§13.1. *Synopsis: Chapter Thirteen.*

This chapter is among the more enigmatic and mathematically forward-looking of Briggs' published work. We have seen in the previous two chapters the use of two methods of sub-tabulation: initially, simple proportion is tried with some success, and then what is now known as the second order forward divided difference method is used for sub-tabulating the logarithms of the first 20,000 numbers, and subsequently from 90,000 to 100,000; however, outside these limits the method fails due to differences of higher orders than the second being encountered.

Now Briggs had found a method that involved correcting what he termed divided differences, suitable for all orders, with which to sub-tabulate the rest of the tables from 20,000 to 89,000. The method is presented mainly for quinquisection, with a little said about trisection, and Briggs notes the generality of the method by applying it to tables of powers of integers, sines, etc. Differences of a given order are corrected by subtracting simple sums of multiples of already correct or corrected differences of higher orders. No hint is given by Briggs of how he came upon the method: though he mentions elsewhere that it originated with his work on tables of sines in the early 1600's. A modern derivation of the method is given in the Notes to the chapter, following Goldstine, and we speculate on how Briggs himself might arrived at the method.

§13.2. ***Chapter Thirteen***. [p.27.]

It is desired to find the logarithms of a Chiliad. Or, for as many equally spaced given numbers as please, together with the logarithms of these; for the individual intervals of these four numbers, to find the logarithms of the intermediate numbers .

The logarithms of intermediate numbers can be calculated in many ways. I especially recommend the present method to be held in esteem; and we shall see about the rest afterwards.

The first, second, third, fourth, fifth, etc. differences of the given logarithms are taken; and the first difference is divided by 5, the second by 25, the third by 125, etc ; with the ratio of the division increasing in the quintuplet ratio: the quotients are called the first, second, third, &c, mean differences; or rather in place of division, it can be done by multiplication of the first given difference by 2, of the second by 4, the third by 8, etc, cutting off one figure in products with the first order, two with the second, three with the third, etc The products which are equal to those quotients are the first, second , third , etc mean differences. For let the given logarithms [Table 13-1] together with the first, second, third, fourth, and fifth differences of these be given, which the given logarithms themselves show by subtraction:

fifth	*fourth*	*third*	*second*	*first*	*logarithms*	*Absolute numbers*
		1151695		102791641337		
	8138		242719568		33253,10371,71106	2115
75		1143557		102548921769		
	8063		241576011		33263,35860,92875	2120
75		1135494		102307345758		
	7988		240440517		33273,58934,38633	2125
		1127506		102066905241		

[Table 13-1]

The required mean differences are placed nearby [Table 13-2], by multiplication of the first differences given by 2, the first with one figure taken away from the products. The remaining means are constructed if the given difference numbers are multiplied by 4, 8, 16, 32, etc.

These means are then to be corrected in this way[1]:

The two most distant means, the fourth and fifth, of course, do not need correcting (because the sixth and seventh differences are zero; but for all the other differences, the correction are made by taking away other more removed and correct differences: as the subtraction of the seventh corrects the fifth: of the sixth the fourth, etc). The fourth and fifth means are therefore taken as the fourth and fifth corrected means; while the third means are corrected, if from the same are taken away three times the fifth corrected means.

Mean differences		
first	20558328267|4	
	20509784353|8	by 2
	20461469151|6	
	2041338104|82	
sec.	9708782|72	
	9663040|44	by 4
	9617620|68	
third	9213|560	
	9148|456	
	9083|952	by 8
	9020|048	
fourth	13|0208	
	12|9008	by 16
	12|7808	
fifth	|02400	by 32
	|02400	

	9213	560 third mean
D		72 3 times fifth correction
	9213	488 third correct mean
	9148	456 third mean
		72 3 times fifth correction
	9148	384 third correct mean
	9083	952 third mean
		72 3 times fifth correction
	9083	880 third correct mean
	9020	048 third mean
		72 3 times fifth correction
	9019	976 third correct mean

From the second means are taken away double the fourth correct means, in addition $1\frac{2}{5}$ of six order is taken away, if any of the sixth [order] are found between these limits.

	9708782	72 second mean
	26	04 twice the fourth correction
	9708756	68 second correct mean
C	9663040	44 second mean
	25	80 twice the fourth correction
	9663014	64 second correct mean
	9617620	68
	25	56 twice the fourth correction
	9617595	12 second correct mean

(true) third corrected means	9213|5	
	9148|4	
	9083|9	D
	9020|0	
sec. corrected	9708756	
	9663014	C

means 9617595

first	20558319053	9	
corrected	20509775205	4	
means	20461460067	7	**B**
	20413372020	2	

From the first means are taken away the third correct mean & $\frac{1}{5}$ of the fifth.

$\frac{1}{5}$ of the fifth mean falls outside the limits and therefore are safely ignored.

B

20558328267	4 first mean
9213	5 third correct mean
	0048 $\frac{1}{5}$ of the fifth mean.
20558319053	9 first correct mean
20509784353	8 first mean
9148	4 third correct mean
20509775205	4 first correct mean
28461169151	6 first mean
9083	9 third correct mean
20461460067	7 first correct mean
2041338104	2 first mean
9020	0
20413372028	2 first correct mean

[Table 13-2]

∵ However, not only are the corrections taken away for logarithms, but also for tangents and secants, etc., and with all equidistant numbers of the same power [2]. But for sines, the differences confined to columns B, D, F, H are added to the differences placed in column A: the rest namely in columns C, E, G, I are taken away from the same.

And by this method all the differences are corrected, and prepared to perform their function. We would be indebted to that same method if there were more different orders, by making a start from the smallest and most removed differences; while the table set out below indicates how much should be taken away from each kind.

20								
19								
18	18 (20)							
17	17 (19)							
16	16 (18)	123 $\underline{1}$ (20)						
15	15 (17)	108 $\underline{0}$ (19)						
14	14 (16)	93 $\underline{9}$ (18)	400 $\underline{4}$ (20)					
13	13 (14)	80 $\underline{6}$ (17)	317 $\underline{2}$ (19)					
12	12 (14)	68 $\underline{4}$ (16)	246 $\underline{4}$ (18)	629 $\underline{64}$ (20)				
11	11 (13)	57 $\underline{2}$ (15)	187 $\underline{0}$ (17)	431 $\underline{20}$ (19)				
10	10(12)	47 $\underline{0}$ (14)	138 $\underline{0}$ (16)	283 $\underline{80}$ (18)	434 $\underline{40}$ (20)			
9	9 (11)	37 $\underline{8}$ (13)	98 $\underline{4}$ (15)	177 $\underline{84}$ (17)	236 $\underline{88}$ (19)			
8	8 (10)	29 $\underline{6}$ (12)	67 $\underline{2}$ (14)	104 $\underline{72}$ (16)	118 $\underline{72}$ (18)	111 $\underline{248}$ (20)		
7	7 (9)	22 $\underline{4}$ (11)	43 $\underline{4}$ (13)	56 $\underline{84}$ (15)	53 $\underline{20}$ (17)	36 $\underline{680}$ (19)		
6	6 (8)	16 $\underline{2}$ (10)	26 $\underline{0}$ (12)	27 $\underline{60}$ (14)	20 $\underline{40}$ (16)	10 $\underline{760}$ (18)	4 $\underline{080}$ (20)	
5	5 (7)	11 $\underline{0}$ (9)	14 $\underline{0}$ (11)	11 $\underline{40}$ (13)	6 $\underline{20}$ (15)	2 $\underline{280}$ (17)	$\underline{500}$ (19)	
4	4 (6)	6 $\underline{8}$ (8)	6 $\underline{4}$ (10)	3 $\underline{64}$ (12)	1 $\underline{28}$ (14)	$\underline{272}$(16)	$\underline{032}$ (18)	$\underline{0016}$ (20)
3	3 (5)	3 $\underline{6}$ (9)	2 $\underline{2}$ (9)	7 $\underline{2}$ (11)	1 $\underline{2}$(13)	$\underline{008}$ (15)		
2	2 (4)	1 $\underline{4}$ (6)	$\underline{4}$ (8)	$\underline{04}$ (10)				
1	1 (3)	$\underline{2}$ (5)						
A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*

