§27.1. *Synopsis: Chapter Twenty - Seven.*

A geometrical construction is given for dividing a line of finite length into sections in the extreme and mean ratio [*i.e.* the golden mean], which is then evaluated from $^1/_2 + \ell.^5/_4$. [Note: $\ell.^5/_4 \equiv \sqrt{^5/_4}$].

A small set of continued proportions is evaluated for successive positive and negative powers of this ratio, and some observations made regarding sums and differences of like powers that give whole numbers: thus, $(^1/_2 + \ell.^5/4)^2 + (^1/_2 + \ell.^5/_4)^{-2} = 3$; $(^1/_2 + \ell.^5/_4)^3 - (^1/_2 + \ell.^5/_4)^{-3} = 4$, etc.

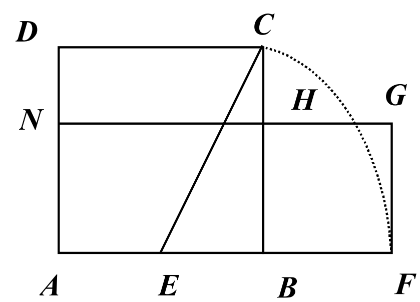
The sequence 1, 1, 2, 3, 5, 8, ... [*i.e.* the Fibonacci sequence] follows, where for any triad of consecutive terms such as 21, 34, 55, the area of the square from the middle term [here 34^2 gives 1156] is one more than the area of the rectangle from the smallest and largest term [here 21×55 gives 1155]. This sequence of natural numbers hence approximates the condition of being in continued proportion, [as $^{55}/_{34} \cong {}^{34}/_{21}$ in our example], and this approximation obviously improves as triads of larger numbers are considered.

Following this, any term in the sequence can be found approximately from a given term using a specified power of the ratio. Thus, $55.(^1/_2 + \ell.^5/4)^3 \cong 233$, while $55.(^1/_2 + \ell.^5/4)^{-3} \cong 13$; being displaced 3 terms to the right and left in the sequence respectively.

Finally, another sequence involving fourth powers of the ratio only are briefly mentioned.

§27.2. Chapter Twenty Seven

To divide a line or number according to the mean and extreme ratio.



[Figure 27-1]

The line is said to be cut in this way according to the third definition of the sixth book of *Euclid*:- The ratio of the whole length to the larger segment is the same as the ratio of the larger segment to the smaller segment. The manner in which this section is taken, is shown in Prop. 11, Book 2, & Prop. 30, Book 6 of *Euclid*. Also, *Ramus*, El.3, Book 14, has shown just how the line can be cut proportionally. So thus very briefly and in the proper manner, the same is indicated here as Euclid sets out in more detail. These segments of the line are asymmetric, both between each other and to the whole length; & therefore, if the length of the whole line is given, none of these segments can be expressed with rational numbers with accuracy at all, either by integers or fractions; but only by irrationals can we get closest to the fractions sought, & to reach the lengths required. Nevertheless, the sections can be defined carefully in terms of numbers from Algebra [*Note:* what Briggs has in mind here is not algebra as we know it, but rather the use of surd numbers of the form $a + \sqrt{b}$, as in Table 27-1], the conventional use of which is possible with hardly any difficulty, except with the customary habit of extracting the square roots of numbers. For let *AB* be of 10 parts, which is bisected in *E*; the line *EC* is drawn to the opposite angle *C* of the square ABCD; *EF*, *EC* are made equal: the line *BF* or *BH* is the larger & *CH* the smaller segment of the line *AB* or *BC* cut proportionally. And *BC, BH, HC* are continued proportionals [*i.e. BC/BH = BH /HC*] : because the square of the larger segment *BG* is equal to the rectangle *DH* described by the whole & the smaller segment[1]. If we wish to express this proportionality with numbers: the square of the line *EB* is 25, & the square of the line *BC* 100: therefore the square of the line *EC* or *EF* is 125: with which no [rational length of] side with the numbers given is possible. Therefore, the true value is more than the segment *BF*, $\ell.125$ - 5. If we wish to reduce this number to an approximate numerical value, then $\ell.125$ is almost $11\underline{180339887499}$ from which, if *EB*, 5 is taken away, then there remains the larger segment *BH* $6\underline{180339887499}$ & the smaller segment *CH* $3\underline{819660112501}$. With the addition of these two lengths; or by taking the smaller from the larger [as *e.g.* the lower row § from upper row §], as you see here; the same ratio between the lengths is always maintained near the true values, with accurate numbers from algebra[2].

