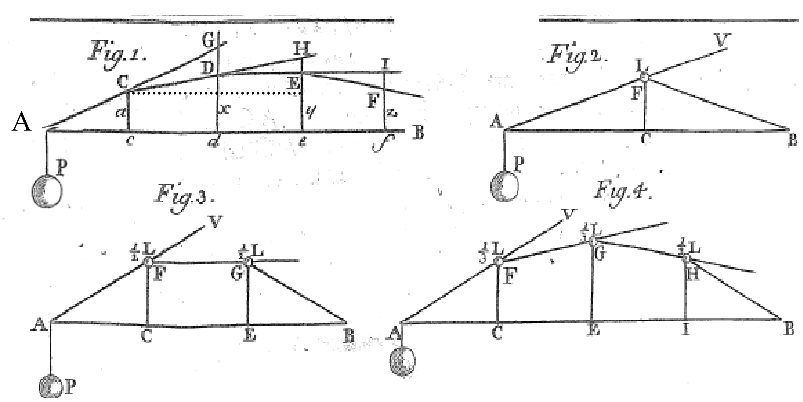
JOHANNE BERNOULLI.

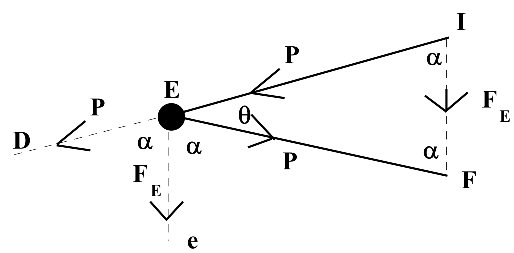
Meditations
On Vibrating Strings,

with small weights set at equal intervals to each other.
Where nevertheless from the principle of the action of the unbalanced forces the number of vibrations for one given oscillation of a pendulum of a given length D is sought. [*see Opus tom. III. p.125, 126.*]

The vibrating string is ACDEF &c., to which at equal distances are joined equal small weights C, D, E, F, &c. which are placed together as in Fig. 1, in order that the individual small weights can return at the same time to the straight line in the situation AB : from which it follows, that the velocities of the individual weights and indeed their forces and the accelerations are in proportion to the distances traveled through: Cc, Dd, Ee, &c. But from the principles of statics [*i. e.* dynamics], the tension of the string is [in the same ratio] to the force by which any small weight such as E is urged towards e, as the sine of the angle DEe is to the sine of the angle DEF, or IEF, that is (on account of the strings in the figure being almost straight, & the intervals between the little weights being equal) as the total sine to FI.

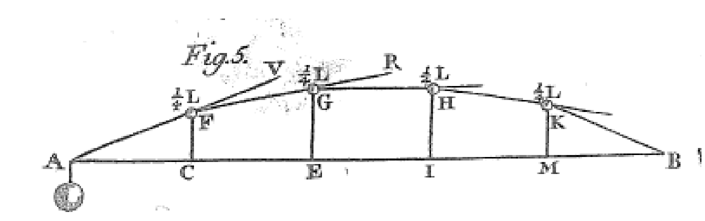


[Note the use of the triangle of forces centred on E: the force triangle and the lines along which these forces act, constitute similar triangles with sides representing distances rather than forces in the second triangle, which are superposed in the explanatory diagram: in this case Δ EFI is viewed as the force triangle for the mass E, which has the sides EF and EI for the tensions P and FI or Ee for the unbalanced force F_E acting on E;
thus, $P/\sin(\alpha) = F_E/\sin(\theta)$.]



Therefore, *from the equality*, the distances Cc, Dd, Ee, &c. are themselves proportionals to DG, EH, FI, &c. respectively.
Now, DG = Gd - Dd = 2Cc - Dd = $2a - x$; HE = He - Ee = $[a + 2(x - a)] - y$ = 2Dd - Cc - Ee = $2x - a - y$; FI = If - Ff = $[x + 2(y - x) - z]$ = 2Ee - Dd - Ff = $2y - x - z$; & hence $2a - x : a = 2x - a - y : x = 2y - x - z : y = 2z - y - t : z =$ &c

[Thus, in an inductive manner, where t is the next displacement not shown on diagram, it is observed that if the vertical displacements of the masses located at x_n are considered as some function $f(x_n)$, then the unbalanced force is proportional to the



difference of the gradients, or to the finite difference form of the second derivative : $2f(x_n) - f(x_{n-1}) - f(x_{n+1})$, for suitable n; as the horizontal distances are equal, they are omitted for simplicity. From the supposition, these finite difference distances are in proportion to the distance from the axis in the same ratio as the force acting on each particle is to the displacement according to the idea of simple harmonic motion : thus, for example, in Fig. 4. ($F_{ext.}$ on F)/a = ($F_{ext.}$ on G)/x = ($F_{ext.}$ on H)/a; while the size of ($F_{ext.}$ on F) α $(2a - x)$, ($F_{ext.}$ on G) α $(2x - 2a)$, and ($F_{ext.}$ on H) α $(2a - x)$, as in part II of the Lemma. Thus, the relations between a, x, y, z, etc. can be established in the parts of this Lemma.]

From which the following Lemmas hold:

I. If there are two small weights, then $x = a, y = 0$, with the others not considered. (Fig. III & IV;)

II. If there are three small weights, then $y = a, z = 0$, with the others not considered; hence $2a - x : a = 2x - 2a : x$, hence $2ax - xx = 2ax - 2aa$ & $x = a\surd 2$.

III. If (Fig. V) there are four small weights, then $y = x, z = a,$ & $t = 0$, with the others not considered; hence $2a - x : a = x - a : x$; hence $2ax - xx = ax - aa$ & $x = \frac{1}{2}a + \sqrt{\frac{5}{4}aa}$.

IV. If there are five small weights, then $z = x, t = a, u = 0$, the rest neglected ; hence $2a - x : a = 2x - a - y : x = 2y - 2x : y$; from these ratios the two equations : $xx = aa + ay$, & $yx = 2ax$ are obtained. From the first equation $y = (xx - aa) : a$; from the latter $y = 2a$; hence $x = a\surd 3$.

V. If there are six small weights, then $z = y, t = x, u = a, s = 0$, with the remainder ignored ; thus $2a - x : a = 2x - a - y : x = y - x : y$. From these two equations, $xx = aa + ay$, & $ay - yx = -ax$. From the latter equation :
$y = ax :(- a + x)$; from the other $y = (xx - aa):a$; hence $aax = x^3 - axx - aax + a^3$, or $x^3 - axx - 2aax + a^3 = 0$.

VI. If there are seven small weights, then $t = y, u = x, s = a, w = 0$, without attending to the rest.. Hence $2a - x : a = 2x - a - y : x = 2y - x - z : y = 2z - 2y : z$; & thus three equations are obtained : $xx = aa + ay$, $xy = ax + az$ & $xz = 2ay$. From the second equation, $z = (xy - ax) : a$, from the third, $z = 2ay : x$; unde $y = axx : (x^2 - 2a^2) : a$; therefore $(x^2 - a^2) : a = ax^2 : (x^2 - 2a^2)$, hence $aaxx = x^4 - 3aaxx + 2a^4$; or $x^4 - 4aaxx + 2a^4 = 0$; and thus $xx = 2aa + aa\surd 2$, & $x = a\surd(2 \pm \surd 2)$; $y = a \pm a\surd 2$; $z = (2a \pm 2a\surd 2) : \surd(2 + \surd 2) = + 4\surd(4 + 2\surd 2)$. Where it is to be noted that the negative sign is not to be squared.

PROBLEM I.

If now the string or thread ALB (Fig. II) is considered without density and all the weight is loaded in the middle in the small weight L ; the tension in the string is P ; the time is sought for the semi-vibration through the distance LC. Let LC = a, AL or LB [AC in text] = b, then AL - AC = ($AL^2 - AC^2$) : (AL + AC) = LC^2 : 2AC = $a^2 : 2b$, & ALB - AB = $a^2 : b$, and the extension of the string will be the change in height for the hanging weight P.

[We may note at this stage the manner in which the problem is set up : the string is considered to be inextensible and massless, and changes in length are achieved by moving the large weight P up or down in a frictionless manner. Each small weight is a s.h.o. with amplitude a, x, y, etc. According to Bernoulli, the acceleration is proportional to the displacement, and the maximum velocity on passing through the equilibrium position is proportional to the maximum displacement, which follows from energy conservation. The maximum speed is that derived from a change in the potential energy (or due to the *action of the unbalanced forces* or *vi vivarum* as termed by Bernoulli, using a word for force that was originally applied in a military sense; the work/energy principle had been around for some time in various vague forms, and as such is to be found in Newton's Principia, but never stated in an outright manner, as is still the case here. Bernoulli has dealt with some problems that can be solved using the principle in an earlier paper, as indicated above.) of a body freely falling under gravity through some height z; if there is only one small weight m, then $mgz = mv^2/2$, and $mgz = Mga^2/b$; hence $v = \surd(2gz)$. The choice is made that $2g = 1$ to simplify the calculation, which means that Bernoulli uses the formula mv^2 for the energy of

motion, and as the time for a quarter period of a vibrating mass or chord is to be taken in the ratio with the quarter period of a pendulum, the pendulum is given the same value of g to enable the quarter period to be evaluated. If there are several small weights, then the potential energy is portioned out according to the amplitudes of the oscillations: thus, if the amplitude of one small mass is k times the amplitude of another equal mass, then the maximum velocity is also k times as large, while the hypothetical distance of free fall goes up by a factor k^2.
The angular velocity ω of the s.h.m. is evaluated from the maximum velocity v from $v = \omega a$, from which the time for the quarter period if found to be $\pi a/2v$.]
Let z be the vertical height through which a freely falling weight acquires a velocity equal to that which L acquires in moving from L to C, which velocity is thus equal to $\sqrt{z}$. The virtual work of the small weight L = Lz is equal to the virtual work of the stretching force $= (a^2 : b) \times$ P ; thus $z = a^2 \times P : b \times L$ [is the equivalent height of the free falling body for the energy transfer]. Whereby truly the force drawing the point L towards C is always in proportion to the distance LC [This force is $2\mathrm{P}z/b$, the vertical components of the tension]; it can be supposed that the diameter of a circle to the circumference to be as 1 to p [i. e. $p = \pi$], & v is the velocity of the point L at C. The time to pass through a distance LC, or the time of a semi-vibration is equal to $ap : 2v = ap : 2\sqrt{z} = p\sqrt{(b.\mathrm{L})} : 2\sqrt{\mathrm{P}}$, & the time of one semi-oscillation of a pendulum of a given length D is equal to $p\sqrt{\tfrac{1}{2}\mathrm{D}}$. Hence $p\sqrt{\tfrac{1}{2}\mathrm{D}}$ is divided by $p\sqrt{(b.\mathrm{L})} : 2\sqrt{\mathrm{P}}$, *i.e.* $\sqrt{(2\mathrm{P.D})} : \sqrt{(b.\ \mathrm{L})}$ gives the number of vibrations of the string during one oscillation of the pendulum D as
$\sqrt{4}\mathrm{D.P} : \sqrt{\mathrm{AB}}.\mathrm{L} = 2\sqrt{(\mathrm{D.P} : \mathrm{AB.L})}$.