[Table 13-3]

The numbers placed in column A describe the first, second, third, etc. mean differences, as far as the twentieth. The numbers in columns B, C, and D, etc., show the kind and amount of the correct difference to be taken away, ∵ from these mean differences, which have been positioned on the same line, with these in column A. For example: from the sixth mean difference are taken away 6 of the correct eighth mean difference; $16\frac{2}{10}$ of the tenth; 26 of the twelfth, etc, by the same method from the first mean difference, are taken away one of the correct third and $\frac{1}{5}$ of the fifth.

After these correct differences have been found, it is in order that each one is placed conveniently in its own location nearby, in order that much of the confusion associated with the multiplication work is avoided. But we will follow this more easily as follows, if we have a sheet of paper divided into spaces by ruled lines in this manner, if the first, third, fifth, seventh, etc. columns are written with a colour different from the others. The given logarithms designated A occupy each fifth place. And the second correct differences C, the fourth E, sixth, eighth, etc. are placed on the same line with the logarithms, and to the left. The correct mean differences, namely the first B, the third D, fifth, seventh, etc., occupy the positions of this space. Finally, the empty spaces are filled in, starting from the left. With the addition of the fourth differences the thirds are completed; the seconds by the additions of the thirds; and thus successively. And between adding we can add or take away unity as required. For with these irrational numbers it suffices to have the closest actual differences to them, as we are not able to find the true numbers with accuracy. The reason for this, I had said in the beginning of this chapter, with the product of the first differences by two, the last place is taken away; here I have taken away nothing. But with the first differences, and the rest, I consider one adjoining place beyond the limits of the constituent numbers: as all turns out surer and more complete. I recommend the same in the making of tables of tangents, secants, and sines.

But with the powers of equally spaced numbers, where all the differences are whole numbers, no less than the given numbers themselves; all the numbers are held between the constituted limits, in which the number is whole for all the diverse differences; beyond which, if the base numbers maintain equal differences, it cannot progress further. For with squares there are two orders of difference, with cubes three, with biquadratics four, etc. And the most remote differences are always equal to each other, and equal to a product formed from the difference of the successive base numbers of the same power, calculated in a continuous product with the indices of the same power and of all the smaller powers. For if the base number difference be 1, the final differences will be with squares 2, cubes 6, biquadratics 24, with fifth powers 120, sixth powers 720, seventh powers 5040, etc. Evidently from the continued product itself: from 1,2 : 2; from 1,2,3: 6; from 1,2,3,4:24; etc. But if the difference of the base numbers is 3, the furthest removed differences of the squares is 18; constructed from the square 9 by 2; of the cubes 162, constructed from the cube 27 by 6. Of the fourth power1944, constructed from the bi-quadratic 81 by 24, etc.

	Difference 4	*Differences 2 & 3.*	*Logarithm & 1st difference*		*absolute number.*
		92135 *D*	205583	190639 *B*	
	97	271445			
		92004	205485	919095	
	97	179441			
		91874	205388	739654	
E 130	97	087567 *C*	3325310371	71106 *A*	2115
		91744	205291	652087	
	96	995824	25515663	36315	16
		91614	205194	656263	
	96	904210	25720858	01941	17
		91484 *D*	205097	752054 *B*	
	96	812726	25925955	77146	18
		91355	205000	939327	
	96	721371	26130956	71079	19
		91226	204904	217956	
E 129	96	630146 *C*	3326335860	92875 *A*	2120
		91097	204807	587810	
	96	539050	26540668	51656	21
		90968	204711	04870	
	96	448082	26745379	56532	22
		90839 *D*	204614	600677 *B*	
	96	357244	26949994	16600	23
		90711	204518	243432	
	96	266533	27154512	40943	24
		90583	204421	976899	
E 128	96	175951 *C*	3327358934	38633 *A*	2125
		90455	204325	80094 9	
	96	085496			26
		90327	204229	71545 2	
	95	995169			27
		9020 0 *D*	204133	72028 2 *B*	

[Table 13-4.]

As with all these powers, so with logarithms, tangents and secants, a few more numbers [*i.e.* rows in the table] will be necessary to continue, without which we cannot reach the highest order differences: as with the given example, from the one part 2110 to 2105, and from the remainder 2130 to 2135. But with sines, if the sines of three equidistant arcs are given, all the differences, or the most removed can be found by the proportional rule, if necessary. For the sine and differences of the second, fourth, sixth, and the eighth orders are in continued proportion, and the first, third, fifth, and the seventh differences are likewise in continued proportion amongst themselves. And as the second order differences among themselves are proportional with the sines themselves from the corresponding place; and in the same way, the fourth, sixth, etc. Thus the first difference between itself and the third, the fifth, seventh, are in proportion with the mean sines of the complementary arc. But I am sensible, as my studies are carried out further, of what rules are allowed for the homogeneous powers [4].

If you decide on another Chiliad added to these, consider the twenty-first, and the fifth part is taken of that initial number: the first number will be 20000, of which the fifth part is 4000; to the logarithm of this number, and one by one to the nearest 200, the logarithm is added to five; the sums will be the logarithms of these groups of five numbers, through the whole of the same Chiliad: namely 20000, 20005, 200010, 200015, etc., and of which the first differences are those themselves in the fifth Chiliad, found within those two hundred logarithms. From which the second differences are sought. Also, the second differences give the third. The fourth truly are very small, which for that reason we can ignore without risk. Then the first difference is multiplied by two, the second difference by four, the third difference by eight. The products are the mean differences, which are arranged in their own locations, with the one place cut off from the second, and with two from the third. But the first are to be kept with their places, while corrected by subtraction of the third. All the rest are completed through addition.

And this method of inserting four logarithms between the two given nearby is called quinquesection: as from a single interval there are made five. Also, general laws can be propounded for trisection and septisection. But of all these quinquesection is the most excellent, as we consider the convenience or ease of use. Nevertheless, it will not be irksome to consider the method of trisection for a few intervals. The first, second, third, etc. of the given differences are selected as earlier. Then, 3 divides the first; 9, the second ; 27, the third ; 81, the fourth; etc, with the divisors increasing by the threefold ratio: the quotients are the first, second, third, fourth, etc, mean differences. These mean differences, as before ought to become smaller in all cases, except with the sines; then when the corrections are made, they are to be transferred into their places: and by starting from the smallest and furthest away, all are to be completed as before, through the addition carried out. But how much and which one of the differences should be taken away, is indicated by the present table. [Table 13-5].

