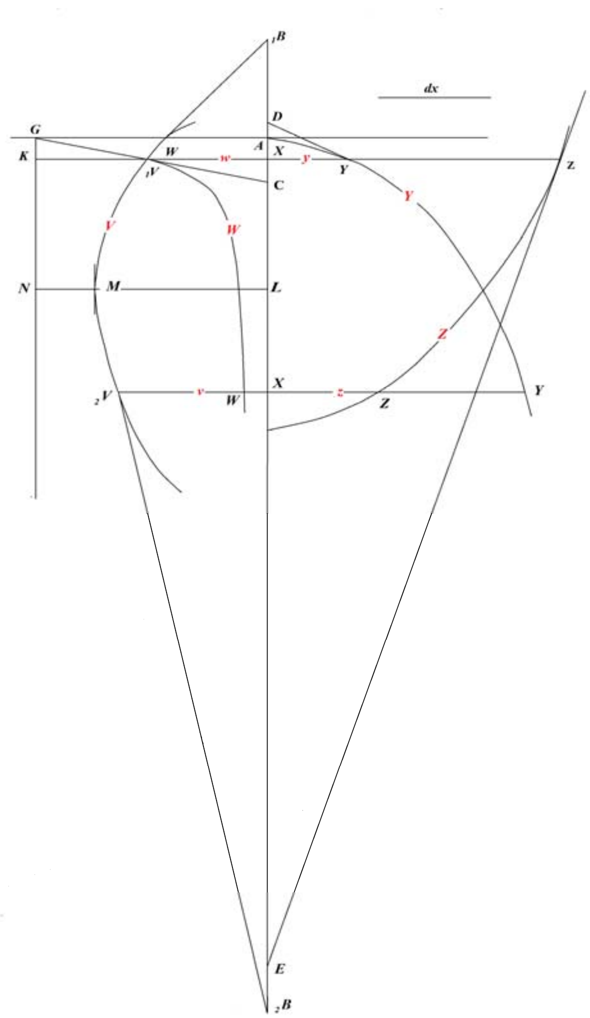
MATHEMATICA; N°. XIII.

A NEW METHOD

FOR FINDING MAXIMA AND MINIMA,

and likewise for tangents, and with a single kind of calculation for these, which is hindered neither by fractions nor irrational quantities.

From Actis Erud. Lips. Oct. 1684. p. 467-473



Let *AX* be the axis, & several curves, such as *VV, WW, YY, ZZ*, of which the ordinates normal to the axis shall be *VX, WX, YX* , *ZX*, which may be called respectively *v*, *w*, *y*, *z*; and the abscissa for the axis *AX* may be called *x*. The tangents shall be *XB, XC, XD, XE* meeting with the axis at the points *B, C, D, E* respectively. Now some right line taken arbitrarily may be called *dx*, and the right line which shall be to *dx*, as *v* (or *w*, *y*, *z*, respect.) is to *VB* (or *WC*, *YD*, *ZE*, respect.) [A confusing error was corrected in 1695 as the lengths *VB, WC, YD, ZE* were used originally.] may be called *dv* (or *dw*, *dy*, *dz*, respect.), or the differentials [Leibiz preferred to call these differences rather than differentials] of *v* (or of *w*, *y*, *z* themselves respect.).

With these put in place the rules of the calculation are as follows: Let *a* be a given constant quantity, *da* will be equal to 0, and $d\overline{ax}$ will be equal to adx: if *y* shall be equal to *v* (or the ordinate of some curve *YY*, equal to the corresponding ordinate of any curve *VV*) *dy* will be equal to *dv*.

Now for *Addition and Subtraction* : if $z-y+w+x$ shall be equal to *v*, $d\overline{z-y+w+x}$ shall be equal to $dz-dy+dw+dx$.

Multiplication : $d\,\overline{xv}$ equals $xdv + vdx$, or by putting y equal to *xv*, *dy* becomes equal to $xdv + vdx$. For by choice there is either a formula, such as *xv,* or a letter such as *y* to be used as a short-cut. It is required to be noted both *x* and *dx* are to be treated in the same manner in this calculation, as *y* and *dy*, or another indeterminate letter with its differential. It should be noted also, a regression [*i.e.* a return to the original expression, or the inverse process of integration] cannot to be given always for a differential equation, unless with a certain caution, about which more elsewhere.

Again for *Division*, $d\frac{v}{y}$ or (on putting z equal to $\frac{v}{y}$) *dz* equals $\frac{\pm vdy \mp ydv}{yy}$.

[*L*. was later to abandon this unnecessary complication of using ambiguous signs for the division rule, as the method moved away from being geometrical to algebraic, where the signs were treated according to the usual rules.]

Because here the *signs* are to be noted properly, since in the calculation of its differential for the simpler letter to be substituted, indeed for the same sign to be kept, and for + *z* write + *dz*, for $-z$ write −*dz*, as will be apparent from the addition and subtraction put in place a little before; but when the value comes to be evaluated critically, or when the relation of *z* to *x* may be considered, then to be apparent, either the value of *dz* shall become a positive quantity, or less than zero, or negative ; because when it happens later, then the tangent *ZE* is drawn from the point *z* not towards *A*, but in the opposite direction, or beyond *X* , that is then when the ordinates *z* themselves decrease, with *x* themselves increasing. And because the ordinates *v* themselves sometime increase, sometimes decrease, sometimes *dv* will be a positive quantity, sometimes negative, and in the first case, the tangent $_1V_1B$ is drawn towards *A*; in the latter $_2V_2B$ in the opposite direction : but neither happens about the middle-point *M* , in which the changes of *v* neither increases nor decreases, but are at rest, and thus as a consequence d*v* equal to 0, and where zero refers to a quantity neither positive nor negative, for + 0 equals − 0: therefore at the position *v* itself, truly the ordinate *LM*, is a *Maximum* (or if the convexity may be turned towards the axis, a *Minimum*) and the tangent of the curve at *M* neither is drawn above *X* to towards *A* everywhere nearer to the axis, nor below *X* in the opposite sense, but parallel to the axis. If *dv* shall be infinite with respect to *dx* , then the tangent is at right angles to the axis, or is the ordinate itself. If *dv* and *dx* are equal, the tangent makes a half right angle to the axis.

