§29.1. Synopsis: Chapter Twenty - Nine.

A number of figures are presented that cannot be constructed with compasses and a straight edge.
The heptagon is the first such figure considered: here Briggs establishes an isosceles triangle in which each base angle is three times the apex angle, in order that the base subtends an angle of $360^0/7$ at the centre of the circumscribed circle. In this case, the base becomes the side of the regular inscribed heptagon, while the equal sides of the triangle of unit length are the diagonals of the heptagon which each subtend three sides of the heptagon in the circumscribed circle. To find the length of the base, Briggs displays considerable geometrical virtuosity in carefully constructing a series of similar triangles that relate the diagonal length to the length of the side of the heptagon. Corresponding sides of these triangles are in continued proportion: thus, lines of lengths $p, q, r,$ and s satisfy the proportionalities $p/q = q/r = r/s = 1/\alpha$, where p is the largest and s the smallest length, while q is the larger mean and r the smaller mean value. Briggs' similar triangles give rise to the condition: $p + s = 2q + r$, which in turn results in to the cubic equation: $1 + x^3 = 2x + x^2$, which is solved numerically, and the root found that corresponds to the ratio of the length of side of the heptagon to the diagonal considered.
A similar method is used for the construction of the regular nonagon. Both figures are scaled to give the circumscribed circle a radius of one. The resulting ratios for the other figures considered are only quoted : these are left has exercises for the interested reader.

§29.2. *Chapter Twenty Nine.* [p.82.]

Concerning the Regular Heptagon, Nonagon, Pentadecagon, and Polygons with 24 & 30 Sides.

The following figures can conveniently be adjoined to these in the above chapter: however, some of them are somewhat more painstaking to analyse, and for these we will be content to the extent of merely showing the lengths of the sides; indeed, from the figures which are expounded here, it is possible for the rest to be completed also, by those who wish to investigate that [aspect] with more care.

[*The heptagon*]

If from four given lines in continuing proportion, the sum of the largest and the smallest lengths is equal to the sum of double the larger mean and the smaller mean : then if the larger sides of a triangle are equal, the base is equal to the larger mean, and each angle at the base is the triple of the remaining angle[1]. In which case, the base is the side of the heptagon inscribed in the circle with the triangle[2].

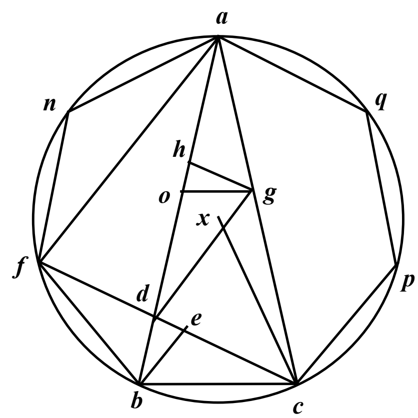
[p.83.] Let *ANFBCPQ* be a regular heptagon inscribed in the circle[3]; & let *AB, BC, BD, DE,* or *HO* be continued proportions: The sum of *AB & HO* is equal to the sum of the lines *AO, HD, BD*; that is, twice the larger mean *BC* and the smaller mean *BD*. Since the triangles *BAC, BCD* are equiangular; it follows that *BC* & *CD* are equal; likewise *AF, AD*, & *FC* are equal. If the angles GAD, GDA are made equal [*i.e.* choose G to insure that this is the case], then the triangles AGD, FBC have equal sides; hence HO, DE are equal in length.

On setting[4] AB equal to 1; then BCis 1 (1) ; BD, 1 (2) ; and DE, 1 (3).

In which case: 1 + 1(3) is equal to 2(1) + 1(2).

continued proportions	AB - -	1000000000000	
	BC - -	445041867913	1 (1)
	BD - -	198062264195	1 (2)
	DE - -	88146000021	1 (3)
	FC - -	801937735805	
	FD - -	356805867892	
continued proportions	XC - -	1000000000000	
	AB - -	1949855824364	
	BC - -	867767478235	
	BD - -	386192859428	
	DE - -	178771991534	

[Table 29-1]



[Figure 29-1]

