§19.1. *Synopsis: Chapter Nineteen.*

Briggs further displays the usefulness of logarithms in calculating unknown quantities for right-angled triangles, set out below. The notes give some assistance in formulating outlines of proofs of the various theorems encountered. We have kept his use of the word *crure*, meaning leg, for what we now call the side of a triangle other than the hypotenuse, as the word *latus* already has a use as the square root of a number.

§19.2. *Chapter Nineteen.*

For a right-angled triangle, given:

1. A Leg and Hypotenuse,

2. A Leg & the sum or difference of the Hypotenuse

& of the remaining leg,

3. Two Legs,

4. A leg and the Area,

5. The area & the sum or difference of the Legs,

6. Hypotenuse & sum or difference of the Legs,

7. Hypotenuse and the Area,

8. The Perimeter & Area,

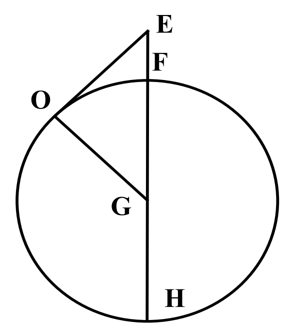
the rest are sought.

1. *With a leg and Hypotenuse given, the remaining legis sought.*

The sum and difference of the hypotenuse and the given side & the logarithms of these are taken: half the sum of the logarithms is the logarithm of the leg sought.

					Logarithms
given	other leg	5	sum	18	125527251
	hypotenuse	13	difference	8	0900308999
			sum		215836250
leg sought	12		half sum		107918125

[Table 19-1]



[Figure 19-1]

For the leg sought is the mean proportion between the sum and difference of the hypotenuse and the other leg, as shown by the first diagram of Ch. 18, & therefore the logarithm of the mean sought is half the sum of the given logarithms, by Ch. 17.[1] For this reason it is allowed: Given HF for the diameter of the earth, & EF the height of the eye above the sea: to find the straight line EO, or the distance of the eye from the point of contact (i.e. the horizon)

[p.53.] Let FH be 6873 miles [really ~7912 miles], EF 2 miles

		logarithms	
HE	6875	383727270	
EF	2	030102999	
	sum	413830269	
	$^1/_2$ sum	206915134	EO 117$\underline{25}$

[Table 19-2]

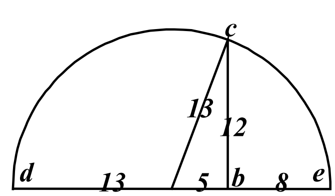
2. *Given a leg & the sum or difference of the hypotenuse & the remaining leg: the remaining leg and the hypotenuse are sought.*

The logarithm of the given side doubled is equal to sum of the logarithms of the sum and difference of the hypotenuse and the remaining leg. Therefore by taking the logarithm of one of these, from double the logarithm of the given leg, there is left the logarithm of the remainder, & the hypotenuse is half of the sum of the found sum and difference: and of the difference of the same, half is the remaining leg[2].

Let the given leg BC be 12, & the sum of the hypotenuse and the remaining leg DB 18, the difference BE of the same is first sought. Then AE or AC, the half sum of the extremes is the hypotenuse, & AB the half difference of the same is the remaining leg.

				Logarithms
DB: sum of the hypotenuse and remaining leg 18				125527251
BC: log. doubled of the given leg 12				215836250
BE : difference of the hypotenuse & remaining leg 8				090308999
BD, 18; BC, 12; BE, 8: continued proportions				
		8		
		$\underline{18}$		
Extremes	sum of	26	13: ½ sum. Hypotenuse	
	Difference of	10	5: ½ difference. Remaining leg	

[Table 19-3]



[Figure 19-2]

3. *With two legs given, the hypotenuse is sought.*

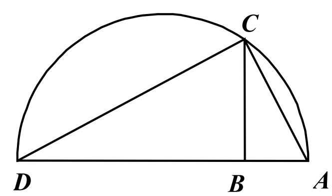
With the third proportion first sought, as by Prop. 11, Book 6, Euclid.

Then the mean proportion is sought between the first & the sum of the first and third proportions: that mean is equal to the hypotenuse sought, by Prop. 31, Book 6, Euclid.[3]

Let the two given sides be AB, 8; BC, 15. The third proportion is DB $28^1/_8$.

		Logarithms	
AB. 8		090308999	
BC. 15 with the log. doubled		235218252	
BD. $28^1/_8$		144909253	
DA. sum of first & third	$36^1/_8$	155780785	
AB. First	8	090308999	
Sum of logs.		246089784	
½ sum		123044892	17. Hypotenuse.