If with the ordinates *v* increasing, the increments themselves of these increase also, or the differences *dv* (or if with *dv* taken positive, also *ddv*, the differences of the differences, are positive, or with these negative, negative also) the curve turns *convex* towards the axis, otherwise *concave* [initially *L.* had these terms in the wrong order, which he subsequently corrected; these results refer to curves along the positive *x* direction, as clearly we have to consider decreases in the ordinate in order to proceed away from the origin in the negative sense]; truly where there is a maximum or minimum increment, or where the increments become increasing from decreasing, or vice versa, where there is a *point of inflection* [*i.e.* a point where opposite curvatures combine], and concavity and convexity may be interchanged between each other, but here the ordinates do not become decreasing from increasing, or vice-versa, for then the concavity or convexity of the curve may remain : but so that the increments may continue to increase or to decrease; truly it cannot happen that the ordinates become decreasing from increasing or vice-versa. And thus a point of contrary curvature [*i.e.* an inflection point] has a place, when neither v nor dv may become 0, yet ddv is 0. From which also a problem of contrary inflection does not have two equal roots, as the problem of maxima, but three equal roots. And all these depend indeed on the correct use of signs.

Moreover meanwhile, *changeable signs* are required to be used, as used recently in *division,* evidently it must be agreed first how they are to be explained. And indeed with increasing x, $\frac{v}{y}$ increases (decreases), the signs in $d\frac{v}{y}$ or in $\frac{\pm vdy \mp ydv}{yy}$ thus must be explained, so that this fraction becomes a positive (negative) quantity. But $\mp$ indicates the opposite of $\pm$, as if the one shall be $+$, the other shall be −, or the opposite. And more ambiguities are able to occur in the same calculation, which I distinguish with brackets, for the sake of an example if there should be $\frac{v}{y}+\frac{y}{z}+\frac{x}{v}=w$, there shall be

$$\frac{\pm\, vdy \mp ydv}{yy}+\frac{(\pm)\, ydz\, (\mp)\, zdy}{zz}+\frac{((\pm))\, xdv\, ((\mp))}{vv}=dw,$$

otherwise the ambiguities arising from different sources may become confused. Where it is to be noted, an ambiguous sign into itself gives $+$ itself, into its opposite gives $-$, into another ambiguity makes a new ambiguity, and depending on both.

Powers: $d\,x^a = a.\,x^{a-1}dx$, e.g. $d\,x^3 = 3.\,x^2 dx$.
$d\frac{1}{x^a}=-\frac{adx}{x^{a+1}}$, e.g. if w shall be $=\frac{1}{x^3}$ it becomes $dw=\frac{-3dx}{x^4}$.
Roots: $d,\sqrt[b]{x^a}=\frac{a}{b}dx\sqrt[b]{x^{a-b}}$ (hence $d,\sqrt[2]{y}=\frac{dy}{2\sqrt[2]{y}}$ for in that case a is 1, and b is 2; therefore $\frac{a}{b}dx\sqrt[b]{x^{a-b}}=\tfrac{1}{2}\sqrt[2]{y^{-1}}$; now y^{-1} is the same as $\frac{1}{y}$ [established by Wallis in his *Arithmetica Infinitorum*], from the nature of the exponents of a geometrical progression, and $\sqrt[2]{\frac{1}{y}}$ is $\frac{1}{\sqrt[2]{y}}$, $d\,\frac{1}{\sqrt[b]{x^a}}=\frac{-adx}{b\sqrt[b]{x^{a+b}}}$.

Moreover the rule of whole powers may be sufficient for determining both fractions as well as roots, for the power shall be a fraction when the exponent is negative, and it is changed into a root when the exponent is a fraction : but I have preferred these same consequences to be deduced from that, as with others remaining to be deduced, since the rules shall be exceedingly general and occurring frequently, and it may be better to ease deliberations by themselves in this complicated matter.

Just as from this known *Algorithm,* thus as I may say of this calculation which I call *differential,* all other differential equations can be found through a common calculation, and both maxima and minima, and likewise tangents are to be had, thus so that there shall be no need to remove fractions or irrationals or other root signs, because the method can still be done following the method produced this far. The demonstration of all will be easy with these things changed, and until now this one matter has not been paid enough consideration : *dx, dy, dv, dw, dz* themselves (each in its own series) can be had as proportional differences of *x, y, v, w, z*, either with momentary increments or decrements. [Note that these differential are made into momentary quantities, and the line labeled *dx* in the diagram above is not used further; it is of course allowed to have a triangle of finite size similar to that involving infinitesimals, and to call the ratio of the sides dy:dx. In addition, the idea of a function is not yet evident, and both abscissas and ordinates are treated in the same manner.]

From which it arises, that it shall be possible for any proposed equation to write the equation of its differential, because it can be done for any *member* (that is with a part, which by addition or subtraction alone, agrees according to the equation being established) by substituting the simpler quantity of the differential of the member; truly for any quantity (which is not itself a member [in as much as it follows by addition or subtraction only, as a term in an expression], but concurs in the same manner as a member being formed), its differential quantity indeed is not simply made from forming the differential quantity of its member being used, but follows the Algorithm prescribed thus far [*i.e.* as in multiplication and division of parts]. Indeed the methods used so far do not have such a transition, for generally they use a right line such as *DX,* even another of this kind, truly not the right line *dy*, which is the fourth proportional with *DX, DY*, *dx* themselves, which changes everything [note: these lines cannot be related to some shown in Fig. 1 with the same labels]; hence they may be arranged first, so that fractions and irrationals (which enter as indeterminate) may be removed; also it is apparent our method extends to transcendental lines, which cannot be recalled to an algebraic calculation, or which are of no certain order, and that with the most universal manner, and not always by succeeding without some particular substitutions;

[Thus *L*. admits he cannot use his simple mechanical methods to resolve transcendental differentiation.] ; just as it may be held in general, to find a *tangent* is to draw a right line, which joins two points of the curve having an infinitely small difference, or the side of an infinite angled polygon produced, which is equivalent to the *curve* for us. But I may note that infinitely small distance by some differential, as *dv*, or by a relation it can express to that itself, that is through a known tangent. Specially, if *y* were a transcending quantity, for example with the ordinate of a cycloid, and it may enter that calculation, with aid of which *z* itself, the ordinate of another curve, may be determined, and *dz* may be sought, or through that the tangent of this latter curve, *dz* by *dy* shall be required to be determined everywhere, because the tangent of the cycloid may be had. But that tangent of the cycloid itself, if it may not yet have been devised, may be able to be found by a like calculation from a given property of the tangent of the circle.