[In modern terms, the equation satisfied by the vibrating mass is :
$m\ddot{x} = -2T.\frac{x}{b}$; giving $\omega = \sqrt{\frac{2T}{mb}} = \sqrt{\frac{2Mg}{mb}} = \sqrt{\frac{M}{mb}} = \sqrt{\frac{\mathrm{P}}{\mathrm{L}b}}$. This is Bernoulli's result, where we note that L is a mass, while P was originally a weight but now P is now a mass times by a unit acceleration to give the correct dimensions. In any case, the velocity of L at C becomes $v = \omega a =$ $v = \omega a = a\sqrt{\frac{\mathrm{P}}{\mathrm{L}b}}$, Bernoulli's result, and the time T/4 becomes $\frac{p}{2}\sqrt{\frac{b\mathrm{L}}{\mathrm{P}}}$. This quantity is now to be compared with the quarter swing of a pendulum of length l, for which $T_{qpen} = \frac{\pi}{2}\sqrt{\frac{l}{g}}$: if in $T_{qpen} = \frac{\pi}{2}\sqrt{\frac{l}{g}}$, we substitute $g = 0.5$, then we obtain
$T_{qpen} = p\sqrt{\frac{D}{2}}$, and the ratio of quarter periods becomes
$T_{qpen} / T_{quarter} = p\sqrt{\frac{D}{2}} / \frac{p}{2}\sqrt{\frac{bL}{P}} = \sqrt{\frac{2PD}{bL}} = 2.\sqrt{\frac{P.D}{AB.L}}$ as required in Bernoulli's formula.]

PROBLEM II.

Now the string AFGB is stretched by the weight P, and is weighed by two equal small weights (Fig. 3), each of which is equal to $\frac{1}{2}L$, and which divide the string into three equal parts, AF, FG, GB. Again let FC = GE = BE = $a^2 : 2b$; and thus AFGB - AB = $a^2 : b$ = for the fall of the weight P. Again let $\sqrt{z}$ = velocity of the point F located in C, or the point G in E; the *vires vivae* of the small weights F and G together = Lz = *vi vivae* of the large stretching weight P = $(aa : b) \times$ P, then $z = a^2.\mathrm{P} : b.\mathrm{L}$. Therefore the remainder is now found as before, for if that quantity is now put into the formula for the number of vibrations, we obtain : $\sqrt{(2D.P : b.L)} = \sqrt{(2D.P : \tfrac{1}{3}AB.L)} = \sqrt{(6D.P : AB.L)}$.

[In this case, the speeds of the masses m on the line AB are both equal to ωa, the total kinetic energy at this point is $m\omega^2 a^2$, which is equal to the change in the potential energy of the large mass M, which in turn is Mga^2/b; hence the frequency of the motion is $\omega = \sqrt{\frac{Mg}{mb}} = \sqrt{\frac{\mathrm{P}}{\mathrm{L}b}}$ and $\mathrm{T}_{\text{quart.}} = \frac{\pi}{2}\sqrt{\frac{\mathrm{L}b}{\mathrm{P}}}$. Again,
$T_{qpen} = \pi\sqrt{\frac{D}{2}}$, giving $T_{qpen} / T_{quarter} = \pi\sqrt{\frac{D}{2}} / \frac{\pi}{2}\sqrt{\frac{bL}{P}} = \sqrt{\frac{2PD}{bL}} = .\sqrt{\frac{6.P.D}{AB.L}}$]

PROBLEMA III.

Now let there be three small weights (Fig. 4), each equal to $\frac{1}{3}L$, again the difference AF - AC = BH - BI is equal to $a^2 : 2b$. But FG - CE = HG - IE = $(3aa - 2aa\surd 2) : 2b$, (from Lemma 2.)
[For AC = CE = EI = $b - a^2/2b$; $x = a\surd 2$, and FG - CE = $(FG^2 - CE^2)/(FG + CE) = (x - a)^2/2(b - a^2/2b) \sim (3a^2 - 2a^2\surd 2)/2b$]. Hence AFGHB - AB = $(4aa - 2aa\surd 2) : b$ is equal to the distance of descent of the weight P holding the string. Now, with $\surd z$ called equal to the velocity of the point F at C, then the velocity of the point G at E = $\surd 2z$ [as the amplitude is increased over a by a factor of $k = \surd 2$] ; then the *action of the unbalanced forces* of all the small weights taken together is equal to $\frac{4}{3}z \times L$ which is equal to the same quantity for the stretching weight, that is
$(4aa - 2aa\surd 2)$ P : b; thus z = $(6aa - 3aa\surd 2)$ P: $2b$L = $(12aa - 6aa\surd 2)$ P: AB × L, and thus $\surd z$, or $v = \surd((12aa - 6aa\surd 2)$ P: AB × L) and the time to travel through the distance FC, or
$ap : 2v = p\,\surd$AB × L : $2\surd(12 - 6\surd 2)$ P. Therefore $p\sqrt{\frac{1}{2}\mathrm{D}}$ divided by $ap : 2v$, i. e.,

$2\sqrt{(6-3\sqrt{2})}\mathrm{P}\times\mathrm{D}:\sqrt{\mathrm{AB}\times\mathrm{L}}:\surd\mathrm{AB}\times\mathrm{L}$, gives the number of vibrations of the string in the time of one swing of the given pendulum D.

[In this case, the speeds of the masses F and H m on the line AB are both equal to ωa, while the speed of the middle mass G is $\omega a\surd 2$; hence the total maximum kinetic energy on crossing AB is $m\omega^2a^2 + m\omega^2a^2$, or $2m\omega^2a^2 = (2L/3)\omega^2a^2$, which is equal to the change in the potential energy of the large mass M, which in turn is $(2 - \surd 2)\,2Mga^2/b = (2 - \surd 2)\,Pa^2/b$; hence the frequency of the motion is

$$\omega = \sqrt{\frac{3(2-\sqrt{2})\mathrm{P}}{2\mathrm{L}b}} = \sqrt{\frac{(12\text{-}6\sqrt{2})\mathrm{P}}{\mathrm{L}\times\mathrm{AB}}} \text{ and } \mathrm{T}_{\text{quart.}} = \frac{\pi}{2}\sqrt{\frac{\mathrm{L}\times\mathrm{AB}}{(12\text{-}6\sqrt{2})\mathrm{P}}}. \text{ Again, } T_{qpen} = \pi\sqrt{\frac{D}{2}}, \text{ giving}$$

$$T_{qpen}/T_{quarter} = \pi\sqrt{\frac{D}{2}}\Big/\frac{\pi}{2}\sqrt{\frac{\mathrm{L}\times\mathrm{AB}}{(12\text{-}6\sqrt{2})\mathrm{P}}} = 2\sqrt{\frac{(6-3\sqrt{2})\mathrm{PD}}{\mathrm{AB}\times\mathrm{L}}}, \text{ as required.]}$$

PROBLEM IV.

Let there be four small weights, each equal to $\frac{1}{4}L$, (see Fig. 5). By supposing for the present that GE = HI = x, with the remaining set up as before, again AF - AC = BK - BM = aa: $2b$; FG - CE = KH - MI = $(xx - 2ax + aa) : 2b$; thus AFGHKB - AB = $(xx - 2ax + 2aa)$: b b is equal to the distance of descent of the weight P holding the string. The velocity of the point F at C equal to $\sqrt{z}$, the velocity of the point G at E is equal to $\frac{x}{a}\sqrt{z}$; and hence the sum of the *actions of the unbalanced forces* of all the small weights is equal to $\frac{aa\ +\ xx}{2aa}\times z\times L$ = same quantity for the weight P = $(xx - 2ax + 2aa)$ P : b hence z = $(2aaxx - 4a^2x + 4a^4)$ P: $(xx + aa).b$L. Whereby $\surd z$ or $v = a\surd(2xx - 4ax + 4aa)$ P: $\surd(xx + aa).\,b$ L; thus the time to pass through FC =
$ap : 2v = p\,\surd(aa + xx).b$L : $2\surd(2xx - 4ax + 4aa)$ P. Hence $p\sqrt{\frac{1}{2}\mathrm{D}}$ divided by $ap : 2v$, *i.e.*,

$\sqrt{(4xx-8ax+8aa)}\mathrm{D}\times\mathrm{P}:\sqrt{(aa+xx)}\ b\mathrm{L}$, which is equal to

$2\sqrt{(5-\sqrt{5})}\ \mathrm{D}\times\mathrm{P}:\sqrt{(5+\sqrt{5})b}\mathrm{L} = 2\sqrt{(25-5\sqrt{5})}\ \mathrm{D}\times\mathrm{P}:\sqrt{(5+\sqrt{5})}\mathrm{AB}\times\mathrm{L}$, [since by Lemma III, $x = \frac{1}{2}a + a\sqrt{\frac{5}{4}}$] gives the number of vibrations of the string in one oscillation of the given pendulum D.

[In this case, the speeds of the masses m, F and K on the line AB are both equal to ωa, while the speed of the middle masses G and H is ωx; hence the total maximum kinetic energy of all the small masses as they cross AB is $m\omega^2a^2 + m\omega^2x^2$, or $(L/4)\omega^2(a^2 + x^2)$, which is equal to the change in the potential energy of the large mass M, which in turn is $(x^2 - 2ax + 2a^2)$ Mg$/2b$ = $(x^2 - 2ax + 2a^2)$ P$/4b$; hence the frequency of the motion is

$$\omega = \sqrt{\frac{(x^2\text{-}2ax\text{+}2a^2)\mathrm{P}}{2b}}\times\sqrt{\frac{4}{\mathrm{L}(a^2+x^2)}} = \sqrt{\frac{(2x^2\text{-}4ax\text{+}4a^2)\mathrm{P}}{\mathrm{L}b(a^2+x^2)}} \text{ and } \mathrm{T}_{\text{quart.}} = \frac{\pi}{2}\sqrt{\frac{\mathrm{L}b(a^2+x^2)}{(2x^2\text{-}4ax\text{+}4a^2)\mathrm{P}}}.$$

Again, $T_{qpen} = \pi\sqrt{\frac{D}{2}}$, giving

$$T_{qpen} / T_{quarter} = \pi\sqrt{\frac{D}{2}} / \frac{\pi}{2}\sqrt{\frac{\mathrm{L}b(\mathrm{a}^2+\mathrm{x}^2)}{(2\mathrm{x}^2\text{-}4\mathrm{ax}+4\mathrm{a}^2)\mathrm{P}}} = 2\sqrt{\frac{(2x^2-4ax+4a^2)\mathrm{PD}}{\mathrm{Lb}(\mathrm{a}^2+\mathrm{x}^2)}},$$

as required, where P and L are taken as masses rather than weights. The final result follows by inserting $x = \frac{1}{2}a + a\sqrt{\frac{5}{4}}$. The corresponding treatments of the last two problems are similar to the above and omitted for brevity; Bernoulli's treatment is clear in any case.]

PROBLEM V.