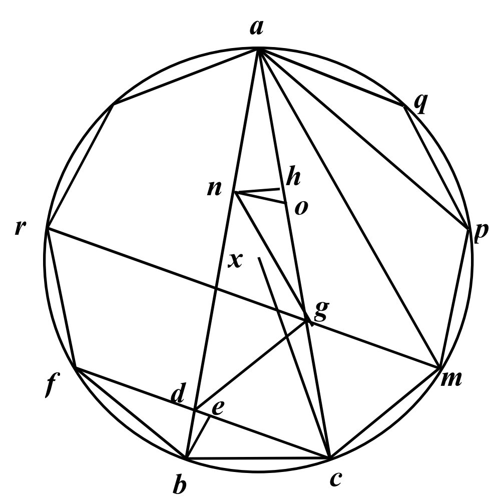
If the radius of the circumscribed circle XC is set to 1:

then BC is the side of the heptagon, 867767478235;

and CF the chord subtending two sides has length 1563662964936. AB the chord subtending three sides is of length 1949855824364 .

[*The nonagon*]

If from four lines in continued proportion, the sum of the maximum and the minimum is equal to three times the larger mean: with the largest sides of the triangle equal, and the base equal to the larger mean, and both the base angles are four times the other angle of the triangle: then the base is the side of a regular nonagon inscribed in a circle along with the triangle.



[Figure 29-2]

Let *APCBR* be the nonagon, and let *AB,BC,BD,DE* be the continued proportionals. The sum of *AC* & *HO* is equal to the sum of the lines *AO, HG*, *GC*: that is, three times the length of the line *BC*[5]. For the angles *BAC, BCD* are equal & therefore *BC, CD* are equal.

And if *RM* be drawn parallel to *FC*, the angles *FCG, CGM* are equal; from which also the angles *GMC* and *GCM* are equal; triangles APM, ARF, RBC are equal as they stand on the periphery. Therefore GC, GM, CM are equal And drawing the line AM, the triangles AMG, AMP are equal angled, and with equal sides. And the triangles AQP, ANG, FBC have equal sides: & the lines AO, HG, CB; likewise HO, DE are equal.

Setting AB, 1; BC, 1(1). BD, 2(2); DE, 1(3); then 3 (2) is equal to 1 + 1(3) [6].

	AB	1000000000000	continued proportions
1 (1)	BC	347299355334	
1 (2)	BD	120614758428	
1 (3)	DE	41889066002	
	CF	652793644666	
	AM	876385241572	

If XC the radius of the circumscribed circle be 1. BC will be 6840402866513
CF subtending two sides - 12855752173732
AM subtending three sides 17320508075688

XC - - -	10000000000000	continued proportions
AB	19696155060245	
BC	6840402866513	
BD	2375646984557	
DE	825053539294	

[Table 29-2]

[p.84.] For the length of the side of the regular Pentadecagon[7] inscribed in the circle with radius 1, the side is found from the root of the three numbers $^{7}/_{4} - \ell.\,^{5}/_{16} - \ell.$ bin $^{15}/_{8} - \ell.\ ^{45}/_{64}$, which is equal to the length of the chord subtended by an angle of 24 degrees, 41582338164.

The side of the 24-gon is $\ell.$ bin 2. - $\ell.$ bin. 2 + $\ell.$ 3; or $\ell.$ bin. $^{1}/_{2} + \ell.\,^{1}/_{2} - \ell.$bin.$^{3}/_{2} - \ell.\ ^{9}/_{8}$; or the root of the three numbers $\ell.$ trin. 2 - $\ell.\,^{1}/_{2} - \ell.\,^{3}/_{2}$, which is equal to the length of the chord subtended by the angle 15^0, 26105238444 .

The side of the 30-gon is $\ell.$ bin $^{5}/_{8} - \ell.$ bin. $^{45}/_{64} - \ell.\ ^{5}/_{16} - {}^{1}/_{4}$; or, $\ell.$ trin. $^{9}/_{4} - \ell.\,^{5}/_{16} - \ell.$bin.$^{15}/_{8} + \ell.\ ^{45}/_{64}$, which is equal to the length of the chord subtended by the angle12^0, 2090569265353.

	Side	Logarithm
Heptagon	867767478235	– 0061596630
Nonagon	6840401866513	– 0164918320
Pentadecagon (15-gon)	41582338164	– 0381091094
24-gon	26105238444	– 0583272335
30-gon	2090569265353	– 0679735439

[Table 29-3]

In these figures, as with those above, the sides can be found, if the radius of the circumscribed circle is given, or the side of any other polygon (from these which are included in this chapter or the previous one) inscribed in the same circle, as that of which the side is sought: a single example of which suffices to show everything.