But it pleases to propose an example of the calculation, where it is to be noted here division is to be designated by me in this manner: x: y, because this is the same as x divided by y, or $\frac{x}{y}$. The *first* or the given equation shall be $x : y + \overline{\overline{a + bxc - xx}}:$ the square from $ex + fxx + ax\sqrt{gg + yy} + yy : \sqrt{hh + lx + mxx}$: equals 0, [*i.e.* in modern terms : $\frac{x}{y}+\frac{(a+bx)(c-xx)}{(ex+fxx)^2}+ax\sqrt{gg+yy}+\frac{yy}{\sqrt{hh+lx+mxx}}=0$;]

expressing the relation between *x* and *y*, or between *AX* and *XY*, with *a, b, c, e, f, g, h, 1, m* themselves put to be given ; the manner is sought of elucidating *YD* from the given point *Y*, which touches the curve, or the ratio of the right line *DX* to the given right line *XY* is sought. In order to abbreviate we will write *n* for $a+bx$; for $c-xx,\ p$; for $ex+fxx,\ q$; for $gg+yy,\ r$; for $hh+lx+mxx,\ s$; the equation becomes $x:y+np:qq+ax\sqrt{r}+yy:\sqrt{s}$ equals 0, which shall be the *second* equation.

Now from our calculation it is agreed $d,x:y$ to be $\pm xdy\mp ydx:yy$;

[i.e. $d\left(\frac{x}{y}\right)=\frac{\pm xdy\mp ydx}{yy}$;]

and similarly $d,np:qq$ to be $(\pm)2npdq(\mp)q(ndp+pdn),:q^3$

[i.e. $d(\frac{np}{qq})=\frac{(\pm)2npdq(\mp)q(ndp+pdn)}{q^3}$;]

and $d,ax\sqrt{r}$ to be $+axdr:2\sqrt{r}+adx\sqrt{r}$; and $d,yy:\sqrt{s}$ to $((\pm))yyds((\mp))4ysdy:2s\sqrt{s}$,

[i.e. $d\left(ax\sqrt{r}\right)=\frac{+axdr}{2\sqrt{r}}+adx\sqrt{r}$; and $d\left(\frac{yy}{\sqrt{s}}\right)=\frac{((\pm))yyds((\mp))4ysdy}{2s\sqrt{s}}$;]

which all the differential quantities thence from $d,x:y$ itself as far as to $d,yy:\sqrt{s}$ in one addition make 0, and they give in this way the *third* equation, indeed thus for the members of the second equation the amounts of their differentials may be substituted. Now *dn* is *bdx*, and *dp* is $-2xdx$, and *dq* is $edx+2fxdx$, and *dr* is *2ydy*, and *ds* is $ldx+2mxdx$. With which values substituted into the third equation the *fourth* equation will be had, where the differential quantities, which remain only, surely *dx, dy*, are found always outside the numerators and roots, and each member is acted on by either *dx*, or by *dy*, always with the rule of homogeneity maintained as regards these two quantities, in whatever manner the calculation may become entangled [*i.e.* all the differentials of a given member are of the same order;]: from which a value can be found always of $dx:dy$ itself of the ratio of *dx* to *dy*, that is a *DX* sought for a *XY* given, which ratio in our calculation here (by changing the fourth equation into the Analogous form) will be as

$$\mp x:yy-axy:\sqrt{r}((\mp))2y:\sqrt{s} \text{ is to}$$

$$\mp 1 : y(\pm) 2np(e+2fx) : q^3 (\mp)\overline{-2nx+pb} : qq + a\sqrt{r}\,((\pm))\,yy\overline{1+2mx} : 2s\sqrt{s}.$$

$$[i.e.\ \mp \frac{x}{yy} - \frac{axy}{\sqrt{r}}((\mp))\frac{2y}{\sqrt{s}} \text{ is to}$$

$$\mp\frac{1}{y}(\pm)\frac{2np(e+2fx)}{q^3}(\mp)\frac{-2nx+pb}{qq} + a\sqrt{r}\,((\pm))\,\frac{yy(1+2mx)}{2s\sqrt{s}}.]$$

Moreover *x* and *y* are given according to a given point *Y*. And the values of the letters *n, p, q, r, s* written above are given in terms of *x* and *y*. Therefore what is sought is found. And this example we have worked out thus only to be involved enough, so that the manner is apparent from the above rules also how it may be used in a more difficult calculation. Now to show it has an outstanding use in confronting more meaningful examples.

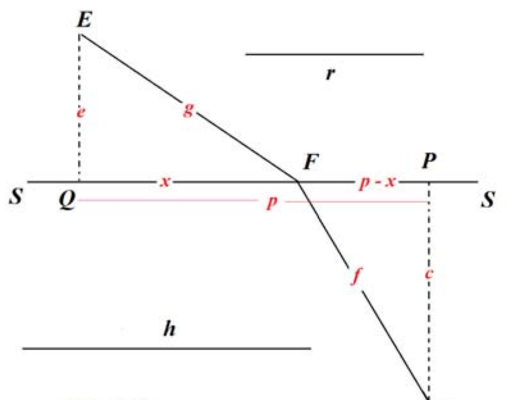
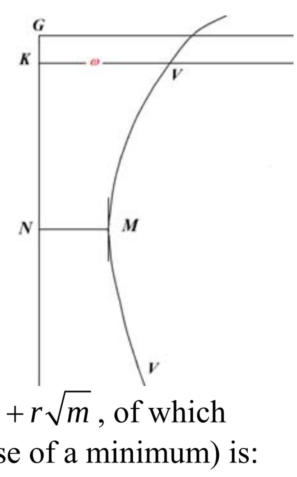


