Chapter Two

§2.1. *Synopsis: Chapter Two.*

The logarithm of one is most conveniently taken as zero. Following this definition, Briggs presents what he calls three axioms, or 'working rules'[†]:

A1: Three sequences of numbers in continued proportion (each forming a G. P.) are given with their common indices: 1, 10, 100, 1000,...., etc.; 1, 3, 9, 27,...., etc; and 1, 2, 4, 8, ..., etc. The common indices for each sequence are taken as: 0, 1, 2, 3,..., etc. These were customary at the time, apart perhaps from taking 0 as the index of 1. Another sequence of numbers called logarithms is appended to each sequence: for powers of ten these are written in the form 0000, 1000,2000, ...etc ; while the other two columns are multiples of 047712 and 03010299, see Table 2-1. Thus, for any row of the table, these latter logarithms are each proportional not only to their displacement or separation from the first row, but also to the whole number logarithms of the first sequence, and also between themselves. The logarithms of 2, *i.e.* 03010299 and 3, *i.e.* 047712 are taken from the accompanying tables of logarithms at the back of the *A. L.*

A2: The logarithm of a product of two factors is equal to the sum of the logarithms of the factors of the product, a result that can be extended to a product of three or more factors.

A3: The logarithm of the quotient of two numbers is equal to the difference of the logarithms of the number to be divided or dividend and the dividing number or divisor. This result can also be extended to division of the dividend by two (or more) divisors.

Briggs notes that the choice of 0 as the logarithm of 1 facilitates the processes of multiplication and division.

[†] Briggs was content to give numerical examples in tables to illustrate the truth of statements he made, which conformed with the custom of the time.

§2.2 *The logarithm of unity shall be zero.*

Although we can fit several kinds of Logarithms to the same numbers in this manner, it will be so much more advantageous, however, to use a single form: namely that which puts zero for the logarithm of unity, because before all others this will provide the most suitable and certainly expedient use for everyone[1]. With this [provision] put in place, three axioms of the greatest importance necessarily follow.

1. First Axiom. The logarithms of all numbers shall be: either, these numbers which are called indices, and with which all the Arithmeticians in their writing are accustomed to adjoin to numbers from unity in continued proportion, and these indices show how far the proportional numbers are from unity; or these other numbers proportional to the usual indices.

A	*B*	*C*	*A*	*B*	*C*	*A*	*B*	*C*
1	0	0000	1	0	000000	1	0	00000000
10	1	1000	3	1	047712	2	1	03010299
100	2	2000	9	2	095424	4	2	06020599
1000	3	3000	27	3	143136	8	3	09030899
10000	4	4000	81	4	190848	16	4	12041199
100000	5	5000	243	5	238561	32	5	15051499
1000000	6	6000	729	6	286272	64	6	18061799

[Table 2-1]

A: These are numbers in continued proportion from unity;

B: The common indices;

C: The logarithms [from Briggs' log tables] proportional to the indices themselves.

Since indeed, as the logarithms so the indices are increasing uniformly from the same beginning, in accordance with the numbers which are in continued proportion from unity [in the columns labelled *A* in Table 2-1], (to which only the indices are customarily associated); for the ratio of the distances, shall themselves be in proportion with the intervals, and by necessity they shall be in proportion among themselves; as we can see from these adjacent numbers [2].

2. Second Axiom. The logarithm of a product is to be equal to the logarithms of the factors. Since indeed from the law of multiplication, the ratio will always be the same of unity to the multiplying number, which is of the multiplied number to the product, and shall agree from the definition of logarithms itself: the logarithms of numbers in proportion are to have equal differences: this should be evident from the second Lemma in chapter one: the logarithms of the first and fourth [proportions], that is, of unity and of the product, to be equal to the logarithms of the second and third [proportions], that is, of the multiplying and of the multiplied numbers. And since the logarithm of

		Logarithms
Factors	1	00000
	3	047712
	27	143136
Composite number	81	190848

[Table 2-2.]

unity is 0, clearly it is the logarithm of the product alone which is to be equal to the logarithms of the factors, as we see with these[3] [Table 2-2]. Because if there are more factors, the logarithm of the product is equal to the logarithms of all of these factors: for if the factors are 2, 8, 64, the number formed from the continued product of these will be 1024; the logarithm 301030 of this product is equal to the sum of the logarithms of these three numbers, from which this product has been produced by multiplication, as you see here [in Table 2-3]:

Since indeed the logarithm of the number *E* 16, the product from the number *A* taken by *B*, is equal to [the sum of the logarithms] of the numbers[4] *AB*, and by the same reason the logarithm of the number *D,* is equal to [the sum of] the logarithms of the numbers *E C*. It is necessary that the logarithm of the number *D* is equal to the logarithms of the numbers A, B, and C[5]. Q.E.D.

