Chapter Six

§6.1. ***Synopsis: Chapter Six.***

In the present chapter, the real business of determining the (base 10) logarithms gets under way. Briggs constructs a sequence of rational logarithms formed from successive square roots of 10, in which a given root is the absolute number (evaluated with much labour), and the corresponding index is the logarithm of this absolute number: what happens if you persist with this sequence of continued mean numbers to ever decreasing values? By persisting with the evaluation of successive square roots of 10, a well-defined pattern begins to emerge, and finally does so at the 54 th root extraction, labeled P in Table 6-2, to the 34 significant figures of the calculation. At this stage, the number of zeros of the continued mean, after the initial 1, is the same as the number of significant places after the zeros [note: 0(15) below represents a string of 15 zeros, etc], and that the remaining digits are in proportion to the index or logarithm of the mean. Reversing the procedure, all the means labeled 54 - 1 in the table are continued squares of the mean P [Table 6-3], and those immediately adjoining P labeled $N, M,$ and L show proportionality between themselves and also their logarithms. Thus, for level P, in modern notation: $10^{1/2^{54}} = r = 1 + \Delta$, where $\Delta =$ 0.0(15)1278191493200323442; with the corresponding logarithm $l = 1/2^{54} =$ 0.0(16)5551115123125782702 12. For the next level N: $r^2 \sim 1 + 2\Delta$; with logarithm $2l$; and subsequently for M: $r^4 \sim 1 + 4\Delta$; with logarithm $4l$; and finally L: $r^8 \sim 1 + 8\Delta$, with logarithm $8l$; . The intermediate values r^3,$3l$; r^5,$5l$; etc, are also readily found. Hence, the numbers Δ and l define a proportionality between the fractional part of the continued mean and the corresponding logarithm, between the values of the roots designated from P to L, and the logarithm of any other small number of the form $1 + \delta$ in this region can be found by proportionality to be $\delta l/\Delta$. All of this has been established arithmetically by Briggs: the links to modern analysis are investigated a little in the notes to the chapter. However, we may note at this stage that from the proportionality, Δ/l can be written in the form $(10^{1/2^{54}} - 1).2^{54} \sim ln(10)$, the natural logarithm of 10.

§6.2. ***Continued Mean Numbers.***

In Chapter 3 above, it was shown that the logarithms of all numbers placed in the same given series of continued proportionals with one and ten are rational[1], and in accordance with the ratio of the given situation these logarithms can be found most easily. Therefore from a large enough series in continued proportion between one and ten, some particular numbers are to be sought, which we shall call continued means. Because any of these is the homogenous mean[2] between one and the next larger number in the same series. As we see with the numbers designated A, B, and C :

A	*B*	*C*
1	1	1
2	$^{2}/_{3}$	2
4	$^{4}/_{9}$	8
16	$^{16}/_{81}$	512
256	$^{256}/_{6501}$	134217728
65536	$^{65536}/_{43046721}$	
4294967296		

[Table 6-1]

where any term in A and B is the square root, and in C the cube root of the next larger number.

