

§11.1.

Synopsis: Chapter Eleven.

Briggs establishes three methods of subtabulation in his growing table of logarithms, in this and in the two following chapters. The first method is simple proportion, which he has discovered does not work very well either for tables of logarithms or any others. However, he has found the best region for interpolation to be where the differences between known logarithms are smallest and changing slowest, for equal increases of the absolute numbers: which occurs near the number 100,000. A method based on this fact is used to find a good approximation to the number corresponding to a known logarithm: by moving the logarithm to this part of the table by adding the logarithm of a selected number, interpolating in the 'linear' region to find the approximate composite number, and then returning to the original position on dividing by the selected number.

§11.2.

Chapter Eleven. [p.21.]

To improve on the accuracy of a number found from the Chiliades by the use of a proportional part.

By these methods we have shown how we can find the logarithms of all numbers, prime and composite, whole and fractional. Of these, we have described thirty Chiliads¹, comprised of two columns. The first on the left-hand side of these contains the absolute numbers in the series of natural numbers increasing from 1 to 20,000: then this interrupted series, from 90,000 to 100,000. The other column near the former, has the logarithms of the absolute numbers, and the differences of these: which not only show the errors, if by chance these have crept in; but also provide a proportional part, as often as necessary; in order that we can find a little more accurately that which is had in the table itself. But that proportional part, which is acquired through these differences, is not absolutely perfect; but always departs a little from that absolute accuracy we are looking for. Indeed, since the absolute numbers increase everywhere uniformly, but the logarithms which are greater have smaller increments there: if for an absolute number, the logarithm of the number nearest should be increased by a proportional part, that part sought interposed by the difference, will always be less than it ought to be. But on the other hand, if for a given logarithm the absolute number is sought, that part will be increased more by the proportional part from the regular amount². The same inconvenience happens with sines, tangents, and secants; and entirely with all tables of numbers, where the differences from the one part are equal and unequal for the other. Nevertheless, where the inequality of the differences is small, there the difference from the

true value will be small. For that reason, the proportional part recedes further from the true value in the earlier Chiliads than in the latter; from which we shall be able to produce [an increase in accuracy] in this way. (How often the need arises to investigate a number sought more carefully, and the leisure there might have been [!])

The given logarithm (of which the absolute number is sought), in the first place drops the characteristic, (which does not change the number sought in any way, but only to move it forward by a higher step to a place more removed from unity) then taken away from the logarithm of ten, 1,00000,00000,0000. For the remainder, not inconveniently, will be called the arithmetical complement. This remainder (by adding first the characteristic seen by us to be the most useful) we seek among the Chiliads, and the nearest smaller logarithm selected, and (with the absolute number of the same logarithm reserved at the time) it is added to the given logarithm. The total by necessity falls into the final Chiliad, in which the corresponding absolute number of the same is taken, which through the added proportional part we can correct. This number, found and corrected, is divided by the first number set aside before; the quotient is the number agreeing with the given logarithm, which we sought in the first place. Take this example of this instruction. The mean proportional between unity and 1200 is sought, of which the logarithm is 3,07918,12460,4762. But the logarithm of the mean sought is half of this, by the definition of the logarithm, Ch.1 & 2; Axiom 2,Ch.2, which is of course, 1,53959,06230,2381. Which with the characteristic removed and taken from 1,00000,00000,0000 leaves 0,46040,93769,7619. The nearest logarithm to this, if we ignore the characteristic, is found in the third Chiliad 3,46029,63267,5746. And for this the absolute number 2886 ought to be set aside across the table; here the logarithm found is added to the given logarithm, and the total except the characteristic is 0,99988,69497,8127. This I seek in the 100th Chiliad, where the nearest smaller logarithm I discover to be 4,99988,27246,5701. And across from that 99973, which is increased by

*We can increase or
decrease the
Characteristic most
often by choice.
Complementary
Arithmetic*

the proportional part to 99973,97261,3004. The quotient of this number (divided by the number 2886 set aside before) is 34641.01615,1422.³

