Chapter Five

§5.1. *Synopsis: Chapter Five.*

The properties of logarithms developed in the first four chapters will be referred to as axioms in the further development of the subject by Briggs. Moreover, the present chapter is a special one, in which Briggs pays homage to his mentor, Napier, whom he held in the highest regard. In it, the method developed by Napier for finding the base 10 logarithms of small primes is demonstrated. The method is highly original and elegant, but very time consuming, because of the excessive amount of arithmetic involved. Apart from confirming later results, it plays no part in the subsequent development of logarithms by Briggs. Initially, a number of simple Lemmas are established numerically. These results are derived for a sequence of numbers in continued proportion by using a numerical example, based on the multiples of 2: as the notation of the index as a power lay in the future, we have decided, for the sake of brevity, to write Briggs' sequence S_2: 1, 2, 4, 8, 16,...., 32768 in the short form 1, 2(1), 2(2), 2(3),2(4), ... ,2(15), where the index in brackets follows the latus, base, or common ratio 2: the latus and index of a number in Briggs' terminology correspond in modern terms to base (or common ratio), and index (or power) respectively of the number; 'latus' is also used for the root of a number. In Briggs' text, the various indices occupy boxes in tables, the method commonly used by him to express some attribute of a number

The lemmas.

L1. *If the sequence S_2 is given in continued proportion, then the two sub-sequences S_8 , S_{32} based on two particular terms 2(3) or 8 and 2(5) or 32 can be found in the original sequence: i.e. S_8:* 1, 8, 64, 512, 4096, 32768 *; or, 1, 8(1), 8(2), 8(3),8(4), 8(5); and S_{32}: 1, 32(1), 32(2), 32(3). In which case, as an example, the number 32768 with latus 2 and index 15 in the original sequence can also be expressed with latus 8 and index 5, or as latus 32 and index 3: i.e. 2(15) = 8(5) =32(3).*

L2. *Given a term M in the original sequence above and the latus n, the integral number of times n can be divided into M and its subsequent quotients, gives the index r of the term M.*

L3. *Given the sub-sequence based on 8 with index 3, and the index 5 associated with it for the term 32768 according to L1, then according to L2, the index associated with the sub-sequence based on 32 and the index 5, is found by sequentially dividing 32768 by 32 an integral number of times, namely 3, as 2(15), 8(5), and 32(3) are all equal to 32768, by L1.*

L4. *The rule for adding indices: The index of a product of two terms in a sequence in continued proportion, such as S_2 , is equal to the sum of the indices of the terms multiplied. E.g. 2(2) $\times$ 2(8) is equal to 2(10), and not all the terms, such as 2(9), have to be tabulated.*

L5. *With a few provisos mentioned by Briggs, we can say: The number of places in a product of natural numbers $M \times N$ is equal to the sum of the places of M and of N taken together. As far as Briggs is concerned, this rule follows from the direct observation of products: occasionally,* 1 *has to be subtracted if the left-most digits of N and M give rise to a single digit in the product.*

[Note: The grand original sequence envisaged by Briggs, has as its common ratio the 54th square root of 10, with the elemental index or logarithm $1/2^{54}$, to be presented in the next chapter.]

The Main Result: In L1, a subset of the numbers in continued proportion is taken to contain 10, multiples of which form a sub-sequence, *and the indices 1, 2, 3, 4,....correspond to the characteristics of the logarithms of 10, 100, 1000, 10000, etc. The number 2 also lies in the original sequence, and has an unknown fractional index called log 2 (being irrational and with characteristic 0), which is to be determined. Now, 2 is formally evaluated to the index of 10(14),* (*in accordance with the number of significant places desired for log 2*), *to produce a very large number BigN, which is not fully evaluated, but is expressed in the form $N \times 10^{M-1}$, where N is a number lying in the interval (1, 10), and M is the number of places in the final product of 2 raised to the power 10^{14}. The accompanying index M, is found by direct computation in a rather clever way using L5 to be* 30102,99956,6399*. To do this, the passage of 2 to this very large index is performed in steps, each comprising four operations called a tetrad, where only a few of the more significant places need be retained, by L5, but the number of places is counted accurately enough to give the equivalent characteristic or index of BigN, which thus has the index of log 2 $\times$ 10(14), to the accuracy desired, while it also has the index M - 1 =* 30102,99956,6398, plus the fractional index of *N. Thus, we find according to L3, the index for log 2 to be 0,*30102,99956,6398, *to this accuracy, where the contribution due to N is negligible, and has not rated a mention by Briggs. The method is also used to find log 7.*

§5.2. [*Lemmas*] [p.5.]