1 (12)				
1 (11)				
1 (10)	$3\,{}^{1}/_{3}$ (12)			
1 (9)	3 (11)			
1 (8)	$2\,{}^{2}/_{3}$ (10)	$3\,{}^{1}/_{9}$ (12)		
1 (7)	$2\,{}^{2}/_{3}$ (9)	$2\,{}^{3}/_{9}$ (11)		
1 (6)	2 (8)	$1\,{}^{6}/_{9}$ (10)	${}^{20}/_{27}$ (12)	
1 (5)	$1\,{}^{2}/_{3}$ (7)	$1\,{}^{1}/_{9}$ (9)	${}^{20}/_{27}$ (11)	
1 (4)	$1\,{}^{1}/_{3}$ (6)	${}^{6}/_{9}$ (8)	${}^{4}/_{27}$ (10)	${}^{1}/_{81}$(12)
1 (3)	1 (5)	${}^{3}/_{9}$ (7)	${}^{1}/_{27}$ (9)	
1 (2)	${}^{2}/_{3}$ (4)	${}^{1}/_{9}$ (6)		
1 (1)	${}^{1}/_{3}$ (3)			
A	*B*	*C*	*D*	*E*

[Table 13-5.]

From the first mean difference a third part of the third corrected difference is taken away. From the Fourth mean difference, to be taken away are ${}^{4}/_{3}$ of the sixth, ${}^{2}/_{3}$ of the eighth, ${}^{4}/_{27}$ of the tenth, and ${}^{1}/_{81}$ of the twelfth corrected differences.

The remaining sections, for which the names are assigned from even numbers, such as bisection, quadrisection, etc., are more difficult. Because also by finding the sub-tangents for a circle we prove: for odd sections, the sought chords may themselves be established with one operation. Indeed for the rest, the even sections, not the chords themselves but only the squares of the chords are expedient.

Example of trisection

Mean Differences found by given division

4^{th} by 81	3^{rd} by 27	2^{nd} by 9	1^{st} by 3
$^{24}/_{81}$	D.4 $^{72}/_{81}$ D.5 $^{63}/_{81}$	$33^{45}/_{81}$ $48^{18}/_{81}$ $65^{45}/_{81}$	$223^{2}/_{3}$ $368^{1}/_{3}$

Here you see with fourth powers.

Differences given

4^{th}	3^{rd}	2^{nd}	1^{st}	biquad ratics	sides
				256	4
24		302		625	5
24	132	434	671	1296	6
24	156	590	1105	2401	7
				4096	8

Third and fourth mean differences need not be corrected. If from the second mean is taken $^{2}/_{3}$ of the fourth, the second correction C remains. If from the first mean is taken $^{1}/_{3}$ of the third, there is left the first correction B. As may be clear from the workings of the table on p.29. [Table 13 - 3]

4^{th} diff.	3^{rd} diff.		2^{nd} diff.		1^{st} diff.	biquad.	roots
$^{24}/_{81}$	$4^{48}/_{81}$	*C*	$33^{29}/_{81}$			625 *A*	5
	D $4^{72}/_{81}$		$37^{77}/_{81}$	- -	$184^{7}/_{81}$	809 $^{7}/_{81}$	5 $^{1}/_{3}$
	$5^{15}/_{81}$		$42^{68}/_{81}$	*B*	$222^{3}/_{81}$	$1031^{10}/_{81}$	5 $^{2}/_{3}$
$^{24}/_{81}$	$5^{39}/_{81}$	*C*	48 $^{2}/_{81}$	- -	$264^{71}/_{81}$	1296 *A*	6
	D $5^{63}/_{81}$		$53^{41}/_{81}$	- -	$312^{73}/_{81}$	$1608^{73}/_{81}$	6 $^{1}/_{3}$
	$6^{6}/_{81}$		$59^{23}/_{81}$	*B*	$366^{33}/_{81}$	1975 $^{5}/_{81}$	6 $^{2}/_{3}$
$^{24}/_{81}$		*C*	$65^{29}/_{81}$	- -	$425^{56}/_{81}$	2401*A*	7

[Table 13-6.]

Table of logarithms found by subtabulation

1	0,00	100001	0,00000,43429,23104
2	0,30102,99956,63981	100002	0,00000,86858,02778
3	0,47712,12547,19662	100003	0,00001,30286,39025
4	0,60205,99903,27962	100004	0,00001,73714,31846
5	0,69897,00043,36012	100005	0,00002,17141,81240
6	0,77815,12503,83644	100006	0,00002,60568,87210
7	0,84509,80400,14257	100007	0,00003,03995,49757
8	0,90308,99869,91943	100008	0,00003,47421,68882
9	0,95424,25094,39324	100009	0,00003,90847,44585
11	0,04139,26851,58226	1000001	0,00000,04342,94265
12	0,07918,12460,47625	1000002	0,00000,08685,88095
13	0,11394,33523,06837	1000003	0,00000,13028,81491
14	0,14612,80356,78238	1000004	0,00000,17371,74453
15	0,17609,12590,55681	1000005	0,00000,21714,66981
16	0,20411,99826,55925	1000006	0,00000,26057,59074
17	0,23044,89213,78274	1000007	0,00000,30400,50733
18	0,25527,25051,03306	1000008	0,00000,34743,41958
19	0,27875,36009,52828	1000009	0,00000,39086,32748
101	0,00432,13737,82645	10000001	0,00000,00434,29447
102	0,00860,01717,61928	10000002	0,00000,00868,58888
103	0,01283,72247,05172	10000003	0,00000,01302,88326
104	0,01703,33392,98780	10000004	0,00000,01737,17759
105	0,02118,92990,69938	10000005	0,00000,02171,47187
106	0,02530,58652,64771	10000006	0,00000,02605,76611
107	0,02938,37776,85210	10000007	0,00000,03040,06031
108	0,03342,37554,86950	10000008	0,00000,03474,35447
109	0,03742,64979,40624	10000009	0,00000,03908,64858

1001	0,00043,40774,79319	100000001	0,00000,00043,42935
1002	0,00086,77215,31227	100000002	0,00000,00086,85890
1003	0,00130,09330,20418	100000003	0,00000,00130,28834
1004	0,00173,37128,09000	100000004	0,00000,00173,71779
1005	0,00216,60617,56507	100000005	0,00000,00217,14724
1006	0,00259,79807,19908	100000006	0,00000,00260,57668
1007	0,00302,94705,53619	100000007	0,00000,00304,00613
1008	0,00346,05321,09508	100000008	0,00000,00347,43558
1009	0,00389,11662,36913	100000009	0,00000,00390,86502
10001	0,00004,34272,76862	1000000001	0,00000,00004,34295
10002	0,0000868502,11648	1000000002	0,00000,00008,68589
10003	0,00013,02688,05227	1000000003	0,00000,00013,02883
10004	0,00017,36830,58464	1000000004	0,00000,00017,37178
10005	0,00021,70029,72230	1000000005	0,00000,00021,71472
10006	0,00026,04985,47390	1000000006	0,00000,00026,05767
10007	0,00030,38997,84812	1000000007	0,00000,00030,40061
10008	0,00034,72966,85363	1000000008	0,00000,00034,74356
10009	0,00039,06892,49911	1000000009	0,00000,00039,08650

[Table 13-7.]

§13.3. *An Extended Note:*

[1] **Briggs' Second Subtabulation Scheme, and Some of the History Surrounding It**.