Factors	composite numbers	Logarithms	Factors	composite numbers	Logarithms
4.4.8.8	1024	3,01029,99566,3981	6.8.9	4320	3,63548,374681492
4.4.7	1120	3,04921,80226,7081	7.7.9.9	4410	3,64443,85894,6784
4.4.8.9	1152	3,06145,24790,8719	7.8.8	4480	3,65127,80139,9814
3.4	1200	3,07918,12460,4762	5.9	4500	3,65321,25137,7534
5.5.7.7	1225	3,08813,60887,0055	8.8.8.9	4608	3,66351,24704,1515
2.7.9	1260	3,10037,05451,1756	2.6.7.7.8	4704	3,67246,73130,6807
4.4.8.8	1280	3,10720,99696,4787	6.8	4800	3,68124,12373,7559
4.4.9.9	1296	3,11260,50015,3457	7.7	4900	3,69019,60800,2851
4.6.7.8	1344	3,12839,92687,1780	5	5000	3,69897,00043,3602
3.5.9	1350	3,13033,37684,9500	7.8.9	5040	3,70243,05364,4552
2.8.9	1440	3,15836,24920,9525	8.8.8	5120	3,70926,99609,7583
3.5	1500	3,17609,12590,5568	8.8.9.9	5184	3,71466,49928,6253
4.6.8.8	1536	3,18639,12156,9549	2.6.7.8.8	5376	3,73045,92600,4576
4.4	1600	3,20411,99826,5592	6.9	5400	3,73239,37598,2297
2.9.9	1620	3,20951,50145,4263	5.5.5.5.9	5625	3,75012,25267,8340
3.8.7	1680	3,22530,92817,2586	7.9.9	5670	3,75358,30588,9291
6.6.6.8	1728	3,23754,37381,4287	8.8.9	5760	3,76042,24834,2321
4.7.8.8	1792	3,25333,80053,2611	8.9.9.9	5832	3,76581,75153,0992
3.6	1800	3,25527,25051,0331	6	6000	3,77815,12503,8364
3.7.9	1890	3,27646,18041,7324	5.5.5.7.7	6125	3,78710,60930,3657
6.6.6.9	1944	3,28869,62605,9026	2.6.8.8.8	6144	3,78845,12070,2345
2	2000	3,30102,99956,6398	7.9	6300	3,79934,05494,5358
6.6.7.8	2016	3,30449,05277,7348	8.8	6400	3,80617,99739,8389
4.8.8.8	2048	3,31132,99523,0379	8.9.9	6480	3,81157,50058,7060
6.6.6	2160	3,33445,37511,5093	9.9.9.9	6561	3,81697,00377,5729
4.7.8	2240	3,35024,80183,3416	2.6.7.8	6720	3,82736,92730,5382
4.8.8.9	2304	3,36248,24747,5117	3.5.5.9	6750	3,82930,37728,3102
6.7.7.8	2352	3,37143,73174,0410	3.6.6.7.9	6804	3,83276,43049,4055
4.6	2400	3,38021,12417,1161	2.6.8.8.9	6912	3,83960,37294,7083
5.7.7	2450	3,38916,60843,6453	7	7000	3,84509,80400,1426
5.7.8.9	2520	3,40140,05407,8154	3.6.7.7.8	7056	3,84855,85721,2376
4.8.8	2560	3,40823,99653,1185	4.4.7.8.8	7168	3,85539,79966,5407
4.8.9.9	2592	3,41363,49971,9855	8.9	7200	3,85733,24964,3127
6.7.8.8	2688	3,42942,92643,8178	3.5.7.7	7350	3,86628,73390,8420
3.9	2700	3,43136,37641,5899	2.6.7.9	7560	3,87852,17955,0121
4.7	2800	3,44715,80313,4222	2.6.8.8	7680	3,88536,12200,3151
4.8.9	2880	3,45939,24877,5923	2.6.8.9.9	7776	3,89075,62519,1822
3	3000	3,47712,12547,1966	4.4.7.7	7840	3,89431,60626,8443
6.8.8.8	3072	3,48742,12113,5948	2.7.7.9.9	7938	3,89971,10945,7114
4.8	3200	3,50514,99783,1990	8	8000	3,90308,99869,9914
4.9.9	3240	3,51054,50102,0661	2.7.8.8.9	8064	3,90655,05191,0146
6.7.8	3360	3,52633,92773,8984	9.9	8100	3,90848,50188,7865
6.8.8.9	3456	3,53857,37338,0686	4.4.8.8.8	8192	3,91338,99436,3175
7.7.8.9	3528	3,54752,85764,5978	3.4.7	8400	3,92427,92860,6188
7.8.8.8	3584	3,55436,80009,9009	3.4.8.9	8640	3,93651,37424,7889
4.9	3600	3,55630,25007,6729	2.7.7.9	8820	3,94546,85851,3128
6.7.9	3780	3,57749,17998,3723	4.4.7.8	8960	3,95230,80096,6212
6.8.8	3840	3,58433,12243,6753	9	9000	3,95424,25094,3932
6.8.9.9	3888	3,58972,62562,5424	2.8.8.8.9	9216	3,96454,24660,7914
7.7.9.9	3969	3,59868,10989,0716	3.7.7.8.8	9408	3,97349,73087,3205
4	4000	3,60205,99913,2796	3.5.7.9	9450	3,97543,18085,0926
7.8.8.9	4032	3,60552,05234,3747	2.6.8	9600	3,98227,12330,3957
8.8.8.8	4096	3,61235,99479,6777	2.6.9.9	9720	3,98766,62649,2628
7.6	4200	3,62324,92903,9790	2.7.7	9800	3,99122,60756,9249

