§18.1. ***Synopsis: Chapter Eighteen.***

Three variations of the Pythagorian Triplet are presented, based on 3 numbers in continued proportion – by means of which right-angled triangles can be constructed, both for sides of lengths either integer or rational, and hence any required triangle.

Subsequently, a number of well-known theorems on solving triangles are presented numerically: the use of logarithms is shown to ease the burden of calculation of the length of an unknown side, the altitude, area, in-radius, etc., of a triangle.

§18.2. *Chapter Eighteen.* [p.46.]

From the given sides of a plane triangle: to find the area, altitude, diameters of the inscribed & circumscribed circles, and any angle.

Before I approach this proposition, I should say a few things beforehand about right angled triangles; as we are able with the help of these to judge more correctly about the truth and certainty of these propositions which are to be presented.

Proclus [410 - 485], in the 47th proposition of the 1st book of Euclid, propounded two methods, by means of which: *with either leg given about the right angle, we can find the other leg and the hypotenuse*; just as in the works of Ramus [Pierre de Ramie, 1515 - 1572], we have the same conclusions 5e, Book 12. Of these methods, one is attributed to Plato, the other to Pythagoras, who were able to propound propositions in general, without distinguishing between cases involving odd and even numbers, such as : *If half the length of the side, &c.* And, *If the square of the length of the first given side , &c* [1]. However, I will propound three general propositions (the latter two methods have emanated from the first of these), by means of which: *from any given side adjacent to a right angle, to find as many right angled triangles as you please.* These propositions are of this kind:

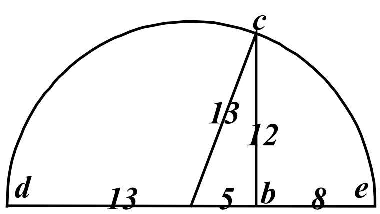
If three straight lines are in continued proportion: the legs of a right angled triangle are:

1. Half the difference of the extremes, and the mean; while the hypotenuse certainly is half the sum of the extremes[2]*,* or

2. The legs are sum of the first & second [proportions]*; the sum of the second and half the third; and the hypotenuse is the sum of the first and second and half the third* [3]*, or*

3. The legs are the difference of the first and second, the difference of the second and half the third; while the hypotenuse is the difference of the second and the sum of the first and half the third [4] .

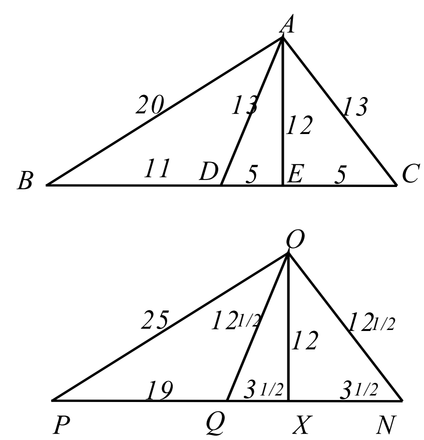
Let the numbers in continued proportion be 8, 12, 18. By the first proposition, the sides sought are 5, 12, 13; by the second proposition they are: 20, 21, 29 and 30, 16, 34; and by the third proposition, they are : 4, 3, 5 & 6, 8, 10. With the continued proportionals 18, 24, 32, the right angled triangles are 7.24.25‖42.40.58| 56.33.65‖6.8.10| 8.15.17 [5].



[Figure 18-1]

The first of these propositions is a logical consequence of this following proposition. In the right angled triangle ABC: the leg BC is the mean proportional between, on either side, the sum BD and the difference BE, of the hypotenuse and the remaining leg AB; and conversely. The second and third propositions are demonstrated from the first. And all of these can easily be shown, as with straight lines geometrically, so also by expressing the measures of the lengths by numbers. Three numbers in continued proportion are found by division. And through logarithms, Axiom 3, Ch. 2. If any number divides the square of the other: the root of the number divided is the mean proportional between the divisor and the quotient. Let the given leg of the right angle be 12, of which the square is 144. Let the divisors be 2, 3, 4, 5, 6, 7, 8, 9, 10, 11. The quotients are 72,48, 36, 28$\underline{8}$, 24, $20^{4}/_{7}$, 18, 16, 14$\underline{4}$, $13^{1}/_{11}$. As you see here * [Table 18-1] The mean is everywhere 12, the extremes are the divisors and the quotients. The numbers in column D are half the sums, in column S half the differences. These columns give us 19 right angled triangles of which the given leg is 12, the remaining leg is had in

[p.47.] column S, certainly with the hypotenuses from the



[Figure 18-2]

directly opposite side in column D. As A 12, 16, 20; B 12, 9, 15; C 12, 5, 13; E 12, $3\underline{5}$, $12\underline{5}$.

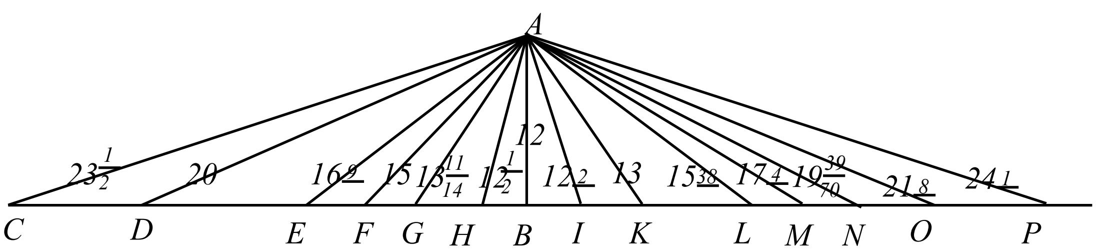
[See Table 18-1.]

To be divided * 144		DD	SS	D	S
2	72	74	70	37	35
3	48	51	45	$25\underline{3}$	$22\underline{5}$
4	36	40	32	20	16 *A*
5	$28\underline{2}$	$33\underline{8}$	$23\underline{8}$	$16^{9}/_{16}$	11
6	24	$20^{4}/_{7}$	$20^{4}/_{7}$	15	9 *B*
7	$20^{4}/_{7}$	$27^{4}/_{7}$	$13^{4}/_{7}$	$13^{11}/_{14}$	$6^{11}/_{14}$
8	18	26	10	13	5 *C*
9	16	25	7	$12\underline{5}$	$3\underline{5}$ *E*
10	$14\underline{4}$	$24\underline{4}$	$4\underline{4}$	$12\underline{2}$	$2\underline{2}$
11	$13^{1}/_{11}$	$24^{1}/_{11}$	$2^{1}/_{11}$	$12^{1}/_{22}$	$1^{1}/_{22}$
14	$10^{2}/_{7}$	$24^{2}/_{7}$	$3^{5}/_{7}$	$12^{1}/_{7}$	$1^{6}/_{7}$
15	$9\underline{4}$	$24\underline{6}$	$5\underline{4}$	$12\underline{3}$	$2\underline{7}$
20	$7\underline{2}$	$27\underline{2}$	$12\underline{2}$	$13\underline{6}$	$6\underline{4}$
25	$5\underline{76}$	$30\underline{76}$	$19\underline{24}$	$15\underline{38}$	$9\underline{62}$
30	$4\underline{8}$	$34\underline{8}$	$25\underline{2}$	$17\underline{4}$	$12\underline{6}$
35	$4^{4}/_{35}$	$39^{4}/_{35}$	$30^{31}/_{35}$	$19^{39}/_{70}$	$15^{31}/_{70}$
40	$3\underline{6}$	$43\underline{6}$	$36\underline{4}$	$21\underline{8}$	$18\underline{2}$
45	$3\underline{2}$	$48\underline{2}$	$41\underline{2}$	$24\underline{1}$	$20\underline{9}$
50	$2\underline{88}$	$52\underline{88}$	$47\underline{12}$	$26\underline{44}$	$23\underline{56}$
Divisors	Quotients	Sum of extremes	Diff. of extremes	Hypot.	Legs

[Table 18-1]

BC	$22\frac{1}{2}$
BD	16
BE	$11\underline{9}$
BF	9
BG	$6\frac{11}{14}$
BH	$3\underline{5}$
BI	$2\underline{2}$
BK	5
BL	$9\underline{62}$
BM	$12\underline{6}$
BN	$15\frac{31}{70}$
BO	$18\underline{2}$
BP	$20\underline{9}$

[Table 18-2.]