Fig. 112

The two points *C* and *E* shall be given (fig.112), and the right line *SS* in the same plane with these; the point *F* in *SS* is sought thus being taken, so that with *CF* and *EF*, the sum of the rectangles *CF* for a given *h*, and *FE* for a given *r*, to be the smallest possible of all, that is if *SS* shall be separating the mediums, and *h* may represent the density of a medium such as water from part *C*, and *r* the density of a medium such as air from *E*, the point *F* is sought such, that the path from *C* to *E* through *F* shall be the most convenient possible of all. We may consider the sum of all the rectangles possible, or all the difficulties of the paths possible, to be represented by the ordinates *KV* themselves, of the curve *VV* normal to the right line *GK*, which we call ω, and the minimum of these *NM* is sought. [The adjoining diagram is taken from fig.111.] Because the points *C* and *E* are given, and the perpendiculars to *SS* shall be given, truly *CP* (which we will call *c*) and *EQ* (as *e*) and besides *PQ* (as *p*), moreover that *QF* itself, which shall be equal to *GN* itself (or *AX*), we will call *x* and *CF, f*, and *EF, g*; there arises *FP*, $p-x$, f is equal to $\sqrt{cc+pp-2px+xx}$ or for brevity $\sqrt{l}$, and *g* is equal to $\sqrt{ee+xx}$ or for brevity $\sqrt{m}$. Therefore we have ω equal to $h\sqrt{l}+r\sqrt{m}$, of which equation the differential equation (on putting $d\omega$ to be 0, in the case of a minimum) is: $+\,hdl : 2\sqrt{l} + rdm : 2\sqrt{m}$ equal to 0, by the rules of calculus treated by us;



[The word *calculus* in Latin just refers to a calculation, as performed in ancient Rome in daily life using small pebbles or calculi; *L* may be using the word in this sense, but here we use it in its modern mathematical sense.];

now *dl* is $-\,2dx\overline{p-x}$, and *dm* is $2xdx$, therefore : $h\overline{p-x} : f$ equals $rx : g$.

$$[i.e.\ \frac{h(p-x)}{f}=\frac{rx}{g}\ ]$$

But if now this may be adapted to optics, and f and g may be made equal, or *CF* and *EF* are equal, because the same refraction remains at the point *F*, however great a length of the right line *CF* may be put, there becomes $h\,\overline{p-x}$ equals *rx*, or $h:r::\ x:p-x$, or *h* to *r* shall be as *QF* to *FP*, that is, the sines of the angles of incidence and refraction *FP* and *QF* will be reciprocally as *r* and *h*, the densities of the mediums, in which shall be the incidence and the refraction. Which density is required to be understood, not however to be with respect to us [*i.e.* the matters we are free to arrange as we wish], but with regard to the resistance which the rays of light cause. [We now consider this 'density' to be the refractive index of the medium.] And thus a demonstration of the calculus is had, shown by us elsewhere in these *Actis* [*A.E.* vol. I, p. 185], when we were explaining the general foundations of Optics, Catoptrics and Dioptrics, which other savants shall have come upon in a roundabout way, and which the skilled will perform after three lines of the calculus. Which at this point I will illustrate by another example.

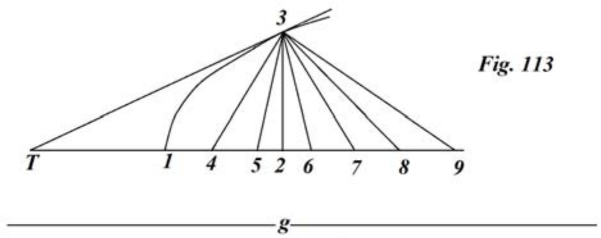


Fig. 113

The curve 133 (fig.113) shall be of such a kind: that from any point of that such as 3, six right lines shall be drawn to six fixed points placed on the axis at 4, 5, 6, 7, 8, 9, the six right lines likewise added together, 34, 35, 36, 37, 38, 39 shall be equal to the given line *g*. The axis shall be *T* 14526789, and 12 shall be the abscissa, 23 the ordinate, and the tangent 3*T*; I say *T*2 shall be to 23 as

$$\frac{23}{34}+\frac{23}{35}+\frac{23}{36}+\frac{23}{37}+\frac{23}{38}+\frac{23}{39}\ \text{is to}\ -\frac{24}{34}-\frac{25}{35}+\frac{26}{36}+\frac{27}{37}+\frac{28}{38}+\frac{29}{39}.$$

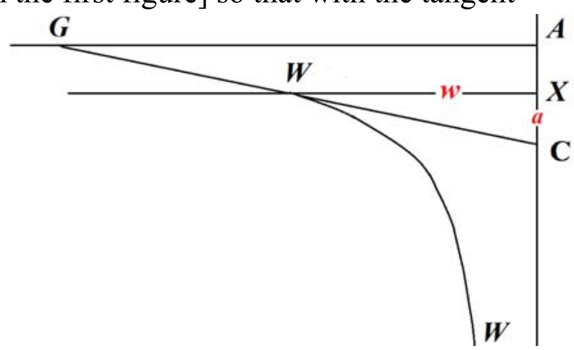
[Note that the 'numbers' written here is actually an early form of the method of naming points by the use of indices; thus 23 is the length of a line section d_i indicated as follows : For if we designate any of the fixed points on the *x*-axis by x_i, then the distance to the point 3 or (*x, y*) is given by $d_i=\sqrt{y^2+(x-x_i)^2}$; hence the problem amounts to finding the function *y* such that $g=\sum_{i=1}^{n}d_i=\sum_{i=1}^{n}\sqrt{y^2+(x-x_i)^2}$. By differentiation we find

$$0=\sum_{i=1}^{n}\left(\frac{y\frac{dy}{dx}+x-x_i}{\sqrt{y^2+(x-x_i)^2}}\right)=\sum_{i=1}^{n}\left(\frac{y\frac{dy}{dx}+x-x_i}{d_i}\right)=\frac{dy}{dx}\sum_{i=1}^{n}\frac{y}{d_i}+\sum_{i=1}^{n}\frac{x-x_i}{d_i}\ \text{etc.}$$

Cleary if $n=2$ we have an ellipse with the foci as the denoted points on the *x*-axis.]