Let the side of the octagon be given as 6: the side of the pentadecagon is required, inscribed in the same circle as the given octagon.

Proportions		*Logarithms*
Given side of octagon	7653668647	– 0,11613,034
Side of octagon taken	6 - - - - - - - -	0,77815,125
Side of given pentadecagon	41582338164	– 0,38109,109
sum of means	- - - - - - - - -	0,39706,016
Side of required pentadecagon	325979658	0,51319,050

[Table 29-4]

§29.3. *Notes On Chapter Twenty Nine.*

[1] AB/BC = BC/BD = BD/DE = $1/\alpha$. The larger mean BC = αAB, and the smaller mean BD = α^2AB; while AB is the largest side, and DE = α^3AB is the smallest side.

[2] The isosceles triangle Briggs has in mind has base BC, equal base angles of $77^1/_7{}^0$, and an apex angle of $25^5/_7{}^0$, corresponding to an angle at the centre of $51^3/_7{}^0 = 360/7^0$.

[3] Having noted that ΔABC is isosceles. ΔAOG is constructed similar to ΔABC with AO = α.AB, and ΔDGH is congruent to this triangle, but inverted. The triangle ΔGHO is congruent to ΔBDE, and ΔCBD is congruent to ΔAOG and ΔDGH. The proof of all of this can be left as an exercise, but it can be seen that these smaller triangles have sides parallel to particular sides or diagonals of the figure.

[4] It can be seen from the diagram that AB can be dissected up into lengths of sides of these lesser triangles; thus :–

AB = AO + DH + BD – OH = α.AB + α.AB + α^2.AB – α^3.AB. Hence, $1 + \alpha^3 = 2\alpha + \alpha^2$.

The cubic $x^3 - x^2 - 2x + 1 = 0$ has to be solved for a positive root α less than one..

The notation in use at the time to describe the powers of a variable, which was not itself written down: thus the cubic would be written as (3) – (2) – 2(1) +1 = 0.

We may solve the cubic conveniently to find : $x_1 = 1.8019377..$, $x_2 = -1.5803129..$,

and $x_3 = 0.4450418..$

As the final root is the only one that satisfies both the requirements of being positive and less than one, it is the required ratio α.

Briggs gives no hint here of how he solved this equation: however, in his subsequent publication, the *Trigonometria Britannica*, he set out a numerical method in great detail where chord lengths corresponding to given angles were determined from cubic, fourth, and fifth power equations. This method is identical in execution to the Newton-Raphson method: over which we must acknowledge that Briggs had priority. How Briggs came upon the method we do not know: there are at least three possibilities:

(*i*) In some undisclosed way, he found the complete method himself;

(*ii*) He was privy to some of the later developments of Arabic mathematics, for Al Tusi had developed the method around 1208 A. D.

[See Roshdi Rashed. Resolution des Equation Numeriques et Algebre: Saraf-al-Din al-Tusi, Viète. *Archive for history of exact sciences*, (1974), 12, pp. 244 - 290.

Also by Rashed: *The Development of Arabic Mathematics: Between Arithmetic and Algebra.* Kluwer, Boston (1996)];

(*iii*) He was able to adapt the workable but clumsy method of 'affected cubes' invented by Viète, which had come from Viète's work in finding the cube and higher order roots of numbers. The last mentioned seems to be the most credible option.

At the time of writing, an article in the *Mathematical Gazette* on the *Trigonometria Britannica* by this writer due to appear in Nov. 2004, explores this possibility.

Courant and Robbins, in their classic book: *What is Mathematics,* (OUP 4th ed., 1981), p.138, give a small discussion on the regular heptagon, and also derive a cubic equation (rather more easily with the help of complex numbers); which they do not solve, but demonstrate that it has no rational roots, and so the sides are not rational and the heptagon cannot be constructed with a straight edge.

A number of articles have been written on the heptagon in recent times, which is a structure rich in identities. For the interested reader, a good source is the article by Bankhoff & Garfunkel, *Math. Mag.* 46, pp. 7 - 19, 1973; while on the web, Eric Weisstein's 'mathworld' on the heptagon at www.mathworld.wolfram.com is well worth investigating.

[5] In the construction, the point N is located to make the angle ∠AGN = ∠BAG; the triangles FBC & ANG are congruent, as the triangles BDE & HNO are also.