[Table 11 - 1.]

$\left. \begin{array}{l} \textit{proportions} \\ 1 \text{-----} \\ \ell.1200 \quad A \\ 1200 \text{-----} \end{array} \right\}$	$\left. \begin{array}{l} \textit{Logarithms} \\ 0 \\ 1,53959,06230,2381 \\ 3,07918,12460,4762 \end{array} \right\}$	To be given
10	1,00000,00000,0000	
A	1,53959,06230,2381	
B	0,46040,93769,7619	Complement
2886 C	3,46029,63267,5746	Nearest to the Complement
Sum of A and C	0,99988,69497,8127	
	42251,2426	Difference
H 99973	0,99988,27246,5701	
	43440,9600	Difference
99974	0,99988,70687,5301	
proportions	$\left\{ \begin{array}{l} 434409600 \\ 422512426 \\ 1 \text{---} \\ \underline{0972613004} \end{array} \right\}$	Given differences
		Proportional part to be added

Product to be divided H 99973972613004 Absolute number increased

Factors Divisor 2886
 Quotient 34641016151422

[Table 11 - 2].

All these are demonstrated by Axiom 2, Ch.2. Indeed $\ell.1200$ is now the other remaining factor: but 2886, with which the logarithms added gives 0,99988,69497,8127 equal to the logarithm of the product, undoubtedly of the number 99973,97261,3004. This product divided by the given factor 2886 gives as the quotient, the remaining factor in question 34641,01615,1422, which is almost equal to the mean proportional sought 34 641016151377546.

But since with division by this method which is, for a number with four places, exceedingly painstaking⁴, I have thought other composite numbers should be selected, which are shown on the preceding page [Table 11-1]. The numbers on the left are factors, from which by continued multiplication, they become those composite numbers; while to the right are the logarithms of the same. For if a number is divided by a composite number, the place of the divisor is taken by the factors of the same, which find the required quotient more easily. But how this ought to be done I show by an example. The mean proportional between 1 and 10800 is sought, the logarithm of this is 4,03342,37554,8695. The logarithm of the mean sought is the half of this, 2,01671,18777,4347, of which the complement is 0,98328,81222,5653. The logarithm found in the table here described

nearest to this is 0,98227,12330,3957 (to which 9600 is situated straight across), which added to that given gives 0,99898,31107,8304, the logarithm of the number 99766, which is corrected by the proportional part: 99766,12651, 6521. This divided by 9600, (or in turn by 2, 6, 8 as I have advised in Lemma 2, Ch. 5) 1039230484547 is the quotient.