[Table 19-3]



[Figure 19-3]

AD, $36^1/_8$; AC, 17; AB, 8 are continued proportions, and ABC is a right angled triangle . We find the same hypotenuse, if the line CB, 15 becomes the first proportion, & BA, 8 the second. The third proportion is $4^4/_{15}$. Then between the sum of the logarithms of the first and the third $19^4/_{15}$ & CB 15 the first, the mean is AC, 17.

Concerning the logarithms of fractions, and of whole numbers to which the fractions are added on, see Ch. 10.

4. *With a leg and the area given: the remaining leg and the hypotenuse are sought.*

The area of the triangle is half the area of the rectangle comprised of the legs, by Prop. 41, Book 1, Euclid. If therefore the given leg is divided into the double of the area, the quotient is the remaining leg. And by the preceding section, with the two legs given, the hypotenuse is found.

[p.54.] Let the given leg be 7, the area 84, the remaining leg is sought.

			Logarithms	
proportions	Divisor	7	084509804	
	Unity	1	000000000	
	Dividend	168	222530928	
	Quotient	24	138021124	required leg 24

[Table 19-4]

5. *Given the area & the sum or difference of the legs, to find the legs.*

The rectangle comprising the legs (which by Prop. 41, Book 1, Euclid is equal to double the area), together with the square of half the difference of the legs, is equal to the square of half the sum of the legs, by Prop. 5, Book 2, Euclid.

Therefore if double the given area is added to the square of half the difference of the legs, the total is the square of half the sum of the same.

But from the square of half the sum of the legs let double the area be taken away, the square of half the difference of the same remains[4].

Let the area be 60, the sum of the legs 23, The square of half the sum $132^1/_4$, from which 120 is taken away, double the area, $12^1/_4$ remains: of which the square root 3½ is half the difference of the legs, and the legs are 15, 8.

11½	half the sum of the legs
3½	half the difference of the legs
$\left.\begin{matrix}15\\8\end{matrix}\right\}$	the legs

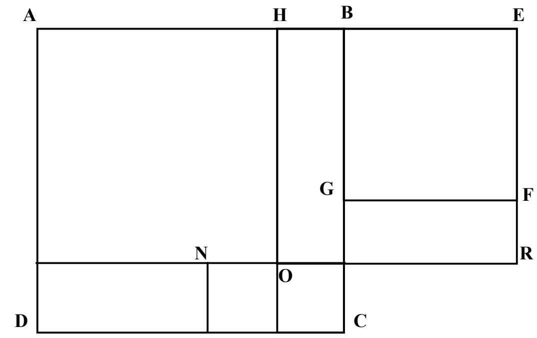
[Table 19-5]

6 *With the hypotenuse and the sum or difference of the legs given: the legs are sought.*

The square of the hypotenuse is equal to the sum of the squares of the sides by Prop. 47, Book 1, Euclid. Therefore with the sum of two squares given, together with the sum or difference of the sides: the sides are sought.

If the sum of the sides are given: *the square of half the sum of the sides is taking away from half the sum of the squares; there is left the square of half the difference of the sides*[5], for let the hypotenuse be 26, the sum of the legs 34, 676 is the sum of the squares from the legs. Half of the sum of the legs 17, of which the square is 289, is taken away from half the sum of the squares 338, & 49 remains, of which the root 7 is half the difference of the legs. The legs are therefore 24, 10.

But given the difference of the two sides: *the squares of half the difference, taken from half the sum of the squares, leaves the square of half the sum of the sides.* For with the same example, half the difference of the sides 7, of which the square 49, taken from half of the square 338, leaves 289, the square of half the sum of the sides 17. Therefore the required legs will be, as before, 24 & 10, which can be shown as follows:

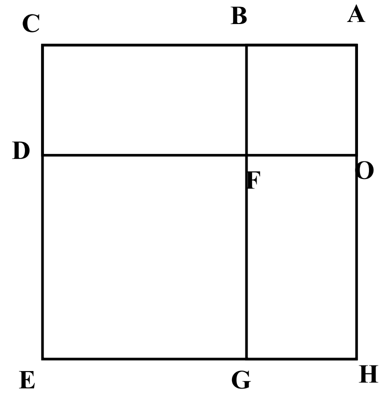


[Figure 19-4]

Let AB be 9, BE 5, of which the squares are AC 81, EG 25, & the sum of the squares is 106 [See Figure 19-4]. Half the sum of the sides is HE 7, of which the square is HR 49, & which doubled is AR, 98. Half the difference of the sides is HB, 2, of which the square doubled is NC, 8. But with the rectangle GR 10, which is included in the rectangle AR; whereas being excluded it is equal to the rectangle DN . Manifestly NC, double the square of the line HB, is to be the difference of the given squares above 106 and AR 98, double the square of the line EH. Therefore the square HR 49, & the square OC 4, are equal to half the sum of the given squares 53. [Thus AC + BF $\rightarrow$ (AO + HR) + (NO + OC) $\rightarrow$ AC + BF again.]