[Figure 18 -3.]

And if two right-angled triangles of this kind are placed thus, so that both sides have the common leg; we have an oblique-angled triangle of the given sides, of which the area can be found from the rational lengths. As [in Fig. 18-2] *BAE, CAE* constitute *BAC, BAD*; in the same way, *OPX, OXN* give *PON, OPQ*.

1. We find the area of a triangle of the given sides by the method of Ch. 9, Book 12 of Ramus. If from half of the combined lengths of the sides, the lengths of the sides are taken away one by one: the square root of the continued product of half perimeter and the factors from the subtractions is the area of the triangle[6]. Let the three sides be 20, 11, 13. [Δ ABD in Fig. 18-2 above, upper

figure], or from the combined sides 44, the semi-perimeter is 22; with the numbers 2, 11, 9 remaining from the subtraction of the sides from the semi-perimeter, the continued product emerges from these four numbers by multiplication. But the logarithm of this product is equal to the sum of the logarithms of the factors by Axiom 2, Ch. 2. Therefore the logarithms are taken which must be for the semi-perimeter and the remaining numbers. The sum of which is the logarithm of the continued product: & half the sum is the logarithm of the square root, or the area sought. For

				Logarithms
20	sides of the triangle	semi-perimeter	22	1,34242,26808
11		numbers remaining	2	0,30102,99957
13			11	1,04139,26852
44	perimeter		9	0,95424,25094
			Product	3,63908,78711
			Area 66	1,81954,39355

[Table 18-3]

[p.48.]

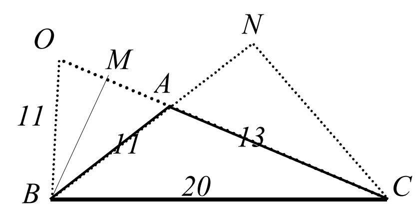
Hence the area of the given triangle is 66 [Fig. 18-2]. We find the same size of area if we multiply the altitude of the triangle AE 12 by halve the base $5^1/_2$, because the rectangle from the whole base and altitude is double the area of the triangle by Prop. 41, Book 1 of Euclid.

2. By finding the logarithm of the area, we can find the altitude of the same triangle, or the perpendicular from any angle to the opposite side: if from the logarithm of the area is taken away the logarithm of half the side on which the perpendicular from the opposite angle is incident, the remainder is the logarithm of the altitude, by Axiom 2, Ch. 2.

It is indeed equal to the product of the base halved by the altitude, and therefore the logarithm of the area is equal to the logarithms of the base halved and the altitude.

	The logarithm	
of the area 66	1,81954,39355	
of half the base $6^1/_2$	0,81291,33566	
of the altitude $10\ ^2/_{13}$	1,00663,05789	10<u>1538461</u>

[Table 18-4]



[Figure 18-4]

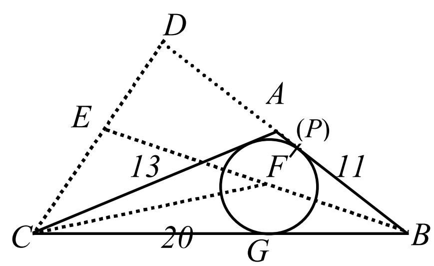
As from the angle B with the base continued, the perpendicular BM shall fall [Fig. 18-4], the logarithm of which is 1,00663,05789, certainly BM itself is nearly $10\underline{1538461}$, or $10^2/_{13}$.

We find the same altitude from geometry, thus[7]: the difference is taken of the squares of the sides BA, BC, comprising the angle ABC, from which the perpendicular falls on the base CA continued if necessary, as here. This difference 279 is divided by the base CA 13, the quotient $21^6/_{13}$ will be CO, the sum of the base segments from the angles O and C, or A and C [if M lies within AC], to the point M, on which the perpendicular falls: from this sum CO, the given base CA 13 is taken away, which is the difference of the segments, leaves AO $8^6/_{13}$ the double of the smaller segment AM $4^3/_{13}$. Then by taking away the square of AM, $17^{152}/_{169}$ from AB squared, 121 (or CM squared from CB squared) there remains $103^{17}/_{169}$, the square of the right line required BM, $10^2/_{13}$.

3. The radius of the inscribed circle is found thus: the logarithm of the semi-perimeter is taken away from the logarithm of the area; the logarithm of the radius required is left. As in the triangle with the sides 20,13,11.

Logarithm of the area found	1,81954,39355	66
semi-perimeter	1,34242,26808	22
radius of inscribed circle for the triangle	0,47712,12547	

[Table 18-5]



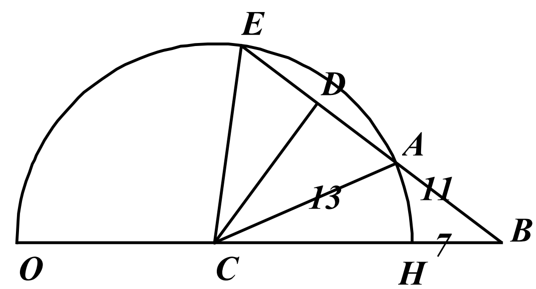
[Figure 18-5]

I assert that 3 is the radius of the inscribed circle.

Indeed the product of the radius and the semi-perimeter is equal to the area of the given triangle . Because with straight lines drawn from the centre F to the angles A, B, & C, the given triangle is divided into 3 triangles: of which the altitudes are equal to the radius of the circle, but the bases are the sides of the given triangle [Fig. 18-5: FA is not drawn]. And with the common altitude FG taken by the half of each of the sides, that is by the semi-perimeter, the products are the areas of the component triangles AFC, AFB, and BFC [note: AFC is written again instead of BFC in the original, and P is omitted from the diagram]. That is, the area of the given triangle.