	D *Numbers from Continued Means between Ten & One.*	*E* *Rational Logarithms*
	10	1,000
1	31622,77660,16837,93319,98893,54	0,50
2	17782,79410,03892,28011,97304,13	0,25
3	13335,21432,16332,40256,65389,308	0,125
4	11547,81984,68945,81796,61918,213	0,0625
5	10746,07828,32131,74972,13817,6538	0,03125
6	10366,32928,43769,79972,90627,3131	0,015625
7	10181,51721,71818,18414,73723,8144	0,00781,25
8	10090,35044,84144,74377,59005,1391	0,00396,25
9	10045,07364,25446,25156,64670,6113	0,00195,3125
10	10022,51148,29291,29154,65611,7367	0,00097,65625
11	10011,24941,39987,98758,85395,51805	0,00048,82812,5
12	10005,62312,60220,86366,18495,91839	0,00024,41406,25
13	10002,81116,78778,01323,99249,64325	0,00012,20703,125
14	10001,40548,51694,72581,62767,32715	0,0000610351,5625
15	10000,70271,78941,14355,38811,70845	0,00003,05175,78125
16	10000,35135,27746,18566,08581,37077	0,00001,52587,89062,5
17	10000,17567,48442,26738,33846,78274	0,00000,76293,94531,25
18	10000,08783,70363,46121,46574,07431	0,00000,38146,97265,625
19	10000,04391,84217,31672,36281,88083	0,00000,19073,48632,8125
20	10000,02195,91867,55542,03317,07719	0,00000,09536,74316,40625
21	10000,01097,95873,50204,09754,72940	0,00000,04768,37158,20312,5
22	10000,00548,97921,68211,14626,60250,4	0,00000,02384,18579,10156,25
23	10000,00274,48957,07382,95091,25449,9	0,00000,01192,09289,55078,125
24	10000,00137,24477,59510,83282,69572,5	0,00000,00596,04644,77539,0625
25	10000,00068,62238,56210,25737,18748,2	0,00000,00298,02322,38769,53125
26	10000,00034,31119,22218,83912,75020,8	0,00000,00149,01161,19384,76562,5
27	10000,00017,15559,59637,84719,93879,1	0,00000,00074,50580,59692,38281,25
28	10000,00008,57779,79451,03051,17588,8	0,00000,00037,25290,29846,19140,625
29	10000,00004,28889,89633,54198,42901,3	0,00000,00018,62645,14923,09570,3125
30	10000,00002,14444,94793,77767,42970,4	0,00000,00009,31322,57461,54785,15625
31	10000,00001,07222,47391,14050,76926,8	0,00000,00004,65661,28730,77392,57812,5
32	10000,00000,53611,23694,13317,14831,4	0,00000,00002,32830,64365,38696,28906,25
33	10000,00000,26805,61846,70731,51508,7	0,00000,00001,16415,32182,69348,14453,125
34	10000,00000,13402,80923,26383,99277,7	0,00000,00000,58207,66091,34674,072265,625
35	10000,00000,06701,40461,60946,55519,6	0,00000,00000,29103,83045,67337,03613,28125
36	10000,00000,03350,70230,79911,91730,0	0,00000,00000,14551,91522,83668,51806,64062,5
37	10000,00000,01675,35115,39815,61857,6	0,00000,00000,07275,95761,41834,25903,32031,25
38	10000,00000,00837,67557,69872,72426,9	0,00000,00000,03637,97880,70917,12951,66015,625
39	10000,00000,00418,83778,84927,59087,9	0,00000,00000,01818,98940,35458,56475,83007,8125
40	10000,00000,00209,41889,42461,60262,5	0,00000,00000,00909,49470,17729,28237,91503,90625
41	10000,00000,00104,70944,71230,25311,0	0,00000,00000,00454,74735,08864,64118,95751,95312
42	10000,00000,00052,35472,35614,98950,4	0,00000,00000,00227,37367,54432,32059,47875,97656
43	10000,00000,00026,17736,17807,46048,9	0,00000,00000,00113,68683,77216,16029,73937,98828
44	10000,00000,00013,08868,08903,721678	0,00000,00000,00056,84341,88608,08014,86968,99414

45	10000,00000,00006,54434,044518586975	0,00000,00000,00028,42170,94304,04007,43464,49707
46	10000,00000,00003,27217,0222592881337	0,00000,00000,00014,21085,47152,02003,71742,24853
47	10000,00000,00001,636085111296427283	0,00000,00000,00007,10542,73576,01001,85871,12426
48	10000,00000,00000,818042555648210295	0,00000,00000,00003,55271,36788,00500,92935,56213
49	10000,00000,00000,409021277824104311	0,00000,00000,00001,77635,68394,00250,46467,78106
50	10000,00000,00000,204510638912051946	0,00000,00000,00000,88817,84197,00125,23233,89053
51	10000,00000,00000,102255319456025921 **L**	0,00000,00000,00000,44408,92098,50062,61616,94526
52	10000,00000,00000,051127659728012947 **M**	0,00000,00000,00000,22204,46049,25031,30808,47263
53	10000,00000,00000,025563829864006470 **N**	0,00000,00000,00000,11102,23024,62515,65404,23631
54	10000,00000,00000,012781914932003235 **P**	0,00000,00000,00000,05551,11512,31257,82702,11815

