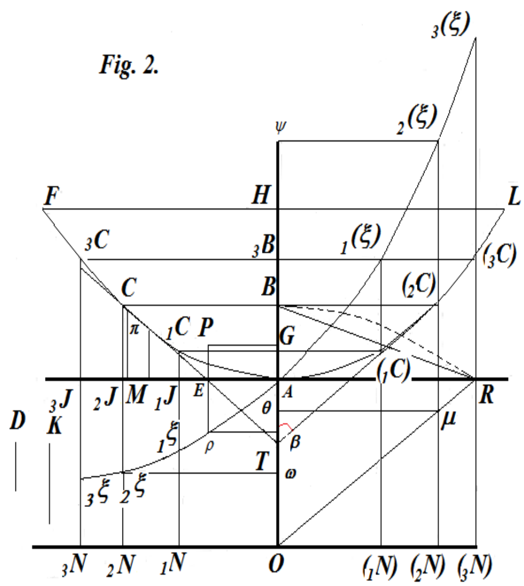
CONCERNING THE CURVE FORMED BY A HEAVY FLEXIBLE CORD DUE TO ITS OWN WEIGHT, AND ITS CONSPICUOUS USE FOR FINDING ANY MEAN PROPORTIONALS AND LOGARITHMS.

W.W. LEIBNIZ.
Act. Erudit. Lips. an. 1691.

The problem of the catenery or rope curve has two uses, the one that analysis or the art of investigating may be augmented, which until now has not been extended far enough to solve such problems, the other in order that the practise of constructing such curves may be advanced. Indeed I found this curve could be prepared in the easiest manner, thus to be in effect a most useful curve, nor following after any of the [known] transcendental curves. For it can be prepared without difficulty from the suspension of a string or rather of a catenary [*i.e.* a chain] (which does not change in length) and it can be described physically according to a certain kind of [geometrical] construction. And once described, with its aid, any number of mean proportionals can be shown, both of logarithms and of the quadrature of the hyperbola. Initially *Galileo* thought about that curve, but did not understand its nature : indeed it is not a Parabola, as he had supposed himself. *Joachim Jung*, an outstanding philosopher and mathematician of our century, who before *Descartes* had many outstanding ideas about improving our knowledge of this curve, entered into with calculations and experiments performed, excluded the parabola, but did not put in place the true curve. From that time the question has been attempted by many, with the solution found by no one, until recently when the occasion of its treatment had been provided to me by a most learned mathematician. For the most *Cel. Jacob Bernoulli*, had on my urging, with a certain kind of my infinitesimal analysis - the differential calculus - applied that happily to certain problems involving a certain expression introduced, now he had desired me to announce publicly in the *Acta* of May of the previous year, p. 218 seq., that I might test, whether or not our kind of calculus could be extended to problems of this kind, such as the finding the curve of the catenary. [*i.e.* James Bernoulli had invited Leibniz to solve this problem of the catenary, and he had given himself a year in which to do it; both Huygens and both the Bernoulli brothers send solutions to the *Acta Erud*. before the year was up; these were published in the *A.E.* after Leibniz's paper had appeared in June 1691. See Aiton's *Biography of Leibniz* p.203 for further details, and the original in Gehardt's MS, vol. 5. p.243, all of which are translated here. Unfortunately, there is a serious error in the MS, arising from a misunderstanding by Gehardt, which resulted in his changing a diagram of Huygens solution.] For the sake of trying out the matter, and in the first place, not only did I succeed in solving this illustrious problem, least I am mistaken, but also I have understood the curve to have many outstanding uses, the matter was done, as in the example of *Blasé Pascal* amongst others. I had invited mathematicians in a predetermined time, to try out their methods for the same enquiry, so that it would be apparent what they might bring to fruition from that, which perhaps others might use, which has been used by me and *Bernoulli*. Within the elapsed time, only two indicated they had attended to the matter, *Christian Huygens,* whose great merit in the *République des Lettres* no one could ignore, and both the very erudite *Bernoulli* himself with his clever young brother, who from these things which he had given, has put into effect that we may hope again for something outstanding from them. Because I have indicated that I consider the matter therefore truly to be tested, here too I extend our account of the calculation, and which before had the greatest difficulty, now can approached. Moreover I am pleased to put in place, what I have found ; what the others have excelled in, will be shown gathered together.