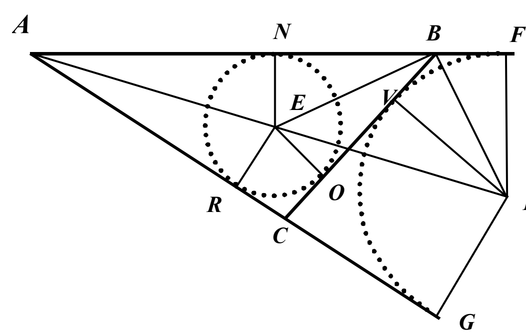
We will find the radius geometrically, thus: the right line CD is drawn perpendicular to the opposite line BA [Figure 18-5]. BD, 16. & CD, 12 are found. Then BFE is drawn bisecting the angle CBD. The centre F lies on the bisecting line, by Prop. 4, Book 4, Euclid. Thirdly, DE is

[p.49.] found: which by Prop. 3, Book 6, Euclid, thus DE itself has the ratio to EC, as DB is to BC. & therefore, as the sum of the sides CB, BD 36, to BD 16; thus the total CD 12, to ED $5^1/_3$. Fourthly, the line BP is sought, from the angle B, to the point of contact P; on which the radius FP falls, perpendicular to the line BA. But BP is 9. For with all the triangles, if whatever line AC 13 is taken away from the semi-perimeter 22, 9 remains, the distance of the opposite angle B from the points of contact G and P [8]. Finally, by the rule of proportions, I seek PF. As BD 16 to DE $5^1/_3$; so BP 9 to PF 3.



[Figure 18-6]

But we can find AD [in Fig. 18-5], either by that which preceded in the second section of this chapter, or just as easily by this method. With centre C and radius CA the semi-circle O E A H is described, & by Prop. 37, Book 3, and Prop. 16, Book 6 Euclid; AB 11, BH 7, BO 33, BE 21 are lengths in proportion. And by subtraction BA 11, AE will be 10, & AD 5 [9].



[Figure 18-7]

It is possible also to find the inscribed radius of the circle by logarithms, thus. By taking away the lengths of the given lines from the semi-perimeter, the logarithms of the remainders are taken, & from the sum is taken away the logarithm of the semi-perimeter: the logarithm of the remaining half is of the radius required[10].

Thus, we can explain this as follows: The triangle ABC are constructed with the given sides AB, 20. BC, 11. AC, 13. & AB, AC, are produced to F & G, thus as AF 22 is equal to the semi-perimeter and the right line AG, & the angles AFD, AGD are right; & with the centre D, radius DF the arc GVF is described touching the lines produced to F and G. The same arc by necessity touches the right line CB at the point V: with CV and VB are equal by the construction to CG, BF. (For if CB cuts the periphery, GVF is larger than CG & BF: if it does not touch the periphery then it is less than CG & BF.) Then with the centre E, the circle NOR is inscribed to the given triangle. AN & AR are 11; BN & BO 9; while CO, CR, BV & BF are 2. And ENB, BFD are similar triangles, because the angles at N and F are right, & both NBO & OBF are equal two right angles by Prop.13, Book 1, Euclid. & NBE, FDB taken together are worth the half of two right angles: and therefore the angles FBD, NEB, likewise the angles NBE and BDF are equal. Therefore, EN, NB: BF, FD are in proportion, and the rectangles EN by FD and NB by BF, equal. But also FA, NA: DF, EN are in proportion; and as DF to EN, thus the rectangle DF, EN to the square EN; & therefore as FA to NA, thus the rectangle EN, FD to the square EN [11].

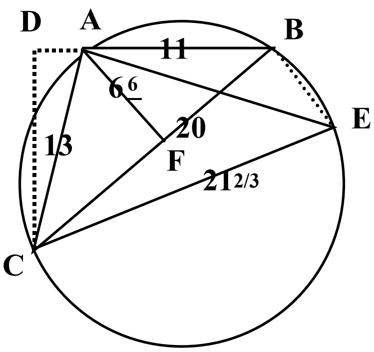
Therefore if the logarithm FA 22 is taken away from the logarithms of NA 11 & rectangle NB,BF; the logarithm of the square EN is left: of which half is the logarithm of the radius EN. The same comes about if the complement of the first logarithm is taken, etc. As we showed in Chapter 15.

[p.50.]

				Logarithms
proportions	AF - - - - - - - - - 22		Complementary arithmetic	8,65757,73192
	AN - - - - - - - - - 11			1,04139,26852
	Oblong 18	NB 9		0,95424,25094
		BF 2		0,30102,99957
	Square	EN 9		10,95424,25095 [10 → 0]
		EN 3		0,47712,12547

[Table 18-6]

Hence undoubtedly, 198 by the continued product from these remainders is equal to the product from the semi-perimeter 22 in the square of the radius EN 9 by Prop.19, Book 7, Euclid . And if the semi-perimeter 22 is multiplied with these, 4356 is the continued product from AF, AN, NB, BF, which is equal to the continued product from AF, AF, EN, EN, the product from the square of AF into the square of EN[12]. And if a square is multiplied by a square, the product is a square: and the square root produced is a factor from the same square. And therefore 66 the root of the continued product of the semi-perimeter and the subtracted lengths, is the product of the semi-perimeter into the radius of the circle: which is always equal to the area of the triangle.



[Figure 18-8]

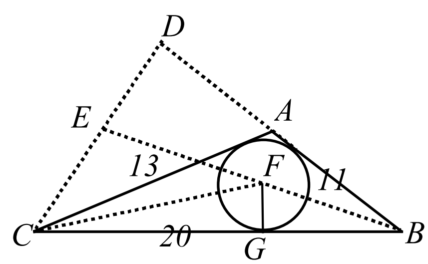
4. The diameter of the circum-circle is found thus. The logarithm of the perpendicular from any vertex to the opposite side is taken away from the logarithms of the sides of the comprised angle, the remaining term is the logarithm of the diameter sought. For in any triangle ABC, the two sides AC, CB are the mean proportionals between the perpendicular CD, and the diameter of the circum-circle CE. Because the angles DAC, BEC are equal by Prop. 22, Book 3, & Prop. 14, Book 1, Euclid, & the angles CDB, CBE are right: therefore the remaining angles DCA, BCE are equal & triangles DCA, BCE are similar & therefore DC, CA: BC, CE are in proportion. Or if the interior perpendicular seems to be better suited, the proportionals will be FA, AB: AC, CE. [Using ΔABF and ΔCAE].

	CD.	12 Compl. Log	8.92081,87539
Proportions	CA.	13 Logarithm	1,11394,33523
	CB.	20 Logarithm	1,30102,99957
21666666	CE	216666666666	(1)1,33579,21019 the whole sum.

CE Diameter $21^2/_3$; see page 37 concerning complementary arithmetic.

[Table 18-7]

5 If we wish to know any angle, the side opposite the same is taken away from the semi-perimeter: the remainder is the segment of one or other of the legs between the same angle & the point in which the circumference of the inscribed circle touches the line. But this segment is to the radius of the inscribed circle, as the radius of the circle to the tangent of half the



[Figure 18-9]

angle sought[13]. As in triangle ABC: by taking the line CA 13 from the semi-perimeter 22, BG or BP 9 is left. I assert.

			Logarithms	
Proport. -ions	BF	9 compl. arith.	904575749	
	GF	3	047712125	
	Radius	100000000000	10000000000	
Tangent	GBF	333333333333	1 | 952287874	18:26:06

Therefore the angle GBF is 18:26:06. & the whole angle GBA 36:52:12. By the same method the remaining angles at C and A can be found.

			Logarithms	
Proport- -ions	CG	11 compl. arith.	8958607314	
	GF	3	047712125	
	Radius	100000000000	10000000000	
Tangent	GCF	272727272727	1 | 9435728569	15:15:18
			Logarithms	
	AP	2 compl. arith.	9698970004	
	PF	3	0477121255	
	Radius	100000000000	10000000000	
Tangent	PAF	150000000000	20176091259	56:18:12
Angles	ACB	30:30:36		
	CAB	112:37:12		
	CBA	36:52:12		
		180:00:00		

[p.51.]