[6] In modern terms, for the first set of proportions, we have

$AB/BC = BC/CD = CD/DE = 1/\alpha$,

and initially set $AB = 1$, then $BC = \alpha$; $BD = BC^2 = \alpha^2$; $DE = BD^2/BC = \alpha^3$.

Briggs' equation then becomes: $AO + HO = 1 + \alpha^3$; while $3.BC = 3\alpha$.

Hence: $1 + a^3 = 3\alpha$.

The cubic $x^3 - 3x + 1 = 0$ may be solved to give the roots:

$x_1 = 1.5320889..$; $x_2 = -1.8793852\ldots$; $x_3 = 0.34729935\ldots$

Again, the positive root less than one is taken for α, which agrees with version Briggs': the same comments apply to Briggs' solution as above for the heptagon.

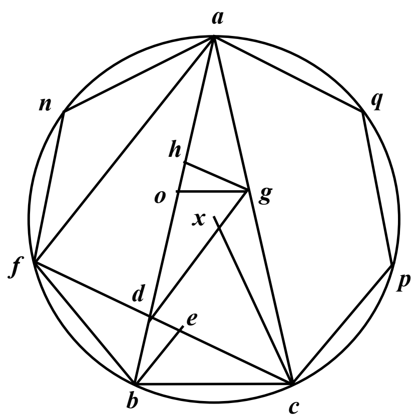
[7] The detailed working of these results, and similar ones in succeeding chapters, have not been included in these brief notes. The reader may refer to a mathematical handbook or the like, or put pen to paper for their evaluation.

§29.4. **Caput XXIX.** [p.82.]

De Septangulo, Nonangulo, Quindeangulo, te multangulis laterum 24 et 30.

Superius figuris poterunt et hae subsequentes non incommode adiungi. sed cum earum quaedam sint paulo magis operosae, content erimus earum latera tantum indicasse: cum ex illis quae hic traduntur, reliqua etiam suppleri possint ab ijs, qui ista scrupulosius indagare voluerunt.

Si e quatuor rectis continue proportionalibus, maxima et minima aequentur mediae maiori duplicatae, et mediae minori : Trianguli crurum maximae aequalium, basis maiori mediae aequalis, uterque angulus ad basim erit triplis relique. Et basis erit latus septanguli in circulum cum triangulo inscripti.



[p.83.]

Esto *ANFBCPQ* septangulum ordinatum circulo inscriptum; & sint *AB, BC, BD, DE,* vel *HO* continue proportiones: erunt *AB et HO* aequales rectis *AO, HD, BD*; id est mediae maiori *BC* duplicatae, et mediae minori *BD*. Sunt enim triangula *BAC, BCD* aequiangula; ed idcirco *BC* & *CD* itero *AF, AD*, & *FC* aequales: et si fiant anguli GAD, GDA aequales, erunt triangula AGD, FBC aequilatera; et HO, DE aequales. Ponatur AB,1; erit BC 1(1) ; BD, 1(2) ; DE, 1(3).
eruntque: 1 + 1(3) et 2(1) + 1(2) aequales.

cont.	AB - -	1000000000000	
prop.	BC - -	445041867913	1 (1)
	BD - -	198062264195	1 (2)
	DE - -	88146000021	1 (3)
	FC - -	801937735805	
	FD - -	356805867892	
cont.	XC - -	1000000000000	
prop.	AB - -	1949855824364	
	BC - -	867767478235	
	BD - -	386192859428	
	DE - -	178771991534	

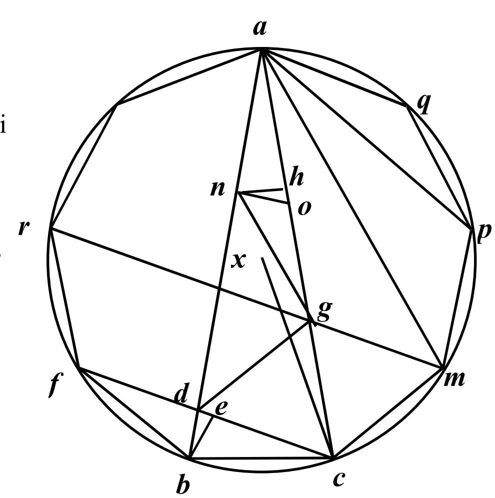
Si Radius circuli circumscripti XC ponatur 1: erit BC latus septangului, 867767478235; CF subtendens duo latera erit 1563662964936. AB subtendens tria latera erit 1949855824364 .