You will be able to consider here how this method works:

		<i>Logarithms</i>		
proportions -	1-----	0	}	Given
	l. 10800 A	2,01671,18777,4347		
	10800 -----	4,03342,37554,8695		
	10	1,00000,00000,0000		
	A	2,01671,18777,4347		
	B	0,98328,81222,5653		Complement
	9600 C	0,98227,12330,3957		Nearest to the Complement
	Sum of A and C	0,99898,31107,8304		Difference
		5507,4025		
	H 99766	4,99898,25600,4279		Difference
		43532,0933		
	99767	4,99898,69131,5212		
	proportions	{ 435310933	}	Given differences
		55074025		
		1-----		
		126516521		
	H 99766126516521			Proportional part to be added Absolute number increased
Factors of the number 96 are 2.6.8, which are divisors of the number H found				
	H to be divided	99766126516521		
	Divisor 8)	124707658145651		First quotient
	Divisor 6)	2 0784609690942		Second quotient
	Divisor 2)	<u>1 0392304845471</u>		Final quotient
		103923048454133		The required mean for finding the square root.

This final quotient is a little more than the required mean

[Table 11-3.]

Here finally we consider the characteristic, hitherto almost ignored for the whole working .

Indeed, since the logarithm of the number A shall have 2 for a characteristic, the first three places sought for the root, designate the units of the whole number, while the following places express the numerator of the fraction added on, as indicated before in Ch.4. Between this and the previous working, this is hardly different, because the number nearest the complement was sought previously among the Chiliades; here however it is sought in the table of composite numbers here described: in that place the first divisor can be a prime number; here truly it will always be a composite number, in place of which all the divisors can be substituted, which allow the same

quotient to be found more easily. As for the divisor 9600: 8, 6, & 2 are substituted, of which the first divides the given number, the second the given quotient, the third that second quotient.

§11.3. Notes On Chapter Eleven.

¹ A Chiliad represents a thousand, we get the prefix 'kilo' from it. Briggs is describing the actual tables of logarithms, which form a large and profoundly impressive part of the *Arithmetica*: which are represented in this book by a few sample pages only.

² Briggs' 'inconvenience' is shown here using co-ordinates: AB is the log curve drawn schematically: δ is the deficiency, if the co-ordinate $\log N_3$ is estimated by proportionality for a known N_3 , from similar triangles;

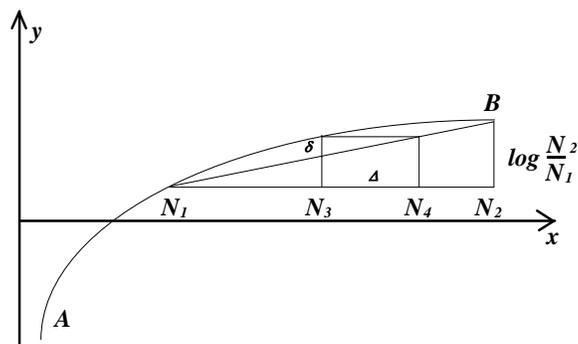


Figure 11 - 1.

while N_4 is the number estimated from proportionality, too large by Δ , for a known log value at the ordinate N_3 .

³ Briggs tries to come to terms numerically with the non-linear nature of the log function in this chapter, in his attempts to evaluate either the logarithm of a number lying between numbers with known logarithms, or the value of a number with a known logarithm similarly placed between known values. Thus, if $N_1 < N_3 < N_2$, the approximate proportionality he considers is:

$$\frac{\log(N_2/N_1)}{N_2 - N_1} \sim \frac{\log(N_3/N_1)}{N_3 - N_1}$$

, with the intention of finding either N_3 or $\log N_3$, one of which is given.

Now, if $N_2 = N_1 + 1$, and $N_3 = N_1 + x$, for $0 < x < 1$, then the approximate proportionality is reduced to $\frac{\log(1+1/N_1)}{1} \sim \frac{\log(1+x/N_1)}{x}$, which is true if the first term only in the Mercator expansion is significant, which is the case for a large enough value of N_1 being chosen, near the end of the log tables.