Briggs had run into trouble interpolating logarithms using his method of forward finite differences, developed in the last chapter. We recall that this method worked well as long as 1st and 2nd order differences only were involved, which corresponded to numbers in the locality of a power of ten, when 14 decimal places were required. Thus, Briggs was able to find the first 20 Chiliades, and the 90 to 100 Chiliades using this easy form of interpolation. The sophisticated alternative method of subtabulation propounded by Briggs in this Chapter is explained in modern terms at some length by Goldstine in *A History of Numerical Analysis...* , pp. 27 - 30. An independent development of interpolation (we call this sub-tabulation when the arguments are equally spaced between known values) was undertaken by Newton some 50 years after Briggs, see D.T.Whiteside, *The mathematical papers of Isaac Newton*, Vol. IV, (C.U.P.), p.45: it appears that Newton was either unaware of Briggs' work (which seems unlikely, as Isaac Barrow was enthusiastic about

Briggs - see the introductory chapter) or simply decided to ignore it, and work things out for himself: the *Arithmetica* was of course available in print; moreover, all of Harriot's scientific work, apart from the *Praxis*, was carefully tucked away in a trunk in the library at Petworth House, where it was to remain until the end of the 18th century. Thus, apart from the tables themselves, the advances made by Briggs and Harriot (in the latter's case in so many different fields of study) were totally ignored, forgotten about, or lost, by those who followed - and this sad state of affairs was to include Briggs' sophisticated method for solving higher order equations numerically (set out in the *Trigonometria Britannica*: an article in the *Mathematical Gazette*, Nov. 2004 by the translator, gives the details), also to be rediscovered by Newton. Can one blame the upheavals of the civil war for all of this neglect? Or did Briggs died untimely before he could publicise his methods?

The Method:

It is appropriate to explain here how the tables 13-3 and 13-5 of corrected differences for 5 and 3 equally spaced points of subdivision may be obtained, following Goldstine. There is no hint given by Briggs as to how he came upon his method, though he does tell us in the preface that he spent a long time perfecting it. It is difficult for us, conditioned as we are to symbolic methods, to imagine that such a scheme could be visualised without a theoretical abstract representation: to help us overcome this barrier, we may note that Briggs worked with actual numbers, in locations or cells in his difference table, which corresponded to the order and point of application of the difference. The repeated application of an arithmetical difference formula generated the various orders of differences: Briggs was always on the look out for relations between the numbers, and one can but presume that eventually he found useful ones. The rather unusual development (in the sense of being out of place) of sub-tabulating the 6^{th} powers of large numbers in the *Trigonometria* may well have been the testing ground for his methods, though the smaller effort with sub-tabulating 4th powers at the end of this chapter may also have been of use. Briggs' way of doing mathematics, without any symbols to speak of, was highly intuitive, and perhaps explains why he never divulged the origins: he had a scheme that just 'worked', and which could be iterated through the orders of differences to give the table of corrections required.

With the reader's forbearance, we will indulge in a little speculation concerning these things, for there were not too many options available in creating an improved scheme. Let us first note that the second order differences scheme in Chapter 11 was *not* extended to higher orders: in spite of this being an almost trivial exercise algebraically - so one has to assume that this scheme was not formulated Briggs from algebraic considerations as we understand them - as its generalisation was not obvious to Briggs. However, Briggs was aware that by making small corrections to the divided differences, the new scheme would work: it is possible that he found his method by a 'brute-force' approach, assuming it worked for all kinds of tables. Perhaps initially he evaluated a set of consecutive powers, say: n^6, $(n+1)^6$,$(n+2)^6$, ..., and found all the divided differences to their vanishing point, for some range of values of n. He might then have compared this completed scheme - which when ran forwards from the higher orders, would necessarily generate the powers of the consecutive numbers - with an incomplete scheme where multiples of say 3 or 5 only were given, and blanks were left in the table for the values to be sub-tabulated. This approach would indicate what numerical corrections were needed to the divided differences for the various orders. One could proceed from the highest orders to the lowest, or vice versa: in the latter case, it would then be a matter of observation to see how the first order difference had to be corrected, assuming the rest corrected, (say by taking a third of the third corrected order for trisection), and subsequently the second, third, and higher orders could be corrected essentially by 'squaring', 'cubing' the difference scheme for the first order: 'squaring' meaning the correction was applied once, and the new values corrected again in the same way, 'cubing' involved three passes over the data, etc. Again, if one chose to proceed down through the orders, the difference needed for the correction could be found from the known values, and provided by multiples of appropriate higher orders. At an early stage it would have become apparent that the odd and even divided differences were corrected amongst themselves. We may contrast these ideas with the following modern account of Briggs' scheme, following Goldstine.

For a suitable function f(x), (such as log *x*) tabulated at the points 5n in a table, where n = 1, 2, 3, 4,... , starting from the first value in the table, it is proposed to find the four intermediate values of f(x), in order that the function is known for all integral values in the table, to a comparable degree of precision. Firstly, we define a shift-right operator E by E f(a) = f(a + 1), for some integer a in the table; then by induction, $E^2f(a) = E.E\,f(a+1) = f(a+2)$, $E^3f(a) = f(a+3)$, etc., as often as required. Similarly, a shift-left operator E^{-1} is defined by $E^{-1}\,f(a) = f(a-1)$, $E^{-2}\,f(a) = f(a-2)$, etc, as often as required, while $E^{-1}E\,f(a) = I\,f(a)$; these operators being associative and commutative, etc. Secondly, we define difference operators of different orders, and specify whither they have been shifted left or right: it is common practise in texts on numerical analysis to define the first order operator by:

$\Delta\,f(a) = \Delta^1\,f(a) = f(a+1) - f(a) = (E - I)f(a)$;

the second order by: $\Delta^2\,f(a) = \Delta^1(f(a+1) - f(a)) = f(a+2) - 2f(a+1) + f(a) = (E - I)^2f(a)$, etc., and in general for n > 2 inductively: $\Delta^n\,f(a) = (E - I)^n\,f(a)$, which can be expanded binomially. Briggs' scheme has added features: we will consider terms such as

$E\,\Delta^1f(a) = E(f(a+1) - f(a)) = f(a+2) - f(a+1)$, which we will call $\Delta^1_1\,f(a)$. Thus, the lower right suffix denotes the shift in the point of application from a. In addition, a special kind of difference is required initially to handle the quotients of the form: $\frac{f(a+5) - f(a)}{5} = {}_5\Delta^1$ introduced by Briggs.

Note that all of these operations can easily be performed arithmetically, and by placing the results in appropriately labeled cells in the table care is taken of the notation problem, rather as a modern computer program does by labeling memory locations, however complex the task may be. Hence, we definite a general difference operator:

$${}_m\Delta^n_p f(a) = \left[\frac{f(a+p+m) - f(a+p)}{m}\right]^n,$$

where n is the order of the difference, m is the size of the step, usually omitted when m = 1, and p is the shift from the argument a, also omitted if p = 0.

Two results A and B are required, the second following from first:

A: $(E^1 + E^{-1} - 2I)\, f(a) = f(a + 1) + f(a - 1) - 2f(a) = \{f(a + 1) - f(a)\} - \{f(a) - f(a - 1)\} = \Delta^2_{-1} f(a)$;

B: $(E^2 + E^{-2} - 2I)\, f(a) = \{(E^1 + E^{-1})^2 - 4I\}f(a) = ((\Delta^2_{-1} + 2I)^2 - 4I)\, f(a) = (\Delta^4_{-2} + 4\Delta^2_{-1})\, f(a)$.