Now there are 5 small weights, each of which is equal to $\frac{1}{5}L$. By supposing again that GE = KM = x, with the rest always the same, HI or y [by Lemma IV] = 2a & $x = a\sqrt{3}$, & AF - AC = BN - BO = aa: $2b$; FG - CE = NK - OM = $(xx - 2ax + aa) : 2b = (4aa - 2aa\sqrt{3}) : 2b$; GH - EI = KH - MI = $(yy - 2xy + xx) : 2b$ $= (7aa - 4aa\sqrt{3}) : 2b$; then on account of which AFGHKNB - AB = $(12aa - 6aa\sqrt{3}) : b$ is equal to the amount of descent of P. Moreover, by taking $\sqrt{z}$ for the velocity of the point F at C, the velocity of the point G at E = $= 3\sqrt{z}$; & the velocity of the point H at I = $= 2\sqrt{z}$. As a consequence the sum of all the *action of the unbalanced forces* of the small weights is equal to $\frac{12}{5} \times z \times L$, which is equal to the same quantity for the stretching weight P = $(12aa - 6a^2\sqrt{3})$ P : b ;
hence z = $(10aa - 5a^2\sqrt{3})$ P: 2b×L = $(30a^2 - 15aa\sqrt{3})$ P: AB × L ; & thus
$\sqrt{z}$ or $v = a\sqrt{(30 - 15\sqrt{3})}$ P: √AB × L, then the time to pass through the length
FC = $ap : 2v = p$√AB × L: 2$\sqrt{(30 - 15\sqrt{3})}$ P. Hence $p\sqrt{\frac{1}{2}\mathrm{D}}$ divided by $ap : 2v$,
i.e. $\sqrt{(60 - 30\sqrt{3})}$ D × P : √AB × L gives the number of vibrations of the string for one oscillation of the given pendulum D.

PROBLEM VI.

Let there be six small weights, each of which is equal to $\frac{1}{6}L$. Now by putting GE = NO = x, HI = KM = y ; AF - AC = BR - BS = $a^2 : 2b$; FG - CE = RN - OM = $(xx - 2ax + aa) : 2b$; GH - EI = NK - OM = $(yy - 2xy + xx) : 2b$; therefore AFGHKNRB - AB = $(2xx - 2ax + 2aa + yy - 2yx) : b$ is equal to the distance for the descending weight P. Indeed by taking $\sqrt{z}$ for the velocity of the point F at C, then the velocity of the point G at E is equal to $x\sqrt{z : a}$; & velocity of the point H in I is equal to $y\sqrt{z} : a$; then the sum of the *actions of the unbalanced forces* of all the small weights is equal to $(aa + xx + yy)$ z × L : $3aa$, which is equal to the *vi vivae* of the stretching weight P = $(2xx - 2ax + 2aa + yy - 2yx)$ P : b ;
& thus √z = $\sqrt{(6aaxx - 6a^2x + 6a^4 - 3aayy - 6aayx)}$ P: $\sqrt{(aa + xx + yy)b\mathrm{L}} = v$. Hence the time to pass through FC = $ap : 2v$
$= ap\sqrt{(aa + xx + yy)b\mathrm{L}} : 2\sqrt{(6aaxx - 6a^2x + 6a^4 - 3aayy - 6aayx)}$ P. Therefore $p\sqrt{\frac{1}{2}\mathrm{D}}$ divided by $ap : 2v$,
that is $\sqrt{(12aaxx - 12a^2x + 12a^4 - 6aayy - 12aayx)}$ D × P : $a\sqrt{(aa + xx + yy)b\mathrm{L}}$ =
[from Lemma V, where $y = ax : (-a + x)$ & $y = (xx - aa) : a$; hence $yy = xx + ax$]
$\sqrt{(18aaxx + 6a^3x + 12a^4 - 12ax^3)}$ D × P : $a\sqrt{(aa + 2xx + ay)b\mathrm{L}}$
$= \sqrt{(126axx + 42aax + 84a^3 - 84x^3)}$ D × P : $a\sqrt{(a^2 + 2axx + axx)}$ AB×L =
[from Lemma V, $x^2 = axx + 2aax - a^2$] $\sqrt{(42axx - 126aax + 168a^3)}$ D × P : $\sqrt{(a^3 + 2axx + axx)}$AB×L
$= \sqrt{(42xx - 126ax + 168aa)}$ D × P : $\sqrt{(2xx + ax + aa)}$ AB×L
gives the number of vibrations of the string for one oscillation of the given pendulum D. After which, for x the value can be substituted fuerit, which is the root of this equation $x^3 - axx - 2aax + a^3 = 0$.

Solutions or the same Problems from the principles of Statics.

LEMMA I.

Let g be the natural force of gravity, by which bodies are naturally set in motion, that is, by which they are urged to fall. Let x be the altitude through which a body falls, v velocity at the end of the descent, t the descent time, M the mass of the weight P; for these $M \times g = P$; $gdx : v = dv$, and thus $\sqrt{(2gx)} = v$.

LEMMA II.

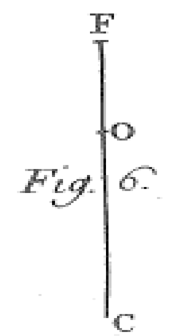
$dx : v = dt = dx : \sqrt{(2gx)}$; hence $t = \sqrt{2x} : \sqrt{g}$.

LEMMA III.

As shown elsewhere, the ratio of the time of descent time through the diameter of some circle is to the time of a semi- oscillation in the descent along the cycloid of the same height as the circle as $1 : \frac{1}{2}\pi$, or $2 : \pi$; the time for a semi-oscillation of a given pendulum of length D is $\frac{\pi}{2}\sqrt{\frac{D}{g}}$; for by the preceding Lemma, the descent time for the diameter of the circle is $\sqrt{(D/g)}$. [This time is $\sqrt{(2D/g)}$, which throws out the previous ratio by a factor of $\sqrt{2}$.]

LEMMA IV.

The point F is pulled towards C by a force which is in proportion to the distance FC (Fig. 6) ; it is to be shown, that whatever the point F may be that begins to move, it always takes the same time to pass through the distance FC. Thus the forces acting on the particle for any distance $= f \times$ FC [by f is to be understood the parameter for that force, in order that the absolute force taken can be increased or decreased] : With these put in place, let the distance of FC from the equilibrium position be taken equal to a, some part of which FO = x, $f \times (a - x)\, dx / v = dv$; thus $v = \sqrt{f(2ax - xx)}$, thus $t = \frac{1}{\sqrt{f}} \int \frac{dx}{\sqrt{(2ax - xx)}}$.



Hence the time for the total distance FC is equal to $\pi / (2\sqrt{f})$.
[The integral can be expressed in the form :
$t = \frac{1}{\sqrt{f}} \int \frac{dx}{\sqrt{(a^2 - (a-x)^2)}} = \frac{-1}{\sqrt{f}} \left[\sin^{-1}\left(\frac{a-x}{a}\right) \right]$ evaluated between 0 and a to give $\pi / (2\sqrt{f})$. As t is independent of the amplitude, then all vibrations have the same period.]

[Note : in what follows, usually the whole expression is included after a √ sign, eliminating the use of brackets.]

PROBLEM I.

AF is produced in the second figure & in the following. The force of the weight P is to the force acting on F towards C, as the sine of the AFC to the sine of the angle VFB, which is equal to the sine of the angle AFC : to the sine of twice the angle FAC = [since FAC is very small] AC : 2FC = $b : 2a$, and thus the force acting on F towards C = $(2a : b)$ P = $2aMg : b$ [by M is understood the mass of the weight P.] Since truly the small weight L, from which gravity is now abstracted, is nevertheless to be considered a little mass acted on by a force towards C, this force is expressed through $f a$ L, according to $2aMg : b = f a$ L, from which $f = 2aMg : b$L and thus by Lemma IV of this section, the time to pass through FC or $[\pi : 2\sqrt{f}] = p\sqrt{b}$L $: 2\sqrt{2gM}$ = the short time for a semi-vibration of the string : thus the time for a semi-oscillation of the given pendulum D, which [by Lemma III] is equal to $\pi\sqrt{D} : 2\sqrt{g}$, is to be divided by the short time for a semi-vibration of the string $p\surd b\mathrm{L} : 2\surd 2gM$; from which comes $\surd(2\mathrm{D.M} : b.\mathrm{L})$ = [where the weights are to be substituted for the masses] $\surd(2\mathrm{D} \times \mathrm{P} : \mathrm{AB} \times \mathrm{L})$ gives the number sought of the vibrations of the string, as was thus determined in the preceding solution by the use of virtual work [i. e. work and energy changes : note that the first solutions were time independent, while the current ones rely on the time being evaluated at some point.]

PROBLEM II.

Now P is in the ratio of the force of the point F towards C (Fig. 3) as the sine of the angle AFC to the sine of the angle VFB or sine of FAC $= b : a$, from which the force of the point F to C $= aMg : b = fa \times \frac{1}{2}\mathrm{L}$, and thus $f = 2Mg : b\mathrm{L}$, & the time for the distance FC $[p : 2\surd f] = \mathrm{P}\surd b\mathrm{L} : 2\surd 2gM$ = the short time for the semi-vibration of the string. The time for the pendulum $p\surd\mathrm{D} : 2\surd g$ is divided by this time $p\surd b\mathrm{L} : 2\surd 2gM$ to give $\surd(2\mathrm{D} \times \mathrm{M} : b \times \mathrm{L}) = \surd(6\mathrm{D} \times \mathrm{P} : \mathrm{AB} \times \mathrm{L})$ for the number of vibrations sought.

PROBLEM III.

The force acting on the point F towards C is called φ here and in the following ; now $\mathrm{P} : \varphi : : S\,\mathrm{AFC} : S\,\mathrm{VFG}$; [for S is understood to mean the sine of the angle]. Truly from the second Lemma set out for the first method, $S\,\mathrm{VFG}$ [since it is very small] =
$2a - a\surd 2$, b is taken for the radius : hence $\varphi = (2a - a\surd 2)\,\mathrm{P} : = (2a - a\surd 2)\,\mathrm{M}g : b = fa \times \frac{1}{3}\mathrm{L}$,
from which $f = (6 - 3\surd 2)\,Mg : b\mathrm{L}$, and the time to pass through $[p : 2\surd f] = \mathrm{P}\surd b\mathrm{L} : (6 - 3\surd 2)\,Mg$ is equal to the small time for the semi-vibration of the string; thus on dividing $p\surd\mathrm{D} : 2\surd g$, by $p\surd b\mathrm{L} : (6 - 3\surd 2)\,Mg$, from which arises $\surd(6 - 3\surd 2)\,\mathrm{D}M : \surd b\mathrm{L} = 2\surd(6 - 3\surd 2)\mathrm{D} \times \mathrm{P} : \surd\mathrm{AB} \times \mathrm{L}$ as before giving the sought number of vibrations.

PROBLEM IV.

Here again $\mathrm{P} : \varphi : : S\,\mathrm{AFC} : S\,\mathrm{VFG}$; Moreover from Lemma III from the above method, $S\,\mathrm{VFG}$ [since it is very small] $= \frac{1}{2}a - \surd\frac{4}{5}aa = (3a - a\surd 5) : 2$, with b taken for the whole sine :
hence $\varphi = (3a - a\surd 5)\mathrm{P} : 2b = (3a - a\surd 5)\,\mathrm{M}g : 2b = fa \times \frac{1}{4}\mathrm{L}$, from which $f = (6 - 2\surd 5)\,Mg : b\mathrm{L}$; and the time to pass through FC $[p : 2\surd f] = p\surd b\mathrm{L} : (6 - 2\surd 5)\,Mg$ = short time taken for the semi-vibration of the string. Hence on dividing $p\surd\mathrm{D} : 2\surd g$ by $p\surd b\mathrm{L} : 2\surd(6 - 2\surd 5)\,Mg$ the result is acquired : $\surd(6 - 2\surd 5)\,\mathrm{D}M : \surd b\mathrm{L} = \surd(30 - 10\surd 5)\,\mathrm{D} \times \mathrm{P} : \surd\mathrm{AB} \times \mathrm{L}$, which will give the number of vibrations similar to above, for
$\surd(30 - 10\surd 5)\,\mathrm{D} \times \mathrm{P} : \surd\mathrm{AB} \times \mathrm{L} = 2\surd(25 - 5\surd 5)\,\mathrm{D} \times \mathrm{P} : \surd(5 + \surd 5)\,\mathrm{AB} \times \mathrm{L}$.