[Table 6-2]

Now the continued means of this kind between ten and one shall be sought, the first of these will be $\ell.10$, assuredly 3<u>1622776016837933199889354</u>, that is the root of 10, or the mean proportion between 10 and 1. Then I look for the root of the root most recently found, that is $\ell\ell.10$: 1<u>77827941003892280119730413</u>, With the third I go on to investigate the root of that root, $\ell\ell\ell.10$: 1<u>3335,21432,16332,40256,65389,31</u> [3]. By maintaining the same method of working, I progress until the whole series of continued means, together with ten, shall contain fifty five separate numbers, which you see marked with the letter *D* [Table 6-2]: with which are placed the rational logarithms agreeing with the same, marked *E,* on the same line[4].

The smallest of the continued mean numbers is: 1<u>0000,00000,00000,01278,19149,32003,23441,65</u>, of which the logarithm has been found by bisection: 0,00000,00000,00000,05551,11512,31257,82702,11815,83. Indeed it is evident by axiom 2, chapter 2, the logarithm of the root is half of the logarithm which is attributed to the square, because the square shall be from the multiplication of the root by itself, and therefore the double of any of these logarithms is next to the smaller. But since the significant places, which after fifteen ciphers have been added to one, shall be the half of the closest preceding places: so in accordance with the continued means, as with the logarithms: the logarithms themselves are seen to keep the same decreasing proportion as those numbers to which they have been adjoined. And therefore, if the numbers which have decreased up to that extent, so that after unity fifteen ciphers have been placed nearest, the remaining significant figures after the added ciphers will present to us the correct logarithms, or the correct nearby ones, through that golden rule of proportion. Because, in order that everything shall be made clearer, it has been considered to add a few numbers in continued proportion together with their logarithms, from those which are nearest the numbers *L, M, N*, & *P*; that keep the ratio between adjoining terms, which is the ratio of one to *P* [Briggs writes ratios in the form denominator: numerator], the minimum from the continued means, which all increase almost equally with the ratio of the separation between each other. Because the logarithms of these too can continue to be made from the strength of the definition: therefore, if a small number should happen to be given precisely that ought to be placed between 1 and the number *L*, although it should not in any way be counted as a number from the proportions, the logarithm of this number will be easily discovered by the rule of proportion. Since in accordance with these small numbers the logarithms may either increase or decrease with the ratio of the significant places, which have been placed just after the ciphers. So, let the given number X be 10000,00000,00000,01. I assert that these four numbers are in proportion:

12781,91493,20032,34416,5 } Significant figures to be added after the 15 zeros.
1 - - - - - - - - - - - - - - -
55511,15123,12578,27021,18158 } Significant figures after the characteristic (which is zero)
43429,44819,03251,804 - - - - - - and the other sixteen ciphers in the logarithms to be written down.

[Table 6-3]

	Numbers in continued proportion greater than unity.	*Logarithm.*
	1 Unity	0,00000
P	10000,00000,00000,01278,19149,32003,23442	0,00000,00000,00000,05551,11512,31257,82702,12
N	10000,00000,00000,02556,38298,64006,47047	0,00000,00000,00000,11102,23024,62515,65404,24
	10000,00000,00000,03834,57447,96009,70815	0,00000,00000,00000,16653,34536,93773,48106,35
M	10000,00000,00000,05112,76597,28012,94747	0,00000,00000,00000,22204,46049,25031,30808,47
	10000,00000,00000,06390,95746,60016,18842	0,00000,00000,00000,27755,57561,56289,13510,59
	10000,00000,00000,07669,14895,92019,43101	0,00000,00000,00000,33306,69703,87546,96212,71
	10000,00000,00000,08947,3404524022,67523	0,00000,00000,00000,38857,80586,18804,78914,83
L	10000,00000,00000,10225,53194,56025,92108	0,00000,00000,00000,44408,92098,50062,61616,95
X	10000,00000,00000,01	0,00000,00000,00000,04342,94481,90325,1804