These Logarithms are found by two particular methods, each of which is set out in an appendix which has been added to the little book: *The Construction of the Magnificent Canon of Logarithms*[1], by the most distinguished inventor of logarithms, the Baron of Merchiston, which I recommend you consult from page 40 as far as page 53. I myself will attempt however to make each method clear. Some lemmas should be set out by me to expedite the first method.

First Lemma[2].

If from a series of numbers from one in continued proportion *A*, any two numbers *C,* 8 and *D,* 32 are selected; and the indices of these are 3 & 5; & two other series of numbers in continued proportion *E* and *F* are established, in which the said numbers *C* & *D* are the first terms after one, and each series is continued until [the term associated with] 3 (The index of the number *C* in the given series *A*) corresponds to the continued product in series *F* ; and in the same manner the [term associated with] the index 5 of the number *D* corresponds to the continued product in series *E*; these products in both series *E* and *F* are equal, as we see here [Table 5-1]. Indeed, each of them is equal to the number found in the given series A, of which the index 15 is reckoned from either index first by the remaining index.

Q	*A*	*B*	*P*	*E*	*M*	*F*	*N*
0	1	0	0	1	0	1	0
	2	1		8	1	32	1
	4	2		64	2	1024	2
	C 8	3	1	512	3	32768	3
	16	4		4096	4		
1	*D* 32	5		32768	5		
	64	6	2				
	128	7					
	256	8					
	512	9	3				
2	1024	10					
	2048	11					
	4096	12	4				
	8192	13					
	16384	14					
3	32768	15	5				

[Table 5-1]

A is the first series of continued proportions given: *B* are the indices of the same series.

E is the second series of continued proportions; *M* are the indices of the same, equal to *P*.

F is the third series, of which the indices are *N*, and these are equal to *Q*.

[p.6.] Here the product is said to be continuous, which for any number is reckoned first into itself, then into this product, and into all their following products in the same way. As from 8, they become successively: 64, 512, 4096, 32768, 262144, 2097152. And these numbers (and however many are produced through the same multiplication) are continued proportions: because all are the nearest equi-multiples of the preceding ones.

Second Lemma.

2 If any number in a series of numbers from unity in continued proportion is divided continually [a] by its own root [b] as many times as possible: the number of divisions [c] is the index of the number divided, showing the difference of the given number from unity, or the number of intervals between unity and the number divided. As for instance, let the given number be 729, and of that [series] the root 3 is given, this root successively divides the given number six times, and the quotients are 243, 81, 27, 9, 3, 1. Therefore the index of the given number is 6, as we see here [Table 5-2]:

1	0
3	1
9	2
27	3
81	4
243	5
729	6

[Table 5-2]

[a] The number is said to be continuously divided: when the same divisor has divided both the given number and quotient of the given number, and any quotient of the quotient successively. And these quotients are continued proportions, because any quotient is to its quotient as the divisor is to one[3].

[b] The root of a given number is the name given to the number nearest to one in the same series of continued proportions with the given number. As for instance, in this series the root is 3.

[c] The quotients themselves are not sought here, but only the number of quotients. Although the quotients themselves may be written down also, from which everything may become clearer.

Third Lemma.

3 With two given numbers placed in the same series of continued proportions, together with the index of one of the numbers; to find the index, or the separation of the number from one, of the remaining number .

As in the first lemma, let the two numbers 8 and 32 be selected from the same given series, and 5 is the given index of the greater number, the remaining unknown index is found thus. Let another series of numbers in continued proportion be made, through the multiplication of the number 8 (of which an index is requested) into itself and into factors from itself, until a number is reached of which the index is equal to the given index 5. As you see here: [Table 5-3]:

1	0
8	1
64	2
512	3
4096	4
32768	5
1	0
32	1
1024	2
32768	3

[Table 5-3]

The last product 32768, by the first lemma, is the same as the product from the other number 32, reckoned as many times into itself, in order that the last product in this third series has that index which we require, which in the first given series was written down in terms of eight. If therefore we divide this number 32768 (by that method described in the second lemma) by the given 32 (which is the root of the third series) as often as it is possible, the three quotients will: 1024, 32, 1. So the index of this product 32768 in this third series therefore is 3, as we see here [bottom of Table 5-3], as in that first series given of eight in which these two given numbers are placed.