PROBLEM V.

The Figure is to be constructed in the mind. Here we have from the preliminary Lemma IV : $S\,\mathrm{VFG}$ to be very small $= 2a - a\surd 3$; and thus $\varphi = (2a - a\surd 3)\,\mathrm{P} : b = (2a - a\surd 3)\,\mathrm{M}g : b = fa \times \frac{1}{5}\mathrm{L}$, from which
$f = (10 - 5\surd 3)\,Mg : b\mathrm{L}$; and the time to pass through FC $[p : 2\surd f] = p\surd b\mathrm{L} : 2\surd\,(10 - 5\surd 3)\,Mg$ is equal to the short time for the semi-vibration of the string : whereby $p\surd\mathrm{D} : 2\surd g$ is divided by this to give
$\surd(10 - 5\surd 3)\,\mathrm{D}M : \surd b\mathrm{L} = \surd(60 - 30\surd 3)\,\mathrm{D} \times \mathrm{P} : \surd\mathrm{AB} \times \mathrm{L}$, for the number sought as above.

PROBLEM VI.

From the preliminary Lemma V, here $S\,\mathrm{VFG}$ [on account of being very small] is equal to $2a - x$; where x is the root of the equation $x^3 - ax^2 - 2ax + a^3 = 0$, and $\varphi = (2a - x)\,\mathrm{P} : b = (2a - x)\,\mathrm{M}g : b = fa \times \frac{1}{6}\mathrm{L}$, from which $f = (12 - 6x)\,Mg : ab\mathrm{L}$; & the time to pass through FC $[p : 2\surd f]$ is equal to $p\surd ab\mathrm{L} : 2\surd\,(12a - 6x)\,Mg$ = the short time for the semi-vibration of the string; $p\surd\mathrm{D} : 2\surd g$ is divided by this, from which $\surd(12 - 6x)\,\mathrm{D}M : \surd ab\mathrm{L} = \surd(84 - 42x)\,\mathrm{D} \times \mathrm{P} : \surd a \times \surd\mathrm{AB} \times \mathrm{L} =$

$\sqrt{(42xx - 126ax + 168aa)}$ D × P : $\sqrt{(2xx + ax + aa)}$AB × L, as multiplication is signified by the cross.

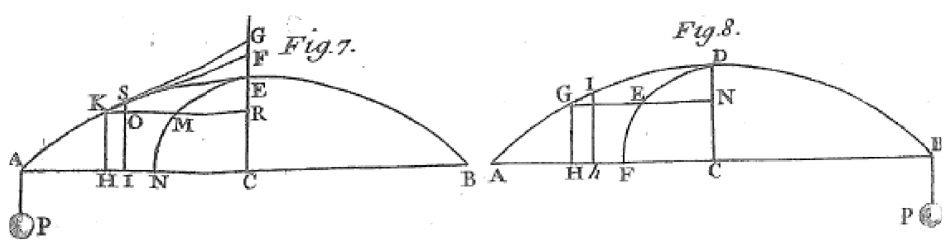
SCHOLIUM.

The theory can be drawn out for any number of small weights : indeed, let the number of small weights be n, & for we have $\varphi = (2a - x)$ Mg: $b = fa \times \frac{1}{n}$ L, from which $f = (2na - ax)\, Mg : ab$L ; & the time to pass through FC [$p : 2\sqrt{f}$] = $p\sqrt{ab}$L : $2\sqrt{(2na - nx)}\, Mg$ = the short time for the semi-vibration of the string. Hence $p\sqrt{D} : 2\sqrt{g}$ is to be divided by this to give : $\sqrt{(2na - nx)}$ DM : $\sqrt{ab}$L = $\sqrt{((n + 1)(2aa - nx))}$ D × P : $\sqrt{(a \times AB \times L)}$ = the number of vibrations of the string for one of the given pendulum D. In which expression is substituted for x the value of that, which should be obtained by adhering to the method used in the preliminary Lemmas. Thus, e.g., let there be seven small weights, in which case $n = 7$, & x [by the preceding Lemma VI] = $a\sqrt{(2 + \sqrt{2})}$, this gives:
$\sqrt{((n + 1)(2na + nx))}$ D × P : (a × AB × L) = $2\sqrt{(28 - 14\sqrt{(2 + \sqrt{2})})}$D × P : $\sqrt{AB}$ × L,
for the number of vibrations sought, which is approximately as $2\sqrt{2}$ D × P : $\sqrt{AB}$ × L, just less.

PROBLEM VII.

Now let there be a musical chord AB uniformly thick, the mass or length of which is L, and the tension from the weight P = Mg ; the number of vibrations for one oscillation of the given pendulum of length D is sought. [As with P, some times we interpret L as a mass, at others as a length, in order to get the correct dimensions; in modern terms we use the line density ρ, in which case the time for a quarter - period for the fundamental note is $T_{1/4} = \frac{L}{2}\sqrt{\frac{\rho}{T}}$, where T is the tension in the chord, and L is the length.

SOLUTION.



A curvilinear shape AEB (Fig. 7) can be assumed for the chord, beyond the straight line position AB, which it must have, in order that any of its points K arrives at the same time at the corresponding point H on the straight line AB, as the middle point E arrives at C : in order that this occurs, the accelerating force by which a point K is directed towards H, is everywhere in proportion to the distance KH. Hence with two tangents drawn close together KG, SF [text has GF] ; and from K and S, KH and SI are connected : from the principles of statics: the weight P or Mg is in the same ratio to the force by which the small length of the chord KS is directed towards H, as the sine of the angle KSO, which is taken as a right angle, to the sine of the angle GKF, [Mg/F = sin(KGF)/sin(GKF) = 1/(FG/KG)] that is, as 1 is to (FG : KG); indeed KF can be considered as perpendicular to the axis CF, and thus that force [F]acting on K is :
FG × Mg : KG = f × KH × dL. & the accelerating force itself f × KH = FG × Mg : KG × dL. Moreover, as FG : KG is to be found, it is noted that the curve AEB belongs to the family of elongated trochoids [A generisation of the cycloid, where the point tracing out the elongated trochoid lies at a small distance a from the centre of the cycloid generating circle of radius $R > a$, while the circumference of the generating circle rolls - in this case - through a quarter turn along a line parallel to AB at a distance R below the centre of the circle. Thus, the parametric equations for the curve, where the origin and axis are as defined below, are : $x = a - a\cos\theta$ and $y = R\theta - a\sin\theta$], *i.e.*, of that kind, as described by the quadrant of the circle EMN, and with KR drawn parallel to the base AC, shall be AC : KR = EMN: EM, the demonstration of which we will add below. Now let EC = a, ER = x, EM = s, EMN = $\frac{1}{2}pa$ [by always realising that 1 is to p as the

diameter to the circumference]; also AC : EMN = n : 1, KR = ns, and the subtangent RG is found to be equal to $s\sqrt{(2ax - xx)} : a$,
[We need to use calculus to prove this last result, and to assume that $R >> a$. Take E as the origin of co-ordinates, FR the positive x-axis and a line through E parallel to KR as the y-axis; then the equation of the trochoid can be expressed parametrically in terms of the angle of rotation θ, which is equal to MRC:
$x = a - a\cos\theta;\ y = R\theta - a\sin\theta$; then of differenting w.r.t. the time t :

$$\frac{dx}{dt} = a\sin\theta.\dot{\theta};\frac{dy}{dt} = R\dot{\theta} - r\cos\theta.\dot{\theta};\text{hence}\ \frac{dy}{dx} = \frac{R - r\cos\theta}{r\sin\theta} \sim \frac{n}{\sin\theta} = \frac{\text{KR}}{\text{GR}};$$

$$\text{hence GR} = \frac{\text{KR}\sin\theta}{\frac{\text{R}}{\text{r}} - \cos\theta} \sim \text{KR}\sin\theta / n = s.\sin\theta = s.\text{MR}/a.$$

Note also that MR = $\sqrt{(2ax - xx)}$.]
CG = $a - x + s\sqrt{(2ax - xx)}: a$, and the differential of this, FG = $-dx + (asdx - sxdx) : a\sqrt{(2ax - xx)} + dx$ = $(asds - sxdx) : a\sqrt{(2ax - xx)}$ [a extra dx seems to be added to ease the calculation, as an approximation] and thus **FG: KG**, or what is the same FG: KR [= ns] = $\mathbf{(a - x)dx : na\sqrt{(2ax - xx)}}$; truly the element KS itself , which is considered to be equal to KO, [as KR = ns]; *i.e.* **KO** = nds = $\mathbf{nadx : \sqrt{(2ax - xx)}}$.
[For $dx/ds = \sin\theta = \sqrt{(2ax - xx)}/a$, from which $ds = adx/\sqrt{(2ax - xx)}$.]
Moreover, AB: HI [KO] = L: dL, from which $\mathbf{dL = KO \times L : AB = nadx \times L : AB\sqrt{(2ax - xx)}}$; hence from which by substitution from above, the force [F] for the acceleration is obtained : $\mathbf{FG \times Mg : KG \times dL}$ = $\mathbf{AB \times (a - x) \times Mg : n^2a^2\ L}$ [since npa = AB, the semi-circumference of the large circle of radius R.]
= $pp(a - x)\text{M}g : \text{AB} \times \text{L} = pp \times \text{KH} \times Mg : \text{AB} \times \text{L} = \boldsymbol{f} \times \mathbf{KH}$; and thus $f = pp \times Mg : \text{AB} \times \text{L}$, and the time to pass through the distance KH [$p : 2\sqrt{f}$, from previously] = $\mathbf{\sqrt{AB \times L} : 2\sqrt{Mg}}$ = short time for the semi-vibration of the musical chord; [In modern terms, the frequency of the note for a string of length L and under a tension T, is given by $T_{1/4} = \frac{L}{2}\sqrt{\frac{\rho}{T}}$, in which case the line density of the chord is 1.]
thus on dividing $\mathbf{p\sqrt{D} : 2\sqrt{g}}$ for the pendulum
by $\sqrt{\text{AB} \times \text{L}} : 2\sqrt{Mg}$, then $p\sqrt{DM} : \sqrt{\text{AB} \times \text{L}} = p\sqrt{D} \times \text{P} : \sqrt{\text{AB} \times \text{L}}$ is obtained; the number of vibrations of the chord for the duration of one oscillation of the pendulum of given length D; as TAYLOR found. *See Meth. Increm*. p. 93. In the same way, I have found the solution from the principle of the *action of the unbalanced forces*, as follows:

Let (Fig. 8) DN = x; NG = $y = n\int(adx : \sqrt{(2ax - xx)})$;

[This follows from the previous method, as in Fig. 7, KO = $nds = dy = nadx : \sqrt{(2ax - xx)}$] ;

DG = s; DC = a. Also, $ds - dy = (ds^2 - dy^2) : (ds + dy) = dx^2 : 2dy$ =

[on account of $y = n\int(adx : \sqrt{(2ax - xx)})$; and $dy/dx << 1$;]

$dx^2 : \frac{2nadx}{\sqrt{(2ax - xx)}} = dx\sqrt{(2ax - xx)} : 2na$; and thus the extension is the arc length DA - the half chord length CA = $\int(dx^2 : 2dy) = \int(dx\sqrt{(2ax - xx)}) : 2na) = \frac{1}{2}\text{CD} \times \text{DEF} : 2n\ \text{CD}$.