The curve is constructed geometrically thus, without the aid of a string or a catenary, and without the supposition of quadratures, from that kind of construction, where nothing could be had in greater and perfect agreement from the analysis according to my sentiments about transcendental curves. There shall be any two right lines, having a certain invariable ratio determined between each other, that here evidently are set forth as D and K (fig. 2; [א & ב are used for D and K in the text, but not in the diagram]), with which ratio once known all the others preceed by ordinary geometry. ON shall be an indefinte right line parallel to the horizontal, and OA perpendicular to that, equal to O_3N, and the vertical line above ${}_3N_3\xi$, which shall be to



OA, as D to K. The mean proportion ${}_1N_1\xi$ is sought between OA and ${}_3N_3\xi$; and between ${}_1N_1\xi$ and ${}_3N_3\xi$, and likewise between ${}_1N_1\xi$ and OA again the mean proportion is sought, and thus again the means and the third proportionals may be sought and found [*i.e.* $\frac{{}_3N_3\xi}{OA}=\frac{D}{K};\frac{OA}{{}_3N_3\xi}=\frac{{}_1N_1\xi}{{}_3N_3\xi}$;etc.], and the curve $\xi\xi A(\xi)(\xi)$ may be described and continued, which will be of such a kind, as for example, with its intervals ${}_3N_1N,\ {}_1NO,\ O_1(N),\ {}_1(N)_3(N)$ etc assumed equal, the ordinates ${}_3N_3\xi,\ {}_1N_1\xi,\ OA,\ {}_1(N)_1(\xi),\ {}_3(N)_3(\xi)$ shall be in a continued geometrical progression, such a curve I am accustomed to call *Logarithmic*. Now with ON, O(N) taken equal, upon N or (N), NC or (N)(C) may be erected equal to half the sum of $N\xi$ and $(N)(\xi)$, and C or (C) will be a point of the catenary curve FCA(C)L, thus any number of points of this can be assigned geometrically.

[Leibniz had been familiar with the logarithmic curve and its inverse for some time, and it is given as one of the examples at the end of his first calculus paper of Oct.1684 published in the *Acta*. We can resurrect a modern log function $y=\log_a x$ from the diagram using a left-handed set of axes [if we rotate by a rt. angle anticlockwise, otherwise], if we set the vertical axis OA as the positive *x* or the abscissa-axis, while the positive *y* ordinate axis is to the right along O(N), and set $\mathrm{OA} = a$, as the logarithm of 1 is zero to come convenient base *a*; conversely, if we consider the horizontal axis O(N) as the abscissa axis *y* to the right from the origin O, and the vertical *x* axis as the +ve ordinate axis, we have a form of an exponential function $y = a^x$ to the same base, and again $\mathrm{OA} = a$. Thus the dual nature of the mean proportionals follows if we consider either exponential terms of which the mean proportional is the square root of the values considered, or logarithmetic terms which follow by interchanging the axes. Finally, on taking half the sum of the right and left-hand portions of the exponential curve, the catenary or cosh curve results, as Leibniz indicates, and which in modern terms can be expressed by $x = \frac{1}{2}\left[e^y + e^{-y}\right] = \cosh y$; where natural logs are considered for convenience, and so the inverse exponential function becomes e^x, as defined as such later by Euler.]