Where it is to be noted: here there are three different series in continued proportion, in all of which one occupies the first place. The first series, from which two given numbers are selected, together with an adjoined index of one of the given numbers. The second series, which is produced by multiplication of that selected number (an index of which is sought) until the index of the last product is equal to the index given of the other number . The third series however can be fully described if necessary, where by the division of the last product found in the preceding series [by the other selected number], a whole number is found finally. In this, the index of the product divided is equal to the number of the quotients, and it is that same index which we seek, which in

[p.7.] the first series is adjoined to that number of which the index is previously unknown, as we see in the first lemma.

Fourth Lemma: Or the method by which any product in a series in continued proportion is found, with the products of most of the intermediaries overlooked.

1	0
2	1
4	2
8	3
16	4
32	5
64	6
128	7
256	8
512	9
1024	10

[Table 5-4]

For a series of numbers from unity in continued proportion, if a number multiplies another, the product is situated in the same continued series: and the index of this series is the sum of the given indices. As for instance in this series: if four multiplies 256, the product is1024, but the index of this product is 10, the sum of the indices 2 and 8, which are adjoined to the factors 4 and 256.

Therefore, if the number is multiplied by itself, the index of the product is the double of the index which is assigned to the base. As for instance16, 16 squared is 256, of which the index of the product is 8, while the index of the base is 4.

Lemma Five:

Or the Method, in which it is possible to have the number of places in a whole product computed, although not all the places are stated.

There are as many places from both factors as there will be in the product. Unless perhaps a product can be expressed by a single place from the first places towards the left in both factors[4]. Whenever this happens, the number of places in the product is one less than the number of places in the factors. For instance, if 68 is

This first series is said therefore to be infinite, not as if it could include all the numbers entirely, but because there are very many. Indeed, we consider 99999,99999,9999 separate numbers to be placed in continued proportion between one and ten, which is evident from the index of ten, which is equal to the number of intervals between the same and one. Between these, among the mean proportions, are to be placed the numbers, i.e. the indices, for two, three, and the rest of the integers less than ten, which are little different from finite numbers; so the indices are themselves taken as finite numbers, as are these same indices for the logarithms[5].

multiplied by 26, the product 1768 is written with four places: as many of course as belong to the factors. But if 14 multiplies 68, the product 952 is allowed with these factors to be written with three places, not less than with those above that have four places.

The first method of finding the Logarithm of any number whatever proposed:

We may place all the numbers, however many there are to be described, in a single series of numbers from unity in continued proportion. And the same indices adjacent to these shall be the logarithms themselves which we seek. And from this infinite series, two numbers are selected, the first of which is ten, the index of which is 1,00000,00000,0000: the other is two (or any other number), the index of which is unknown and sought. Besides this first series that we call infinite, from which these two numbers are selected, a second series by the third lemma ought to be established by successive multiplication of two into itself and into the products formed from itself. And this series is continued, until the index of the last product is equal to the given index of ten in that first infinite series. However, in order that the insurmountable weariness of such a great number of multiplications may be avoided, most of the intermediate numbers are disregarded: following the laws of numbers in continued proportion, by the fourth lemma of this chapter, particular products are sought, together with their indices, until the last product required with its own index is found. It will not be necessary to write out the entire product , but only a few places towards the left, with the rest truncated and discarded. Thus, nevertheless, as by the fifth lemma, [p.8.] the number of all the places is known, by which the whole product could be described, if the whole number should be set out.

Taking therefore two by itself makes four, of which the index is 2. Four by itself makes 16, of which the index is 4. Likewise 16 by itself makes 256, of which the index is 8, then this product multiplied by 4 makes 1024, of which the index 10 is equal to the indices of the factors. Now, the number of places in this last product is 4, which are all most clearly seen by the fourth and fifth lemmas closely preceding, as you see here [in Table 5-4]:

[A] 1	0		
2	1		
4	2	1	
16	4	2	First Tetrad
256	8	3	
1024	10	4	
10,48576	20	7	
109,9511627776	40	13	Second Tetrad
12089,25819,61463	80	25	
12676,50600,22823	100	31	
16069,38044,25899	200	61	
25822,49878,08685	400	121	Third Tetrad
66680,14432,87940	800	241	
10715,08607,18618	1000	302	
11481,30695,27407	2000	603	
13182,04093,43051	4000	1205	Fourth Tetrad
17376,62031,93695	8000	2409	
19950,63116,87912	1000	3011	
	Indices	Number of places	