[Table 18-8]

We can also find any angle otherwise, through the altitude of the triangle found, by section 4 of this chapter, [Figure 18-8]. It is by means of the perpendicular from any angle to the opposite side. For as the common leg of the vertical angle & the angle sought, to the perpendicular of the same vertical angle ; so the total sine to the sine of the angle sought[14]. For in the same triangle, if the angle CBA is required: they are

				Logarithms	
proport -ion	CB common leg	20	compl. arith.	8698970004	
	perpendicular	12		1079181246	
	Sine Total	10000000000		10000000000	
	Sine CBA	6000000000		**1**| **9**778151250	36:52:12

[Table 18-9]

Or, with the base BD made continuous, (in proportion) to the side DC: so the same radius of the circle, to the tangent of the angle required DBC.

				Logarithms	
proport.-ion	BD	16	compl. arith.	87958800173	
	CD	12		10791812460	
	Radius	10000000000		100000000000	
	Tangent DBC	7500000000		1| 98750612633	36:52:12

[Table 18-10]

Or, with the base BD made continuous, (in proportion) to the line BC: so the radius of the circle, to the secant of the angle DBC sought .

				Logarithms	
proport.-ion	BD	16	compl. arith.	87958800173	
	BC	20		13010299957	
	Radius	10000000000		100000000000	
	Secant DBC	12500000000		200969100130	36:52:12

[Table 18-11]

Or, for seeking the opposite angle: as the diameter CE of the circumscribed circle to CA, so the radius of the in-circle, to the sine of the angle sought .

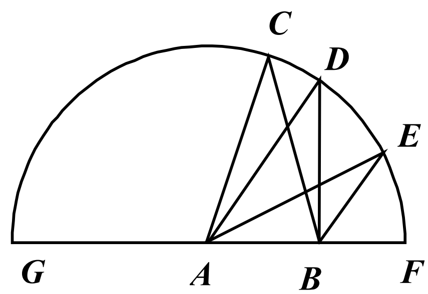
				Logarithms	
proport-ion	CE	$21^2/_3$	compl. arith.	86642078980	
	CA	13		11139433523	
	Radius	10000000000		100000000000	
	Sine ABC	6000000000		197781512503	36:52:12

[Table 18-12]

But since any angle and its complement of two right angles have the same sine, as 63:0. & 117:0, then in order that we can be certain that the angle sought is acute or obtuse, it is noted that the two angles of any triangle opposite the smaller sides are always acute: but the third angle, that the largest side subtends, can sometimes be acute, or sometimes right or obtuse. Therefore in order that we know what kind of angle the largest side is placed opposite, the largest sum and difference is taken of the sides, and the mean (or the smallest) logarithm of these: and if the half of these logarithms is equal to the logarithm of the third side, the maximum angle of the opposite side is right. But if the logarithm of the third side is more than half, the angle will be acute: but if less, obtuse. [A result that we would now relate to the cosine rule]. *As in the triangles CAB, DAB, EAB,* [Fig. 18-10] *let the sides CA, DA, EA, be of 10 parts: AB, 8: BC, 7: BD, 6: BF, 5: GB is the sum of* [p.52.] *the sides from the angle A, 18: BF the difference of the same sides 2, half the sum of the logarithms 077815125 is equal to the logarithm of the side BD, & therefore BD is by Ch. 17 the mean proportional between GB & BF, & the angle ABD is right, by Prop. 13, Book 6, Euclid. But the logarithm of the line CB is more than half the sum of the logarithms, & ABC is acute. On the other hand, ABE is obtuse, & the logarithm of the line BE is less than half of the same.*

		Logarithms
sum of the sides GB	18	1,25527,251
difference of the sides BF	2	0,30102,999
sum of the logarithms		1,55630,250
half of their sum		0,77815,125
BC	7	0,84509,804
BD	6	0,77815,125
BE	4	0,69897,000

[Table 18-13]



[Figure 18-10]

These sines, tangents and secants, with the aid of which the angles are found, are found in the works of Palatine, or elsewhere from a Canon of Triangles as you please. Or if we are satisfied with the logarithms themselves, the Canon of Triangles by Edm. Gunther recently published by us will make amends abundantly; or that which I have prepared, [which was posthumously published as the *Trigonometria Britannica*] *& sometime I hope to be given the light by me; the same previous book will be better in both ways, both for the sines and tangents themselves, and for the logarithms of the same. However, the whole sine* [i.e. the sine of 90^0] *is understood to be written with eleven places, of which the logarithm is 10* [comp. arith.] *in the place of the characteristic, as we have shown in Ch. 4. Thus the rest of the sines are set out all the way to 5^0 45 minutes. & the tangents from 45^0, 0 min to 5^0,43 min, have 9 for the characteristic. Beyond this all the way to 0^0,35 min, the characteristic of these is 8. But for the secants as far as 84^0, 16 min., & the tangents from 45^0, 0 min to 84^0, 18 min, the characteristic is 10. And beyond those limits, nearly as far as the end of the quadrant, 11.*

§18.3. *Notes on Chapter 18.*

1 See Heath, Vol. 1, *Euclid's Elements*, e.g. (Dover) p. 104, for notes on the commentaries on *The Elements* by these authors mentioned by Briggs, and other early writers.

2 Briggs sets the lengths in continued proportion to be $a, b, c,$ where $a/b = b/c$. The mean proportional is b, while a and c are the extremes. The triangle has sides of length $(c - a)/2$, $\sqrt{(ac)}$, and $(a + c)/2$; by writing c as b^2/a and removing fractions, it is an easy matter to show that this set of numbers is equivalent to the Pythagorian Triplet:

$(b^2 - a^2, 2ab, b^2 + a^2)$, with $b > a$.

We may also note that Briggs favours using the Theorem of Pythagoras in the proportional form: $(c - a)/b = b/(c + a)$, where (a, b, c) is a Pythagorean Triplet.

3 The triplet is now $a + b$, $b + c/2$, $a + b + c/2$, where a, b, c are as above: in a similar manner, this set of lengths of sides is equivalent to the triplet $((a + b)^2 - a^2, 2a(a + b), (a + b)^2 + a^2)$.

4 The triplet is now $b - a$, $c/2 - b$, $a + c/2 - b$: This triplet being equivalent to the triplet: $(b - a)^2 - a^2, 2a(b - a), (b - a)^2 + a^2)$, where $b > a$.

There is hence nothing extra gained from propositions 2 and 3: there is obviously an indefinite number of equivalent ways of stating the Pythagorian Triplet. For, by starting from $(m^2 - n^2, 2mn, m^2 + n^2)$, $m > n$ as the basic triplet, and inserting simple sums and differences of a and b, other triplets can be produced. E.g. letting $m = a + b$ and $n = b$ gives eventually $b + c, b + a/2, a/2 + b + c$, etc.

5 The proportionals are allowed to increase or decrease, so that 32, 24, 18 is used to construct the 3rd and 5th triplets from the second set of ratios, while 18, 12, and 8 are used in the first to generate the extra triangles.