Otherwise if the curve may be constructed physically with the aid of a hanging string or chain, with its aid any number of mean proportions can be shown, and the logarithms of these given numbers found or the numbers of the given logarithms. Thus if the logarithm of the number $\mathrm{O}\omega$ may be sought, with the logarithm of OA (as of unity, and which I call the *parameter*) to be equal to zero [recall at this time there was no standard notation for functions of variables]; or, because it amounts to the same, if the logarithm of the ratio between OA and $\mathrm{O}\omega$ is sought, the third proportion $\mathrm{O}\psi$ is taken of $\mathrm{O}\omega$ and OA, and OB as half the sum of the abscissas $\mathrm{O}\omega$ and $\mathrm{O}\psi$, for the abscissa, corresponding to the ordinate BC of the curve of the catenary, or ON will be the *Logarithm sought of the given number.* Conversely, with the Logarithm ON given, thence it is required to cut twice the vertical length NC drawn to the catenary curve into two equal parts, so that the mean proportional between the the segments shall be equal to the given OA (of unity) (because it is most easy) and the two segments will correspond to the number sought for the given logarithm, the one greater, the other less than unity. Otherwise: it is found, as has been said, NC or OR (thus with the point R taken on the horizontal AR, so that we may have OR equal to OB or NC) will be the sum and difference of the right lines OR and AR with the two corresponding to the logarithm of the given number, one greater and the other less than one. [As *Parmentier* points out in *Naissance du Calcul.*. p.193, at this stage *Leibniz* regards the logarithms of both a number and its inverse as equal.] For the difference of OR and AR themselves is $\mathrm{N}\xi$, and the sum of these is $(\mathrm{N})(\xi)$; so that in turn OR is half the sum, and AR half the differnce of $(\mathrm{N})(\xi)$ and $\mathrm{N}\xi$ themselves. [From the equality of OB and OR, it follows in modern terms that $\mathrm{OR} = \cosh x$ and from the rt. angled triangle, OAR, $\mathrm{AR} = \sqrt{1 - \cosh^2 x} = \sinh x$]

The solutions of the main problems follow, which are accustomed to be proposed about curves.

To draw the tangent to the curve at a given point C. R is taken on the horizontal AR through the vertex A, so that OR may be given equal to OB, and CT drawn antiparallel to OR (crossing with the axis AO at T) will be the tangent sought. I call the lines OR and TC themselves to be *antiparallel* as an abbreviation because here, if the parallel lines AR and BC do not make certain angles [*i.e.* ARO and BCT] the same, but still with the complements themselves ARO and BCT being present for a right angle. And the right angled triangules OAR and CBT are similar.

[Thus the tangent sought is $\mathrm{AR}/1 = \sinh x = dx/dy$.]

To find the right line equal to an arc of the catenary. By describing a circle with centre O and radius OB, which cuts the horizontal through A at R, AR will be equal to the given arc AC. It is apparent also from the aforesaid that $\psi\omega$ to be equal to the catenary CA(C). If the catenary CA(C) shall be equal to twice the parameter, that is if AC or AR shall be equal to OA, the inclination of the catenary at C or the angle BCT becomes 45 degrees, and thus the angle CT(C) is right. [*i.e*. the known integral cosh *x* of the known derivative sinh *x*]

To find the quadrature of the area between the catenary and a right line or right lines. Evidently with the point R found as before, the rectangle OAR is equal to the four sided figure AONCA. From which the quadrature of any other parts is easy to find. It is apparent also, the arcs to be in proportion to the areas of the four sided shapes.

To find the centre of gravity of the catenary, or of any of its parts. To the arc AC or the right line AR, of the ordinate BC, with the parameter OA, the fourth proportional found $\mathrm{O}\Theta$ is added to the abscissa OB, and half the sum OG will give G the centre of gravity of the catenary CA(C). Again the tangent CT may cut the horizontal line through A at E, and the rectangle GAEP can be completed, and P will be the centre of gravity of the arc AC. The distance of the centre of gravity of any other arc such as $\mathrm{C_1C}$ from the axis is AM, with πM put to be perpendicular to the horizontal passing through the highest point, dropped from the point π of the intersection of the tangents $\mathrm{C}\pi$, ${}_1\mathrm{C}\pi$, just as its centre may be had easily from the centres of the arcs AC and $\mathrm{A_1C}$. Hence the maximum possible drop may be had BG, to the centre of the rope or catenary or the flexible curve without stretchin of any kind, suspended from the two ends C and (C), having the given length $\psi\omega$; indeed whatever other figure it may assume, the centre of gravity will descend less than if it may be curved in our form CA(C).

To find the centre of gravity of the figure, comprising the curve of the catenary and either a right line or right lines. Oß may be taken half of OG, and the rectangle ßAEQ may occupy the space, Q will be the centre of gravity of the four sided figure AONCA. From which it will be easy to find the centre of gravity of any other area bounded by the curve of the catenary and a right line or right lines. Hence again that well-known path is followed, because as we have noted, not only the four sided figure such as AONCA is to be proportional to arc AC but also the distances of both the centres of gravity from the horizontal line through O, certainly OG and Oß, are to be proportionals, since that will be the double of this always; and the distances from the axis OB, surely PG and Qß thus are to be proportionals, as they shall clearly be equal.