The logarithm of the number *X* is therefore 0,00000,00000,00000,04342,94481,90325,1804.

Of these proportional numbers[5], the first place is one, the remainder of the places, the zeros as well as significant places, express the numerator for us, [consisting] of the parts to be adjoining one, of which the denominator itself is that one itself, and zeros, for these remaining places equal to the number. The number X is not to be counted among these numbers in proportion:

nevertheless I have carefully computed the logarithm of this number to be expressed ; because the logarithms which are to be sought by the proportionality rule, may be found with the help of this more easily than by any other way. With the remaining logarithms of these proportional numbers, nothing will be possible without multiplication and division; but here by multiplication alone the whole work has been completed.

§6.3. *Notes On Chapter Six.*

[1] Logarithms with finite decimal expansions; Briggs used the numbers $\sqrt{10}, \sqrt{\sqrt{10}}$ with the logs equivalent to 0.5, 0.25. In the present case each log is half the preceding one, and so they are in proportion.

[2] For any given positive numbers *a, b*, and *c* satisfying the relation $a/b = b/c$, $b = \sqrt{ac}$ is the geometric or homogeneous mean.

[3] We need to make some comments on Briggs' Table of continued means at this point. These concern certain non-fatal flaws. First, there is an arithmetical mistake in the evaluation of $\sqrt{\sqrt{10}}$ at the 19th decimal place: it should be 1<u>778279410038922801**22542119519**</u>, with the correct digits highlighted here, while $\sqrt{\sqrt{\sqrt{10}}}$ should be 1<u>3335,21432,16332,40256,**7593171530**</u>, an error which slowly makes its way through the rest of the table before disappearing. (Napier suffered from the same kind of difficulty: a silly arithmetical error was made in one of his preparatory tables that reduced the accuracy of all his subsequent calculations.) Secondly, and most unfortunately for Briggs, he did <u>not</u> have to carry the working to 30 places or so, to obtain the final accuracy of 14 places in these horrendous repeated square root extractions, over which he laboured, before finding a time - saving square root algorithm, which is the subject of Chapter Seven. (It may be mentioned here, if it needs to be said at all, that Henry Briggs was one of the most prodigious calculators who ever lived, but not being in the category of the idiot-savant). In fact, 14 or15 places are sufficient throughout, as long as one does not consider the increasing string of consecutive zeros as places.

Unknown to Briggs, his work can be viewed as the evaluation of the first 29 places in a series expansion of 10^x, where $x = {}^1/_2{}^n$, and

n = 1, 2, 3,....53, 54, in succession: from Briggs' table, the square and higher order terms are observed to uncouple from the linear term in the decimal sum of the expansion for small x, when the number of consecutive zeros in the expansion is the same or greater than the number of non-zero places. Thus, for 14 figure accuracy, 28 or 29 places are finally required, but the first 14 can be ciphers, and only 14 or 15 places in the root expansion need to be considered. Knowing this, would have saved Briggs an untold amount of labour, for the root extraction (e.g. by completion of the square) obviously becomes much more laborious as the number of places increases. He was never aware of this state of affairs, though some elementary experimentation would have convinced him of its truth; and no commentator on Briggs' work has ever observed this 'obvious' fact, to this writer's knowledge - considering the length of time that logarithms were in common use (some 350 years), and one could imagine that someone would have pored over the numbers in the table construction, and discovered at least part of the truth. (See this writer's article in the *Mathematical. Gazette* for July, 2001).