[The integral has the value $a^2\pi/4$ = CD × arc DEF/2, which is evaluated by elementary means by putting $a - x = a\cos\theta$.]
Hence DA - CA = $\frac{1}{4}$DEF : n = DEF : $4n$ = AC: $4n^2$ = AB : $8n^2$; hence 2DA - 2CA = arc ADB - AB
= AB : $4n^2$ = the difference in length between the arc and the chord.
The radius of curvature at G, for the element position Gg or with constant ds, is generally $dsdy : ddx$ = [for curves with maximum elongation where *dy = ds; these are Bernoulli's brackets*] $dy^2 : ddx$. Hence in the case of the Troichoid family with the maximum elongation,
where $dy = ds = nadx : \sqrt{(2ax - xx)}$ = constant [R; on taking the next derivative] :
[for $ddy = \frac{naddx}{\sqrt{(2ax - x^2)}} - \frac{na((a - x)dx^2}{(2ax - x^2)^{3/2}} = 0$; hence $naddx\sqrt{(2ax - x^2)} - \frac{na((a - x)dx^2}{(2ax - x^2)^{1/2}} = 0$; now we call the second derivative d^2y/dx^2]

$naddx\,\surd(2ax - xx) - (na^2dx^2 - naxdx^2) : \surd(2ax - xx) = 0$, hence $ddx = (a - x)dx^2 : (2ax - xx)$; and thus the radius of the osculating $dy^2 : ddx = \frac{nnaadx^2}{2ax-xx} : \frac{(a-x)dx^2}{(2ax-xx)} = \frac{nnaa}{a-x}$, *i.e.* the radius of the osculating circle varies inversely as GH. [We give this calculation in a modern setting, *i.e.* the axes are rotated for the present through 90^0: the curvature or the inverse of the radius κ of the osculating circle for a plane curve $y(x)$ is given by $1/\kappa = (d^2y/dx^2)/(1 + (dy/dx)^2)^{3/2}$; in the present case,

$$\frac{dy}{dx} = \frac{nadx}{\sqrt{(2ax-x^2)}};\ \frac{d^2y}{dx^2} = \frac{-na(a-x)}{(2ax-x^2)^{3/2}};\ \text{and}\ 1 + (\tfrac{dy}{dx})^2 = \frac{2ax+n^2a^2-x^2}{2ax-x^2};$$

$$\text{hence}: \frac{1}{\kappa} = \frac{-na(a-x)}{(2ax-x^2)^{3/2}} \times \frac{(2ax-x^2)^{3/2}}{(n^2a^2+2ax-x^2)^{3/2}} \sim \frac{a-x}{n^2a^2},$$ where only the term n^2a^2 is retained in the 2nd term.]

Now the hanging weight stretching the chord is P ; the weight of the chord AB itself is L; the velocity of the point D when it arrives at C due to the vibration is equal to √ S [the distance S is to be understood as the distance through which a weight falls freely from rest to reach the velocity of the point D at C] : the velocity of any point G at H will be equal to GH √S : DC = $(a - x)$√S : a, and thus
[Before proceeding, we note that Bernoulli has set $2g = 1$, as a comparison is to be made with a pendulum treated in the same manner; hence, $v^2 = 2gS$ or $v = \surd S$; thus the velocity of any other point G on crossing the line AB at H is proportional to the amplitude GK of the vibration at that point : thus, $v_H = (GH/DC)\surd S$; in addition a term in proportion to the kinetic energy mv^2, which Bernoulli calls the *'moving force'* where m is the mass of the small element Hh, is given by the first term that follows. We note also from above that Hh $= dy = nds = nadx/\surd(2ax - xx)$.]

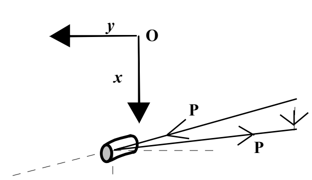
$$\frac{(a-x)^2}{aa}S \times \frac{Hh}{AB} \times L = \frac{(a-x)^2}{aa} \times S \times \frac{dy.L}{AB} = \frac{(a-x)^2}{a} \times S \times \frac{ndx.L}{AB.\sqrt{(2axx-xx)}},$$ which is equal to the *'moving force'* of the small part of the chord Gg or Hh in H = $\frac{nS.L}{a.AB} \times \frac{(a-x)^2dx}{\sqrt{2ax-xx}}$; this becomes on integration for the whole chord : [Note that $\int \frac{(a-x)^2dx}{\sqrt{2ax-x^2}} = \int (a-x)d\{\sqrt{(2ax-xx)}\}dx$]

$$\frac{nS\times L}{a.AB} \times ((a-x)\sqrt{(2ax-xx)} + \int dx\sqrt{(2ax-xx)}) =$$

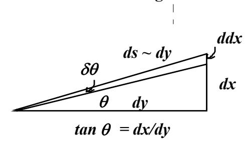
$\frac{2nS.L}{a.AB} \times \frac{1}{2} \times a \times DEF = \frac{nS.L.DEF}{AB} = \frac{1}{2}L \times S$ = the amount of the *action of the unbalanced forces* for the whole chord [as n.DEF = AB/2]. Moreover, this is equal to the amount of the *action of the unbalanced forces* [or work done]of the descending weight P by (AB : $4n^2$)= P.AB : $4n^2$
[As shown above for the difference of ADF and AB].
Hence, S = P.AB : $2n^2 \times$ L = 2P.DEF^2 : L × AB & √ S = DEF × √(2P : L.AB) [On substituting for n^2.].
Hence the time is found for the string to pass from D to C to be equal to √(L.AB:2P) [our formula $v = a\omega$].
But the time of the semi-oscillation of the simple pendulum the length of which is C = DEF × √2C: DC; therefore in order that these two times are equal, it is required that
√(L.AB:2P) = DEF × √2C: DC ; thus C = DC^2.AB.L : 4DEF^2.P. Hence the number of vibrations of the chord in the time for one vibration of the pendulum of given length D is equal to :
2DEF×√(D × P): DC × √(AB.L), = [by supposition 2DEF: DC = p] p√(D.P : AB.L) as TAYLOR has, for whom L & N are what AB & L are for me.

It follows from this demonstration, as was asserted above, that the figure of an elongated trochoid is established in a vibrating chord ADB [see the preceding figure].

It has been shown above, that the sine of the contact angle in some point of the chord G is proportional to the length to be traversed GH. Now with the same symbols retained, which we used above, the sine of the contact angle is equal to $ddx : ds$ = [on account of the figure with the maximum elongation, and consequently, $ds = dy$] $ddx : dy$, with dy kept constant of course : moreover, the length to be traversed GH $= a - x$. Hence ($ddx : dy$) to $a - x$ is in a constant ratio. This ratio is as dy to $nnaa$;



Force triangle for the element ds.



Difference of the gradients across the element ds.

[For from the radius of curvature calculation, we have seen, on reverting to the original axes as defined in the text : $\frac{d^2x}{dy^2} = \frac{ddx}{dy} \sim: \frac{1}{\kappa} = \frac{a-x}{n^2a^2}$, hence $\frac{ddx}{a\text{-}x} = \frac{dy}{n^2a^2}$.]

and

nnaaddx : dy = ady - xdy; [or divided by *dy*] *nnaaddx* : $dy^2 = a - x$. The other member is to be multiplied by *dx*, & *nnaadxddx* : $dy^2 = adx - xdx$ is obtained; and with integration performed : $nnaadx^2 : 2dy^2 = ax - \frac{1}{2}xx$, or $nnaadx^2 = (2ax - xx)dy^2$ hence *nadx* : $\sqrt{(2ax - xx)} = dy$,

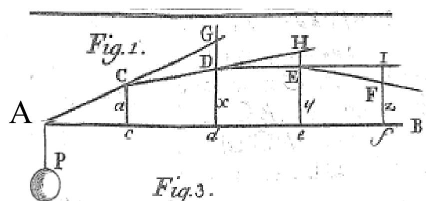
& $n\int(adx : \sqrt{(2ax - xx)}) = y$. Thus y [NG] : $\int(adx : \sqrt{(2ax - xx)})$ [the arc DE] $= n : 1$; , that is, with the line NG attached to the arc DE in a constant ratio, and one that is very large. The ratio AC to CD is indeed very large by hypothesis]. Hence the ratio AC to the quarter of DEF is also very large. But AC : DEF = *n* : 1. [*as demonstrated*.] Whereby the proposition is agreed upon.

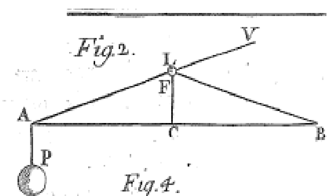
NO. CXL.
JOHANNIS BERNOULLI

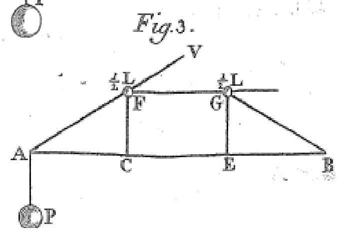
MEDITATIONES

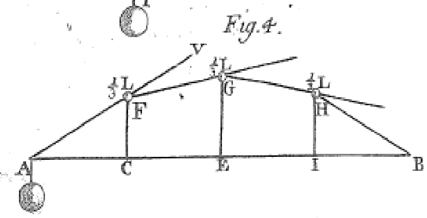
DE CHORDIS VIBRANTIBUS,

cum pondusculis aequali intervallo a se invicem dissitis,
Ubi nimirum ex principio virium vivarum quaeritur numerus vibrationum chorda pro una oscillatione Penduli data longitudinis D(see p.125, 126).







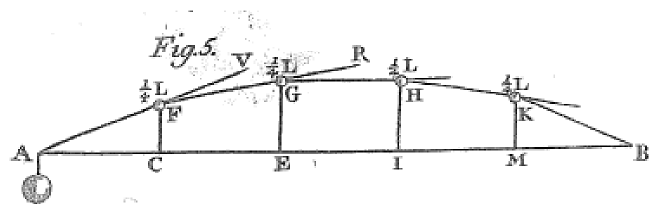


Chorda vibrans ACDEF &c., cui ad distantias aequals affixa sunt ponduscula aequalia, C, D, E, F, &c. in eam se componere debet figuram, ut singula ponduscula simul perveniant in situm rectilineum AB : unde sequitur, singulorum velocitates, adeoque & vires acceleratrices, proportionales esse debere longitudinibus percurrendis Cc, Dd, Ee, &c. Sed per principia statica, tensio chordae est ad vim qua ponusculum quodvis, exempli gratia, E, urgetur versus e, ut sinus anguli DEe ad sinum anguli DEF, vel IEF, id est, [ob figurum chordae fere rectam, & pondusculorum intervalla aequalia] ut sinus totus ad FI. Ergo, *ex aequo*, distantiae Cc, Dd, Ee, &c. proportionales sunt ipsis DG, EH, FI, &c. respective. Jam DG = Gd - Dd = 2Cc - Dd = $2a - x$; HE = He - Ee = 2Dd - Cc - Ee = $2x - a - y$; FI = If - Ff = 2Ee - Dd - Ff = $2y - x - z$; & unde $2a - x : a = 2x - a - y : x = 2y - x - z : y = 2z - y - t : z =$ &c. Hinc sequentia fluunt Lemmata.