$^{11}/_{2} + \ell.^{125}/_{4}$	$11\underline{0901699437495}$ ∴
$^{7}/_{2} + \ell.^{45}/_{4}$	$6\underline{8541019662497}$
$2 + \ell.5$	$4\underline{2360679774998}$ *
$^{3}/_{2} + \ell.^{5}/_{4}$	$2\underline{6180339887499}$
$^{1}/_{2} + \ell.^{5}/_{4}$	$1\underline{6180339887499}$ §
1	$1\underline{00}$
$\ell.^{5}/_{4} - ^{1}/_{2}$	$\underline{6180339887499}$ §
$^{3}/_{2} - \ell.^{5}/_{4}$	$\underline{3819660112501}$
$\ell.5 - 2$	$\underline{2360679774998}$ *
$^{7}/_{2} - \ell.^{45}/_{4}$	$\underline{1458980337503}$
$l.^{125}/4 - ^{11}/_{2}$	$\underline{0901699437495}$ ∴

[Table 27-2]

But it is permissible for these absolute numbers to be lacking somewhat from the true values: indeed, we can arrive almost at the same conclusion, by the continued addition of the two nearby whole numbers. With any two whole numbers [from the series shown] taken for the first two terms: with the square of the middle number taken from the oblong comprising the first and third numbers, serving in the same manner everywhere for the whole series[3]; it is allowed that the numbers themselves be made as large as possible: & therefore any three are close, very little departing from that proportion which we seek.

1
2
3
5
8
13
21
34
55
89
144

2, 3, 5: square 9; rectangle 10;
21, 34, 55: square 1156; rectangle 1155;
55, 89, 144: square7921; rectangle 7920.

The important use of these sections is in Geometry. *If the radius of a circle is cut proportionally, the side of the decagon inscribed in the same circle is the larger segment;* 8.e.18 of Ramus. Also: *the diagonal subtended by two adjacent sides of the inscribed pentagon is the larger segment.* Again, if a dodecahedron & an icosahedron are inscribed in a cube: then *the side of the dodecahedron is the smaller segment, and the side of the icosahedron the larger segment, for the sections of the side of the cube.*

			logarithms
Extreme and Mean of the ratio	Terms of	1000000000000000	1000
		618033988749894	079101235975002
	Diff. of Logs	[Being log τ]- - - - - - -	020898764024998

[Table 27-3]

For any given number cut proportionally [according to the golden section], you want to know the larger segment, from the logarithm of the given number, by taking away the logarithm of the difference of the ratio of the extreme and the mean [log τ]: the remainder is the logarithm of the larger segment. For let 55 be the given number:

	logarithms
Being given 55	174036268949424
Difference of extreme and mean ratio	020898764024998
33 9918693812442	153137504924426

The larger segment is almost 34.

[Table 27-4]

And if we want to know some other more distant term of the series in continued proportion, with neither the larger nor smaller number given: the said difference is multiplied by the number of the intervals, between the given number and the sought; the product, with the logarithm of the given number added, gives the logarithm of the larger number sought; the same subtracted leaves the logarithm of the smaller section[4]. For let the given number be 55 & the third larger from the given number sought, the [log of the] difference triplicated is 062696292074994; which added to logarithm of the given, gives 23673256102441 8, the logarithm of the number 23298373876244 ; the same taken away from the given, leaves 11133996874430, to which corresponds 1298373876249 . Consult Ch. 17 about this.

With these numbers, & in general with all numbers (integers, or fractions: absolute or algebraic) which come from the addition of the two nearest numbers; for as many considered as you please of the adjoining members, being an odd number, with the non-integer fourth power ratio , the sums of the extremes will be as five, nine, thirteen; with the remainder, the various differences of the means will be as seven, eleven, fifteen [Briggs may be indicating here forming a sequence from fourth powers of τ : ... τ^{-15}, τ^{-11}, τ^{-7}, τ^{-3}, τ^{1}, τ^{5}, τ^{9}, τ^{13},...].

§27.3. *Notes On Chapter Twenty Seven.*

[1] The length BH is required that satisfies the relation BC/BH = BH /CH ; if BH is called x, and CH is taken to be 1, then $(1 + x)/x = x/1$, and the positive value of the ratio is determined to be $\frac{1+\sqrt{5}}{2}$, the 'golden ratio', often denoted τ, associated with the Fibonacci sequence. Briggs defines log τ as the difference of the logarithms of the extreme (BC) and mean (BH) lengths.

[2] (Briggs' explanation is elaborated on here a little more: the ratios are evidently terms in continued proportion formed from the powers, both negative and positive, of the golden ratio τ).....