[4] Another table is included here, comparing some of Briggs' values, labeled B, with the true values. In addition, the decimal point is used.

B1	3.16227766016837933199889354	
1	3.16227766016837933199889354443	0.5
B2	1.778279410038922801**19730413**	
2	1.77827941003892280122542119519	0.25
B3	1.333521432163324025**65389308**	
3	1.33352143216332402567593171530	0.125
B4	1.154781984689458179**661918213**	
4	1.15478198468945817966648288730	.625e-1
B5	1.074607828321317497**21****38176538**	
5	1.07460782832131749721594153196	.3125e-1
B10	1.002251148292912915465 6**117367**	
10	1.00225114829291291546567363887	.9765625e-3
B15	1.000070271789411435538 81**170845**	
B20	1.0000021959186755542033 17**07719**	
20	1.00000219591867555420331713751	.95367431640625e-6

B25	1.000000068622385621025737187**482**	
25	1.000000068622385621025737189369	.298023223876953125e-7
B30	1.0000000021444494793777674297**04**	
30	1.000000002144449479377767429764	.931322574615478515625e-9
B35	1.00000000006701404616094655519**6**	
35	1.000000000067014046160946555199	.2910383045673370361328125e-10
B40	1.00000000000209418894246160262**5**	
40	1.000000000002094188942461602626	.9094947017729282379150390625e-12

[Table 6-4]

5 log X should be 0,00000,00000,00000,04342,94481,90325,17999,0..., but the result given is accurate to 16 real places.

Thus: 0.0000,00000,00000,012781,91493,20032,34416,5, (or $10^{1/2^{54}} - 1$) is to 0.0000,00000,00000,01, as

0.00000,00000,00000,05551,11512,31257,82702,12 (or $1/2^{54}$) is to log X,

i.e. log X = 0,00000,00000,00000,04342,94481,90325,18(04) , where we have introduced the decimal point, from which the result follows, and where the last two places are incorrect.

<u>Some Additional Notes on Briggs' Table of Continued Means and Rational Logarithms</u>.

Briggs' fundamental method for finding logarithms is a perfectly sound, though an extremely tedious, numerical algorithm. It was the engine which enabled Briggs to commence building his tables of logarithms, by finding first the logs of prime numbers, using methods to be described in the next and later chapters. He was to invent several schemes, each in turn served him well, but to be superceded by another one more convenient and less time - demanding, as the framework for calculated logarithms was built up. Thus, the second scheme uses difference equations for easing the troublesome root extractions, and ways were found to reduce the number of roots to be extracted. The final two schemes were powerful methods of subtabulation, for filling in gaps in the tables, by means of which the logarithms of numbers advanced down the page . We have already indicated some of the inefficiencies associated with the first construction, which led to many more calculations being performed than were necessary. One wonders at the tenacity of Briggs to keep going with his calculations, especially at the beginning, when there was little hope of ever completing a table of logarithms by such long time consuming processes as were then at his disposal. Thus, it presumably took several months to calculate the logarithm of 2: he does mention these trying circumstances, or of his desire to ease the labour.

Archimedes had been among the first in antiquity to consider what might be loosely called limiting processes, as in his early integration techniques for finding areas and volumes, and in the continued approximations for the ratio of the circumference to the diameter of a circle using inscribed and circumscribed regular figures, with increasing number of sides, which we now call π. Napier and Briggs were instrumental in developing a new form of limiting process, which in Briggs' approach was not based on geometrical considerations, and which enabled the calculation of logarithms. Let us now consider Table 6-1 in some detail.