And the rule will be the same, with so many terms continued, if not six, but ten, or more fixed points may be supposed ; such a kind as produced following the methods of the tangents from the calculus to be better with irrationalities removed, as it would become a most tedious and finally an insurmountable amount of work, if plane rectangles or even volumes, according to all the two or three [subscripts] possible, should be composed from these right lines which must be equated to a given quantity, in which everything is more involved ; and likewise it is with the belief that the use of our method facilitates the resolution of the rarest example. And these indeed are only the beginnings of a certain kind of a much more sublime geometry, extending to the most difficult and most beautiful problems, also with each a mixture of mathematics, which without our differential calculus, or something similar, will not be able to be treated with equal facility, but blindly.

It pleases to add the solution of a problem as an appendix, which De Baune proposed to Decartes to attempt himself, in Vol. 3 of his letters, but which he did not solve : To find the line of such a kind *WW*, [adapted from the first figure] so that with the tangent *WC* drawn to the axis, *XC* shall always be equal to the same constant right line *a*. Now *XW* or *w* shall be to *XC* or *a,* as *dw* to *dx*; therefore if *dx* (which can be taken by choice) may be assumed constant or always the same, truly *b*, or if *x* itself or if *AX* may increase uniformly, *w* will be made equal to $\frac{a}{b}dw$, [*i.e.* $w = \frac{a}{b}dw$], and

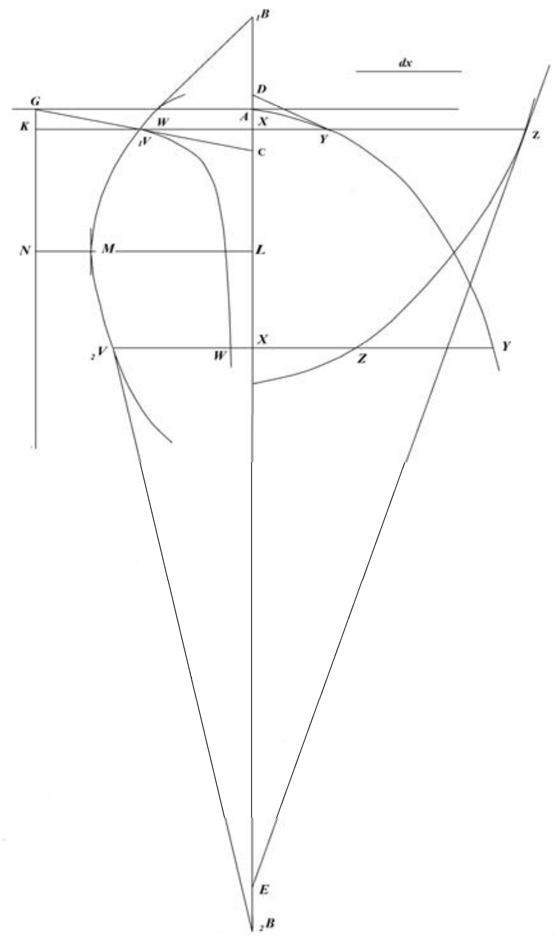


the ordinates *w* themselves which will proportional to their increments, or differentials, from *dw*, that is if the *x* [abscissas] shall be in an arithmetic progression, the *w* [ordinates] shall be in a geometric progression, or if *w* shall be numbers, *x* will be their logarithms: therefore the line *WW* is logarithmic.
[Thus, we have the differential equation

$\frac{dw}{dx} = -\frac{w}{a}$, or $\frac{-x}{a} = \ln w - \ln A$, giving $w = Ae^{\frac{-x}{a}}$. Hence, under the term *logarithmic*, the inverse or exponential function must be included; which Leibniz has realized, but the inverse relation at the time did not have a name as such, to be set out a little later by Johann Bernoulli, see *e.g.* his *Opera Omnia*, vol. I, p.179.]

MATHEMATICA ; N°. XIII.

NOVA METHODUS
PRO MAXIMIS ET MINIMIS,
itemque tangentibus, quae nec fractas, nec irrationales quantitates moratur, & singulare pro illis calculi genus.
Ex Actis Erud. Lips. ann. 1684.