I. Si duo sint ponduscula, erit $x = a, y = 0$, reliqua non considerantur.

II. Si tria sint ponduscula, erit $y = a, z = 0$, reliquis non consideratis; adeoque $2a - x : a = 2x - 2a : x$, unde $2ax - xx = 2ax - 2aa$ & $x = a\surd 2$.

III. Sint quatuor sint ponduscula, erit $y = x, z = a$, & $t = 0$, non consideratis reliquis ; adeoque $2a - x : a = x - a : x$; unde $2ax - xx = ax - aa$ & $x = \frac{1}{2}a + \sqrt{\frac{5}{4}aa}$.



IV. Sint quinque ponduscula, erit $z = x, t = a, u = 0$, reliquis negectis ; adeoque $2a - x : a = 2x - a - y : x = 2y - 2x : y$; hinc duae aequationes habentur, $xx = aa + ay$, & $yx = 2ax$. Ex priori aequatione est $y = (xx - aa) : a$; ex posteroiri, $y = 2a$; unde $x = a\surd 3$.

V. Si sex sint ponduscula, erit $z = y, t = x, u = a, s = 0$, reliquis neglectis ; adeoque $2a - x : a = 2x - a - y : x = y - x : y$. Hinc duae aequationes, $xx = aa + ay$, & $ay - yx = -ax$. Ex posteriori aequatione $y = ax :(- a + x)$; ex altera $y = (xx - aa) : a$; unde $aax = x^3 - axx - aax + a^3$, seu $x^3 - axx - 2aax + a^3 = 0$.

VI. Si septem sint ponduscula, erit $t = y, u = x, s = a, w = 0$, non attento ad reliqua. Adeoque $2a - x : a = 2x - a - y : x = 2y - x - z : y = 2z - 2y : z$; & ita tres habentur aequationes $xx = aa + ay$, $xy = ax + az$ & $xz = 2ay$. Ex aequatione secunda $z = (xy - ax) : a$, ex tertia $z = 2ay : x$; unde $y = axx : (x^2 - 2a^2) : a$;

igitur $(x^2 - a^2) : a = ax^2 : (x^2 - 2a^2)$ unde $aaxx = x^4 - 3aaxx + 2a^4$; seu $x^4 - 4aaxx + 2a^4 = 0$; adeoque $xx = 2aa + aa\sqrt{2}$, & $x = a\sqrt{(2 \pm \sqrt{2})}$; $y = a \pm a\sqrt{2}$; $z = (2a \pm 2a\sqrt{2}) : \sqrt{(2 + \sqrt{2})} = + 4\sqrt{(4 + 2\sqrt{2})}$. Ubi notandum signa inferiora huc non quadrare.

PROBLEMA I.

Sit nunc chorda vel filum ALB omnis crassitiei expers oneratum in medio ponduscula L ; sitque filum tensum a pondere P ; quaeritur tempus semivibrationis per LC. Esto LC = a, AL vel AC = b, erit AL - AC = $(AL^2 - AC^2) : (AL + AC) = LC^2 : 2AC = a^2 : 2b$, & ALB - AB = $a^2 : b$ = descensui ponderis P filum tendentis. Sit z = altitudini verticali per quam grave libere descendens acquirit velocitatem aequale illi quam habet punctum L quando pervenit in C, quae velocitas adeo erit = $\sqrt{z}$. Erit vis viva pondiusculi L = Lz = vi vivae ponderis tendentis = $(a^2 : b) \times P$; unde $z = a^2 \times P : b \times L.$ Quia vero vis trahens punctum versus C semper est proportionalis distantiae LC; erit supponendo diametrum circuli ad ejus circumferentiam ut 1 ad p, & v velocitatem puncti L in C, tempus per LC seu tempus semivibrationis $= ap : 2v = ap : 2\sqrt{z} = p\sqrt{b}.L : 2\sqrt{P}$, & tempus unius semioscillationis penduli datae longitudinis D, = $p\sqrt{\tfrac{1}{2}D}$. Ergo $p\sqrt{\tfrac{1}{2}D}$. divisum per $p\sqrt{b}.L : 2\sqrt{P}$, hoc est, $\sqrt{2P.D} : \sqrt{b}. L$, dabit numerum vibrationum fili durante una oscillatione penduli D = $\sqrt{4D.P} : \sqrt{AB..L} = 2\sqrt{(D.P : AB.L)}$.

PROBLEMA II.

Sit nunc filum AFGB tensum a pondere P, & oneratum duobus ponduscculis aequalibus (Fig. 3), quorum unumquodque = $\tfrac{1}{2}L$, & quae dividant filum in tres partes aequales, AF, FG, GB. Sit iterum FC = GE = BE = $a^2 : 2b$; adeoque AFGB - AB = $a^2 : b$ = descensui ponderis P. Sit iterum $\sqrt{z}$ = ve. locitati puncti F in C, vel puncti G in E ; erunt vires vivae ponduscculorum F & G simul = Lz = vi vivae ponderis tendentis P = $(aa : b) \times P$, unde $z = a^2.P : b.L$. Reliqua ergo jam ponendum sit pro numero vibrationum $\sqrt{(2D.P : b.L)} = \sqrt{(2D.P : \tfrac{1}{3}AB.L)} = \sqrt{(6D.P : AB.L)}$.

PROBLEMA III.

Sit jam tria ponduscula (Fig. 4) $= \tfrac{1}{3}L$, erit rursus AF - TAB ? AC = BH - BI = $a^2 : 2b$. Sed FG - CE = HG - IE = (ex. Lemm. 2.) $(3aa - 2aa\sqrt{2}) : 2b$. Hinc AFGHB - AB = $(4aa - 2aa\sqrt{2}) : b$ = descensui ponderis P. filum tententis. Erit nunc, vocata $\sqrt{z}$ = velocitate puncti F in C, velocitas puncti G in E = $\sqrt{2z}$; unde quantitas virium vivarum omnium ponduscculorum simul = $\tfrac{4}{3}z \times L$ = vivae ponderis tendentis = $(4aa - 2aa\sqrt{2})$ P : b , adeoque z = $(6aa - 3aa\sqrt{2})$ P: $2b$L = $(12aa - 6aa\sqrt{2})$ P: AB × L, & sic $\sqrt{z}$, vel $v = \sqrt{((12aa - 6aa\sqrt{2})\ P : AB \times L)}$ & tempus per FC, seu $ap : 2v = p\sqrt{AB \times L} : 2\sqrt{(12- 6\sqrt{2})\ P}$. Igitur $p\sqrt{\tfrac{1}{2}D}$ divisum per $ap : 2v$, hoc est, $2\sqrt{(6-3\sqrt{2})}P \times D : \sqrt{AB} \times L,$ dabit numerum vibrationum fili in una oscillatione penduli dati D.

PROBLEMA IV.

Sint ponduscula quatuor , singula (Fig. 5) $= \tfrac{1}{4}L$. Supponendo tandisper GE = HI = x, reliquis ut prius manentibus, erit iterum AF - AC = BK - BM = $aa : 2b$; FG - CE = KH - MI = $(xx - 2ax + aa) : 2b$; unde AFGHKB - AB = $(xx - 2ax + 2aa) : b$ = descensui ponderis P. vocetur velocitas puncti F in C $= \sqrt{z}$, velocitas puncti G in E = $= \tfrac{x}{a}\sqrt{z}$; ac proinde summa virium vivarum omnium ponduscculorum = $\tfrac{aa\ +\ xx}{2aa} \times z \times L$ = vi vivae ponderis $= (xx - 2ax + 2aa)$ P : b ; igitur z = $(2aaxx - 4a^2x + 4a^4)$ P: $(xx + aa).b$L Quare $\sqrt{z}$ vel $v = a\sqrt{(2xx - 4ax + 4aa)}$ P: $\sqrt{(xx + aa)}$. b L; itaque tempus per FC = $ap : 2v = p\sqrt{(aa + xx)}.b$L : $2\sqrt{(2xx - 4ax + 4aa)}$ P. Ideoque $p\sqrt{\tfrac{1}{2}D}$ divisum per $ap : 2v$, hoc est, $\sqrt{(4xx - 8ax + 8aa)}D \times P : \sqrt{(aa + xx)}\ b$L = [quia per Lemma III, $x = \tfrac{1}{2}a + a\sqrt{\tfrac{5}{4}}$]

$2\sqrt{(5-\sqrt{5})}\ \mathrm{D}\times\mathrm{P} : \sqrt{(5+\sqrt{5})b}\mathrm{L} = 2\sqrt{(25-5\sqrt{5})}\ \mathrm{D}\times\mathrm{P} : \sqrt{(5+\sqrt{5})}\mathrm{AB}\times\mathrm{L}$,

dabit numerum vibrationum fili in una oscillatione penduli dati D.

PROBLEMA V.

Sint jam quinque ponduscula, quorum unumquoque $=\frac{1}{5}L$. Supponendo iterum GE = KM = x, reliquis semper manentibus, erit HI seu y [per Lemma IV] = 2a & $x = a\sqrt{3}$, & AF - AC = BN - BO = $aa : 2b$; FG - CE = NK - OM = $(xx - 2ax + aa) : 2b = (4aa - 2aa\sqrt{3}) : 2b$; GH - EI = KH - MI = $(yy - 2xy + xx) : 2b$ = $(7aa - 4aa\sqrt{3}) : 2b$; quocirca unde AFGHKNB - AB = $(12aa - 6aa\sqrt{3}) : b$ = descensui ponderis P. Est autem, sumta $\sqrt{z}$ pro velocite puncti F in C , velocitas puncti G in E = $=3\sqrt{z}$; & velocitas puncti H in I = $=2\sqrt{z}$. Per consequens aggregatum virium vivarum omnium ponduscolorum $=\frac{12}{5}\times z\times L$ = vi vivae tendentis ponderis P = $(12aa - 6a^2\sqrt{3})$ P : b ;
proinde z = $(10aa - 5a^2\sqrt{3})$ P: 2b×L = $(30a^2 - 15aa\sqrt{3})$ P: AB × L ;
& ita $\sqrt{z}$ vel $v = a\sqrt{(30 - 15\sqrt{3})}$ P: √AB × L, unde tempus per FC = $ap : 2v = p\sqrt{\mathrm{AB}\times\mathrm{L}}$: $2\sqrt{(30 - 15\sqrt{3})}$ P. Ideoque $p\sqrt{\frac{1}{2}\mathrm{D}}$ divisum per $ap : 2v$, hoc est $\sqrt{(60 - 30\sqrt{3})}\ \mathrm{D}\times\mathrm{P} : \sqrt{\mathrm{AB}\times\mathrm{L}}$
dabit numerum vibrationum fili in una oscillatione penduli dati D.

PROBLEMA VI.