Si e quatuor rectis continue proportionalibus, maxima et minima aequentur mediae maiori triplicatae : Trianguli crurum maximae aequalium, basis mediae maiori aequalis, uterque angulus ad basim est quadruplus reliqui : et basis erit latus Nonanguli in circulum cum triangulo inscripti.

Esto *APCBR* Nonangulum et sint *AB,BC,BD,DE* continue proportionales. erunt *AC* & *HO* aequales rectis *AO, HG*, *GC*: id est rectae BC triplicatae. Sunt enim *BAC, BCD* aequiangula; et idcirco *BC, CD* aequantur. et si ducatur *RM* parallela rectae *FC*, erunt anguli *FCG, CGM* aequales; quibus etiam aequantur *GMC, GCM*; quia peripheriae in quas instituant, APM, ARF, RBC sunt aequales. Sunt igitur GC, GM, CM aequales. et ducta recta AM, erunt triangula AMG, AMP aequiangula, et aequilatera. Et Trianglula AQP, ANG, FBC aequilatera: et rectae AO, HG, CB; item HO, DE aequales.

Ponatur AB, 1; BC, 1(1). BD, 2(2); DE, 1(3); erunt 3 (2) et 1 + 1(3) aequalia.

	AB	1000000000000
1 (1)	BC	347299355334
1 (2)	BD	120614758428
1 (3)	DE	41889066002
	CF	652793644666
	AM	876385241572



[Figure 29-2]

Si XC Radius circuli circumscripti sit 1. erit BC 6840402866513

CF subtendens duo latera -	12855752173732
AM subtendens tria latera	17320508075688

XC - - -	10000000000000	cont.
AB	19696155060245	prop.
BC	6840402866513	
BD	2375646984557	
DE	825053539294	

[p.84]

Latus Quindecanguli inscripto circulo cuius radius est 1, est latis trinomij $^{7}/_{4}$ - $\ell.^{5}/_{16}$ - $\ell.$ bin $^{15}/_{8}$ - $\ell.$ $^{45}/_{64}$, cui aequatur subtensa 24 graduum, 41582338164.

Latus multanguli laterum 24, est $\ell.$ bin 2. - $\ell.$ bin. 2 + $\ell.$ 3; vel $\ell.$ bin. $^{1}/_{2}$ +$\ell.^{1}/_{2}$ - $\ell.$bin.$^{3}/_{2}$ - $\ell.$ $^{9}/_{8}$; Vel $\ell.$ trin. 2 - $\ell.^{1}/_{2}$ - $\ell.^{3}/_{2}$, cui aequatur subtensa 15 graduum, 26105238444 .

Latus multanguli laterum 30. est $\ell.$ bin $^{5}/_{8}$ - $\ell.$ bin. $^{45}/_{64}$ - $\ell.$ $^{5}/_{16}$ - $^{1}/_{4}$; vel, $\ell.$ trin. $^{9}/_{4}$ - $\ell.^{5}/_{16}$ - $\ell.$bin.$^{15}/_{8}$ + $\ell.$ $^{45}/_{64}$, cui aequatur subtensa 12 graduum 2090569265353.

	Latera	Logarithmi.
Septanguli	867767478235	– 0061596630
Nonanguli	6840401866513	– 0164918320
Quindecanguli	41582338164	– 0381091094
24-anguli	26105238444	– 0583272335
30-anguli	2090569265353	– 0679735439

In his figuris, ut in superioribus, poterunt latera inveniri, si datus fuerit radius circuli circumscripti, vel latus cuiuscumque multanguli (ex ijs quae hoc capite vel superiore continentur) eidem circulo inscripti cum eo, cuius quaeritur latus. quod unico exemplo ostendisse sufficiet

Esto latus Octanguli datum 6: quaeritur latus quindecanguli, eidem circulo cum dato Octangulo inscripti.

Proportiones		*Logarithmi.s*
Latus octaanguli datum	7653668647	– 0,11613,034
Latus Octanguli sumptum	6 - - - - - - - -	0,77815,125
Latus Quindecanguli datum	41582338164	– 0,38109,109
aggregatum mediorum	- - - - - - - - -	0,39706,016
Latus Quindecanguli quaesitum	325979658	0,51319,050