We note that these are central differences, as opposed to the forward differences of the last chapter. Now A and B are latent in Briggs' numerical scheme, and he used their intuitive equivalents to generate the columns of corrected differences in his table. We now show, using A and B, how to write the above original 5 space difference ${}_5\Delta^1$, and the corresponding higher order differences, ${}_5\Delta^2, {}_5\Delta^3$, etc., in terms of single space differences, which will eventually be used to generate the required intermediate values :

$$
\begin{aligned}
{}_5\Delta^1 f(a) &= \frac{f(a+5)-f(a)}{5} = \frac{(E^5 - I)}{5} f(a) = \frac{(E-I)(E^4+E^3+E^2+E+I)}{5} f(a) \\
&= \frac{(E-I)E^2(E^2+E^1+I+E^{-1}+E^{-2})}{5} f(a) = \tfrac{1}{5}\Delta^1_2(\Delta^4_{-2} + 4\Delta^2_{-1} + 5I + \Delta^2_{-1})f(a) = (\tfrac{1}{5}\Delta^5 + \Delta^3_1 + \Delta^1_2)f(a).
\end{aligned}
$$

We can write this difference equation in the form:

${}_5\Delta^1 f(a) = (\frac{1}{5}\Delta^5 + \Delta^3_1 + \Delta^1_2)f(a)$, or $\Delta^{(1)} f(a) = ({}_5\Delta^1_{-2} - \frac{1}{5}\Delta^5_{-2} - \Delta^3_{-1})f(a)$. The final $\Delta^{(1)}$ bracket notation shows the first order has been corrected, while the shift operator can act on the whole scheme to change the point of application of the difference operators up or down the same column: only the relative positions matter; also, the higher order differences are already corrected. Higher orders are obtained in an iterative manner, and are continued until the region is reached where the finite differences are zero. Consider the second order values:

$$
{}_5\Delta^2 = (\tfrac{1}{5}\Delta^5 + \Delta^3_1 + \Delta^1_2)^2 = (\Delta^2_4 + 2\Delta^4_3 + 1\tfrac{2}{5}\Delta^6_2 + +\tfrac{2}{5}\Delta^8_1 + \tfrac{1}{25}\Delta^{10});
$$

$$
\text{or, } \Delta^{(2)}_4 = {}_5\Delta^2 - (2\Delta^4_3 + 1\tfrac{2}{5}\Delta^6_2 + +\tfrac{2}{5}\Delta^8_1 + \tfrac{1}{25}\Delta^{10}), \text{ as required.}
$$

Three point subtabulation is effected in the same manner, which we briefly illustrate:

$$ {}_3\Delta^1 \mathrm{f(a)} = \frac{\mathrm{f(a+3)-f(a)}}{3} = \frac{(E^3-\mathrm{I})}{3}\mathrm{f(a)} = \frac{(E-\mathrm{I})(\mathrm{E}^2+\mathrm{E}+\mathrm{I})}{3}\mathrm{f(a)} $$

$$ = \frac{(E-\mathrm{I})\mathrm{E}(\mathrm{E}^1+\mathrm{I}+\mathrm{E}^{-1})}{3}\mathrm{f(a)} = \tfrac{1}{3}\Delta_1^1(\Delta_{-1}^2+3\mathrm{I})\mathrm{f(a)} = (\tfrac{1}{3}\Delta_0^3+\Delta_1^1)\mathrm{f(a)}.;\mathrm{or} $$

$$ \Delta_1^{(1)} = {}_3\Delta^1 - \tfrac{1}{3}\Delta_0^3 $$

Higher orders again follow by iterative means:

$$ {}_3\Delta^1 = (\tfrac{1}{3}\Delta_0^3+\Delta_1^1) $$

e.g. for the second order ${}_3\Delta^2 = (\tfrac{1}{3}\Delta_0^3+\Delta_1^1)^2 = \tfrac{1}{9}\Delta_0^6+\Delta_2^2+\tfrac{2}{3}\Delta_1^4$; hence:

$$ \Delta_2^{(2)} = {}_3\Delta^2 - (\tfrac{2}{3}\Delta_1^4+\tfrac{1}{9}\Delta_0^6). $$

The given orders are then corrected numerically from higher to lower ones, and the other values inserted from corrected or initially correct higher orders, as we observe in Table 13-6, where A, B, and C refer to corrected values from the original uncorrected table.

We now elaborate a little on the 5-section scheme, by giving Table 13-2 a modern setting, and examine how the second tabulation scheme works in practise in this case. According to Briggs' example, the corrections required are as follows
(i.e. to convert the various orders of differences into useful differences that can be used to insert the values of the logarithms into the empty slots in the table with steadily increasing accuracy):
For the 5th & 4th orders, no change needed, as higher orders are negligible at the required precision for the example provided. Hence, $\Delta^{(5)} = \Delta^5 = 0$ and $\Delta^{(4)} = \Delta^4$; while for 3rd order, no change is needed here either, as the 5th order is zero in this part of the table to this precision; otherwise:

$\Delta^{(3)} = {}_5\Delta^3 - 3.\Delta^{(5)}$; (See Table 13-3). However, for 2nd order:

$\Delta^{(2)} = {}_5\Delta^2 - 2\Delta^{(4)} - 1.4\Delta^{(6)} = {}_5\Delta^2 - 2.\Delta^{(4)}$ in this case, and for the 1st order:

$$ \Delta^{(1)} = {}_5\Delta^1 - \Delta^{(3)} - 0.2\Delta^{(5)} = {}_5\Delta^1 - \Delta^{(3)} $$

Perusal of Table [13-4] shows how these means occupy various levels called *A*, *B*, *C*, *D*, & *E* for orders 0, 1, 2, 3, 4, & 5 respectively: these are used in an iterative scheme to produce the intervening logarithms, as we show here, where the Zeroth order difference is the logarithm itself :

$$\Delta^{(2)}_{j-2} + \Delta^{(3)}_{j-2} = \Delta^{(2)}_{j-1}\,; \quad \Delta^{(1)}_{j-1} + \Delta^{(2)}_{j-1} = \Delta^{(1)}_{j}\,; \quad \Delta^{(0)}_{j} + \Delta^{(1)}_{j} = \Delta^{(0)}_{j+1}$$

It is seen that 3 rows are involved at any one time in this scheme, corresponding to the temporary indices j - 1, j, and j + 1. To illustrate these differences, consider part of Table 13-4 (the cross arrows indicate subtractions): Table 13-4A :-

4th C.D.	2nd & 3rd C.D		1st C.D & log			number
$\Delta^{(4)}_{-1}$ 12(9)	$\Delta^{(2)}_{-1}$ $\Delta^{(3)}_{-1}$	9672137(1) – 9122(6) =	$\Delta^{(0)}_{-1}$ $\Delta^{(1)}_{-1}$	332613095671079 + 20490421795(6) =	j = -1	2119
$\Delta^{(4)}_{0}$ 12(9)	$\Delta^{(2)}_{0}$ $\Delta^{(3)}_{0}$	9663014(6) – 9109(7) =	$\Delta^{(0)}_{0}$ $\Delta^{(1)}_{0}$	332633586092875 + 20480758781(0) =	j = 0	2120
$\Delta^{(4)}_{1}$ 12(9)	$\Delta^{(2)}_{1}$ $\Delta^{(3)}_{1}$	9653905(0) – 9096(8) =	$\Delta^{(0)}_{1}$ $\Delta^{(1)}_{1}$	332654066851656 + 20471104876(0) =	j = 1	2121
$\Delta^{(4)}_{2}$ 12(9)	$\Delta^{(2)}_{2}$ $\Delta^{(3)}_{2}$	9644808(2) – 9083(9) =	$\Delta^{(0)}_{2}$ $\Delta^{(1)}_{2}$	332674537956532 + 20461460067(7) =	j = 2	2122

Further Historical Developments: This complex scheme proposed by Briggs to extend the log tables by subtabulation between known logarithms to cover the missing Chiliades was never implemented by him, although in his advice to the reader at the start of the *Arithmetica* he was keen that others should undertake this task. Adrian Vlacq, a Dutch bookseller and publisher, who was to finance and oversee the production of the first complete set of Briggs' logarithms for the first hundred Chiliades to fewer places, got round the computational difficulties with the help of his associate, the mathematically inclined Ezechiel de Decker, by reducing the accuracy of the logarithms from 14 to 10 places, in which case the previous scheme of Briggs could still be used. See Evert M. Bruins: *On the History of Logarithms*..... *Janus*, LXVII (1980), pp. 241 - 258. Furthermore, Vlacq's 2nd edition of Briggs' book omitted completely Briggs' original Chapters 12 & 13, [Thus, in Vlacq's edition, chapter 12 corresponds to Briggs' original Chapter 14,] a fact that

Briggs was to bemoan in his subsequent work, the *Trigonometria Britannica*, with just cause, for this was the main thrust of his development – as we shall see shortly. Also, material was moved around inappropriately by Vlacq in the new edition – thus, the section on compound interest, being an application, was presented at the end of Chapter 7, long before the main development was complete; while most of Briggs' occasional typographical errors, were copied without correction.