To find the volume and surface area of the solids, generated in some manner about a fixed right line, arising from the rotation of the curve of the catenary and a right line or right lines. This is found from the two preceding problems, as noted. Thus if the catenary CA(C) may be rotated about the axis AB, the surface generated will be equal to the area of a circle, of which the radius may be put equal to twice the square root of the rectangle EAR. The other surfaces as well as the volume generated can be measured just as well in the manner indicated. I will disregard many theorems and problems, which either may be contained in what we have said, or thence may be derived without much trouble on account of brevity. Thus with two points taken on the catenary, such as C and ${}_1C$, the tangents of which cross each other at π, perpendiculars may be sent from the points of the curve ${}_1C$, π, C to the horizontal line AEE, ${}_1C_1J$, πM, CJ : there becomes

$$ {}_1JJ.AC - {}_1CC.{}_1JM = {}_1BB.OA. $$

Infinite series also can be usefully employed. Thus if the parameter OA shall be one, and the arc AC or the right line AH may be called a, and the ordinate BC may be called y, there becomes $y = \frac{1}{1}a - \frac{1}{6}a^3 + \frac{3}{40}a^5 - \frac{5}{112}a^7$ etc., which series can be continued easily according to the rule. Likewise with each curve determined given, from what has been said, the remainder can be found. Thus with the vertex A given, and with some other point C, and with the length AH of the catenary AC intercepted, the parameter AO or the point O of the curve can be found O: for since B is given, BR may be joined, and from R the right line $R\mu$ may be drawn thus so that the angle $BR\mu$ shall be equal to the angle RBA, and $R\mu$ itself (produced) will cross the axis BA (produced) at the point O sought.

I consider to have encompassed the main point from these, from which the remaining matters about this curve, where there is a need, can be deduced easily. I have refrained from adding demonstrations, for the sake of avoiding prolixity, especially since our new analytical calculus set out in these Actis arises to be understood at once.

[The demonstrations of these properties of the catenary depend on Leibniz's calculus, which clearly at this point he was not willing to divulge fully; most of these are presented as notes in the *Naissance du Calcul*…, Ch. X; the modern reader may wish to consult the chapter on the catenary and note that the catenary is the evolute of the tractrix and vice versa, which can be found in Lockwood's *Book of Curves*, Ch. 13, or one can look in Wikipedia, where many other references and details are given.]

Additional Documents.

The Solution of the Rope Problem, shown by Johanne Bernoulli, Basil. Med. Cand.

Act. Erud. Lips. an. 1691.

It is almost a year, since on conversing with my cel. brother mention might have been made about the nature of the curve which a rope forms suspended between two fixed points. We were amazed the matter set out on a daily basis had excited the attention of no one so far of all with eyes and hands. The problem may be seen to be extraordinary