[Table 5-5]

Now these four numbers 4, 16, 256, 1024 constitute the first tetrad. Then a different tetrad is to be constructed, of which the first number is formed by multiplying the last number of the preceding tetrad by itself. The second likewise is the square of the first, and the third the square of the second; but the fourth is formed by multiplication of the first term of this tetrad by the third, and of these the four indices are 20, 40, 80, 100. There are in 7 places in the first of these, 13 in the second, 25 in the third, and 31 in the fourth. In the same way I have completed the remaining tetrads until the last tetrad has the index[6] 1,00000,00000,0000. But the number of places in the same fourth member is 30102,99956,6399. And according to this method we have had to place nearly fifty seven particular numbers in the second series *. The first of which is 2, which is the base and the root of the rest. But the last has the same index which we assigned to ten in that first infinite series. The whole third series of integers could be written down by the second lemma, if we had the final product of the second series, which in general is equal to the final product of the third series, by the first lemma. But the number of places in this final product and of each series is known, and the same index of this product in the third series, and of two in that first infinite series, is only exceeded by one. For, in order that we may find the index in the third series, the last product is to be continually divided by the given number ten, of which the index also has been given. Since if ten divides any number, the quotient is the tenth part of the
[p.9.] dividend, it is clearly the same as the number to be divided, with only a single place removed (as 4357 divided by 10 gives the quotient 435); and therefore the number of places in quotients, or of numbers divided, will only be one place less from the number of places in the dividends. But the

** But not indeed these entire numbers, for neither is this necessary or possible; but for each of these only some places to the left, which will show us the number of places in the product. For it is with the places of these first tetrads that we must exercise a little care, because otherwise the true number of places in the final product cannot be had. It will not be possible to have the number in the final product other than by experimenting: we see by the numbers designated B, if one is taken from the final number of the first Tetrad 1024, then it will arrive at a number of places in the final number of tetrad three as 301, which ought to have been as it was found before, 302.*

B

1023	10	4
1046529	20	7
1095223	40	13 Tetr.2
1199513	80	25
1255332	100	31
1575870	200	61
2483363	400	121 tetr.3.
6167075	800	241
9718508	1000	301
	Indices	Number of places

[Table 5-6]

C		
1	0	
7	1	1
49	2	2
2401	4	4 Ist Tetrad
4764801	8	7
282475249	10	9
79792266297612	20	17
63668057609090	40	34 2nd Tetrad
40536215597144	80	68
32344765096247	100	85
	Indices	Number of places

[Table 5-7]

number of quotients shows the distance of the dividend from unity, or the index of the same dividend in this third series: which is the same as that index in the first infinite series, and to the base of the second series 2 ought to be assigned. The same as that which we require: the logarithm of 2 is indeed 0,30102,99956,6398.

The same method by which we have found the logarithm of 2, will furnish a method for finding the logarithm of any other number, such as seven, the beginning of the investigation of which we see in ***C***. As the number of places in the last product (of which the Index is 10000,00000,00000) is 84509,80400,1427, from which one is taken away, is the logarithm of seven.

And this is the first method by which we can find the logarithm of any number whatsoever proposed. The other method which follows seems easier.

§5.3. *Notes On Chapter Five*

[1] This appendix can be found in W. R. MacDonald's translation of Napier's Latin text, '*Mirifici Logarithmorum Canonis Constructio*', with the above English title (Blackwood, Edinburgh, 1889), which in turn may be had as a reprint by Dawsons of Pall Mall, London, 1966. An addition to Napier's Appendix with the portly title: 'Some Remarks by the learned Henry Briggs on the foregoing Appendix' show how the subject matter was evolving at this critical time, as Briggs had undertaken to see to the London printing of Napier's explanation of his own logarithms, and Briggs had decided to add some of his own thoughts on the matter.

Briggs presents here the method due to John Napier for finding log 2, an alternative to Briggs' main development that follows in the next two chapters, and useful for providing an independent check. The method must date from the time of collaboration of these two developers of logarithms, from around 1615 - 1616; it is perhaps Napier's last contribution, as he and Briggs cast around for a suitable algorithm on which to base their calculations of base 10 logs. As with Napier's other related work, it shows a penetrating insight into the properties of logarithms.