Sunto ponduscula sex, quorum singula $=\frac{1}{6}L$. Positis nunc GE = NO = x, HI = KM = y ; erit AF - AC = BR - BS = $a^2 : 2b$; FG - CE = RN - OM = $(xx - 2ax + aa) : 2b$; GH - EI = NK - OM = $(yy - 2xy + xx)$: $2b$; propterea AFGHKNRB - AB = $(2xx - 2ax + 2aa + yy - 2yx) : b$ = descensui ponderis P. Est vero sumta $\sqrt{z}$ pro velocite puncti F in C , velocitas puncti G in E $= x\sqrt{z : a}$; & velocitas puncti H in I = $y\sqrt{z} : a$; unde summa virium vivarum omnium ponduscolorum = $(aa + xx + yy)$ z × L : $3aa$ = vi vivae tendentis ponderis P = $(2xx - 2ax + 2aa + yy - 2yx)$ P : b ;
& ideo $\sqrt{z} = \sqrt{(6aaxx - 6a^2x + 6a^4 - 3aayy - 6aayx)}$ P: $\sqrt{(aa + xx + yy)b}$L = v. Hinc tempus per FC = $ap : 2v$ = $ap\sqrt{(aa + xx + yy)b}$L : $2\sqrt{(6aaxx - 6a^2x + 6a^4 - 3aayy - 6aayx)}$ P. Idcirco $p\sqrt{\frac{1}{2}\mathrm{D}}$ divisum per $ap : 2v$, hoc est $\sqrt{(12aaxx - 12a^2x + 12a^4 - 6aayy - 12aayx)}$ D × P : $a\sqrt{(aa + xx + yy)b}$L = [ob Lemma V, ubi $y = ax : (-a + x)$ & $y = (xx - aa) : a$;
indeque $yy = xx + ax$] $\sqrt{(18aaxx + 6a^3x + 12a^4 - 12ax^3)}$ D × P : $a\sqrt{(aa + 2xx + ay)b}$L
= $\sqrt{(126axx + 42aax + 84a^3 - 84x^3)}$ D × P : $a\sqrt{(a^2 + 2axx + axx)}$ AB×L =
[ob Lemma V, $x^2 = axx + 2aax - a^2$] $\sqrt{(42axx - 126aax + 168a^3)}$ D × P : $\sqrt{(a^3 + 2axx + axx)}$AB×L
= $\sqrt{(42xx - 126ax + 168aa)}$ D × P : $\sqrt{(2xx + ax + aa)}$ AB×L dabit numerum vibrationum fili in una oscillatione penduli dati D. postquam pro x substitutus fuerit ejus valor, qui est radix hujus aequationis $x^3 - axx - 2aax + a^3 = 0$.

Solutiones eorundem Problematum ex principiis Staticis.

LEMMA I.

Sit vis gravitas naturalis g, qua corpora naturaliter animantur, hoc est, ad descensum urgentur. Sit x altitudo per quam descendit, v velocitas in fine descenscus, t tempus descensus, M massa ponderis P; erit $M\times g$ = P; $gdx : v = dv$, adeoque $\sqrt{2gx} = v$.

LEMMA II.

$dx : v = dt = dx : \sqrt{2gx}$; adeoque $t = \sqrt{2x} : \sqrt{g}$.

LEMMA III.

Quia alibi demonstratum, tempus descensus naturalis per diametrum alicuis circuli ad tempus semioscillationis in cycloide de aeque altae cum circulo ut $1 : \frac{1}{2}p = 2 : p$; erit tempus semi-oscillationis penduli datae longitudinis D, $p\sqrt{D} : 2\sqrt{g}$; est enim per Lemma praecends, tempus descendus per diametrum = $\sqrt{D} : \sqrt{g}$.

LEMMA IV.

Tendat punctum F ad C, viribus quae sunt proportionales distantiis FC ; demonstratum est, undecunque punctum F incipiat moveri, aequalibus semper temporibus percurrere distantum FC. Sit itaque vis qua in qualibet distantia urgentur = $f \times$ FC [per f intelligo parametrum illius vis, ut vis absolute sumta augeri & minui possit] : His positis, sit distantia FC a puncto quietis sumta = a, pars quaelibet FO = x, erit $f \times (a - x)\,dx : v = dv$; adeoque $v = \sqrt{f(2ax - xx)}$, atque $t = \frac{1}{\sqrt{f}}\int\frac{dx}{\sqrt{(2ax-xx)}}$. Ergo tempus per totam FC, = $p : 2\sqrt{f}$.

Fig. 6.

PROBLEMA I.

Producatur AF in secunda figura & sequentibus. Vis ponderis P est ad vim qua punctum F versus C urgetur, ut sinus anguli AFC ad sinum ang. VFB = sin. ang. AFC : sin, dupli ang. FAC = [quia FAC pro infinite parvo habetur] AC : 2FC = $b : 2a$, adeoque vis qua punctum F versus C urgetur = $(2a : b)$ P = $2aMg : b$ [intelligo per M massam ponderis P.] Quia vero pondusculum L, a cuius gravitate nunc abstrahitur, considerando tantum ejus massulam, urgeri debet ad C, vi quae exprimitur per fa L, erit $2aMg : b = fa$ L, unde $f = 2aMg : b$L adeoque per Lemma IV huius, erit tempus per FC [$p : 2\sqrt{f}$] = $p\sqrt{b}$L : $2\sqrt{2gM}$ = tempusculo semivibrationis fili : dividendo itaque tempus semioscillationis penduli data D, quod [per Lemma III] = $p\sqrt{}$D : $2\sqrt{}$g, per tempusculum semivibrationis fili $p\sqrt{b}$L : $2\sqrt{2gM}$; quod provenit $\sqrt{}$(2D.M : b.L) = [substituendo pro massis pondera] $\sqrt{}$(2D × P : AB × L) dabit numerum quaesitum vibrationum fili, prorsus ut in solutione praecedente per vires vivas eruta.

PROBLEMA II.

Nunc est P ad vim puncti F versus C ut sinus anguli AFC ad sinum VFB seu sin.FAC = $b : a$, unde vis puncti F ad C = $aMg : b = fa \times \frac{1}{2}$ L, adeoque $f = 2Mg : b$L, & tempus per FC [$p : 2\sqrt{f}$] = P$\sqrt{b}$L : $2\sqrt{2gM}$ = tempusculo semivibrationis fili. Divisum itaque tempus $p\sqrt{}$D : $2\sqrt{}$g, per $p\sqrt{b}$L : $2\sqrt{2gM}$; dabit $\sqrt{}$(2D ×M : b × L) = $\sqrt{}$(6D × P : AB × L) pro numero vibrationum quaesto.

PROBLEMA III.

Vocetur hic & in sequentibus φ vis puncti Fversus C ; erit jam P : φ : : S AFC : S VFG ; [per S intelligo sinum anguli]. Ex vero ex Lemmate secundo priori methodo praemisso, S VFG [quia infinite parvus] = $2a - a\sqrt{2}$, sumto b pro radio : hinc φ = $(2a - a\sqrt{2})$ P : = $(2a - a\sqrt{2})$ Mg : $b = fa \times \frac{1}{3}$ L, unde $f = (6 - 3\sqrt{2})\,Mg : b$L, tempusque per FC [$p : 2\sqrt{f}$] = P$\sqrt{b}$L : $(6 - 3\sqrt{2})\,Mg$ = tempusculo semivibrationis fili; adeoque dividendo $p\sqrt{}$D : $2\sqrt{}$g, per $p\sqrt{b}$L : $(6 - 3\sqrt{2})\,Mg,$ quod orietur $\sqrt{(6 - 3\sqrt{2})}\,$DM : $\sqrt{b}$L = $2\sqrt{(6 - 3\sqrt{2})}$D × P : $\sqrt{}$AB × L dabit quaesitum numerum, ut ante.

PROBLEMA IV.

Hic iterum P : φ : : S AFC : S VFG ; Est autem, ex Lemmate III pro superiori methodo, S VFG [quia infinite parvus] = $\frac{1}{2}a - \sqrt{\frac{4}{5}aa} = (3a - a\sqrt{5}) : 2$, sumto b pro sinu toto :
hinc $\varphi = (3a - a\sqrt{5})P : 2b = (3a - a\sqrt{5})\ Mg : 2b = fa \times \frac{1}{4}L$, ex quo $f = (6 - 2\sqrt{5})\ Mg : bL$; ac tempus per FC [$p : 2\sqrt{f}$] = $p\sqrt{b}L : (6 - 2\sqrt{5})\ Mg$ = tempusculo semivibrationis fili. Hinc dividendo $p\sqrt{D} : 2\sqrt{g}$ per $p\sqrt{b}L : 2\sqrt{(6 - 2\sqrt{5})\ Mg}$ acquiritur $\sqrt{(6 - 2\sqrt{5})\ DM} : \sqrt{b}L = \sqrt{(30 - 10\sqrt{5})\ D \times P} : \sqrt{AB \times L}$, quod dabit numerum vibrationum conformem superiori, nam
$\sqrt{(30 - 10\sqrt{5})\ D \times P} : \sqrt{AB \times L} = 2\sqrt{(25 - 5\sqrt{5})\ D \times P} : \sqrt{(5 + \sqrt{5})\ AB \times L}$.

PROBLEMA V.

Figura in mente concipienda est. Hic habemus ex Lemmate IV praeliminari, S VFG infinite parvum = $2a - a\sqrt{3}$; adeoque $\varphi = (2a - a\sqrt{3})\ P : b = (2a - a\sqrt{3})\ Mg : b = fa \times \frac{1}{5}L$, unde $f = (10 - 5\sqrt{3})\ Mg : bL$; ac tempus per FC [$p : 2\sqrt{f}$] = $p\sqrt{b}L : 2\sqrt{(10 - 5\sqrt{3})\ Mg}$ = tempusculo semivibrationis fili : quare dividendo $p\sqrt{D} : 2\sqrt{g}$ per hoc, prodit $\sqrt{(10 - 5\sqrt{3})\ DM} : \sqrt{b}L = \sqrt{(60 - 30\sqrt{3})\ D \times P} : \sqrt{AB \times L}$, pro numerum quaesito ; ut supra.

PROBLEMA VI.

Ex Lemmate V praeliminari, hic erit S VFG [ob infinite parvum] = $2a - x$; ubi x est radix hujus aequationis $x^3 - ax^2 - 2ax + a^3 = 0$, erit $\varphi = (2a - x)\ P : b = (2a - x)\ Mg : b = fa \times \frac{1}{6}L$, unde $f = (12 - 6x)\ Mg : abL$; & tempus per FC [$p : 2\sqrt{f}$] = $p\sqrt{ab}L : 2\sqrt{(12a - 6x)\ Mg}$ = tempusculo semivibrationis fili. Dividendo $p\sqrt{D} : 2\sqrt{g}$ per hoc, oritur $\sqrt{(12 - 6x)\ DM} : \sqrt{ab}L = \sqrt{(84 - 42x)\ D \times P} : \sqrt{a} \times \sqrt{AB \times L} = \sqrt{(42xx - 126ax + 168aa)\ D \times P} : \sqrt{(2xx + ax + aa)AB \times L}$, ut multiplicandi per crucem patebit.

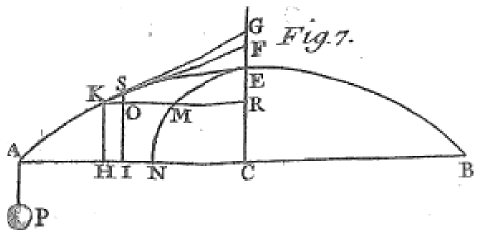
SCHOLIUM.