6 From the half-perimeter s, where $2s = (a + b + c)$ is taken the length of side a to give the factor $(s - a)$: the square root of the factors by continuation from the half perimeter and the subtractions of the 3 sides will be the area of the triangle. This is Heron's formula $A = \sqrt{[s(s - a)(s - b)(s - c)]}$.

[7] This follows readily, Fig. 18-4, from OC.CA = (CM + OM)(CM - OM) $= (BC^2 - BM^2) - (AB^2 - BM^2) = BC^2 - AB^2$, etc.

[8] In modern standard notation, where a, b, and c are the lengths of the sides of the triangle opposite the angles A, B, and C, if the lengths of the tangents from A, B, and C to the in-circle are x, y, and z, where $2s = a + b + c$ then $s = x + y + z$ and $x = s - a$, $y = s - b$, and $z = s - c$.

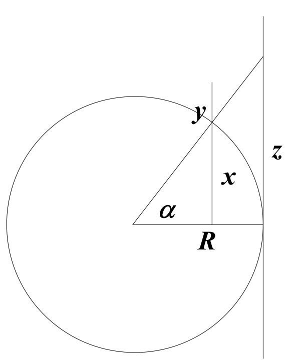
[9] According to Euclid, BA.BE = BO.BH (Prop. 37, Book 3), where BE is the unknown. The other proposition (Prop. 16, Book 6), relating the products of the diagonals to the sum of the products of the opposite sides of a cyclic quadrilateral, does not seem to have been used here.

[10] Briggs makes use of the standard result $A = rs = \sqrt{[s(s - a)(s - b)(s - c)]}$.

[11] Briggs is commenting that the similar triangles NEB and FBD give NE/BF = NB/FD, or EN.FD = BF.NB, and as NE and FD are parallel, the triangles ANE and AFD are also similar, with equal ratios of sides. Finally, $FA/NA = FD/EN = [BF.NB/NE]/NE = (BF.NB)/EN^2$ $(= DF.EN/EN^2)$. The reason for doing this is to produce via geometry the lengths FA, NA, NB, and BF that correspond to s and the differences $(s - a)$, $(s - b)$, and $(s - c)$ respectively, used to find EN = r from $s/(s - a) = (s - b)(s - c)/r^2$

[12] i.e. the factors are $s - a$, $s - b$, and $s - c$, where $(a, b, c) = (11, 13, 20)$; $s = 22$, and $(x, y, z) = (11, 9, 2)$ are the lengths of the tangents; 11.9.2 being the product of half the perimeter into the square of the radius EN 9: $r^2 s = (s - a)(s - b)(s - c)$, by Prop. 19, Book 7, Euclid.

[13]



The tangent was defined at the time as the ratio of z to R, where R = 10000000000 in the case of Briggs' tables.

The sine was the ratio of x to R, while the secant was defined by the ratio R to y.

[14] i.e., from the sine rule, BC/CD (or AB/AF) = $\sin(\pi/2)/\sin(CBA)$ in Figure 18-8.

§18.4. **Caput XVIII.** [p. 46.]

Datis trianguli plani lateribus: invenire Area, Altitudinem, Diametros circulorum inscripti & circumscripti, & quemlibet angulum.

Antequam hanc propositionem aggrediar, liceat pauca de Triangulo rectangulo praefari; ut eius ope possimus, de veritate & certitudine eorum quae tradentur rectius iudicare.

Proclus ad quadragesimam septimam propositionem primi libri Euclidis, duos modos tradidit, quibus possimus, *dato altero crure circa rectam angulum, invenire crus reliquum & hypotenusam*. Horum unus tribuitur Platoni, alter Pythagorae, eosdem habemus apud Ramum tanquam consectaria 5.e.12.lib. qui sine distinctione numeri paris & imparis, generalius proponi potuissent. *Si dimidius numeri pro crure, &c.* Et, *Si quadratus numeri pro crure primo dati, &c.* Ego vero tres tradam generales propositiones (ex quarum prima isti modi fluxerunt) per quas poterimus *dato quolibet latere circa rectum, invenire quotlibet triangula rectangula.* Hae sunt huiusmodi.

Si tres recta sunt continue proportionales: erunt trianguli rectanguli.

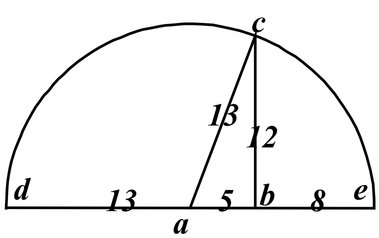
1. *Crura; semissis differentia extremarum, & media; hypotenusa vero erit semissis summa extremarum.*
2. *Crura; prima & secunda, secunda & semissis tertia; hypotenuse vero prima & secunda & semissis tertia.*
3. *Crura; Differentia prima & secunda, Differentia secunda & semissis tertia; hypotenusa vero erit Differentia secunda & aggregati e prima & semisse tertia.*

Ut sunto continue proportionales 8, 12, 18. per primam erunt latera quaesita 5, 12, 13; per secundum, 20, 21,& 29, & 30, 16, 34; per tertiam 4, 3, 5 & 6, 8, 10.

Continue proportionales 18, 24, 32, triangula rectangula erunt

7.24.25‖42.40.58|56.33.65‖6.8.10|8.15.17 ·

Harum prima est consectarium huius sequentis proportionis. In triangulo rectangulo ABC: crus utrumlibet BC est medium proportionale inter BD summam, & BE differentiam hypotenusae AC, & reliqui cruris AB & contra. Secunda & tertia demonstrantur per primam, atque haec omnia, tam in lineis rectis, quam in numeris, linearum mensuras exprimentibus, ostendi poterunt facillime. Numeri tres continue proportionales per divisionem inveniuntur. & per Logar.3.ax.c.2. Si numerus quilibet quadratum alterius diviserit: latus divisi est medium proportionale inter divisorem & quotum.

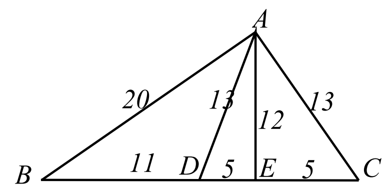


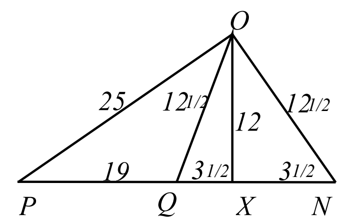
Esto datum crus anguli recti 12, eius quadratum144. Sunto divisores 2, 3, 4, 5, 6, 7, 8, 9, 10, 11. Quoti erunt 72,48, 36, 28, 24, $20\,^{4}/_{7}$, 18, 16, 144, $13\,^{1}/_{11}$. ut hic vides *Medius ubique est 12, extremi sunt divisores &quoti. Columna *D* sunt semisses summarum, columna S semisses differentiarum, hae columnae dant nobis 19 triangula rectangula: quorum crus datum est 12, reliquum crus habetur in columna S, hypotenusa vero e regione in columna D. ut A 12, 16, 20; B 12, 9, 15; C 12, 5, 13; E 12, 35, 125.