Briggs had inadvertently left the door open for such a development, by not overseeing something along the same lines himself. He felt that his remaining years would be better spent in finishing off the production of his tables of the trigonometric functions then in use, with their logarithms, rather than completing the tables of logarithms of natural numbers; also, following Vieta, he wanted to set in motion a move to decimal parts of the degree rather than the use of minutes and seconds. All of which came to nothing. There were of course ways round the missing logarithms in doing calculations with the original tables, so Briggs does not appear to have been much fussed about it. In this translator's view, history has not treated Briggs fairly: often in historical reports on the history of Logarithms - such as the article in the 9th edition of the *Encyclopedia Britannica* on Tables, written in the 1880's, Vlacq is given all the credit for producing the first set of complete log tables, while Briggs is given scant attention.

Walter Warner, the associate of Harriot (who extracted the material for the *Praxis* from Harriot's manuscripts), with the assistance of Dr Pell, did eventually produce a canon of 100,000 Logarithms in the 1630's, one presumes according to the Briggs recipe, and also, according to Aubrey, with antilogarithms. On the death of Warner in 1640, this canon was to be printed; but a part became lost, and the project eventually was abandoned. See John W. Shirley's, *Thomas Harriot,* (Oxford, 1972), p. 117 for details: Such was the end of the grand scheme.

Other Notes On Chapter Thirteen.

[2] In the *Trigonometria*, Briggs sub-tabulates between sixth powers as an illustration of the power of his method, before applying a modified method to his table of known sines.

[3] We would call this the base number of a sequence of powers. Thus, in the products n^r, where the index r =2, 3, 4, etc., and n has unit increment, the final differences are equal to r!

[4] Briggs applies his difference method to his table of known sines and finds the tables of even differences of any order are each proportional to the original sines, and also amongst themselves; while the odd differences of any order are similarly proportional to the cosine, given by the first order. Intuitively, we may now regard these small divided differences as giving the approximate gradients associated with the function tabulated, and with the various derived functions, from which the results found follow for all the functions considered, though this difference in behaviour between say the table of sines and that of logarithms may have been puzzling at the time.

§13.4. **Caput XIII.** [p.27.]

Logarithmos Chiliadum quae desiderantur invenire. Vel datis quotlibet numeris aequidistantibus, una cum eorum Logarithmis quatuor numerorum, pro singulis intervallis intermediorum Logarithmos invenire.

Logarithmi intermedii pluribus modis haberi poterunt. ego hunc imprimis amplectendum censeo; de reliquis postea videbimus.

Sumantur differentiae primae, secundae, tertiae, quartae, &c. datorum Logarithmorum; & dividantur primae per 5, secundae per 25, tertiae per 125, &c.; divisoribus quintupla ratione crescentibus: quoti appellentur differentiae mediae, primae, secundae, tertiae, &c. vel potius loco divisionis, fiat multiplicatio datarum primarum per binarium, secundarum per quaternarium, tertiarum per octonarium, &c. amputaris unica nota in factis a primis, duabus in proximis, tribus in tertiis, &c. hi facti qui sunt illis quotis aequales, erunt differentiae mediae primae, secundae, tertae, &c. ut sunto dati hi Logarithmi una cum eorum differentiis primis, secundis, tertiis, quadratis, quintis, quas ipsi dati Logarithmi per subductionem ostendunt.

quintae	*quartae*	*tertiae*	*secundae*	*primae*	*logarithmi.*	*numeri absoluti.*
		1151695		102791641337		
	8138		242719568		33253,10371,71106	2115
75		1143557		102548921769		
	8063		241576011		33263,35860,92875	2120
75		1135494		102307345758		
	7988		240440517		33273,58934,38633	2125
		1127506		102066905241		

Proximo in loco sunt differentiae mediae quaerendae, per multiplicationem, primarum datarum per binarium, facti dempta nota unica erunt primae. reliquae mediae fiunt si datas multiplicent 4.8.16.32. &c.

Deinde hae mediae sunt corrigendae ad hunc modum.

Duae species remotissimae, scil. quartae & quintae corrigi nequeunt (quia sexte & septimae sunt nullae: omnis autem differentiarum correctio sit per subductionem differentiarum alternarum magis remotarum & correctarum: ut subductio septimarum corrigit quintas: sextarum quartas, &c.) sumuntur idcirco quartae & quintae mediae, pro quartis & quintis correctis.

[p.28.]

tertiae autem mediae corriguntur si ab iisdem auferantur tres Quintae correctae.

Differentiae mediae		
primae	20558328267|4	
	20509784353|8	per 2
	20461469151|6	
	2041338104|82	
secundae	9708782|72	
	9663040|44	per 4
	9617620|68	
tertiae	9213|560	
	9148|456	
	9083|952	per 8
	9020|048	
E quartae	13|0208	
	12|9008	per 16
	12|7808	
quintae	|02400	per 32
	|02400	

	9213	560 tertia media
D		72 tres quintae correctae
	9213	488 tertia correcta
	9148	456 tertia media
		72 tres quintae correctae
	9148	384 tertia correcta
	9083	952 tertia media
		72 tres quintae correctae
	9083	880 tertia correcta
	9020	048 tertia media
		72 tres quintae correctae
	9019	976 tertia correcta

A secundis mediis sunt auferendae duae quartae correctae, debuit insuper auferri $1\frac{2}{5}$ sexte, si ullae sextae intra hos limites inventae fuissent.

	9708782	72 secunda media
	26	04 duae quartae correctae
	9708756	68 secunda correcta
C	9663040	44 secunda media
	25	80 duae quartae correctae
	9663014	64 secunda correcta
	9617620	68 secunda media
	25	56 duae quartae correctae
	9617595	12 secunda correcta

tertiae correctae	9213|5		
	9148|4		
	9083|9	D	
	9020|0		
secundae correctae	9708756		
	9663014	C	
	9617595		
primae correctae	20558319053	9	
	20509775205	4	
	20461460067	7	B
	20413372020	2	

A primis mediis auferendae sunt una tertia correcta & $\frac{1}{5}$ quintae.

$\frac{1}{5}$ quintae cadit extra limites & idcirco poterit tuto negligi.

	20558328267	4 prima media.
	9213	5 una tertia correcta.
		0048 $\frac{1}{5}$ quintae.
B	20558319053	9 prima correcta.
	20509784353	8 prima media.
	9148	4 tertia correcta.
	20509775205	4 prima correcta.
	28461169151	6
	9083	9 tertia correcta.
	20461460067	7 prima correcta.
	2041338104	2
	9020	0
	20413372028	2 prima correcta.

Atque ad hunc modum sunt omnes differentiae correctae, & ad suum munus exequendum paratae. eodem modo uti deberemus si plures fuissent differentiarum species, incipiendo a minimis & remotissimis. Quot autem in unaquaque specie sint auferendae, indicat haec subiecta tabella:

[p.29.]