Napier had noted that for any number a > 1, $a^{1/2^n} \to 1$ as n becomes large. Now Briggs or Napier (we are never told exactly whose idea it was) had discovered a new limiting process related to this one, by inquiring how the sequence defined by $b_n = a^{1/2^n} - 1$, ($n = 1, 2, ..., 54$) compared to the n^{th} square root index or logarithm $l_n = \frac{1}{2^n}$ as n increases, initially applying the method to $a = 10$. In Table 6-1, Briggs labourously calculated the sequence of continued square roots of ten to 32 places: When $n = 51$, and the number of non-zero places is almost the same as the number of leading zeros, the table shows that for this choice of a, $b_{n+1} / b_n \sim \frac{1}{2}$, while $l_{n+1} / l_n = \frac{1}{2}$, and hence the two sequences are in proportion for this value of n onwards, to the number of significant places considered. We have seen how Briggs used these results in the present synopsis. We may also consider taking the natural logarithms of the defining equation $1 + b_n = 10^{1/2^n}$:

$ln(1 + b_n) = (b_n - b_n^2 / 2 + b_n^3 / 3 - ..) = \frac{1}{2^n} \ln(10)$. Now, b_n has been selected so that the squared and higher order terms in this series lie beyond most of the significant figures considered (there may be a problem with the last few figures: Briggs normally worked to a few more places, then rounded his numbers); hence, $ln(1+b_n) = b_n = \frac{1}{2^n} ln(10) = l_n\, ln(10)$: showing the residual logarithms l_n or $\frac{1}{2^n}$ to be proportional to b_n, and $b_n/l_n = ln(10)$ is the constant of proportionality, as we found previously, or $1/ln(10) = log(e)$, as found by Briggs for the inverse ratio, where *log* signifies base 10. Briggs was unaware of course of the power series expansion for $ln(1 + x)$, but he was to arrive at the same result just the same numerically. We should note that Briggs stopped his square root sequence around the optimal point, where there were the most significant figures in proportion: stopping at a value of n much less than 54 gives less figures in proportion, due to higher order term contributions, while values of n greater than 54 have less significant figures, due to the increase in the number of leading zeros, all of this assuming 32 places altogether.

The evaluation of the logarithms of prime numbers is next considered in chapter 7, using the same scheme, starting with 2. In general, if p is such a prime, then $p = 10^q$, where q is the (unknown) logarithm of p; after extracting sufficient square roots, say m, to reach the region of proportionality as above, we arrive at an equation of the form $p^{\frac{1}{2^m}} = 1 + p_m$. The value of m is chosen in order that the residual value p_m lies within the proportionality region $b_{n+1} < p_m < b_n$, and so the residual logarithm q_m lies within the region $l_{n+1} < q_m < l_n$, with $l_n = \frac{1}{2^n}$, etc, and the value q_m calculated from: $p_m / b_n = q_m/l_n$, giving
$q_m = p_m \,.\, (l_n/b_n) = c_m.\, log(e)$. Finally, the required logarithm $q = 2^m.q_m$ follows by squaring $(1 + p_m)$ m times to get back to p, and doubling the corresponding residual logarithm q_m m times, as indicated. In actuality, however, Briggs had modified the form of p, in order that less roots have to be extracted, as we find in the next chapter: thus 2 is changed into $1.024 = 2^{10}/1000$.

Over the centuries, a number of mathematicians have considered Briggs' work with interest. For example, Edmond Halley (see *Roger Cotes - natural philosopher*, by R. Gowing, C.U.P. (1983), pp. 23 - 24) was to write Briggs' rule scheme analytically in the form :

$m(a^{1/m} - 1) \rightarrow \ln(a)$, as m becomes 'very large'; being the definition of the natural logarithm of a in the limit, and Briggs' rule results on setting $m = 2^n$. Thus, Briggs ' method may be considered in retrospect as a way of finding the base 10 logarithm of a number by first finding its natural logarithm, and then changing to base 10. H. Goldstine, *A History of Numerical Analysis...*, Springer-Verlag (1977), p.14, also provides a brief account of Briggs' table, and a number of references dealing with further developments, for the interested reader. In conclusion, Briggs must have been very happy with his first result, which agreed with the previous calculation of log 2 in chapter 5; it sets Henry Briggs apart as a skilled practitioner in the art of numerical mathematics - certain of the later chapters will only serve to enhance this claim. However, it was in the field of Euclidean geometry that his contemporaries held him in awe: perhaps they were not fully aware of his talents. Unfortunately for mathematicians such as Briggs and Harriot, the times were such that new ideas were easily stolen if divulged and passed off as one's own: consider the well-known case of Cardan and Tartaglia. Thus occasionally ideas went to the grave with their discoverer, impeding the natural development of the subject, to await rediscovery by someone else at a later time, as the unfolding story will reveal in the case of Briggs.