7. *Given the Hypotenuse & the Area to find the Legs.*

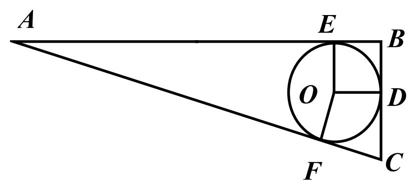
The area doubled is equal to the area of the rectangle contained by the legs by Prop. 41, [p.55.] Book 1, Euclid. And the square of the hypotenuse is equal to the sum of the squares of the legs, by Prop. 47, Book 1, Euclid. Therefore, the sum of the squares & the rectangle from the containing legs are given. But double the rectangle according to this method, with the square of the difference of the sides, is equal to the sum of the squares, by Prop. 7, Book 2, Euclid. Therefore if from the square of the hypotenuse four times the area [of the triangle] is taken away, there is left the square of the difference of the legs. And by the preceding section, *given the hypotenuse & the difference of the legs, the legs can be found.* Let the hypotenuse be 13, the area 30. Let the rectangle contained by the legs be AD or BH 60, & let the legs be AC, AB. The sum of the squares is 169. This sum is equal to the two rectangles AD, BH, with the square of the line BC [the square DG], the difference of the sides AC, AB. Taking away AD, BH 120, from the sum of the squares AE, OB 169: the square GD 49 is left, of which the root BC 7, is the difference of the sides[6].



[Figure 19-5]

8. *With the Perimeter and the Area given, to find the Legs.*

The area of the triangle is equal to the product of the semi-perimeter by the radius of the inscribed circle: as shown in Section 3, Ch. 18.



[Figure 19-6]

And within the right-angled triangle, *the diameter of the inscribed circle is equal to the difference of the hypotenuse & the sum of the legs*. As in triangle ABC, EB, DB are equal to the radii EO, OD. Therefore, if the semi-perimeter divides the area, the quotient is the radius OE. & by taking away EB & BD, equal to the diameter, from the total perimeter, half the remainder is the hypotenuse AFC, or AE, DC. Therefore given the hypotenuse, and the sum of the legs then by Section 6, the legs are found

Let the right-angled triangle ACB have perimeter 60, & area 120: if the semi-perimeter is divided into the area, the quotient is EO 4, the radius of the inscribed circle. And by taking away 8 from the perimeter, there remains 52. Of which half , 26 is the hypotenuse AC, & the sum of the sides AB, BC is 34.

[For 60 = $a + b + c$; and $b = a + c - 8$; hence 52 = $(a + c - 8) + b = 2b$, etc.]

§19.3. *Notes on Chapter 19*

Briggs' logarithmic-arithmetical schemes for producing numerical answers to propositions, have as much to do with algebra as geometry: we present some hints here that could be part of algebraic proofs of the theorems, mainly to aid the reader in quickly surmising the content of the work.

In what follows, (a, b, c) are the sides of a right angled triangle with hypotenuse c, and $a < b$.

1 The sum $(c + a)$ and the difference $(c - a)$ are formed, from which the product $(c + a)(c - a) = b^2$ is used to evaluate b. As Briggs would have it: $(c + a): b :: b : (c - a)$; i.e. b is the mean proportional between the sum and difference of the extremes

2 The quantities given are BC = a; BD = $c + b$ are given, Fig. 19-2. The difference $c - b$ has to be found, (or vice versa). Here $a^2 = (c + b)(c - b)$, which can be expressed in the proportional form: $(c + b)/a = a/ (c - b)$. The quantity $(c - b)$ is found using logs from $a^2/(c + b)$. The extremes are $(c + b)$ and $(c - b)$: half the sum and difference being c and b respectively.