[2] The derivation by numerical example is related to the associative law of multiplication for indices, though this of course is not the reason for presenting this lemma: which establishes that a series of numbers in continued proportion, such as multiples of 2, contains sub-series, such as multiples of 8 or of 32, based on terms in the original series. The first term after 1 in any such series is called the *latus* by Briggs, especially if he is considering multiples of this number; on the other hand, he also calls this number the *root,* if a root of a subsequent term has been found. Here we shall be content to call this first term the *base* rather than latus, while maintaining the root interpretation when needed. The terms of each sub-series have their own indices, here the primes 3 and 5 have been selected for convenience, and lemma one explores how these are connected. See the above synopsis for more details on the lemmas.

[3] Thus, for a given N and d, $N/d = Q_1$; $Q_1/d = Q_2 = N/d^2$, … ,; and finally, $Q_{n-1}/d = Q_n = N/d^n = 1$, for some index n.

[4] A single digit number can be formed in products of the numbers 0 – 9 present in the two left-most places, such as 1 × 6, in the example, with any carried numbers included.

[5] The remarks in this box illustrate the way that Briggs and his fellow mathematicians in the 17th century had of thinking about logarithms: as being a set of whole numbers ranging between 0 and 99999,99999,9999 , each of which corresponds to a member of the series of numbers in continued proportion between 1 and 10; the characteristic is an indicator of which interval the number lies in, whither between 0 and 1, 1 and 10, etc. This is why these adjoined numbers were called logarithms or 'ratio numbers': this is what Napier had in mind. There was thus no need for a decimal point with logarithms, as they were originally envisaged. Indeed, natural sines, tangents, and secants were treated in a corresponding manner without the use of fractions, by considering a circle with a very large radius.

[6] That is, in words: 2 is to be multiplied by itself some very large number of times: always, however being a multiple of ten, in order that the indices of this sub-series formed from the last term of each tetrad are 10, 100, 1000, 10,000,......;and finally1,00000,00000,0000 ; i.e. to give the increasingly large numbers 1024 with 4 indices, 12676.... with 31 indices, 10715... with 302 indices, 19950... with 3011 indices, and finally a number with 30102,99956,6399; of each of which we need not be fully aware of the digits, only a few at the start of each being sufficient, in order that the number of places may be counted for each number. The actual values of the powers of 2, when expressed as powers of ten, result in a number between 1 and 10 with one less place or index: thus, 1024 with 4 places is equal to 1.024×10^3 , etc - which is why one is taken from the number of places in the text. When 2 has been raised to the 10^{14} power, the number of places has risen to M ; while the gargantuan number BigN, to which it corresponds can be written in the form $N \times 10^{M-1}$. It was known to Napier and Briggs in an equivalent form that $a^{1/n} \rightarrow 1$ for any positive number a as n

becomes large. Hence, in the present case, only the power of 10 has any relevance, and we can say: (index of 2) × 100000,00000,0000 = (index of *N*) + 30102,99956,6398, where the index of *N*, which is a logarithm with zero characteristic can be ignored; by continually dividing by ten, or equivalently taking the 10^{th} root of the power of 2 successively, we reach the final result.

Thus, Briggs considers the first series of continued proportions to be the one generated in the next chapter, namely his table of logarithms; the second series is generated by multiples of 2 up to the power 10^{14}, in order to give the correct number of places for the logarithm of 2 finally; while the third series is taken to be powers of 10, where the contribution from the actual value of *N* is negligible.

§5.4. **Caput V.** [p.5.]

Hi Logarithmi duobus praecipue modis inveniri poterunt, quorum uterque a clarissimo inventore Barone Merchistonij traditur in appendice, quam libello de Mirifici Logarithmorum canonis constructione subjunxit, quam consulas suadeo a pagina 40 usque ad 53. Ego utrumque modum ut potero conabor illustrare.

ad priorem modum expediendum, sunt mihi quaedam Lemmata praemittenda.

Lemma primum.

Si e serie numerorum ab unitatie continue proportionalium *A*, sumantur duo quilibet numeri *C* 8 & *D* 32, & eorum Indices 3 & 5; & instituantur duae aliae continue proportionalium series *E* & *F*, in quibus dicti numeri *C* & *D* proximo ab unitate loco ponantur, & continuentur utraque series donec 3 (Index numeri *C* in data serie *A*) respondeat continue facto in serie *F*; & eodem modo 5 Index numeri *D* respondeat continue facto in serie *E*; erunt hi facti in utraque *E* & *F* aequales. ut hic videmus. Eorum enim uterque aequatur numero in data serie *A* reperto, cuius Index est 15 factus ab altero Indicum priorum in reliquum ducto.