$^{11}/_2 + \ell.^{125}/_4$	[$\tau^5 = 3 + 5\tau$]	110901699437495 ∴
$7 + \ell.^{45}/_4$	[$\tau^4 = 2 + 3\tau$]	68541019662497
$2 + \ell.5$	[$\tau^3 = 1 + 2\tau$]	42360679774998 *
$^3/_2 + \ell.^5/_4$	[and BH is $\tau^2 = 1 + \tau$]	26180339887499
$^1/_2 + \ell.^5/_4$	then *CH is* τ]	16180339887499 §
1	[If BC = 1]	100
$\ell.^5/_4 - {}^1/_2$	[or, $BH = \tau^{-1} = -1 + \tau$]	6180339887499 §
$^3/_2 - \ell.^5/_4$	[and $CH = \tau^{-2} = 2 - \tau$]	3819660112501
$\ell.5 - 2$	[$\tau^{-3} = -3 + 2\tau$]	2360679774998 *
$^7/_2 - \ell.^{45}/_4$	[$\tau^{-4} = 5 - 3\tau$]	1458980337503
$\ell.^{125}/4 - {}^{11}/_2$	[$\tau^{-5} = -8 + 5\tau$]	0901699437495 ∴

[Table 27-2A]

Therefore, it will be more than the segment *BF*, ℓ.125 - 5. & if we wish to reduce this number to the absolute, it will be ℓ.125, 11180339887499 almost, from which if *EB*, 5 is taken : there will remain the larger segment *BH*, 6180339887499 & the smaller *CH* 3819660112501, BH^2/BC. These lengths by addition are seen to be close to the two required segments, [In Table 27-2A, the length BC is taken as 1 rather than 10].

This process can be continuing on either side, using **§, *,** and $\therefore$ in the table, as it is observed here, the same ratio will be maintained always, by numbers accurate by algebra near the true values.

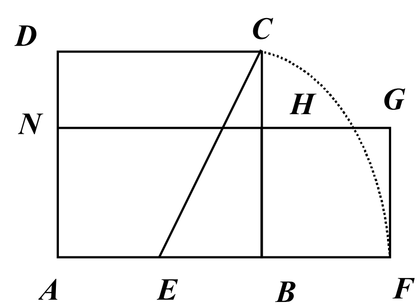
[3] If p, q, r are 3 successive natural numbers, we require $pr - q^2 = 1$. The Fibonacci sequence as defined by $f_0 = f_1 = 1$; $f_{n+1} = f_n + f_{n-1}$, for $n > 1$ satisfies this condition, and it can be shown that $f_{n+1}f_{n-1} - f_n^2 = 1$, and hence for large n, $f_n/f_{n+1} \sim f_{n-1}/f_n \sim 1/\tau$, the approximation improving as n increases. Hence, the larger length BH of the section in the unit segment BC is τ^{-1} and the smaller length CH is $1 - \tau^{-1} = 2 - \tau$, etc.

[4] The larger segment is $55/\tau \cong 34$, while the smaller segment $55/\tau^2 \cong 21$.

§27.4. **Caput XXVII.** [p.76.]

Lineam vel numerum secare secundum mediam & extremam rationem.

Linea recta ad hunc modum secari dicitur apud Euclem tertia definitione sexti libri, cum eadem fuerit ratio totius ad maius segmentum, quae est maioris segmenti ad minus. quomodo haec sectio fieri debeat, ostenditur 11.p.2.lib. & 10.p.6.lib. Euclidis. & Ramus, 3. el.14.lib. ostendit quomodo secari possit linea recta Proportionaliter.ita enim brevissime & non improprie, idem significat quod Euclides pluribus exprimit. Haec segmenta sunt, tam toti quam inter se asymmetra, & idcirco, si detur numero totius rectae longitudo, eius segmenta nullo modo numeris absolutis, vel integris vel fractis exprimi poterunt accurate; sed, ut in irrationalibus, poterimus proxime ad partium quaesitarum veras & quaesitas longitudines accedere, poterunt tamen earum longitudines numeris Algebricis accurate definiri, quorum usus vulgaris vix ullus esse poterit, nisi usitato more ad absolutos reducantur. ut esto *AB* partium 10, quae bisecetur in *E*; & ducatur rectae *EC* ad angulum quadrati oppositum *C* ; & fiant *EF*, *EC* aequales: recta *BF* vel *BH* erit maius & *CH* minus segmentum rectae *AB* vel *BC* proportionaliter sectae. Eruntque *BC, BH, HC* continue proportionales : quia *BG* Quadratum maioris segmenti aequatur rectangulo *DH*, comprehenso a tota & minor segmento. In numeris si ista [p.77.] exprimere velimus, Quadratum rectae *EB* est 25, & Quadratum rectae *BC* 100: erit igitur Quadratum *EC* vel *EF* 125: cuius nullum latus in numeris dati poterit. erit igitur maius segmentum *BF*, ℓ.125 - 5. & si ad absolutos hunc numerum reducere voluerimus, erit ℓ.125 11180339887499 sere. a quo si auferatur *EB*, 5 segmentum maius *BH* 6180339887499 & segment minus *CH* 3819660112501. Hae rectae duorum proximorum additione, vel subductione minoris a maiore, utrinque continuatae, quoad visum fuerit; eandem perpetuo servabunt rationem. ut hic vides, in numeris Algebricis accurate, in absolutis vero proxime.