Sit axis *AX*, & curvae plures, ut *VV, WW, YY, ZZ*, quarum ordinatae, ad axem normales, *VX, WX, YX , ZX*, quae vocentur respective, *v, w, y, z*; & ipsa *AX* abscissa ab axe, vocetur x. Tangentes sint *VB, WC, YD, ZE* axi occurrentes respective in punctis *B, C, D, E*. Jam recta aliqua pro arbitrio assumta vocetur dx, & recta quae sit ad *dx*, ut *v* (vel *w*, vel *y*, vel *z*) est ad *VB* (vel *WC*, vel *YD*, vel *ZE*) vocetur *dv* (vel *dw*, vel *dy*, vel *dz*) sive differentia ipsarum *v* (vel ipsarum *w*, aut *y*, aut *z*). His positis calculi regulae erunt tales. Sit *a* quantitas data constans, erit *da* aequalis 0, & $d\overline{ax}$ erit aequ *adx*: si sit *y* aequ. *v* (seu ordinata quevis curvae *YY*, aequalis cuivis ordinatae respondenti curvae *VV)* erit *dy* aequ. *dv*. Jam *Additio & Subtractio* : si sit $z-y+w+x$ aequ. *v*, erit $d\overline{z-y+w+x}$ aequ. $dz-dy+dw+dx$. *Multiplicatio,* $d\overline{xv}$ aequ. $xdv+vdx$, seu posito *y* aequ. *xv*, fiet *dy* aequ. $xdv+vdx$. In arbitrio enim est vel formulam, ut *xv*, vel compendio pro ea literam, ut *y*, adhibere. Notandum & *x* & *dx* eodem modo in hoc calculo tractari, ut *y* & *dy*, vel aliam literam indeterminatam cum sua differentiali. Notandum etiam, non dari semper regressum a differentiali Aequatione, nisi cum quadam cautione, de quo alibi. Porro *Divisio*, $d\frac{v}{y}$ vel (posito *z* aequ. $\frac{v}{y}$) *dz* aequ. $\frac{\pm vdy \mp ydv}{yy}$ Quoad *Signa* hoc probe notandum, cum in calculo pro litera substituitur simpliciter ejus differentialis, servari quidem eadem signa, & pro + *z* scribe + *dz*, pro – *z* scribe −*dz*, ut ex additione, & subtractione paulo ante posita apparet; sed quando ad exegesin valorum venitur, seu cum consideratur ipsius *z* relatio ad *x*, tunc apparere, an valor ipsius *dz* fit quantitas affirmativa, an nihilo minor, seu negativa ; quod posterius cum fit, tunc tangens *ZE* ducitur a puncta *z* non versus *A* , sed in partes contrarias, seu infra *X* , id eft tunc cum ipsae ordinatae *z* decrescunt crescentibus *x*. Et quia ipsae ordinata *v* modo crescunt, modo decrescunt, erit d*v* modo affirmativa modo negativa quantitas, & priore casu, ${}_1V_1B$ tangens ducitur versus *A*; posteriore ${}_2V_2B$ in partes aversas: neutrum autem fit in media circa *M* , quo momenta ipsae *v* neque crescunt , neque decrescunt, sed in statu sunt, adeoque fit d*v* aequ. 0, ubi nihil refert quantitas fit ne affirmativa an negativa, nam + 0 aequ. − 0: eoque in loco ipsa *v*, nempe ordinata *LM*, est *maxima* (vel si convexitatem axi obverteret, *Minima*) & tangens curvae in *M* neque supra *X* ducitur ad partes A ibique axi propinquat, neque infra *X* ad partes contranas, sed est axi parallela. Si *dv* sit infinita respectu ipsius *dx* , tunc tangens est ad axem recta, seu est ipsa ordinata. Si *dv* et *dx* aequales, tangens facit angulum semirectum ad axem. Si crescentibus ordinatis *v*, crescunt etiam ipsa earum incrementa vel differentiae *dv* (seu si positis *dv* affirmativis, etiam *ddv*, differentiae differentiarum, sunt affirmativae, vel negativis, negativae) curva axi obvertit *concavitatem,* alias *convexitatem* : ubi vero est maximum vel minimum incrementum, vel ubi incrementa ex decrescentibus fiunt crescentia, aut contra, ibi est *punctum flexus contrarii,* et concavitas atque convexitas inter se permutantur, modo non et ordinatae ibi ex crescentibus fiant decrescentes, vel contra, tunc enim concavitas aut convexitas maneret: ut autem incrementa continuent crescere aut decrescere, ordinatae vero ex crescentihus fiant decrescentes, vel contra , fieri non potest. Itaque punctum flexus contrarii locum habet, quando neque v neque dv existente 0, tamen ddv est 0. Unde etiam problema flexus contrarii non duas , ut problema maximae, sed tres habet radices aequales. Atque haec omnia quidem pendent a recto usu signorum.

Interdum autem adhibenda sunt *Signa ambigua,* ut nuper in *Divisione,* antequam scilicet constet quomodo explicari debeant. Et quidem si crescentibus *x*, crescunt (decrescunt) $\frac{v}{y}$, debent signa $d\frac{v}{y}$ seu in $\frac{\pm vdy \mp ydv}{yy}$ ita explicari, ut haec fractio fiat quantitas affirmativa (negativa). Significat autem $\mp$ contrarium ipsius $\pm$, ut si hoc sit $+$, illud sit −, vel contra. Possunt et in eodem calculo occurrere plures ambiguitates, quas distinguo parenthesibus, exempli causa si esset $\frac{v}{y}+\frac{y}{z}+\frac{x}{v}=w$, foret

$$\frac{\pm\, vdy \mp ydv}{yy}+\frac{(\pm)\,ydz\,(\mp)\,zdy}{zz}+\frac{((\pm))\,xdv\,((\mp))}{vv}=dw\,,$$

alioqui ambiguitates ex diversis capitibus ortae confunderentur. Ubi notandum, signum ambiguum in se ipsum dare $+$, in suum contrarium dare − , in aliud ambiguum formare novam ambiguitatem ex ambabus dependentem.

Potentiae: $d\,x^a = a.\,x^{a-1}dx$, exempli gratia $d\,x^3 = 3.\,x^2dx$.
$d\frac{1}{x^a} = -\frac{adx}{x^{a+1}}$, ex. gr. si *w* sit $=\frac{1}{x^3}$ fiet $dw=\frac{-3dx}{x^4}$.

Radices: $d,\sqrt[b]{x^a} = \frac{a}{b}dx\sqrt[b]{x^{a-b}}$ (hinc $d,\sqrt[2]{y} = \frac{dy}{2\sqrt[2]{y}}$ nam eo casu *a* est 1, et *b* est 2; ergo $\frac{a}{b}dx\sqrt[b]{x^{a-b}} = \frac{1}{2}\sqrt[2]{y^{-1}}$ est; jam y^{-1} idem est quod $\frac{1}{y}$, ex natura exponentium progressionis Geometricae, et $\sqrt[2]{\frac{1}{y}}$ est $\frac{1}{\sqrt[2]{y}}$, $d\frac{1}{\sqrt[b]{x^a}} = \frac{-adx}{b\sqrt[b]{x^{a+b}}}$. Suffecisset autem regula potentiae integrae tam ad fractas tam ad radices determinandas , potentia enim sit fracta cum exponens est negativus, et mutatur in radicem cum exponens est fractus : sed malui consequentias istas ipse deducere, quam aliis deducendas relinquere, cum sint admodum generales et crebro occurrentes, et in re per se implicita praestet facilitati consulere.