Now consider the logarithm A in Table 11-2, corresponding to the unknown positive number a , or $\sqrt{1200}$, of which the best estimate is required. We cannot ignore powers of ten in our analysis:

hence, the number $a/10$ lies between 1 and 10; the complementary number b (also known only by its logarithm B) satisfies $ab/10 = 10$. A number c (2886) is selected from the Chiliads which has a logarithm C just less than the complement $\log B$; the product $ac/10^4$ is formed, which from its logarithm is just less than 10. By inspection of the Chiliades, the numbers h (99973) and $h + 1$ (99974) with known logarithms $\log H$ and $\log (H + 1)$ satisfy $\log H < \log (ac) < \log (H + 1)$ and hence $h < ac < h + 1$. The above approximate proportionality is set up: $\frac{\log((h+1)/h)}{1} \sim \frac{\log(ac/h)}{ac-h}$;

Figure 11 - 2 shows the appropriate numerical ratios, from which $AE = x < 1$ is found, where $ac = h + x$, and $a = (h + x)/c = 9973.972613004/2886 = 34.641016151422$. (The true value being 34.6410161513775), as Briggs now asserts).

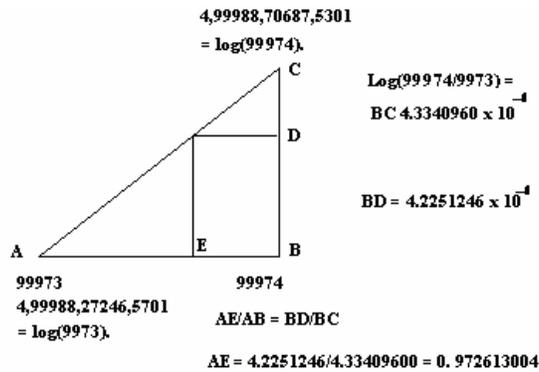


Figure 11 - 2.

[Note : in this diagram, $\log(9973)$ should read $\log(99973)$; my thanks to a reader for pointing out typo's in this diagram.]

⁴ Another example is shown, where a series of divisions by the smaller factors of a composite number is easier to perform than the long division process considered above.

§11.4.

Caput XI. [p.21.]

Numerum in Chiliadibus repertum, per partem proportionalem emendare.

Ostendimus quibus modis omnium numerorum primorum & compositorum, integrorum & partium, Logarithmos invenire poterimus. horum, triginta Chiliades his descriptas habemus, duabus columnis comprehensas. Earum prior versus sinistram, continet numeros absolutos naturali serie crescentes, ab Unitate ad 20000: deinde hac serie interrupta, a 90000 ad 100000. Altera columna priori contigua, habet numerorum absolutorum logarithmos, eorumque differentia. quae non solum ostendunt menda, si quae forte irrepserint, sed etiam partem subministrant proportionalem, quoties usus postulabit; ut aliquid quaerimus accuratius eo, quod in ipso abaco habetur. illa autem pars proportionalis, quae per has acquiritur differentias, non est absolute perfecta; sed semper aliquantulum recedit ab ea quam quaerimus accurata

veritate. cum enim numeri absoluti ubique crescant aequaliter, Logarithmi autem quod sunt maiores eo minora habent incrementa: si pro numero absoluto, Logarithmus numeri proxime, sit per partem proportionalem augendus, pars illa per differentiam interiectam quaesita, erit semper minor quam oportuit. contra autem, si pro Logarithmo dato quaeratur numerus absolutus, is per partem proportionalem auctus erit iusto maior. Idem accidit incommodi in Sinubus, Tangentibus Secantibus; omninoque in omnibus tabulis numerorum, ubi differentiae ex altera parte aequales reliquae inaequales. Veruntamen quo minor est differentiarum inequalitas, eo minor erit aberratio a vero. Ea de causa, pars proportionalis longius recedit a vero in prioribus Chiliadibus, quam in posterioribus, ad quas (quoties opus postulat, & otium fuerit, numerum quaesitum accuratius investigare) provocare poterimus, ad hunc modum.