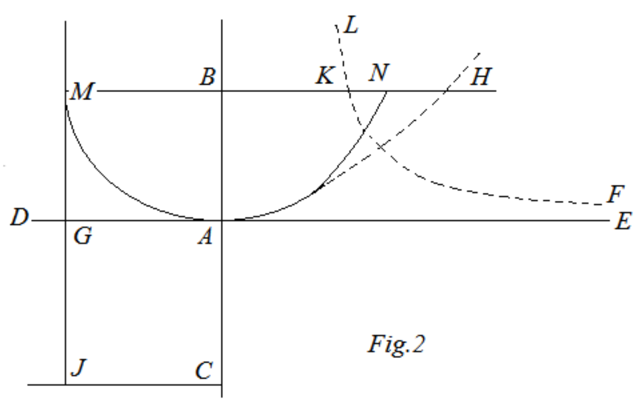


Fig.2

and useful, but moreover we have wished to touch on a foreseen difficulty ; and thus we have established that proposed publicly in the *Actis Eruditorum*, to see whether any would dare to solve the problem : for we did not know, because now thence from the time of Galileo it had been a source of trouble between geometers. Meanwhile this problem had been considered worthy to be prepared to be solved by the great mathematician Leibniz himself, and happily he indicted not long after [See *Act. Erudit. Lips*. an. 1690, p. 360.] how with his key he had opened up his approach to the problem, yet by conceding for some time, within which if no one could solve it, his own solution would be published. That put it in mind, that I might attack the problem anew, because there indeed done is was done successfully, so that I had found the solution of that just a short time before the limit put in place for the appearance of his own solution complete and of various kinds, such as before I would never indeed have dared to hope. But our rope curve is not to be found geometrically, but from a consideration of these matters, which are called mechanical, as the nature of which determined curve cannot be expressed by an algebraic equation, neither depending on a relation of the curve to a right line, nor is it found according to a relation of the intervals of the curve to rectilinear intervals, thus so that according to the rectification of that other curve being described, or the quadrature of the curve may be supposed, as may be clear from the following constructions.

Construction. I. With the normals drawn CB, DE (fig. 2) intersecting each other at A, and with the centre C taken somewhere on the axis CB, and with vertex A, the equilateral hyperbola described AH, and the curve LKF may be constructed, which shall be such that CA everywhere shall be the mean proportional between BH and BK;
the rectangle CG becomes equal to the area EABKF, and with JG and HB produced, the point of concurrence M will be on the funicular curve MAN.

Construction. II. With the equilateral hyperbola BG described as at first to the axis BA (fig.3), the Parabola BH may be constructed to the same axis, the latus rectum of which may be equal to the square of the latus rectum of the transverse hyperbola, and the applied ordinate HA may be produced to produced to E, thus so that the right line GE shall be equal to the parabolic curve BH; I say the point E to be on the funicular curve EBF.

From these it is apparent, the nature of this curve EBF cannot be had through a geometrical equation, unless at the same time the rectification of the parabolic line may be given. But I am pleased to omit the demonstration of this, and of the preceding construction, lest either I may have snatched away to soon the reward of the first discovery from the most celebrated man, or clearly I may take the opportunity of suppressing this matter based on his own discovery : here it will suffice, if I add the more remarkable properties of this curve :

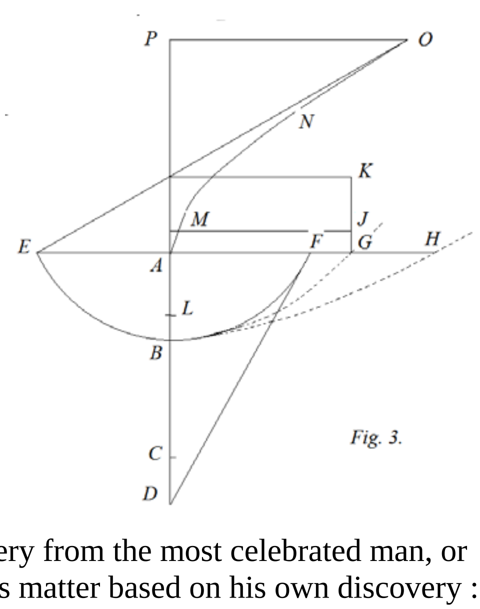


Fig. 3.

I. With the tangent FD drawn (fig.3), the curve will satisfy AF.AD :: BC.BF.

2. AE or AF is equal to the parabola arch BH, with the right line AG subtracted.
3. The curve BE or BF is equal to the right line AG, i.e. the portions of the funicular curve make an equilateral hyperbola to the applied axis : a conspicuous property of this curve.
4. The area of the funicular form BAE or BAF is equal to the rectangle under BA and AF, to half the rectangle under CB and FG.
5. The curve MNO, from whose evolute the funicular curve BE is described, is the third proportional to CB and AG.
6. Truly the evolute right line EO is the third proportional to CB and CA.
7. The right line BM taken as far as the beginning of the curve MNO is equal to CB itself.
8. MP is the double of BA itself.
9. The rectangle under CB and PO is twice the area of the hyperbola ABG.
10. The right line CP is bisected at the point A.
11. The curve EB is to the curve MNO, as the right line CB to the right line AG.
12. If the two rectangles AJ and AK may be adjoined to AG, of which the one for that rectangle AJ is taken under the transverse semi-latus rectum CB and with the half-length FG, the other AK which itself is equal to the area of the hyperbola BGA, and of the different lengths KJ is taken on the axis from the vertex B equal to BL, where the point L will be the centre of gravity of the funicular curve EBF.
13. If above EF infinitely many curves are understood to be described themselves equal to the catenaries [or funiculars] EBF, and these may be extended in right lines, and at the individual points of the individual extensions right lines may themselves be connected equal respectively to the distances from the line EF, it will be the maximum of all of the areas which thus they may affect that arises.