20	
19	
18	18 (20)

∵ Non pro Logarithmis tantum sunt omnes auferendae, sed etiam pro Tangentibus, Secantibus, &., omnibus numerorum aequidistantium potestatibus homogeneis. At pro Sinubus, differentia quae columnis B, D, F, H continentur, sunt addendae differentiis mediis in column A positis: reliquae vero in columnis C, E, G, I sunt ab iisdem auferendae.

17	17 (19)							
16	16 (18)	123 1 (20)						
15	15 (17)	108 0 (19)						
14	14 (16)	93 9 (18)	400 4 (20)					
13	13 (14)	80 6 (17)	317 2 (19)					
12	12 (14)	68 4 (16)	246 4 (18)	629 64 (20)				
11	11 (13)	57 2 (15)	187 0 (17)	431 20 (19)				
10	10(12)	47 0 (14)	138 0 (16)	283 80 (18)	434 40 (20)			
9	9 (11)	37 8 (13)	98 4 (15)	177 84 (17)	236 88 (19)			
8	8 (10)	29 6 (12)	67 2 (14)	104 72 (16)	118 72 (18)	111 248 (20)		
7	7 (9)	22 4 (11)	43 4 (13)	56 84 (15)	53 20 (17)	36 680 (19)		
6	6 (8)	16 2 (10)	26 0 (12)	27 60 (14)	20 40 (16)	10 760 (18)	4 080 (20)	
5	5 (7)	11 0 (9)	14 0 (11)	11 40 (13)	6 20 (15)	2 280 (17)	500 (19)	
4	4 (6)	6 8 (8)	6 4 (10)	3 64 (12)	1 28 (14)	272(16)	032 (18)	0016 (20)
3	3 (5)	3 6 (9)	2 2 (9)	7 2 (11)	1 2(13)	008 (15)		
2	2 (4)	1 4 (6)	4 (8)	04 (10)				
1	1 (3)	2 (5)						
A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*

Numeri in columna *A* positi, designant differentias medias, primas, secundas, tertias, &c., usque ad vicesimas. Numeri vero in columnis *BCD* &c., ostendunt, quot & quales differentiae correctae auferendae, ∵ ab illis differentiis mediis, quae in columna A in eadem cum illis linea sitae sunt. Exempli causa: a differentia media sexta, auferendae sunt differentiae correctae octavae 6; decime $16\frac{2}{10}$, duodecimae 26, &c. eodem modo e differentia prima media, auferendae sunt differentiae correctae tertia 1, & quinque $\frac{1}{5}$.

Postquam inventae fuerint hae differentiae correctae, proximum est ut unaquaeque suo loco commode locetur. ut in tam multiplici negotio, omnis quantum fieri poterit confusio vitetur. hoc autem facilius assequemur si chartam habeamus in areolas, lineis rectis ad hunc modum distinctam; & si differentiae primae, tertiae, quintae, septimae, diverso a ceteris colore scribantur. Dati Logarithmi signati A quintum quemque locum occupant. Differentiaeque Secundae C, Quartae E, Sextae, Octavae, &c. in iisdem lineis cum Logarithmis locantur sinistrorsum. Differentiae vero Primae B, Tertiae D, Quintae, Septimae, &c. medias sedes possident cuisque spatii. Tandem sedes vacuae sunt implendae incipiendo a sinistris. Additione quartarum differentiarum tertiae perficiuntur, Tertiarum additione Secundae: & sic deinceps. Et inter addendum poterimus unitatem addere vel detrahere prout res postulabit. In his enim irrationalibus sufficit differentias habere veris proximas cum ipsas accurate veras invenire non possumus. Ea de causa, cum in initio huius Capitis dixeram, factis a differentiis primis per binarium, ultimam notam esse auferendam; hic nullam abstuli. sed in differentiis primis, relinquisque, unicam notam adijciendam putavi ultra limites constitutos: ut omnia certiora eveniant & perfectiora. Idem in Tangentibus, Secantibus, & Sinubus faciendum suadeo. at in potestatibus numerorum aequidistantium, ubi omnes differentiae sunt rationales, non minus quam ipsi dati numeri; omnia poterunt intra constitutos limites contineri, in quibus omnibus est definitus differentiarum diversarum numerus, ultra quem, si latera aequales servent differentias, progredi non poterit. ut in Quadratis duae sunt differentiarum species, in Cubis tres, in Biquadratis 4, &c. Suntque semper differentiae remotissimae aequales inter se, & aequales facto a

[p.30.]

potestate homogenea differentiae laterum, ducta in continue factum ab Indicibus eiusdem potestatis & omnium inferiorum. ut si laterum differentia sit 1 differentiae ultimae erunt, in Quadratis 2, in Cubis 6, in Biquadratis 24, in (5) 120, in (6) 120, in (7) 5040, &c. ipsi scilicet continue facti ab 1, 2...2. ab 1,2,3. 6. ab 1,2,3,4 24. &c.
sed si differentia laterum sit 3: differentia remotissima, Quadratorum erit 18, factus a quadrato 9 in 2. Cuborum 162, factus a cubo 27 in 6. Bi2quadratorum 1944 factus a biquadrato 81 in 24. &c.

	Differ. 4	Differ. 2 & 3.	Logarithm & Differentiae 1		nu. abs.
		92135 D	205583	190639 B	
	97	271445			
		92004	205485	919095	
	97	179441			
		91874	205388	739654	
E 130	97	087567 C	3325310371	71106 A	2115
		91744	205291	652087	
	96	995824	25515663	36315	16
		91614	205194	656263	
	96	904210	25720858	01941	17
		91484 D	205097	752054 B	
	96	812726	25925955	77146	18
		91355	205000	939327	
	96	721371	26130956	71079	19
		91226	204904	217956	
E 129	96	630146 C	3326335860	92875 A	2120
		91097	204807	587810	
	96	539050	26540668	51656	21
		90968	204711	04870	
	96	448082	26745379	56532	22
		90839 D	204614	600677 B	
	96	357244	26949994	16600	23
		90711	204518	243432	
	96	266533	27154512	40943	24
		90583	204421	976899	
E 128	96	175951 C	3327358934	38633 A	2125
		90455	204325	80094 9	
	96	085496			26
		90327	204229	71545 2	
	95	995169			27
		9020 0 D	204133	72028 2 B	

In omnibus his tam Potestatibus, quam Logarithmis, Tangentibus, Secantibus; necesse erit aliquot plures numeros continuare, sine quibus ulteriores differentias assequi non poterimus. ut in dato exemplo, ex altera parte 2110. 2105. ex reliqua 2130.2135. At in Sinubus, si tres sinus arcuum aequidifferentium dati fuerint, omnes differentiae vel remotissimae per proportionis regulam inveniri poterunt, si opus fuerit. Sunt enim Sinus & Differentiae Secundae, Quartae, Sextae, Octav continue proportionales. & differentiae Primae, Tertiae, Quintae, Septimae, sunt etiam continue proportionales. Atque ut Secundae differentiae sunt inter se proportionales Sinubus ipsis quibus situ respondent, eodemque modo Quaertae, sextae, &c. sic Primae inter se & Tertiae, quintae, Septimae, sunt proportionales Sinubus complementorum arcuum mediorum.

Sed sentio me studio elatum longius progressum quam Homogeneorum leges patiuntur.