Ex cognito hoc velut *Algorithmo,* ut ita dicam, calculi hujus, quem voco *differentialem,* omnes aliae aequationes differentiales inveniri possunt per calculum communem, maximaeque et minimae, itemque tangentes haberi, ita ut opus non sit tolli fractas aut irrationales aut alia vincula, quod tamen faciendum fuit secundum Methodos hactenus editas. Demonstratio omnium facilis erit in his rebus versato et hoc unum hactenus non satis expensum consideranti, ipsas *dx, dy, dv, dw, dz*, ut ipsarum *x, y, v, w, z* (cujusque in sua serie) differentiis sive incrementis vel decrementis momentaneis proportionales haberi posse. Unde fit, ut proposita quacunque aequatione scribi possit ejus aequatio differentialis, quod fit pro quolibet *membro* (id est parte , quae sola additione vel subtractione ad aequationem constituendam concurrit) substituendo simpliciter quantitatem membri differentialem, pro alia vero quantitate (quae non ipsa est membrum , sed ad membrum formandum concurrit) ejus quantitatem differentialem ad formandam differentialem quantitatem ipsius membri adhibendo non quidem simpliciter, sed secundum Algorithmum hactenus praescriptum. Editae vero hactenus methodi talem transitum non habent, adhibent enim plerumque rectam ut *DX,* vel aliam hujusmodi, non vero rectam dy, quae ipsis *DX, DY*, *dx* est quarta proportionalis, quod omnia turbat; hinc praecipiunt, ut fractae et irrationales (quas indeterminatae ingrediuntur) prius tollantur; patet etiam methodum nostram porrigi ad lineas transcendentes, quae ad calculum Algebraicum revocari non possunt, seu quae nullius sunt certi gradus, idque universalissimo modo, sine ullis suppositionibus particularibus non semper succedentibus, modo teneatur in genere, *tangentem* invenire esse rectam ducere, quae duo curvae puncta distantiam infinite parvam habentia jungat, seu latus productum polygoni infinitanguli, quod nobis *curvae* aequivalet. Distantia autem illa infinite parva semper per aliquam differentialem notam, ut *dv* , vel per relationem ad ipsam exprimi potest, hoc est per notam quandam tangentem. Speciatim, si esset *y* quantitas transcendens, exempli causa ordinata cycloeidis, eaque calculum ingrederetur, cujus ope ipsa *z*, ordinata alterius curvae, esset determinata, et quaereretur *dz*, seu per eam tangens hujus curvae posterioris, utique determinanda esset *dz* per *dy*, quia habetur tangens cycloeidis. Ipsa autem tangens cycloeidis , si nondum haberi fingeretur, similiter calculo inveniri posset ex data proprietate tangentium circuli.

Placet autem exemplum calculi proponere, ubi notetur me divisionem hic designare hoc modo: *x*: *y*, quod idem est ac *x* divis. per *y* seu $\frac{x}{y}$. Sit aequato *prima* seu data

$x:y+\overline{\overline{a+bx}\,\overline{c-xx}}:$quadrat. ex $ex+fxx+ax\sqrt{gg+yy}+yy:\sqrt{hh+lx+mxx}$: aequ. 0, exprimens relationem inter *x* et *y* seu inter *AX* et *XY*, posito ipsas *a, b, c, e, f, g, h, 1, m* esse datas; quaeritur modus ex dato puncto *Y* educendi *YD*, quae curvam tangat, seu quaeritur ratio rectae *DX* ad rectam datum *XY*. Compendii causa pro $a+bx$ scribamus *n*; pro $c-xx,\ p$; pro $ex+fxx,\ q$; pro $gg+yy,\ r$; pro $hh+lx+mxx,\ s$; fiet $x:y+np:qq+ax\sqrt{r}+yy:\sqrt{s}$ aeqti. 0, quae sit aequatio *secunda.* Jam ex calculo nostro constat $d,x:y$ esse $\pm xdy+ydx:yy$; et similiter $d,np:qq$ esse $(\pm)2npdq(\mp)qndp+pdn,:q^3$ et $d,ax\sqrt{r}$ esse $+axdr:2\sqrt{r}+adx\sqrt{r}$; et $d,yy:\sqrt{s}$ esse$((\pm))yyds((\mp))4ysdy:2s\sqrt{s}$, quae omnes quantitates differentiales inde ab ipso $d,x:y$ usque, ad $d,yy:\sqrt{s}$ in unum additae facient 0, et dabunt hoc modo aequationem *tertiam,* ita enim pro membris secundae aequationis substituuntur quantitates eorum differentiales. Jam *dn* est *bdx*, et *dp* est $-2xdx$, et *dq* est $edx+2fxdx$, et *dr* est *2ydy*, et *ds* est $ldx+2mxdx$. Quibus valoribus in aequatione tertia substitutis habebitur aequatio *quarta,* ubi quantitates differentiales, quae solae supersunt, nempe *dx*, *dy*, semper reperiuntur extra nominatores et vincula, et unumquodque membrum afficitur vel per *dx*, vel per *dy*, servata semper lege homogeneorum quoad has duas quantitates, quomodocunque implicatus sit calculus : unde semper haberi potest valor ipsius $dx:dy$ seu rationis *dx* ad *dy*, hoc est *DX* quaesitae ad *XY* datam, quae ratio in hoc nostro calculo (mutando aequationem quartam in Analogiam) erit ut

$$\mp x:yy-axy:\sqrt{r}((\mp))2y:\sqrt{s}\ \text{ est ad}$$

$$\mp 1:y(\pm)2npe+2fx,:q^3(\mp)-2nx+pb:qq+a\sqrt{r}((\pm))yyl+2mx:2s\sqrt{s}.$$

Dantur autem *x* et *y* ex dato puncto *Y*. Dantur et valores supra scripti literarum *n, p, q, r, s* per *x* et *y*. Habetur ergo quaesitum. Atque hoc exemplum satis implicatum ideo tantum ascripsimus, ut modus superioribus regulis in calculo etiam difficiliore utendi appareret. Nunc praestat usum in exemplis intellectui magis obviis ostendere.