§6.4. **Caput VI. [p.9.]**

Numeri continue Medii.

Supra ostendimus Cap. 3 rationales esse Logarithmos omnium numerorum positorum in eadem continue proportionalium serie cum Unitate & Denario, eosque pro situs dati ratione facillime posse inveniri. Quaerantur idcirco e satis magna continue proportionalium serie, inter Unitatem & Denarium, aliquot praecipui , quos appellare poterimus continue Medios; quia eorum quilibet est medius homogeneus inter unitatem & numerum in eadem serie ab unitate proxime remotiorem. ut videmus in numeris *ABC* signatis.

A	*B*	*C*
1	1	1
2	2/3	2
4	4/9	8
16	16/81	512
256	256/6501	134217728
65536	65536/43046721	
4294967296		

.Ubi in A & B quilibet est latus Quadraticum, & in C latus Cubicum proxime remotioris. [p.10.]

	D	*E*
	Numeri continue Medij Denarium & Unitatem.	*Logarithmi rationales*
	10	1,000
1	31622,77660,16837,93319,98893,54	0,50
2	17782,79410,03892,28011,97304,13	0,25
3	13335,21432,16332,40256,65389,308	0,125
4	11547,81984,68945,81796,61918,213	0,0625
5	10746,07828,32131,74972,13817,6538	0,03125
6	10366,32928,43769,79972,90627,3131	0,015625
7	10181,51721,71818,18414,73723,8144	0,00781,25
-	- - - - - - - - - - - - - -	- - - -
-	- - - - - - - - - - - - - -	- - - -
-	- - - - - - - - - - - - - -	- - - -
50	10000,00000,00000,204510638912051946	0,00000,00000,00000,88817,84197,00125,23233,89053
51	10000,00000,00000,102255319456025921 **L**	0,00000,00000,00000,44408,92098,50062,61616,94526
52	10000,00000,00000,051127659728012947 **M**	0,00000,00000,00000,22204,46049,25031,30808,47263
53	10000,00000,00000,025563829864006470 **N**	0,00000,00000,00000,11102,23024,62515,65404,23631
54	10000,00000,00000,012781914932003235 **P**	0,00000,00000,00000,05551,11512,31257,82702,11815

[p.11.] Quaerantur autem huiusmodi Continue Medii inter Denarium & Unitatem, quorem primus erit ℓ.10, nempe 31622776016837933199889354, id est latus denarii, vel medius proportionalis inter 10. & 1. deinde quaero latus lateris nuperrime inventi, id est
$\ell\ell$.10: 177827941003892280119730413, tertio pergo investigare latus istius lateris,
$\ell\ell\ell$.10: 13335,21432,16332,40256,65389,31. eodemque servato operationis modo progrtedior, donec tota series continue mediorum, Una cum Denario, contineat numeros distinctos quinquaginta quinque, quos vides signari litera *D*: quibus e regione locantur logarithmi rationales iisdem convenientes, signati *E*.

