{11}{64}(9) + 7\frac{105}{28}(10)$

Sexta $13\frac{9}{16}(6) + 81\frac{3}{8}(7) + 296\frac{87}{128}(8) + 834\frac{43}{128}(9) + 1953\frac{285}{512}(10)$

Septima $122\frac{1}{16}(7) + 1510\frac{67}{128}(8) + 11475\frac{72}{128}(9) + 68372\frac{79}{2048}(10)$

Octava $1937\frac{95}{128}(8) + 47151\frac{93}{128}(9) + 706845\frac{1493}{8192}(10)$

Nona $54902\frac{89}{128}(9) + 2558465\frac{23587}{32768}(10)$

Decima 2805527 (10)

Horum omnium specimen in numeris hic positis exhibetur. In istis autem extremis cavendum, propius accedens vel longius recedens ab Unitatis loco quam oportuit. Cum enim Unitas, Latus, Quadratus, Cubus, &c. sint continue proportionales; si Latus sit pars centisima Unitatis, secundum ab ea occupans locum: Erit Quadratus pars centesima Lateris, in loco quarto ab Unitate ponendus. & quo longius progredimur, eo remotius locandae sunt potestates subsequentes, ut vides∵.

	10000,00000,00000	Unitas.
Latus	15116,46599,90567,29504,88	(1)
Quadratus	22850,75443,00638,16726	(2)
Cubus	34542,26523,94854,62	(3)
Biquadratis	52215,69780,2288	(4)
	78931,68205	(5)
∵	1,19316,81	(6)
	18036	(7)

$\frac{1}{2}$ (2) - - - 11425,37721,50319,08363 *B*

¶ $\frac{1}{2}$ (3) - - - 17271,13261,97427,3

$\frac{1}{8}$ (4)-- -- -- -- 6526,96222,5

$\frac{1}{2}(3) + \frac{1}{8}(4)$ ---- 17271,19788,93649,8 *C*

$\frac{7}{8}$ (4) - -- ----- - - 45688,73557,7

$\frac{7}{8}$ (5) ------------ 69065,2

$\frac{7}{16}$ (6) ------------------522

$\frac{7}{8}(4) + \frac{7}{8}(5) + \frac{7}{16}(6)$ 45689,42623,4 *D*

$2\frac{5}{8}$ (5) ------2,07195,66

7 (6) ---------8,351

$2\frac{5}{8}(5) + 7(6)$ ---2,07204,01 *E*

[p.17]

in exemplo adscripto videmus. in quo numeri continue Medii, istas nobis per Divisionem & Subductionem ministrant Differentias, donec perventum sit ad †. postea vero, hae Differentiae pariunt ipsos continue medios: nam 65, pars tricesima secunda numeri *E* 207, ablata e numero 2855589, parte decima sexta numeri *D* 4568943, relinquit 2855524 numerum *D*, vel differentiam quartam. Eadem haec differentia quarta *D*, ablata e parte octava *C* prioris, relinquit ipsam Differentiam tertiam *C*. quae ablata e quadrante superioris *B*, relinquit *B*. atque eadem *B* ablata e semisse prioris A, relinquit ipsam A. quae adjecta Unitati, est medius quaesitus 1000,07558,20443,63012,14290,760. Eodem modo reliquos omnes Medios subsequentes per huiusmodi Differentias inveniemus, progrediendo donec tandem omnes differentiae paulatim deficiant praeter primas, quae cum reliquae defecerint, omnino aequabuntur semissibus primarum praecedentium. reliqua vero sunt peragenda per proportionalis regulum: ut supra.