		Logarithms
A	2	030103
B	8	090309
C	64	180618
D	1024	301030
E	16	[120412]

[Table 2-3.]

3.Third Axiom: The logarithm of the dividend [*i.e.* the number to be divided] is equal to [the sum of] the logarithms of the divisor and of the quotient. But this by necessity is inferred from the preceding [axiom], because the product from the divisor taken with the quotient is itself the dividend: and therefore, as one is to the divisor, thus the quotient is to the dividend. As, if 4 divides 128, the quotient will be 32 [see Table 2-4[6]].

			Logarithms
		1	000000
Factors	Divisor	4	060206
	Quotient	32	150515
Product	Dividend	128	210721

[Table 2-4]

Because if after the first division has been carried out, the quotient itself is divided, the logarithm of the number divided first will be equal to [the sum of] the logarithms of the other divisor and of the latter quotient, because the latter divisor and quotient are substituted for the place of the first quotient. As, if 7 divides 105, the quotient will be 15. Then let 5 divide this quotient, the latter quotient will be 3. I assert

		Logarithms
Divisor	7	0845098
Divisor	5	0698970
Latter quotient	3	0477121
Dividend	105	2021186

[Table 2-5]

that the logarithm of the number 105 to be equal to the logarithms of the numbers 7, 5, and 3 as you see here in Table 2-5 [7]:

And if we decree the Logarithm of one to be zero then by necessity these three [axioms] follow.

§2.3 *Notes On Chapter Two.*

[1] Napier and Briggs had agreed that the original formulation of log tables by Napier, although a great achievement, did lead to some unfortunate consequences when doing calculations, see the Introductory Chapter. The worst of these perhaps being that the number with the Napierian logarithm of zero was 10^7 (= w), while the Napierian log of one was a very large number, ~ 1.6×10^8. This comes from the circumstance that Napier's logs from a modern perspective are related to the time of an exponential decay from an initial large value, as may be seen simply by writing the Napierian logarithm t of z in the inverse form $z = w.\exp(-t/w)$: for Napier had developed his logarithms from the numerical analysis of his kinematic model, and came very close to discovering the exponential function. Thus, he was the first person to use the concept of 'half-life', to aid with his table construction. Napier's logarithms were hence actually tables of anti-logarithms, for he found the number that corresponded to the given logarithm, rather than vice-versa, a thought given to me by D.T. Whiteside. See the article *Napier's Logarithms*, by this writer in the February 2000 issue of the *American Journal of Physics* for more details of this approach.

[2] For a given column *A*, each row of columns *B* and *C* are in proportion. Note that the numbers representing the logs in the first column C, such as 1000, 2000, etc. for the powers of 10, are called *rationals* by Briggs, as they refer to whole numbers: the left-hand digit will be the characteristic, and the correct vertical placing of columns of numbers becomes important, in order that the characteristics are in the same column when doing calculations. Until this time, Arithmeticians or mathematicians would only have written the indices in the columns B corresponding to the powers of 10, 3 and 2: now an extra column C is introduced where the logarithms of 10, 3 and 2 are placed

(in the second row). These are multiplied by the indices to give the logarithms of the various powers.

[3] For $1 : a :: b : p$, or $1/a = b/p$: note that Briggs writes his ratios in this order, giving the product $p = a \times b = p \times 1$, to which the definition of logarithms and the 2nd Lemma in Ch. 1 can be applied sequentially, using only positive numbers:

$\log a - \log 1 = \log p - \log b$, or $\log p = \log a + \log b$, as $\log 1 = 0$. By 'log *a*', and so on is merely meant, 'the logarithm associated with the number *a*', and it does not mean the logarithm is a function of *a*.

[4] There is a gap in the table opposite 16 in the original work, which should contain the log of 16, i.e.120412, inserted here in parenthesis.

[5] Thus, in modern terms, $D = (AB)C$, and $\log D = \log (AB) + \log C = (\log A + \log B) + \log C$: the associative rule for the multiplication of numbers, and the addition of their logs. Briggs again uses an example to establish a theorem.

[6] In modern terms, if $Q = D/d$, then $1/d = Q/D$, etc.