Numerorum continue mediorum minimus est
10000,00000,00000,01278,19149,32003,23441,65, huius Logarithmus bisecando inventus est 0,00000,00000,00000,05551,11512,31257,82702,11815,83. patuit enim per 2. ax.c.2. Logarithmum Lateris, dimidium esse illius Logarithmi qui quadrato tribuitur, quia ex multiplicatione lateris in seipsum sit Quadratus. & idcirco quilibet horum logarithmorum duplus est proxime minoris. Cum autem notae significativae, quae unitati post quindecim cyphras sunt adjectae, sint dimidiae notarum proxime praecedentium; tam in continue mediis, quam in logarithmis: ipsi Logarithmi proportionem eandem decrescendo servare videntur, quam illi numeri quibus sunt adjuncti. & idcirco, si

qui numeri, eo usque [sic:eousque] decreverint, ut post unitatem cyphrae quindecim in proximo locatae fuerint, reliquae notae significativae post cyphras adjectae, veros Logarithmos vel veris proximos nobis exhibebunt, per proportionis illam auream regulam. quae omnia ut fiant magis manifesta, numeros aliquot continue proportionales una cum suis Logarithmis adscribere visum est, ex illis qui sunt proximi numeris *LMN* & *P*: ea inter proximos servata ratione, quae est Unitatis ad P, minimum e continue mediis. qui omnes fere crescunt aequaliter, pro ratione distantiae inter se. quod eorum etiam logarithmi ex definitionis vi facere tenentur. Idcirco, si contigerit dati numerum adeo exiguum, ut intra Unitatem & numerum L locati debeat, licet non sit ullo modo censendus in numero proportionalium, erit tamen eius Logarithmus inventu facilis, per proportionalis regulum. Cum in his numeris adeo exiguis crescant aut decrescant Logarithmi pro ratione notarum significativarum, quae proximo post cyphras loco sitae sunt. ut sit datus numerus *X* 10000,00000,00000,01. aio hos quator numeros esse proportionales.

12781,91493,20032,34416,5 } Notae significativae post quindecim cyphras.
1 - - - - - - - - - - - - - - - } unitati adijciendeae
55511,15123,12578,27021,18158 } Notae significativae post Characteristicam (quae est cyphra)
43429,44819,03251,804 - - - - - - & alias sexdecim cyphras in Logarithmis scribendae.

	Numeriin continue proportionel supra Unitatem.	*Logarithmi.*
	1 Unitas	0,00000
P	10000,00000,00000,01278,19149,32003,23442	0,00000,00000,00000,05551,11512,31257,82702,12
N	10000,00000,00000,02556,38298,64006,47047	0,00000,00000,00000,11102,23024,62515,65404,24
	10000,00000,00000,03834,57447,96009,70815	0,00000,00000,00000,16653,34536,93773,48106,35
M	10000,00000,00000,05112,76597,28012,94747	0,00000,00000,00000,22204,46049,25031,30808,47
	10000,00000,00000,06390,95746,60016,18842	0,00000,00000,00000,27755,57561,56289,13510,59
	10000,00000,00000,07669,14895,92019,43101	0,00000,00000,00000,33306,69703,87546,96212,71
	10000,00000,00000,08947,3404524022,67523	0,00000,00000,00000,38857,80586,18804,78914,83
L	10000,00000,00000,10225,53194,56025,92108	0,00000,00000,00000,44408,92098,50062,61616,95
X	10000,00000,00000,01	0,00000,00000,00000,04342,94481,90325,1804

[p.12.] Est igitur numeri X dati Logarithmus 0,00000,00000,00000,04342,94481,90325,1804. Horum numerorum proportionalium prima nota est unitas, reliquae omnes notae subsequentes, tam cyphrae quam significativae, exprimunt nobis Numeratorem partium unitati adijciendarum, quarum Denominator est illa ipsa unitas, & cyphrae, reliquis illis notis numero aequales. Numerus X non est inter hos proportionales numerandus: eius tamen Logarithmum diligentius exprimendum putavi; quia Logarithmi, qui per proportionis regulum quaerendi sunt, huius ope facilius inveniantur quam alterius cuiusvis. cum reliqui horum proportionalium Logarithmi, nihil possint sine multiplicatione & divisione; hic autem sola multiplicatione totum negotium absolvit.