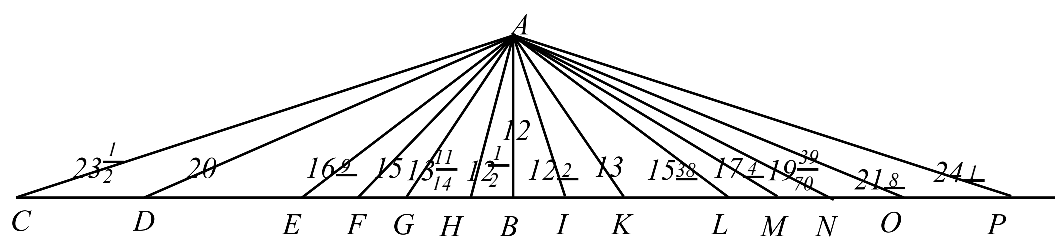
[p.47]

Dividendi * 144		DD	SS	D	S
2	72	74	70	37	35
3	48	51	45	$25\underline{3}$	$22\underline{5}$
4	36	40	32	20	16 *A*
5	$28\underline{2}$	$33\underline{8}$	$23\underline{8}$	$16^{9}/_{16}$	11
6	24	$20^{4}/_{7}$	$20^{4}/_{7}$	15	9 *B*
7	$20^{4}/_{7}$	$27^{4}/_{7}$	$13^{4}/_{7}$	$13^{11}/_{14}$	$6^{11}/_{14}$
8	18	26	10	13	5 *C*
9	16	25	7	$12\underline{5}$	$3\underline{5}$ *E*
10	$14\underline{4}$	$24\underline{4}$	$4\underline{4}$	$12\underline{2}$	$2\underline{2}$
11	$13^{1}/_{11}$	$24^{1}/_{11}$	$2^{1}/_{11}$	$12^{1}/_{22}$	$1^{1}/_{22}$
14	$10^{2}/_{7}$	$24^{2}/_{7}$	$3^{5}/_{7}$	$12^{1}/_{7}$	$1^{6}/_{7}$
15	$9\underline{4}$	$24\underline{6}$	$5\underline{4}$	$12\underline{3}$	$2\underline{7}$
20	$7\underline{2}$	$27\underline{2}$	$12\underline{2}$	$13\underline{6}$	$6\underline{4}$
25	$5\underline{76}$	$30\underline{76}$	$19\underline{24}$	$15\underline{38}$	$9\underline{62}$
30	$4\underline{8}$	$34\underline{8}$	$25\underline{2}$	$17\underline{4}$	$12\underline{6}$
35	$4^{4}/_{35}$	$39^{4}/_{35}$	$30^{31}/_{35}$	$19^{39}/_{70}$	$15^{31}/_{70}$
40	$3\underline{6}$	$43\underline{6}$	$36\underline{4}$	$21\underline{8}$	$18\underline{2}$
45	$3\underline{2}$	$48\underline{2}$	$41\underline{2}$	$24\underline{1}$	$20\underline{9}$
50	$2\underline{88}$	$52\underline{88}$	$47\underline{12}$	$26\underline{44}$	$23\underline{56}$
Divisores	Quoti	Sumae extrem.	Differae extrem.	Hypo-tenusae.	Crura.

Et si duo huiusmodi triangula rectangula ita locentur, ut id sit crus comune utrinque; habebimus triangulum obliqangulum datorum laterum, cuius area erit rationalis: *BAE, CAE* constituunt *BAC, BAD*; eodem modo, *OPX, OXN* dabunt *PON, OPQ*.







1. Aream trianguli datoram laterum inveniemus per 9.c.12.lib.Rami. Si de dimidio collectorum laterum latera sigillatim subducantur: latus continue facti e dimidio & reliquis erit area trianguli. Sunto tria latera 20.11.13 perimeter, seu latera collecta 44, semiperimeter 22, numeri reliqui subductis lateribus e semiperimetro 2.11.9 continue factus provenit ex horum quatuor numerorum multiplicatione, huius autem facti Logarithmus aequatur Logarithmis facientium per 2.ax.2.c sumantur igitur Logarithmi qui semiperimetro & numeris reliquis debentur. horum summa erit Logarithmi qui semiperimetro & numeris reliquis debentur. horum summa erit Logarithmus continue facti:& semissis summa erit Logarithmus lateris seu area quaesitae. Ut

					Logarithmi
20		semiperimeter.		22	1,34242,26808
11	latera trianguli.			2	0,30102,99957
13			numeri reliqui.	11	1,04139,26852
44	perimeter.			9	0,95424,25094
				facti.	3,63908,78711
				Areae 66	1,81954,39355

[p.48] Erit igitur area dati triangulii 66 . Eandem area quantitatem inveniemus si altitudinem trianguli AE 12 multiplicemus per dimidiatam $5^{1}/_{2}$, quia rectangulum e tota base 7 altitudine est duplum trianguli per 41.p.1.lib.Euclid.

2. Invento areae Logarithmo, poterimus altitudinem eiusdem trianguli vel perpendicularem ab angelo quolibet in latus oppositum invenire: si e Logarithmo areae auferatur Logarithmus dimidiati

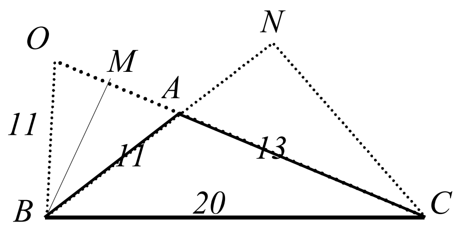
lateris, in quod incidit perpendicularis ab angulo opposito, reliquis erit Logarithmus altitudinis per 2.ax.2.cap.

Est enim aequalis facto e base dimidiata in altitudinem, & idcirco area Logarithmus, dimidiatae basis & altitudinis Logarithmus equabitur.

	Logarithmus.	
areae 66	1,81954,39355	
dimidiatae $6^{1}/_{2}$	0,81291,33566	
altitudinis $10\,^{2}/_{13}$	1,00663,05789	10$\underline{1538461}$

Ut ab angulo B,in basim continuatam, cadat perpendicularis BM, eius Logarithmus erit 1,00663,05789, ipsa vero BM erit10$\underline{1538461}$ proxime, vel $10^{2}/_{13}$.

Eandem altitudinem Geometrice inveniemus sic. Sumatur differentia quadratorum e lateribus BA, BC, comprehendentibus angulum ABC, a quo perpendicularis cadit in basim CA continuatam

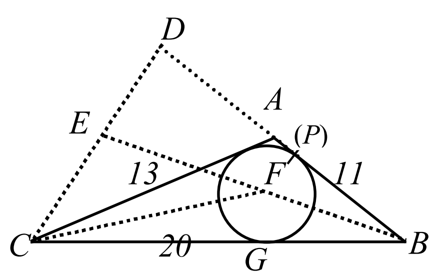


si opus fuerit ut hic, hanc differentiam 279 dividat basis CA 13, quotus $21^{6}/_{13}$ erit CO, summa segmentorum basis ab angulis O, C, vel A , C ad punctum M, in quod incidit perpendicularis :ab hac summa CO, auferatur data basis CA 13 quae est differentia segmentorum, restabit AO $8^{6}/_{13}$ dupla minoris segmenti AM $4^{3}/_{13}$: deinde ablato quadrato AM $17^{152}/_{169}$ e quadrato AB 121 (vel quad. CM e quad. CB) restabit$103^{17}/_{169}$, quadratum rectae BM quaesitae, $10^{2}/_{13}$.