$^{11}/_2 + \ell.^{125}/_4$	1<u>10901699437495</u> ∴
$^{7}/_2 + \ell.^{45}/_4$	68541019662497
$2 + \ell.5$	42360679774998 *
$^{3}/_2 + \ell.^{5}/_4$	26180339887499
$^{1}/_2 + \ell.^{5}/_4$	16180339887499 §
1	100
$\ell.^{5}/_4 - ^{1}/_2$	6180339887499 §
$^{3}/_2 - \ell.^{5}/_4$	3819660112501
$\ell.5 - 2$	2360679774998 *
$^{7}/_2 - \ell.^{45}/_4$	1458980337503
$l.^{125}/4 - ^{11}/_2$	0901699437495 ∴

Licet autem hi numeri absoluti nonnihil deficiant a vero: poterimus tamen in integris, additione
perpetua duorum proximorum, pervenire eo quo volumus sere. cum sumptis duobus quibuscunque pro 1
primis: differentia Quadrati e medio & oblongi comprehensi a primo & tertio, servatur eadem ubique 2
per totam seriem; licet ipsi numeri plurimum augeantur. & idcirco tres quilibet proximi, parum 3
admodum recedunt ab illa quam quaerimus proportione. ut 5

2, 3, 5: Quadratum 9; oblongum 10; 8
21, 34, 55: Quadratum 1156; oblongum 1155; 13
55, 89, 144: Quadratum 7921; oblongum 7920. 21

34

Huius sectionis magnus est apud Geometras usus. *Si Radius circuli secetur proportionaliter,* 55
maius segmentum est latus Decanguli eidem circulo inscripti 8.e.18 *Rami. & Latus Quinquanguli est* 89
maius segmentum subtendendis duo latera Quinquanguli. & si Dodecahedrum & Icosahedrum 144
inscribantur Cubo: *Latus Dodecahedri est minus segmentum, Latus icosahedri maius segmentum Lateris Cubici.*

			logarithmi.
Rationis extremae & Mediae	Termini	1000000000000000	1000
		618033988749894	079101235975002
	Differentia Logarith.		020898764024998

Si numeri cuiuscunque dati proportionaliter secti: maius segmentum scire cupias; e dati numeri Logarithmo, auferatur Differentia rationis extremae & mediae : reliquus erit Logarithmus segmenti maioris. ut esto datus numerus 55:

		logarithmi.
Datus 55		174036268949424
Diff.	extremae & mediae rationis	<u>020898764024998</u>
33<u>9918693812442</u>		153137504924426

maius segmentum erit 34 sere.

[p.78.]

Et si terminum aliquem remotiorem in serie continue proportionalium, sive maiorem dato sive minorem scire velimus; multiplicetur differentia dicta, per numerum intervallorum, inter datum numerum & quaesitum; factus, dati numeri Logarithmo additus, dabit numeri quaesiti maioris Logarithmum; idem ablatus relinquet Logarithmum minoris. Ut esto Datus 55 & quaeratur tertius a dato maior, Differentia triplicat erit 062696292074994; qui dati Logarithmo additus, dat 236732561024418, Logarithmum numeri 232<u>98373876244</u> ; idem a dato ablatus, relinquit 11133996874430, cui respondet 12<u>98373876249</u> . hac de rem consule cap.17.

In his numeris, & omnino in omnibus (integris, vel partibus: absolutis vel Algebricis) qui ex continua duorum proximorum additione proveniunt, si sumantur quotlibet proximi, quorum numerus est impar, & ratio ad quaternarium superparticularis, ut quinque, novem, tredecim, extremorum Summa; in reliquis, ut septem, undecim, quindecim, Differentia, erit multiplex medij.