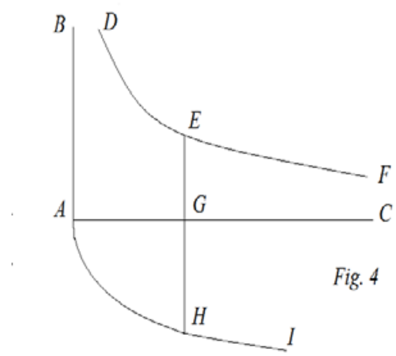


Fig. 4

My honorable brother had begun to consider also how to extend this theory to ropes of unequal thickness, of which he obtained an algebraic equation expressed between the thickness and the length, and he noted one case, in which the problem could be solved by a simple mechanical curve, truly if the curved figure may be supposed ABDEG (fig. 4), whose applied line GE shall be inversely in the square root ratio of the abscissa AG, and that shall be with all its flexion applied, that is, if the weight of the rope AG may be considered at its individual points with respect to the lines GE, or (which is just the same) with the differences of the applied lines GH on the parabola AHI, or finally with the parts of the cycloid curve AHI (of which the vertex is A) and that thus may be understood to be the weight suspended, thus so that the point A shall be the lowest of all (which shall be, where it may have another rope connected to the part A of the same length and at equal distances from the A equally weighed): then it demands an equilateral hyperbola ABC whose vertex is A be constructed to the axis AG (fig. 5), and the applied line BD be produced to E, thus so that the rectangle under the half latus rectum or with the transversal and the line DE shall be equal to the area ADB, and the point E is shown to be to the curve sought AEF, as in the ratio the weight of the said rope forms, truly the curve itself AE to be the third proportional to the right line or the transverse side of the hyperbola and its applied line DB; the tangent EH to be taken the 3rd of the four proportionals to the semi-latus rectum, the abscissa AD and the applied line DB, etc. But to be found, which is remarkable, this curve AEF to be that itself, of which from the evolute the other BE, as may be formed of uniform thickness, will be described, and therefore with the same curve MNO.

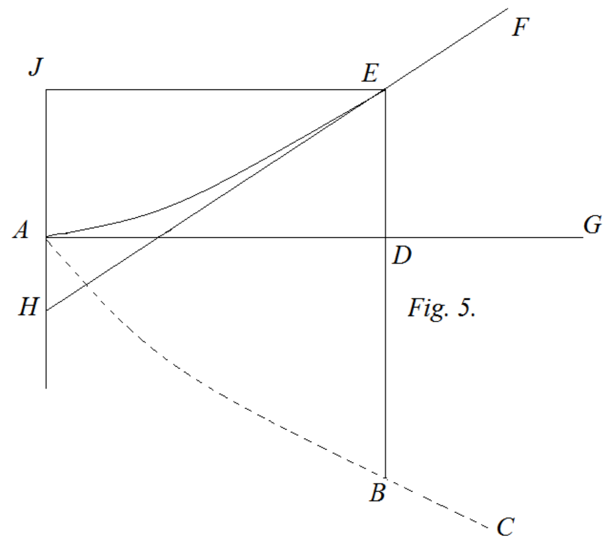


Fig. 5.

It is convenient to note, because if it may be considered to examine this by experiments, a chain must be selected before a rope, on account both of the excessive lightness as well as of the rigidity we may be caught off guard with that. Besides, anyone who will want to enlarge and perfect this material, will be able to investigate the nature of the curve, as it refers to the hypothesis of a rope at a finite distance from the centre of the earth, either if it may be supposed in addition to be extended by its own weight, or by some other kind of weight : or also just as in turn that may be considered to be burdened, so that it may refer to a parabolic curve, a hyperbola, a circle, or any other given curve ; indeed the matter generally is in the strength [of the weight].