3. Radius circuli inscripti sic invenitur. Auferatur Logarithmus semiperimetri, e Logarithmo areae; restabit Logarithmus radii quaesiti. ut in triangulo cuius latera 20.13.11.

areae inventae Logarithmus found	1,81954,39355	66
semiperimetri	$\underline{1,34242,26808}$	22
radij circuli triangulo inscripti	0,47712,12547	

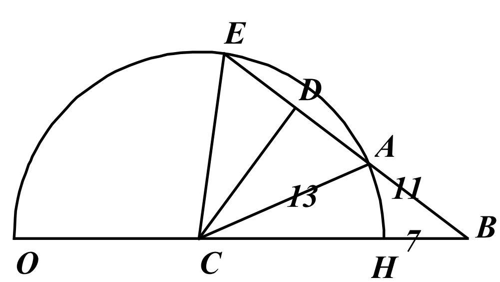
Aio radium inscripti circuli FG esse 3. Est enim factus a radio in semiperimetrum aequalis area



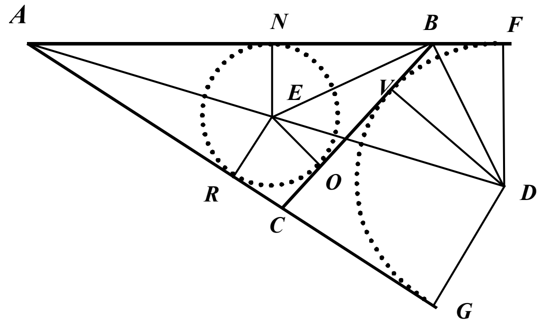
trianguli dati, quia ductis rectis a centro F ad angulos ACB, secabitur datum triangulum, in tria triangula: quorum altitudines aequantur radio circuli, bases autem sunt latera dati trianguli. & communis altitudo FG ducta in semisses singulorum laterum, id est in semiperimetrum, dabit areas triangulorum componentium AFC, AFB, AFC. id est aream dati trianguli.

Geometrice, radium sic inveniemus: ducatur recta CD perpendicularis lateri opposito BA, & quaerantur BD, 16, & CD, 12. deinde ducatur BFE bisecans angulum CBD. centrum F erit in bisecante. per 4.p.4.lib.Eucl. tertio, [p.49.] quaerenda est DE, quae per 3.p.6.lib.Eucl. ita se habet ad EC,ut DB ad BC. & idcirco ut summa laterum CB, BD 36 , ad BD 16; sic CD 12, ad ED $5^{1}/_{3}$. Quarto, quaeritur recta BP ab angulo B, ad punctum contactus P; in quod cadit radius FP, perpendicularis lateri BA. est autem BP is 9. nam in omnibus triangulis si latus quodlibet AC 13 auferatur e semiperimetro 22, restabit 9 distantia anguli oppositi B a punctus contactuum GP. Tandem per proportionis regulam quaero PF. ut BD 16 ad DE $5^{1}/_{3}$; sic BP 9 ad PF 3.

Poterimus autem invenire AD, vel per ea quae praecesserunt in secunda huius Capitis sectione, vel non minus commode ad hunc modum: Centro C radio CA describatur peripheria OEAH, & per 37.p.3.lib. & 16.p.6.lib.Eucl. erunt AB 11. BH11. BH 7.BO 33.BE 21 proportionales. & ablata BA 11, AE erit 10, & AD 5.



Poterit etiam radius inscripti circuli inveniri per Logarithmos, sic. Ablatis lateribus datis e semiperimetro, sumantur Logarithmi reliquorum; & e summa auferatur Logarithmus semiperimetri: semissis reliqui erit Logarithmus radii quaesiti.



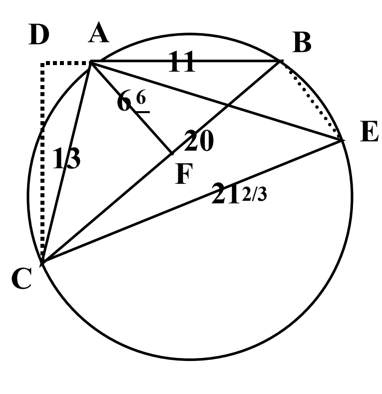
Hoc ita se habere ad hunc modum demonstrare poterimus. Fiat ABC triangulum datorum laterum AB, 20. BC, 11. AC, 13. & producantur AB, AC, ad F & G, ita ut AF sit 22, aequalis semiperimetro & aequalis rectae AG, & fiant anguli AFD, AGD recti; & centro D, radio DF describatur peripheria GVF tangens latera AB, AC producta in F & G. Eadem peripheria necessario continget rectam CB in puncto V: cum CV, VB aequentur ex fabrica rectis CG, BF. (nam CB si secet peripheriam GVF erit maior quam CG, BF: quod si non pertingat ad peripheriam erit minor quam CG,BF.) Deinde centro E, inscribatur dato triangulo circulus NOR . eruntque; AN, AR 11. BN, BO 9. CO, CR, BV, BF, 2. Et ENB, BFD similia triangula, quia anguli ad N & F recti, & NBO, OBF aequalis duobus rectis per 13.p.1.lib.Eucl. & NBE, FDB simul sumti valent dimidium duorum rectorum; & idcirco FBD, NEC, item NBE et BDF aequalia. Sunt igitur EN, NB: BF, FD proportionales & rectangula EN in FD , & NB in BF aequalia. Sunt autem FA, NA: DF, EN proportionales. & ut DF ad EN , sic rectangulum DF, EN; ad Quadratum En. & idcirco ut Fa ad NA, sic rectangulum EN, FD (vel ei aequale NB, Bf) ad Quadratum EN.

Idcirco si Logarithmus FA 22 auferatur e Logarithmis NA 11 & rectanguli NB,BF; restabit Logarithmus quadrati EN : cuius semissis erit Logarithmus radii EN. Idem eveniet si sumatur primi Logarithmi complementum &c. ut cap.15. ostendimus.

				Logarithmi.
proport.	AF - - - - - - - - - 22		Compl. Arith.	8,65757,73192
	AN - - - - - - - - - 11			1,04139,26852
	Oblongum 18	NB 9		0,95424,25094
		BF 2		0,30102,99957
	Quadratum	EN 9		10,95424,25095 [10 → 0]
		EN 3		0,47712,12547

[p.50.]

Hinc manifestum est, 198 continue factum a reliquis 11.9.2 aequari facto a semiperimetro 22 in quadratum radii EN 9. per 19.p.7.l.Eucl. 7 si semiperimeter 22 hos aequales factos multiplicet, erit 4356 continue factus ab AF, AN, NB, BF, aequalis continue facto ab AF, AF, EN, EN, vel facto a quadrato AF in quadratum EN. & si quadratus quadratum multiplicet factus erit quadratus: & latus facti erit factus a lateribus eorundem quadratorum. & idcirco 66 latus continue facti a semiperimetro & reliquis, erit factus a semiperimetro in radium circuli inscripti: qui semper aequatur areae trianguli.