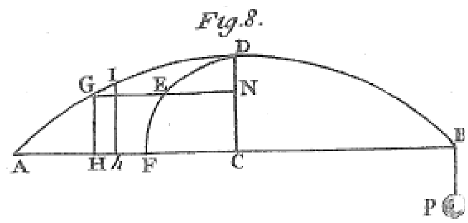
Res generaliter tractari potest pro quocunque numero ponduscolorum : sit enim numerus ponduscolorum n, & habitur erit $\varphi = (2a - x)\ Mg : b = fa \times \frac{1}{n}L$, unde $f = (2na - ax)\ Mg : abL$; & tempus per FC [$p : 2\sqrt{f}$] = $p\sqrt{ab}L : 2\sqrt{(2na - nx)\ Mg}$ = tempusculo semivibrationis fili. Ergo dividendo $p\sqrt{D} : 2\sqrt{g}$ per hoc, prodibit $\sqrt{(2na - nx)\ DM} : \sqrt{ab}L = \sqrt{((n + 1)(2aa - nx))\ D \times P} : \sqrt{(a \times AB \times L)}$ = numero qui quaeritur vibrationum fili oscillante semel pendulo dato D. In qua expressione pro x substituendus est ejus valor, qui quaeri debet per methodum in Lemmatibus praeliminaribus adhibitam. Sic, exempli gratia, se septem sint ponduscula, in quo casu $n = 7$, & x [per Lemma praeliminare VI] = $a\sqrt{(2 + \sqrt{2})}$, erit $\sqrt{((n + 1)(2na + nx))\ D \times P} : (a \times AB \times L) = 2\sqrt{(28 - 14\sqrt{(2 + \sqrt{2})})D \times P} : \sqrt{AB \times L}$, numero quaesito vibrationum, qui quam proxime accidit ad $2\sqrt{2}\ D \times P : \sqrt{AB \times L}$ justo minorem.

PROBLEMA VII.

Esto nunc chorda musica AB uniformiter crassa, cujus quantitas materiae = L, eaque tensa a pondere P = Mg. quaeritur numerus vibrationum in una oscillatione penduli dati D.

SOLUTIO.





Induat chorda, extra situm rectilineum AB figuram curvilimeam AEB, quae ea esse debet, ut quodlibet ejus punctum K eodem tempusculo perveniat ad punctum correspondens H in situ rectilinea, quo punctum medium E pervenit ad C : id quod facit, ut vis acceleratrix, qua punctum K versus H urgetur, ubique sit proportionalis distantiae KH. Ductis ergo duabus tangentibus proximis KG, GF; & ex K & S applicatis, KH, SI : erit ex principio statico pondus P seu Mg, ad vim qua particula chordae KS versus H urgetur, ut sinus anguli KSO, qui pro recto habetur, ad sinum ang. GKF, hoc est, ut 1 ad FG : KG; potest enim KF considerari tanquam perpendicularis ad axem CF. erit itaque vis illa in K = FG × Mg : KG = f × KH × dl. & inde vis ipsa acceleratrix seu f × KH = FG × Mg : KG × dl.. Ut autem determinetur FG : KG, notandum est, curvam AEB esse trochiodis sociam elongatam, hoc est, ejus naturae, ut descripto quadrante circuli EMN, & ducta KR parallela basi AC, sit AC : KR = EMN: EM, cujus demonstrationem infra adjeciemus. Sit nunc EC = a, ER = x, EM = s, EMN = $\frac{1}{2}pa$ [intelligo semper 1 as p ut diametrum ad circumferentiam]; sit etiam AC : EMN = n : 1, erit KR = ns, & reperiatur subtangens RG = $s\sqrt{(2ax - xx)} : a$,
CG = $a - x + s\sqrt{(2ax - xx)} : a$, ejusque differentialis FG = $-dx + (asdx - sxdx) : a\sqrt{(2ax - xx)} + dx =$
$(asds - sxdx) : a\sqrt{(2ax - xx)}$ adeoque FG: KG, vel quod idem est FG: KR = $(adx - xdx) : na\sqrt{(2ax - xx)}$; ipsum vero elementum KS, quod censetur aequale ipsi KO, = nds = $nadx$: $a\sqrt{(2ax - xx)}$. Est autem AB: HI [KO] = L: dL, unde dL = KO × L : AB × = $nadx$ × L.AB$\sqrt{(2ax - xx)}$; quibus ergo substitutis in vi acceleratrice habetur FG × Mg : KG × dL =
AB × $(a - x)$ × Mg: n^2a^2 L [quia npa = AB] $pp(a - x)$Mg : AB × L = pp × KH × Mg : AB × L = f × KH ; adeoque f = pp × Mg : AB × L , & tempus per KH [$p : 2\sqrt{f}$] = $\sqrt{}$ AB × L : 2 $\sqrt{}$ Mg = tempusculo semivibrationis chordae musicae; diviso itaque $p\sqrt{D} : 2\sqrt{g}$ per $\sqrt{}$ AB × L : 2 $\sqrt{}$ Mg
acquiritur $p\sqrt{DM}$: $\sqrt{}$ AB × L = $p\sqrt{}$ D × P : $\sqrt{}$ AB × L ; numerus vibrationum chordae durante una oscillatione penduli datae longitundinis D; quemadmodum invenit TAYLORUS. Vid. *Meth. Increm*. p. 93. Ex sicuti ego quoque inveni ex principio virium vivarum, ut sequitur:

Sit (Fig. 8) DN = x; NG = $y = n\int(adx : \sqrt{(2ax - xx)})$; DG = s; DC = a.

Est $ds - dy = (ds^2 - dy^2)$: $(ds + dy) = dx^2 : 2dy$ = [ob y = $y = n\int(adx : \sqrt{(2ax - xx)})$;] dx^2 :
$\frac{2\pi adx}{\sqrt{(2ax - xx)}} = dx\sqrt{(2ax - xx)} : 2\pi a$; adeoque DA - CA =

$\int(dx2 : 2dy) = \int(dx\sqrt{(2ax - xx)}) : 2na) = \frac{1}{2}\mathrm{CD} \times \mathrm{DEF} : 2\pi$ CD = $\frac{1}{4}$ DEF : n =
DEF : $4n$ = AC^2: $4n^2$ = AB : $8n^2$; hinc 2DA - 2CA = ADB - AB = AB : $4n^2$ = differentiae inter arcum & chordam. Radius osculi in G, posito elemento Gg vel ds constante, est generaliter $dsdy : ddx$ = [in curvis maxime elongatis ubi $dy = ds$] $dy^2 : ddx$. Ergo in hoc casu Troichoidis sociae maxime elongatae, ubi $dy = ds$ = $nadx : \sqrt{(2ax - xx)}$, erit $naddx\sqrt{(2ax - xx)} - (na^2dx^2 - naxdx^2) : \sqrt{(2ax - xx)} = 0$, unde
$ddx = (a - x)dx^2 : (2ax - xx)$; adeoque radius osculi
$dy^2 : ddx = \frac{nnaadx^2}{2ax - xx} : \frac{(a - x)dx^2}{(2ax - xx)} = \frac{nnaa}{a - x}$, hoc est, radii osculi sunt reciproce ut GH. Sit nunce pondus tendus tendens chordam, P ; pondus ipsius AB, L; velocitas puncti D cum vibrando venerit in C = $\sqrt{}$ S [intelligendo per S spatium per quod grave libere descendens acquirit velocitatem puncti D in C] : erit puncti cujuslibet G in H velocitas = GH $\sqrt{}$S : DC = $(a - x)\sqrt{}$S : a, adeoque

$\frac{(a-x)^2}{aa}\text{S}\times\frac{\text{Hh}}{\text{AB}}\times\text{L}=\frac{(a-x)^2}{aa}\times\text{S}\times\frac{dy.\text{L}}{\text{AB}}=\frac{(a-x)^2}{a}\times\text{S}\times\frac{ndx.\text{L}}{\text{AB}.\sqrt{(2axx-xx)}}$ = vi vivae particulae chordae G*g* vel H*h* in H = $\frac{nS.L}{a.AB}\times\frac{(a-x)^2 dx}{\sqrt{2ax-xx}}$; id quod integrando habetur

$\frac{nS\times L}{a.AB}\times((a-x)\sqrt{(2ax-xx)}+\int dx\sqrt{(2ax-xx)})$ = [pro tota chorda]

$\frac{2n\text{S.L}}{a.\text{AB}}\times\frac{1}{2}\times a\times\text{DEF}=\frac{n\text{S.L.DEF}}{\text{AB}}=\frac{1}{2}\text{L}\times\text{S}$ = quantitati virium vivarium totius chordae. Haec autem est aequalis vi vivae ponderis P descendentis per AB : an^2 = P.AB : $4n^2$. Ergo S = P.AB : $2n^2$ × L = 2P.DEF^2 : L × AB & √ S = DEF × √(2P : L.AB). Hince invenitur tempus per DC = √(L.AB:2P). Est autem tempus semioscillationis penduli simplicis cujus longitudino sit C = DEF × √2C: DC; ut igitur haec duo tempora sint aequalia, faciendum est √(L.AB:2P) = DEF × √2C: DC ; unde C = DC^2.AB.L : 4DEF^2.P. Ergo numerus vibrationum chordae in tempore unius vibrationis penduli datae longitudinis D = 2DEF×√(D × P): DC × √(AB.L), = [supposito 2DEF: DC = *p*] *p*√(D.P : AB.L) ut habet TAYLORUS, cui L & N sunt quod mihi AB & L.

Sequitur demonstratio ejus, quod supra asseritur, chordam vibrantem ADB [vid. fig. praeced.] induere figuram sociae Trochiodis elongatae.

Ostensum est in superioribus, sinum anguli contactus in puncto chordae quocuncue G proportionalem esse longitudini per currendae GH. Jam retentis iisdem symbolis, quibus supra usi sumus, erit sinus anguli contactus = $ddx : ds$ = [ob figuram maxime elongatam & consequenter $ds = dy$] $ddx : dy$, positis nimirum dy constantibus : longitudino autem percurrenda GH = $a - x$. Ergo $ddx : dy$ ad $a - x$ in ratio constante. Sit illla ratio ut dy ad $nnaa$; eritque $nnaaddx : dy = ady - xdy$; [seu divid. per dy] $nnaaddx : dy^2 = a - x$. Multiplicetur utrumque membrum per dx, & habebitur $nnaadxddx : dy^2 = adx - xdx$; sumtisque integralibus $nnaadx^2 : 2dy = ax - \frac{1}{2}$xx, seu $nnaadx^2 = (2ax - xx)dy^2$ unde $nadx : \sqrt{(2ax - xx)}$ = dy,

& $n\int(adx:\sqrt{(2ax-xx)})=y.$ Est itaque y [NG] ; $\int(adx:\sqrt{(2ax-xx)})$[arc. DE] = $n:1\frac{1}{2}$, id est, applicata NG ad arcum DE in ratione constante, eaque valde magna. Est enim ratio AC ad CD valde magna [per hyp.]. Ergo etiam ratio AC ad quadrantem DEF valde magna erit. Sed AC : DEF = *n* : 1. [*per demonstr.*] Quare constat propositum.