3 If a (= AB) and b (= BC) are the lengths of the two sides, Fig. 19-3, the third proportional x (= BD) is found from $a/b = b/x$ to be $x = b^2/a$, or from the similar triangles ΔABC and ΔBCD. The proportionality used is now: $a/c = c/(a + b^2/a)$,

from AB/AC = AC/(AD) = AC/(AB + BD), (the similar triangles are now ΔABC and ΔACD), from which the hypotenuse c is found. The same argument works if the proportions are chosen in decreasing rather than in increasing order, as Briggs illustrates here finally.

4 Let a and b be the legs, then the area of the triangle $A = ab/2$, and $(b - a)^2/4 + ab = (b + a)^2/4$.

5 $(a + b)$ is given: $(a^2 + b^2)/2 - (a + b)^2/4 = (a - b)^2/4$, from which both results follow.

Briggs uses a modified form of the Theorem of Pythagoras: for $c^2 = a^2 + b^2$ may be written in the form $c^2/2 = [(a+b)/2]^2 + [(a-b)/2]^2$. Thus the areas of the squares AC (81) and BF (25) is the total area corresponding to c^2 (106): the squares HR and AO each represents half the sum of the sides squared $[(a + b)/2]^2$, while OC represents half the difference squared $[(a - b)/2]^2$. Doubling these areas must give the original total area c^2: this is the case as double HR gives the rectangle AR, while doubling OC gives the rectangle NC. The rectangle GR can then be moved to the equal rectangle DN, both of width b and height (a - b) to complete the solution.

[6] $b^2 = (b - a)^2 + 2ab - a^2$: thus, CH = DG + AD + BH – OB, etc.)

§19.4. **Caput XVIIII**. [p.52.]

In triangulo rectangulo, Datis

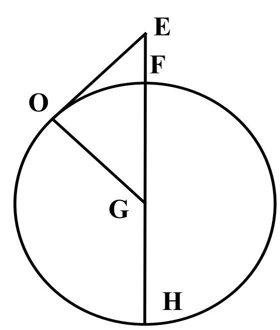
1. Crure & Hypotenusa,
2. Crure & summa vel differentia Hypotenusae & reliqui cruris,
3. Duobus cruribus,
4. Crure & Area,
5. Area & summa vel differentia crurum,
6. Hypotenusa & summa vel differentia crurum,
7. Hypotenususa & Area,
8. Perimetro & Area,

quaeruntur reliqua.

1. *Datis crure & Hypotenusa, quaeritur crus reliquum.*

Sumantur summa & differentia Hypotenusae &dati cruris, & earum logarithmi:semissis summae logarithmorum erit logarithmus cruris quaesiti.

					Logarithmi.
ut sunto	crus alterum	5	summa	18	125527251
	hypotenusa	13	differentia	8	0900308999
			summa		215836250
crus quaesitum 12			$^1/_2$ summa		107918125



Est enim crus quaesitum medium proportionale inter summam & differentiam hypotenusae &dati cruris, ut ostenditur per primum diagramma Capitis 18, & idcirco logarithmus medij quaesiti est semissis summae datorum Logarithmorum, per cap. 17.[1]

Hinc licebit: Datis CE diametro Terrae, & AC altitudine oculi supra mare: invenire rectam AB, vel distantiam oculi a puncto contactus.

[p.53.]

Sit FH millarium 6873, EF 2.

		logarithmi.	
erunt HE	6875	383727270	
EF	2	030102999	
	summa	413830269	
	$^1/_2$ summa	206915134	ab 117$\underline{25}$

2. *Dais crure & summa vel differentia hypotenusae &reliqui cruris:quaeruntur crus reliquum & hypotenusa.*

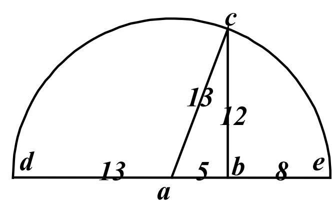
Logarithmus dati cruris duplicatus, aequatur logarithmis summae & differentia Hypotenusae & reliqui cruris. ablato igitur harum alterius Logarithmo, e duplicato dati cruris Logarithmo, restabit reliquae Logarithmus, & inventarum summae & differentiae semissis erit Hypotenusa: differentiae autem earumdem semissis erit crus reliquum.

Ut si datum sit crus AB 12, & summa Hypotenusae & reliqui cruris DB 18, primo quaerenda est BE differentia eorundem. Deinde AE vel AC,semissis summae extremarum erit Hypotenusa, & AB semissis differentiae earundem erit crus reliquum.