Datus Logarithmus (cuius numerus absolutus quaeritur) in primis deponat Characteristicam (quae numerum quaesitum nullo modo mutare potest, sed tantum promovere in gradum altiorem & ab Unitatem loco remotiorem.) deinde auferare e Logarithmo Denarii 1,00000,00000,0000. reliquis non incommode vocari poterit complementum Arithmeticum. hunc reliquum (addita prius Characteristica quae nobis visa fuerit commodissima) quaeramus inter Chiliades. sumaturque Logarithmus proxime minor. & (numero eiusdem absoluto ad tempus seposito) additur dato Logarithmo. totus incidet necessario in ultimam Chiliadem, in qua sumendus est numerus absolutus eidem congruens. quem per partem proportionalem adiectam emendare poterimus. hunc numerum inventum & emendatum, dividat prior numerus antea sepositus; quotus erit numerus dato Logarithmo congruens, quem in primis quaerebamus. huius praecepti hoc cape exemplum. Quaerendus sit medius proportionalis inter Unitatem & 1200 huius Logarithmus est 3,07918,12460,4762. medii autem quaesiti Logarithmus est huius dimidius, ex definitione Logarithmorum, cap.1. & 2.ax.cap.2. nempe 1,53959,06230,2381. qui deposita Characteristica ablatus e 1,00000,00000,0000 relinquit 0,46040,93769,7619. cui proximus, si Characteristicam negligamus, in Chiliade tertia reperitur 3,46029,63267,5746. eique e regione numerus absolutus seponendus 2886; hic Logarithmus repertus addatur dato, totus praeter Characteristicam erit 0,99988,69497,8127. hunc quaero in Chiliade centesima, ubi proxime minorem invenio 4,99988,27246,5701. eique e regione 99973, qui per partem proportionalem auctus sit 99973,97261,3004. huius numeri (divisi per 28860, numerum ante sepositum) quotus erit 34641,01615,1422.

[p.22.]

Factors	numeri comp.	Logarithmi.	Factors	numeri comp.	Logarithmi.
4.4.8.8	1024	3,01029,99566,3981	6.8.9	4320	3,63548,37468,1492
4.4.7	1120	3,04921,80226,7081	7.7.9.9	4410	3,64443,85894,6784
4.4.8.9	1152	3,06145,24790,8719	7.8.8	4480	3,65127,80139,9814
3.4	1200	3,07918,12460,4762	5.9	4500	3,65321,25137,7534
5.5.7.7	1225	3,08813,60887,0055	8.8.8.9	4608	3,66351,24704,1515
2.7.9	1260	3,10037,05451,1756	2.6.7.7.8	4704	3,67246,73130,6807
4.4.8.8	1280	3,10720,99696,4787	6.8	4800	3,68124,12373,7559
4.4.9.9	1296	3,11260,50015,3457	7.7	4900	3,69019,60800,2851
4.6.7.8	1344	3,12839,92687,1780	5	5000	3,69897,00043,3602
3.5.9	1350	3,13033,37684,9500	7.8.9	5040	3,70243,05364,4552
2.8.9	1440	3,15836,24920,9525	8.8.8	5120	3,70926,99609,7583
3.5	1500	3,17609,12590,5568	8.8.9.9	5184	3,71466,49928,6253
4.6.8.8	1536	3,18639,12156,9549	2.6.7.8.8	5376	3,73045,92600,4576
4.4	1600	3,20411,99826,5592	6.9	5400	3,73239,37598,2297
2.9.9	1620	3,20951,50145,4263	5.5.5.5.9	5625	3,75012,25267,8340
- - - -	- - - -	- - - - - - - -	- - - -	- - - -	- - - - - - - -
7.8.8.9	4032	3,60552,05234,3747	2.6.8	9600	3,98227,12330,3957
8.8.8.8	4096	3,61235,99479,6777	2.6.9.9	9720	3,98766,62649,2628
7.6	4200	3,62324,92903,9790	2.7.7	9800	3,99122,60756,9249

[p.23.]

prop.	{	1-----	0	}	dati
		ℓ.1200	A 1,53959,06230,2381		
		1200-----	3,07918,12460,4762		
		10	1,00000,00000,0000		
		A	1,53959,06230,2381		
		B	0,46040,93769,7619		
		2886	C 3,46029,63267,5746		
		Summa A, C	0,99988,69497,8127		
		H	99973 0,99988,27246,5701		
			43440,9600		
	99974 0,99988,70687,5301	Differentia			

Complementum.
Complemento proximus
Differentia.
Differentia