Q	*A*	*B*	*P*	*E*	*M*	*F*	*N*
0	1	0	0	1	0	1	0
	2	1		8	1	32	1
	4	2		64	2	1024	2
	C 8	3	1	512	3	32768	3
	16	4		4096	4		
1	*D* 32	5		32768	5		
	64	6	2				
	128	7					
	256	8					
	512	9	3				
2	1024	10					
	2048	11					
	4096	12	4				
	8192	13					

A: series prima continue proportionalium data.
B Indices eiusdem seriei.
E series secunda continue proportionalium .
M Indices eiusdem, quibus aequantur *P*.
F series tertia, cuius Indices *N*, ijsque aequales *Q*.

[p.6.]

Continue factus hic dicitur, qui sit a numero aliquo ducto in seipsum primo, deinde in hunc factum, eodemque modo in factos suos omnes subsequentes. ut ab 8 fiunt continue 64, 512, 4096, 32768, 262144, 2097152. Atque hi numeri (& quotquot continue fiunt per multiplicationem eiusdem) sunt continue proportionalium: quia omnes sunt proxime praecedentium aequemultiplices.

Lemma Secundum.

2. Si numerus quilibet in serie numerorum ab Unitate continue proportionalium, divitatur continue[a] per suum latus [b] quoties fieri poterit: divisionum numerus [c] erit Index divisi, ostendens dati numeri ab unitate distantiam, vel numerum intervallorum inter Unitatem & Divisum. ut esto datus numerus 729, eiusque latus datum sit 3, hic latus continue dividet datum numerum sexies, eruntque quoti 243.81.27.9.3.1. erit igitur 6. Index dati numeri. ut hic vides:

1	0
3	1
9	2
27	3
81	4
243	5
729	6

[a] Numerus dicitur continue dividi; cum idem divisor diviserit, & datum, & dati quotum, & quoti deinceps quotum quemlibet. Atque hi quoti sunt continue proportionales, quia quilibet est ad suum quotum ut Divisor ad Unitatem.

[b] Latus cuiusque numeri dicitur numero Unitati proximus, in eadem serie continue proportionalium cum dato.ut in haec serie, Latus est 3.

[c] Hic non quaeruntur ipsi quoti, sed tantum numerus quotorum. Licet ipsi etiam quoti sint adscripti, quo omnia fiant magis manifesta.

Lemma Tertium.

3 Datis duobus numeris in eadem continue proportionalium serie positis, una cum Indice alterius datorum: reliqui Indecem vel distantiam ab unitate invenire.

Sunto ut in primo Lemmate dati duo numeri ex eadem serie data desumpti 8 & 32, sitque Index majoris numeri datus 5, reliqui Index ignotus sic invenietur. Fiat per multiplicationem numeri 8 *(cuius Index quaeritur) in seipsum & in factos a seipso, series altera numerorum continue proportionalium, donec perventum fuerit ad numerum cuius Index sit aequalis Indici dato 5. ut hic vides:

1	0
8	1
64	2
512	3
4096	4
32768	5

Ultimus factus 32768 erit per primum Lemma idem cum facto ab altero numero 32 toties in seipsum ducto, ut factus ultimus in hac serie tertia habeat illum quem quaerimus Indicem, qui in prima serie data debet adscribi octonario. Si igitur hunc numerum 32768 dividamus (eo quo dictum est modo in Lemmate secundo) per 32 datum (qui latus est tertiae seriei) quoties fieri proterit, erunt quoti tres 1024.32.1. Erit idcirco 3, Index tam huius facti 32768 in hac tertia serie, ut hic videmus, quam octonarii in serie illa prima facta data, in qua siti sunt illi dati duo numeri.