[p.31]

Si Chiliadem aliam istis adijciendam censes, puta vicesimam primam, sumenda est pars quinta numeri illius a quo initium sumpturus es: primus numerus erit 20000, cuius pars quinta 4000, huius Logarithmo, & ducentis proximis, sigillatim addatur Logarithmus quinarij, summae erunt Logarithmi, quini cuiusque numeri, per totam Chiliadem: nempe 20000, 20005, 200010, 200015, &c. horum autem differentiae primae, sunt illae ipsae quae in Chiliade quinta, intra illos ducentos Logarithmos inveniuntur. unde petendae erunt differentiae secundae. differentias etiam tertias dabunt secundae. Quartae vero sunt perexiguae, quas ea de causa tuto negligere poterimus. deinde binarius, quaternarius, octonarius multiplicent differentias primas, secundas, tertias. Facti erunt differentiae mediae, quae locandae sunt suis locis, amputatis e secundis unica nota, e tertiis duabus. Primae autem sunt a suis sedibus arcendae, donec per subductionem Tertiarum fuerint correctae. reliqua omnia sunt per additionem peragenda.

Atque hic modus interferendi quatuor logarithmos inter duos proximos datos, appellari poterit Quinquesectio: cum ex unico intervallo fiant quinque. Poterunt etiam leges tradi generales pro Trisectione & Septisectione. harum autem omnium praestantissima est Quinquesectiono, sive commoditatem spectemus sive facilatatem.

Trisectionis tamen modum tradere paucis non pigebit. Sumantur datorum differentiae Primae, Secundae, Tertiae, &c. ut prius. Deinde, Primas dividat, 3; Secundas, 9; Tertias, 27; Quartas, 81, &c. Crescentibus divisoribus ratione tripla: Quoti erunt Differentiae Mediae Primae, Secundae, Tertiae, Quartae, &c. Hae differentiase mediae, sunt ut antea minuendae in omnibus, praeterquam in Sinubus; deinde ubi correctae fuerint, sunt in suas sedes transferendae: & incipiendo a minus & remotissimis, omnia sunt ut antea per additionem perficienda. Quantum autem unicuique differentiae detrahendum sit, indicat praesens Tabella.

A Differentia prima media, auferendus est triens Tertiae correctae.

A Differentia quarta media, auferendae sunt $^4/_3$ Sextae, $^2/_3$ Octavae, $^4/_{27}$ Decimae, $^1/_{81}$ Duodecimae correctarum.

Reliquae sectiones quibus a numeris paribus nomina tribuuntur, ut Bisectio, Quadrisectio, &c. sunt magis difficiles. Quod etiam in inventione Subtensarum in Circulo experimur: cum sectiones ab imparibus numeris denominatae, ipsas quaesitas subtensas unica operatione exhibeant. reliquae vero a paribus denominatae, non ipsas subtensas, sed subtensarum tantum Quadrata expediant.

1 (12)				
1 (11)				
1 (10)	$3^1/_3$ (12)			
1 (9)	3 (11)			
1 (8)	$2^2/_3$ (10)	$3^1/_9$ (12)		
1 (7)	$2^2/_3$ (9)	$2^3/_9$ (11)		
1 (6)	2 (8)	$1^6/_9$ (10)	$^{20}/_{27}$ (12)	
1 (5)	$1^2/_3$ (7)	$1^1/_9$ (9)	$^{20}/_{27}$ (11)	
1 (4)	$1^1/_3$ (6)	$^6/_9$ (8)	$^4/_{27}$ (10)	$^1/_{81}$(12)
1 (3)	1 (5)	$^3/_9$ (7)	$^1/_{27}$ (9)	
1 (2)	$^2/_3$ (4)	$^1/_9$ (6)		
1 (1)	$^1/_3$ (3)			
A	*B*	*C*	*D*	*E*

Examplum trisectionis hic vides in biquadratis.

Diff. mediae per divisione datarum.

4th by 81	3rd by 27	2nd by 9	1st by 3
	D.4 $^{72}/_{81}$		
	D.5 $^{63}/_{81}$		

Differenctiae datae.

4th	3rd	2nd	1st	biquadratics	sides
				256	4
				625	5
				1296	6
				2401	7
				4096	8

[p.32.]

4th diff.	3rd diff.	2nd diff.		1st diff.	biquad.	roots
					625 *A*	5
					809 $^7/_{81}$	5 $^1/_3$
					1031 $^{10}/_{81}$	5 $^2/_3$
					1296 *A*	6
					1608 $^{73}/_{81}$	6 $^1/_3$
					1975 $^5/_{81}$	6 $^2/_3$
					2401*A*	7

Tabula inventioni Logarithmorum inserviens

1	0,00	100001	0,00000,43429,23104
2	0,30102,99956,63981	100002	0,00000,86858,02778
3	0,47712,12547,19662	100003	0,00001,30286,39025
4	0,60205,99903,27962	100004	0,00001,73714,31846
5	0,69897,00043,36012	100005	0,00002,17141,81240
6	0,77815,12503,83644	100006	0,00002,60568,87210
7	0,84509,80400,14257	100007	0,00003,03995,49757
8	0,90308,99869,91943	100008	0,00003,47421,68882
9	0,95424,25094,39324	100009	0,00003,90847,44585
11	0,04139,26851,58226	1000001	0,00000,04342,94265
12	0,07918,12460,47625	1000002	0,00000,08685,88095
13	0,11394,33523,06837	1000003	0,00000,13028,81491
14	0,14612,80356,78238	1000004	0,00000,17371,74453
15	0,17609,12590,55681	1000005	0,00000,21714,66981
16	0,20411,99826,55925	1000006	0,00000,26057,59074
17	0,23044,89213,78274	1000007	0,00000,30400,50733
18	0,25527,25051,03306	1000008	0,00000,34743,41958
19	0,27875,36009,52828	1000009	0,00000,39086,32748
101	0,00432,13737,82645	10000001	0,00000,00434,29447
102	0,00860,01717,61928	10000002	0,00000,00868,58888
103	0,01283,72247,05172	10000003	0,00000,01302,88326
104	0,01703,33392,98780	10000004	0,00000,01737,17759
105	0,02118,92990,69938	10000005	0,00000,02171,47187
106	0,02530,58652,64771	10000006	0,00000,02605,76611

107	0,02938,37776,85210	10000007	0,00000,03040,06031
108	0,03342,37554,86950	10000008	0,00000,03474,35447
109	0,03742,64979,40624	10000009	0,00000,03908,64858
1001	0,00043,40774,79319	100000001	0,00000,00043,42935
1002	0,00086,77215,31227	100000002	0,00000,00086,85890
1003	0,00130,09330,20418	100000003	0,00000,00130,28834
1004	0,00173,37128,09000	100000004	0,00000,00173,71779
1005	0,00216,60617,56507	100000005	0,00000,00217,14724
1006	0,00259,79807,19908	100000006	0,00000,00260,57668
1007	0,00302,94705,53619	100000007	0,00000,00304,00613
1008	0,00346,05321,09508	100000008	0,00000,00347,43558
1009	0,00389,11662,36913	100000009	0,00000,00390,86502
10001	0,00004,34272,76862	1000000001	0,00000,00004,34295
10002	0,0000868502,11648	1000000002	0,00000,00008,68589
10003	0,00013,02688,05227	1000000003	0,00000,00013,02883
10004	0,00017,36830,58464	1000000004	0,00000,00017,37178
10005	0,00021,70029,72230	1000000005	0,00000,00021,71472
10006	0,00026,04985,47390	1000000006	0,00000,00026,05767
10007	0,00030,38997,84812	1000000007	0,00000,00030,40061
10008	0,00034,72966,85363	1000000008	0,00000,00034,74356
10009	0,00039,06892,49911	1000000009	0,00000,00039,08650