	Logarithmi.
DB: summa hypotenusae & reliqui cruris 18	125527251
BC: dati cruris 12 Logarithmus duplicatus	215836250
BE : differenctia Hypotenusae & reliqui cruris 8	090308999

BD, 18; BC, 12; BE, 8: continued proportionales

		8	
		$\underline{18}$	
Extrem	summa	26	13: ½ sum. Hypotenusa
-arum	Differentia	10	5: ½ differentiae. Crus reliquum



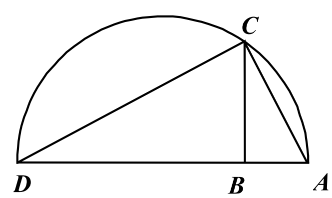
[Figure 19-2]

3. *Datis duobus cruribus, quaeritur hypotenusa.*

Primo quaeritur tertium , per 11.p.6.lib.Eucl.
Deinde inter primum & summam primi & tertia, quaeritur medium proportionale : medium illud aequabitur Hypotenusae quaesitae, per p.31.lib. 6.Eucl.

Ut sunto dadt duo crura AB, 8. BC, 15. Tertium proportionale erit DB 28$^1/_8$.

		Logarithmi.	
AB. 8		090308999	
BC. 15 eius Logarith. duplicatus		235218252	
BD. 28$^1/_8$		144909253	
DA.summa primum &tertij	36$^1/_8$	155780785	
AB. primum	8	090308999	
Logarithmorum summa.		246089784	
½ summae		123044892	17. Hypotenusa.



AD, 36$^1/_8$; AC, 17; AB, 8 sunt continue proportionales, & ABC triangulum rectangulum . Eandem Hypotenusam inveniremus, si recta CB, 15 fiat prima proportionalium, & BA, 8 secundum. Tertia erit 4$^4/_{15}$. Deinde inter summam primae & tertia 19$^4/_{15}$ & CB 15 primam, media erit AC, 17.

De logarithmis partium, & numerorum integrorum quibus partes sunt adiunctae, vide Cap.10.

4. *Datis Crure & Area : quaeruntur crus reliquum & Hypotenusa.*

Area trianguli est semissis rectanguli comprehensi a cruribus. per.p.41l.1, Eucl. Si igitur datum crus aream duplicatam diviserit, quotus erit crus reliquum , & per precedentem sectionem, datis duobus cruribus, invenitur Hypotenusa.

Ut sunto datum crus 7, area 84, quaeritur crus reliquum.

[p.54.]

			Logarithmi.	
proport.	Divisor	7	084509804	
	Unitas	1	000000000	
	Dividendus	168	222530928	
	Quotus	24	138021124	crus quaesitum 24

5. *Datis Area & summa vel differentia crurorum, invenire crura.*

oblongum a cruribus comprehensum (cuius per 41.p.1.li.Eucl. aequatur area duplicata), una cum quadrato semissis differentiae crurum, aequatur quadrato semissis summae crurum. per 5.p.2.lib.Eucl.

Idcirco si datae areae duplicatae addatur quadratum semissis differentiae crurum, totum erit quadratum semissis summae eorundem.

Sin e quadrato semissis summae crurum auferatur area duplicata, restabit quadratum semissis differentiae eorundem.

Ut sunto Area 60, summa crurum 23, quadratum semissis summae132$^{1}/_{4}$, a quo auferatur 120, area duplicata, restabit 12$^{1}/_{4}$: cuius latus 3½ est semissis differentiae crurum eruntque 15. 8.

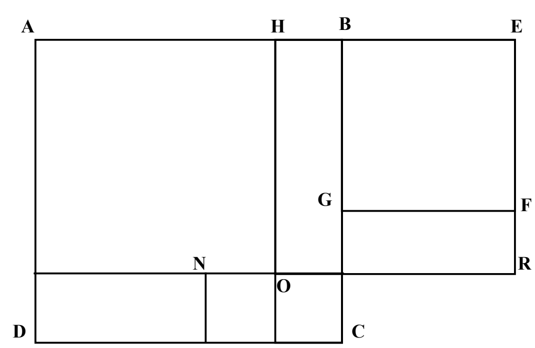
11½ semissis summae crurum,
3½ semissis differentiae crurum,
15 } crura
8 }

6 *Datis hypotenusa & summa vel differentia crurum:quaeruntur crura.*

Quadratum Hypotenusae aequatur quadratis crurum per 47.p.1.lib.Eucl. Data idcirco duorum quadratorum summa, una cum summa vel differentia laterum; quaeruntur latera.