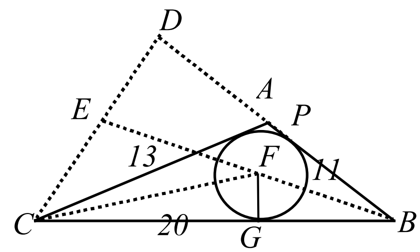
4. Diameter circuli circumscripti sic habetur. Auferatur Logarithmus perpendicularis ab angelo quolibet in latus oppositum, e Logarithmus laterum eundem angulum comprehendentium, reliquus erit Logarithmus diametri quaesiti. Sunt enim in quolibet triangulo ABC; duo latera AC, CB, media proportionalia, inter CD perpendicularem, & CE diametrum circuli circumsvcripti. Quia anguli DAC, BEC aequantur per 22.p.3.lib. & 14.p1.lib.Eucl. & anguli CDB, CBE recti. & idcirco DCA, BCE reliqui aequales: & trianguli DCA, BCE similia. erunt igitur DC, CA : BC, CE proportionales. Vel si interior perpendicularis

magis commoda videatur, erunt FA, AB: AC, CE proportionales.

Prop.	CD.	12	Complem.Log	8.92081,87539
	CA.	13	Logarithmus	1,11394,33523
	CB.	20	Logarithmus	1,30102,99957
21666666	CE	216666666666		(1)1,33579,21019 the whole sum.

CE Diameter $21^2/_3$; vide paginam 37 de compl, arith.

5. Si angulum quemlibet scire velimus, auferatur latus eidem oppositum e semiperimetro: reliquum erit segmentum alterutrius cruris inter angulum eundem & punctum in quo peripheria circuli inscripti tangit latus. Erit autem hoc segmentum ad radium circuli inscripti, ut radius circuli ad tangentem semissis anguli quaesiti. ut in triangulo ABC : ablato latere CA 13 e semiperimetro 22, restabit BG vel BP 9. aio



			Logarithm.	
Proport.	BF	9 compl. arith.	904575749	
	GF	3	047712125	
	Radius	100000000000	10000000000	
Tangens	GBF	333333333333	1 | 952287874	18:26:06

Erit igitur angulua GBF 18:26:06. & totus angulus GBA 36:52:12. Eodem modo inveniri poterunt reliqui anguli ad C & A .

			Logarithm.	
Proport- -ions	CG	11 compl. arith.	8958607314	
	GF	3	047712125	
	Radius	100000000000	10000000000	
Tangens	GCF	272727272727	1 | 9435728569	15:15:18
			Logarithms	
	AP	2 compl. arith.	9698970004	
	PF	3	0477121255	
	Radius	100000000000	10000000000	
	Tangens PAF	150000000000	20176091259	56:18:12
Anguli	ACB	30:30:36		
	CAB	112:37:12		
	CBA	36:52:12		
		180:00:00		

[p.51.]

Poterimus etiam angulum quemlibet aliter invenire, per altitudinem trianguli inventam, per 2.sect. huius Cap. id est per perpendicularem ab angulo quolibet in latus oppositum. Nam ut crus commune anguli verticalis & anguli quaesiti, ad perpendicularem ab eodem angulo verticali; sic sinus totus ad sinum anguli quaesiti. Ut in eodem triangulo, si quaeratur angulus CBA: erunt

				Logarithmi.	
pro- port.	CB crus commone	20	compl. arith.	8698970004	
	CD perpendiculis	12		1079181246	
	Sinus Totus	10000000000		10000000000	
	Sinus CBA	6000000000		**1**| **9**778151250	36:52:12

Vel, ut BD basis continuata, ad latus BC: sic radius circuli, ad secantem anguli quaesiti DBC.

				Logarithmi.	
pro-	BD	16	compl. arith.	87958800173	
port	CD	12		10791812460	
	Radius	10000000000		100000000000	
	Tangens DBC	7500000000		1|98750612633	36:52:12

Vel, ut BD basis continuata, ad latus BC:sic radius circuli, ad secantem angluli quaesiti DBC.

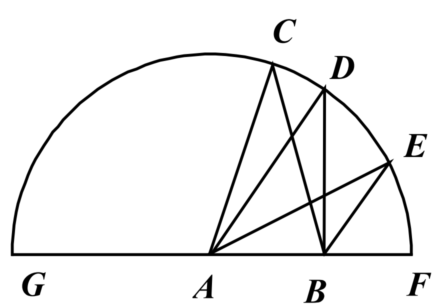
				Logarithmi.	
pro-	BD	16	compl. arith.	87958800173	
port.	BC	20		13010299957	
	Radius	10000000000		100000000000	
	Secans DBC	12500000000		200969100130	36:52:12

Vel, ut CE diameter circuli circumscripti ad CA latus quaesito angulo oppositum: sic radius circuli, ad sinum anguli quaesiti.

				Logarithmi.	
pro-	CE diameter	$21^2/_3$	compl. arith.	86642078980	
port	CA	13		11139433523	
	Radius	10000000000		100000000000	
	Sinus ABC	6000000000		197781512503	36:52:12

Cum autem idem sit sinus anguli cuiuscunque & complementi ad duos rectos, ut 63:0. & 117:0, ut certius scire possimus sitne angulus quaesitus acutus an obtutus:notandum, duos angulos cuiuslibet trianguli, duobus minoribus lateribus oppositos semper esse acutos: tertiam autem angulum, quem maximum latus subtendit, posse quandoque esse acutum, quandoque vero rectum vel obtusum. Ut igitur scire possimus, qualis sit angulus qui maximo lateri opponitur, sumantur summa & differentia maximi Lateris & medii (vel minimi) earumque Logarithmi: & si horum semissis aequetur Logarithmo tertii lateris, angulus maximo lateri oppositus est rectus. sin Logarithmus tertii lateris fuerit maior semisse, angulus erit acutus: sin minor, obtutus. ut in trianguli As in the triangles CAB, DAB, EAB, sint latera CA, DA, EA, aequalia, partium 10: AB, 8: BC, 7: BD, 6: BE, 5: erit GB summa laterum angulorum ad A, 18: BF differentia eorundem laterum 2, semissis summa logarithmorum 077815125 aequalis logarithmo lateris BD, & idcirco BD est per Cap. 17 media proportionalis inter GB & BF, & angulus ABD rectus, per 13.p.6.i.Eucl. est autem logarithmus rectae CB, maior est semisse summa Logarithmorum; & ABC acutus. ut ABE est obtusus, & Logarithmus rectae BE minor eodem semisse.

		Logarithmi.
summa laterum GB	18	1,25527,251
differentia laterum BF	2	0,30102,999
summa logarithmorum		1,55630,250
semissis		0,77815,125
BC	7	0,84509,804
BD	6	0,77815,125
BE	4	0,69897,000



Sinus illi & tangentes secantesque, quorum ope anguli inveniuntur, quaerendi sunt in Opere Palatino, vel alio quolibet Canone Triangulorum.vel si ipsis Logarithmis contenti simus, Canon Triangulorum a M. Edm. Guntero nuper editus nobis abunde satisfaciet, vel ille quem paratum habeo, & aliquando me in lucem daturum spero, utroque modo idem praestabit. sive per ipsos Sinus & Tangentes, sive per eorum Logarithmos. Cum autem sinus Totus undecim notis scribi intelligitur, eius Logarithmus loco characteristicae habebit 10, ut Capite 4 ostendimus. Sic reliqui *sinus usque ad sinum gr. 5. 45^{m}. & Tangentes a gradibus 45. 0 ad 5.43, habent 9 pro characteristica.infra vero usque ad 0.35, eorum characteristica est 8. secantium autem usque ad 84.16, & Tangentium a grad 45.0 ad 84.18, characteristica est 10. atque ultra hos terminos, sere usque ad finem Quadrantis, 11.*