1	0
32	1
1024	2
32768	3

Ubi notandum, tres esse hic diversas continue proportionalium series, in quibus omnibus primum locum occupat Unitas. Prima, e qua desumpti sunt dati duo numeri, una cum Indice alteri datorum adjuncto. Secundo, quae sit per multiplicationem illius numeri (cuius Index quaeritur) donec ultimi facti Index aequetur Indici alterius numeri dato. Tertia vero qua per divisionem Facti in praecedente serie ultimo inventi, integra, si opus esset, describi posset. in hac, Index facti divisi aequatur numero quotorum, estque idem Index ille quem quaerimus, qui in prima [p.7.] serie debet adjungi illi numero cuius Index antea erat ignotus. ut videmus in Lemmate primo.

Lemma quartum, seu Modus, quo factus quilibet in serie continue proportionalium inveniri possit, omissis factis plurimus intermediis.

1	0
2	1
4	2
8	3
16	4
32	5
64	6
128	7
256	8

In serie numerorum ab unitate continue proportionalium , si numerus quilibet alterum multiplicet, factus in eadem serie continuata situs erit: eiusque Index erit aggregatum Indicum datorum. ut in hac serie; si quaternarius multiplicet 256, factus erit 1024, huius autem facti Index erit 10, summa Indicum 2 & 5, qui factoribus 4 & 256 adjunguntur. Idcirco, si numerus seipsum multiplicet, Index facti erit duplus Indicis qui lateri tribuitur. ut 16, 16 ^{ies} sunt 256, cuius facti Index erit 8. lateris autem Index est 4.

Lemma quintum,

seu Modus, quo numerus notarum in omnifacto haberi poterit, licet notae omnes non sint adscriptae.

Quot sunt notae in utroque factore, totidem erunt in facto. nisi forte factus a primis notis versus sinistram in utroque factore, unica nota exprimi poterit. hoc quoties contigerit, numerus notarum in facto, unitate superatur a numeris notarum in factoribus. ut si 68 multiplicentur per 26, factus 1768 notis scribitur quatuor: totidem scilicet quot insunt factoribus. at si 14 multiplicent 68, factus 952 tribus scribitur notis, licet in his factoribus, non minus quam in illis, sint quatuor notae.

Primus modus inveniendi Logarithmum cuiuscunque numeri propositi.

Ponamus omnes quotquot sunt numeros, in unica serie numerum ab unitate continue proportionalium esse descriptos, & Indices iisdem appositos esse hos ipsos quos quaerimus Logarithmos. atque ex hac infinita serie, sumantur duo numeri, quorum primus sit Denarius, cuius Index est 1, 00000,00000,0000: reliquus fit Binarius (vel alius quilibet) cuius Index ignoratur & quaeritur. Praeter hanc primam seriem quam appellamus infinitam, e qua desumpti sunt hi duo numeri, secunda series per tertium Lemma institui debet per continuam multiplicationem Binarii in seipsum & in factos a seipso. Et continuanda est haec series, donec Index ultimi facti, aequetur Denarii Indici dato in prima illa serie infinita. verum ut insuperabile illud tot multiplicationum taedium vitetur, plurimis intermediis numeris omissis, secundum continue proportionalium leges, per quartum Lemma huius capitis, praecipui facti quaerantur, una cum eorum Indicibus, donec ultimus factus quaesitus cum suo indice inventus fuerit. quem integrum adscribere non erit necessarium, sed tantum aliquot notas versus sinistram, amputatis &abjectis reliquis: ita tamen ut per quintum Lemma, notus sit numerus omnium [p.8.] notarum, quibus totas factus, si integer poneretur, describi debuit.

Haec prima serie ideo dicitur infinita, non quod omnes omnino numeros comprehendat, sed quod plurimos. cogitamus enim inter Unitatem & Denarium sitos esse 99999,99999,9999. distinctos numeros continue proportionales, quod ex Indice denarius qua aequatur numero intervallorum inter eundem & Unitatem sit manifestum. inter hos inter medios siti sunt numeri binario, ternario, reliquisque integris denario minoribus fere aequales, qui cum tantillum distant ab integris, tam ipso pro integris sumuntur, quam eorum Indices pro eorundem Logarithmis.

ductus igitur binarius in seipsum facit quatuor, cuius Index 2. Quaternarius in seipsum facit 16, cuius Index 4. 16 item in seipsum facit 256, cuius index 8, deinde hic factus ductus in 4 facit 1024, cuius Index 10 aequatur Indicibus facientium. notae autem in hoc facto ultimo sunt 4. quae omnia sunt manifestissima per quartum & quintum Lemmata proxime praecedentia. ut hic vides:

1	0		
2	1		
4	2	1	
16	4	2	Tetras prima
256	8	3	
1024	10	4	
10,48576	20	7	
109,9511627776	40	13	Tetras secunda
12089,25819,61463	80	25	
12676,50600,22823	100	31	
16069,38044,25899	200	61	
25822,49878,08685	400	121	Tetras tertias
66680,14432,87940	800	241	
10715,08607,18618	1000	302	
11481,30695,27407	2000	603	
13182,04093,43051	4000	1205	Tetras quarta
17376,62031,93695	8000	2409	
19950,63116,87912	1000	3011	
	Indices	Numerus notarum	

Isti autem quatuor numeri 4.16.256.1024. constituunt Tetradem primam. Deinde alia Tetras instituenda est, cuius primus numerus sit multiplicatione ultimi numeri Tetradis praecedentis in seipsum. Secundus item est quadratus primi, & tertius secundi; quartus autem sit per multiplicationem primi, huius Tetradis in tertium. atque horum quatuor Indices sunt 20.40.80.100. Sunt autem in horum primo notae septem, in secundo 13, in tertio 25, in quarto 31. Eodem modo reliquas Tetradas absolvo donec ultimae Tetradis numerus quartus habeat indicem 1,00000,00000,0000. numerus autem notarum in eodem quarto erit 30102, 99956, 6399. Atque ad hunc modum habemus circiter quinquaginta septem praecipuos numeros in serie secunda ponendos. * Horum primus, qui latus est et radix reliquorum, est binarius. Ultimus autem habet eundem Indicem quem in prima illa serie infinita Denario assignavimus.

** non illos integros quidem, neque enim est hoc vel necesse vel possibile, sed uniuscuiusque aliquot notas versus sinistrum, quae nobis indicare poterunt numerum notarum in unoquoque. istas autem primas cuiusque tetradis notas paulo diligentius describendas curavimus, quia versus notarum numerus in ultimo facto alter haberi non poterit. ut experiundo videmus in numeris B signatis, ubi unitate detracta ab ultimo numero Tetradis primae 1024, provenit numerus notarum in ultimo numero Tetradis tertia 301; qui esse debuit ut antea inventus fuit 302.*

B

1023	10	4	
1046529	20	7	
1095223	40	13	Tetr.2.
1199513	80	25	
1255332	100	31	
1575870	200	61	
2483363	400	121	Tetr.3.
6167075	800	241	
9718508	1000	301	
	Indices	Numerus notarum	

Tertia series per secundum lemma describi potuisset integra, si haberemus ultimum factum secundae seriei, qui omnino aequalis est ultimo facto huius tertiae seriei per primum Lemma. numerus autem notarum in hoc ultimo facto utriusque seriei notus est, idemque Indicem huius facti in tertia serie, & Binarii in illa prima serie infinita, superat tantum unitate. nam ut hunc Indicem in tertia serie inveniamus, continue dividendus est ultimus factus per Denarium datum numerum, cuius Index etiam datus fuit. Quod si Denarius quemlibet diviserit, Quotus erit pars decima divisi, id est idem plane cum diviso, dempta tantum unica nota (ut 4357 divisi per 10 dant [p.9.] quotum 435) idcirco numerus quotorum vel divisionum, unitate tantum minor erit numero notarum in diviso. quotorum autem numerus ostendit distantiam divisi ab Unitate, vel eiusdem divisi Indicem in hac tertia serie: qui idem est cum illo Indece in prima illa serie infinita, qui lateri secundae seriei, id est Binario debet adscribi. Quique est idem ille quem quaerimus Binarii Logarithmus. nempe 3012, 99956, 6398.

Idem modus per quem invenimus hunc Binarii Logarithmum, alterius cuiuscunque numeri Logarithmum suppeditabit. ut Septenarii. cuius disquisitionis principium vides in numeris *C*. numerus autem notarum in ultimo facto (cuius Index erit 10000,00000,00000) erit 84509,80400, 1427, qui dempta unitate aequatur Logarithmo Septenarii. atque hic modus primus esto, per quem cuiuscunque numeri

C			
1	0		
7	1	1	
49	2	2	
2401	4	4	Tetras prima
4764801	8	7	
282475249	10	9	
79792266297612	20	17	
63668057609090	40	34	Tetras secunda
40536215597144	80	68	
32344765096247	100	85	
	Indices	Numerus notarum	

propositi Logarithmum invenire poterimus. Sequitur alter qui videtur facilior.