Si data fuerit laterum summa : *auferatur quadratum semissis summae laterum e semisse summae quadratorum; reliquum erit quadratum semissis differentiae laterum,* ut sit Hypotenusa 26, summa crurum 34, erit summa quadratorum e cruribus 676. semissis summae crurum 17, eius quadratum 289: quo ablato e semisse summae quadratorum 338, restabit 49 , cuius latus 7 est semissis differentae crurum. erunt igitur crura 24, 10.

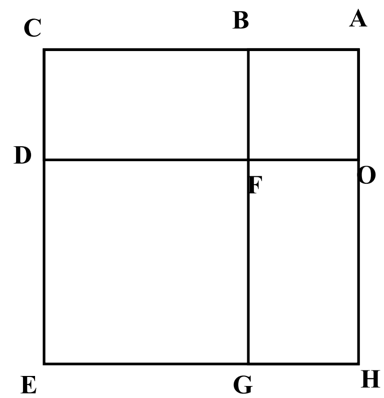
Sin data fuerit laterum differentia: *quadratum semissis differentiae, ablatum e semisse summae quadratorum, relinquit quadratum semissis summae laterum.* Ut in eodem exemplo semissis differentiae laterum 7, eius quadratum 49, ablatum e semisser quadratorum 338; relinquit 289, quadtatum semissis summae laterum 17. erunt igitur crura quaesita, ut antea 24 & 10, quae ita poterunt demonstrari. Sunto AB 9, BE 5, eorum quadrati AC 81, EG 25; quadratorum summae erit 106. Semissis summae laterum erit HE 7, cuius quadratum HR 49, quod duplicatum est AR, 98. Semissis differentiae laterum est HB, 2, cuius quadratum duplicatum est NC, 8. Cum autem GR oblongum 10, quod includitur in oblongo AR; aequatur oblongo DN ; manifestum est NC, duplicatum quadratum rectae HB, esse differentiam datorum quadratorum106, supra AR 98, duplicatum quadratum rectae EH. idcirco quadratum HR 49, & quadratum OC 4, aequantur semissi summae datorum quadratorum 53.



7. *Datis Hypotenusa & Area: quaeruntur crura.*

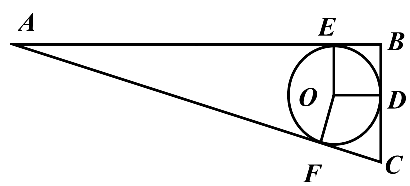
Area duplicata aequatur oblongo a cruribus comprehenso, per 41.p.1.lib. Eucl. 7 Quadratum hypotenusae aequatur quadratis crurum per 47.p.1.lib.Eucl. Dantur igitur Summa quadratorum & Oblongum cruribus comprehensum. At huiusmodi oblongo duo, cum quadrato differentiae laterum, aequantur summae

[p.55.]



quadratorum, per 7.p.2.lib.Eucl. Idcirco si e quadrato Hypotenusae auferatur quaduplum Area; restabit quadratum differentiae crurum. & per precedentem sectionem, *data Hypotenusa & differentia crurum, inveniuntur crura.* Ut sit Hypotenusa 13, Area 30. oblongum a cruribus comprehensum sit AD vel BH 60, & sint crura AC, AB. Summa quadratorum erit 169. Huic summae aequantur oblonga duo AD, BH, cum quadrato rectae BC differentiae laterum AC, AB. Auferatur AD, BH 120, e summa quadratorum AE, OB 169: restabit quadratum GD, 49, cuius latus BC 7, est differentia laterum.

8. *Datis Perimetro & Area invenire Latera.*



*Area trianguli aequatur facto a semiperimetro ducta in radium circuli inscripti:*ut ostendi 3 sectione, Cap. 18.

Et in triangulo rectangulo, *diameter inscripti circuli, aequatur differentia Hypotenusa & summae crurum*. Ut in triangulo ABC, EB, DB aequantur radii EO, OD. Idcirco, si semiperimeter diviserit Aream, quotus erit semidiameter OE. & ablatis, EB, BD diametro aequalibus, e tota perimetro, semissis reliqui erit AFC Hypotenusa , vel AE, DC. Dantur igitur Hypotenusa, & summa crurum: & per 6 sectionem inveniuntur crura.

Ut esto Trianguli rectanguli ACB perimeter 60, & Area 120: si semiperimeter aream diviserit, quotus erit EO 4, radius inscripti circuli. & ablatis 8 e perimetero, restabunt 52. cuius semissis 26 est Hypotenusa AC, & summa crurum AB, BC erit 34.