[7] And again, if $Q_1 = D/d_1$, and $Q_2 = Q_1/d_2$, then

$\log D = \log Q_1 + \log d_1 = \log Q_2 + \log d_2 + \log d_1$.

§2.4. Caput II. [p.2.]

Logarithmus unitatis sit 0.

Licet autem possimus Logarithorum plures species ijsdem numeris ad hunc modum aptare, erit tamen commodissimum unica tantum uti specie, eaque quae cyphram ponit pro Logarithmo unitatis. quod ea prae caeteris omnibus usum preabeat ad omnia accommodatissimum & maxime expeditum. quo posito tria maximi momenti axiomata necessario consequuntur.

1. Primum. Omnium numerorum Logarithmos esse, vel eos numeros qui Indices appellantur, & apud omnes Arithmeticos numeris ab unitate continue proportionalibus adjuncti solent, eorumque ab unitate distantiam ostendunt: vel hisce usitatis Indicibus proportionales.

A	*B*	*C*	*A*	*B*	*C*	*A*	*B*	*C*
1	0	0000	1	0	000000	1	0	00000000
10	1	1000	3	1	047712	2	1	03010299
100	2	2000	9	2	095424	4	2	06020599
1000	3	3000	27	3	143136	8	3	09030899
10000	4	4000	81	4	190848	16	4	12041199
100000	5	5000	243	5	238561	32	5	15051499
1000000	6	6000	729	6	286272	64	6	18061799

A sunt numeri continue proportionales ab unitate.
B Indices vulgares.
C Logarithmi ipsis Indicibus proportionales.
Cum enim tam Logarithmi quam indices in numeris ab unitate continue proportionalibus (quibus solis Indices adjungi solent) ab eodem principio aequaliter crescentes, pro ratione distantiae, sint ipsis intervallis proportionales, inter se proportionales erunt necessario. ut in hisce appositis numeris videmus.

2. Secundum. Logarithmum facti aequari Logarithmis facientium. Cum enim ex multiplicationis lege, eadem semper sit ratio Unitatis ad Multiplicantem, quae est Multiplicati ad Factum, & ex ipsa Logarithmorum definitione constet, [p.3.] proportionalium Logarithmos esse aequidifferentes: patet per secundum Lemma, Logarithmos primi & quarti, id est unitatis & facti, aequali Logarithmis secundi & tertij, id est, multiplicantis & multiplicati. Et cum unitatis Logarithmus sit 0, manifestum est facti solius Logarithmum, aequari Logarithmis facientum. ut in his videmus.

		Logarithmi.
	1	00000
Factores	3	047712
	27	143136
factus	81	190848

Quod si factores plures fuerint; eorum omnium Logarithmis aequatur Logarithmus facti. ut sint factores 2.8.64 numerus continue factus ab ijdem 1024, huius logarithmus 30103 aequatur Logarithmis trium illorum, ex quorum multiplicatione hic factus proveniebat. ut hic vides:

		Logarithmi.
A	2	030103
B	8	090309
C	64	180618
D	1024	301030
E	16	[120412]

Cum enim Logarithmus numeri E 16, facti ab A numero ducto in B, aequatur Logarithmis numerorum AB, & eadem de causa Logarithmus numeri D, aequetur Logarithmis numerorum EC, necesse est Logarithmum numeri D, aequetur Logarithmis numerorum A, B, C. quod erat demonstrandum.

Tertium. Logarithmum divisi aequari logarithmis divisoris & quoti. Hoc autem necessario concluditur ex praecedente, quia factus a divisore ducto in quotum est ipse divisus: & idcirco, ut unitas ad divisorem, sic quotus ad divisum. ut sit dividendus 128, divisor 4, quotus erit 32.

			Logarithmi.
		1	000000
Factores	Divisor.	4	060206
	Quotus.	32	150515
factus	Divisus.	128	210721

Quod si post peractam primam divisionem, Quotus ipse divisus fuerit, Logarithmus numeri primo divisi aequabitur Logarithmis divisoris utriusque & posterioris quoti. quia divisor & quotus posteriores substituuntur loco prioris quoti. ut sit dividendus 105, divisor 7, quotus erit 15. hunc quotum dividat 5, quotus erit 3. aio Logarithmum numeri 105 aequari Logarithmis numerorum 7.5.3. ut hic vides:

	Logarithmi.
Divisor 7	0845098
Divisor 5	0698970
Quotus posterior 3	0477121
Divisus 105	2021186

Atque haec tria necessario consequuntur si statuamus Logarithmum unitatis esse cyphram.