Data sint duo puncta *C* et *E* (fig.112), et recta *SS* in eodem cum ipsis plano; quaeritur punctum *F* in *SS* ita sumendum, ut junctis *CF*, EF, sit aggregatum rectangulorum *CF* in datam *h*, et *FE* in datam *r* , omnium possibilium minimum, hoc est si *SS* sit mediorum separatrix , et *h* repraesentet densitatem medii ut aequae a parte *C* , et *r* densitatem medii ut aeris a parte *E*, quaeritur punctum *F* tale, ut via a *C* ad *E* per *F* sit omnium possibilium facillima. Ponamus omnia ista rectangulorum aggregata possibilia, vel omnes viarum possibilium difficultates, repraesentari per ipsas *KV*, curvae *VV* ordinatas ad rectam *GK* normales, quas vocabimus ω, quaerique minimam earum *NM*. Quia dantur puncta *C* et *E*, dabuntur et perpendiculares ad *SS*, nempe *CP* (quam

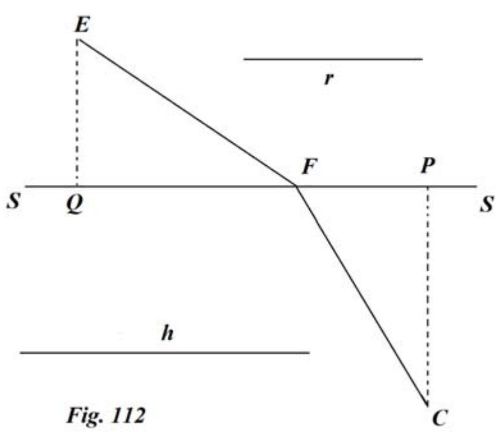


Fig. 112

vocabimus *c*) et *EQ* (quam *e*) et praeterea *PQ* (quam *p*), ipsa autem *QF*, quae sit aequalis ipsi *GN* (vel *AX*), vocabimus *x* et *CF, f*, et *EF, g*; fiet *FP*, $p-x, f$ aequ. $\sqrt{cc+pp-2px+xx}$ seu compendio $\sqrt{l}$, et *g* aequ. $\sqrt{ee+xx}$ seu compendio $\sqrt{m}$. Habemus ergo ω aequ. $h\sqrt{l}+r\sqrt{m}$, cujus aequationis aequatio differentialis (posito $d\omega$ esse 0, in casu minimae) est 0 aequ. $+hdl:2\sqrt{l}+rdm:2\sqrt{m}$ per regulas calculi nostri traditas; jam *dl* est $-2dx\overline{p-x}$, et *dm* est $2xdx$, ergo fit: $h\overline{p-x}:f$ aequ. $rx:g$. Quodsi jam haec accommodentur ad dioptricam, et ponantur *f* et *g* seu *CF* et *EF* aequales, quia eadem manet refractio in puncto F, quantacunque ponatur longitudo rectae CF, fiet $h\ \overline{p-x}$ aequ. *rx*, seu $h:r::\ x:p-x$, seu *h* ad *r* ut *QF* ad *FP*, hoc est sinus angulorum incidentiae et refractionis *FP* et *QF* erunt reciproce ut *r* et *h*, densitates mediorum, in quibus fit incidentia et refractio. Quae densitas tamen non respectu nostri, sed respectu resistentiae quam radii lucis faciunt, intelligenda est. Et habetur ita demonstratio calculi, alibi a nobis in his ipsis Actis exhibiti, quando generale Opticae, Catoptricae et Dioptricae fundamentum exponebamus, cum alii doctissimi Viri multis ambagibus venati sint quae hujus calculi peritus tribus lineis imposterum praestabit. Quod alio adhuc exemplo docebo. Sit curva 133 (fig.113) talis naturae: ut a puncto ejus quocunque ut 3 ductae ad sex puncta fixa in axe posita 4, 5, 6, 7, 8, 9, sex rectae 34, 35, 36, 37, 38, 39 simul additae, sint rectae datae *g* aequales. Sit axis *T* 14526789, et 12 sit abscissa, 23 ordinata, quaeritur tangens 3 *T*; dico fore *T* 2 ad 23 ut

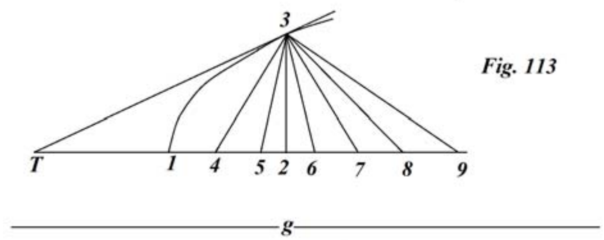


Fig. 113

$$\frac{23}{34}+\frac{23}{35}+\frac{23}{36}+\frac{23}{37}+\frac{23}{38}+\frac{23}{39} \text{ est ad } -\frac{24}{34}-\frac{25}{35}+\frac{26}{36}+\frac{27}{37}+\frac{28}{38}+\frac{29}{39}.$$

Eademque erit regula, continuatis tantum terminis, si non sex, sed decem, vel plura puncta fixa supponerentur, qualia secundum methodos tangentium editas calculo praestare sublatis irrationalibus, taediosissimae et aliquando insuperabilis operae foret, ut si rectangula plana vel solida secundum omnes biniones vel terniones possibiles ex rectis illis composita datae quantitati aequari deberent, in quibus omnibus, et multo implicatioribus, methodi nostrae eadem est opinione multo major rarissimique exempli facilitas. Et haec quidem initia sunt tantum Geometriae cujusdam multo sublimioris, ad difficillima et pulcherrima quaeque etiam mistae Matheseos problemata pertingentis, quae sine calculo nostro differentiali, aut simili, non temere quisquam pari facilitale tractabit. Appendicis loco placet adjicere solutionem Problematis, quod *Cartesius* a *Beaunio* sibi propositum Tom. 3. Epist. tentavit, sed non solvit : Lineam invenire *WW* talis naturae, ut ducta ad axem tangente *WC*, sit *XC* semper aequalis eidem rectae constanti *a*. Jam *XW* seu *w* ad *XC* seu *a*, ut *dw* ad *dx*; ergo si *dx* (quae assumi potest pro arbitrio) assumatur constans sive semper eadem, nempe *b*, seu si ipsae *x* sive *AX* crescant uniformiter, fiet w aequ. $\frac{a}{b}dw$, quae erunt ipsae w ordinatae ipsis dw, suis incrementis sive differentiis proportionales, hoc est si x sint progressionis arithmeticae, erunt w progressionis Geometricae, seu si w sint numeri, x erunt logarithmi: linea ergo WW logarithmica est.