Christian Huygens, D. in Z., the solution of the same problem.

If the chain CVA (fig. 6) [This diagram has been redrawn from Huygens original in his *Oeuvres*, Vol. X. Letters, app. to no. 2680, page 95; the original in the *M. S.* is wrong, though the current *AE* 1691 version, also published by Olms, is correct] shall be suspended from the strings FC, EA joined on each side and the strings without weight, thus so that the heads C and A shall be equal in height, and the angle of inclination of the strings produced CGA, and the position of the whole chain may be given, of which the vertex shall be V, and with the axis VB.

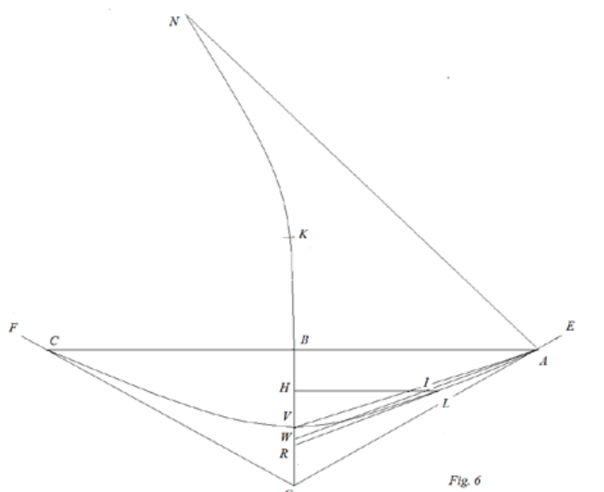


Fig. 6

l. Hence it will be allowed to find the tangent at any given point of the chain. Just as if the point given shall be L, from which the applied line drawn LH may divide the axis BV equally. Now if the angle CGA shall be 60°, the right line AW equal to $\frac{3}{2}$AB will be

inclined to the axis at the point A, for which with the parallel drawn LR touches the curve at the point L. Likewise if the sides GB, BA, AG
shall be of the parts 3, 4, 5, AW will require to be of the parts $4\frac{1}{2}$.

[Thus, for the whole half-curve, the vertical component of the tension is equal to the weight of the whole half curve acting through the mid-point L, while at the vertex V the whole tension acts along the horizontal, and this has the constant value T_o for the horizontal component of the tension at any point on the curve. If the tension at A is T, and the length of the right- hand half of the curve is *s*, and if the weight per unit length is *w*, and AE is inclined to the horizontal AB at an angle φ then

$ws = T\sin\varphi$ and $T_o = T\cos\varphi$, then $T_o \tan\varphi = ws = W$, the weight of the half-curve; Now $\tan\varphi = \frac{BG}{BE}$, for the slope of the curve at L, and we may also put $T_o = wc$, the length of chain hanging vertically that gives the same horizontal tension as in the diagram, and now we have the intrinsic equation for the catenary: $s = c\tan\varphi$; this last equation give $ds = c\sec^2\varphi . d\varphi$, and we :

$\frac{dy}{ds} = \sin\varphi$; hence $dy = ds\sin\varphi = c\sec^2\varphi d\varphi \times \sin\varphi = -c \times \frac{d\cos\varphi}{\cos^2\varphi}$,

and on integrating with the constant taken as zero:

$y = c\sec\varphi.$ By integrating, in a similar manner, $x = c\log(\sec\varphi + \tan\varphi).$

Now $T = T_o\sec\varphi$ and hence $\frac{y}{c} = \frac{T}{T_0}$ or $T = wy$: thus the tension along the curve is proportional to the vertical height, and has the minimum value T_o when $\varphi = 0$ and $y = c$ at the vertex.

Again, from $\sec^2\varphi = 1 + \tan^2\varphi$ we have $\frac{y^2}{c^2} = 1 + \frac{s^2}{c^2}$, or $y^2 = c^2 + s^2$.

The right triangles quoted, *i.e.* in the $(1, \sqrt{3}, 2)$ triangle and the (3,4,5) triangle are used as examples. For the former triangle, we have

$c = y\cos\varphi = \sqrt{3} \times \frac{1}{2} = \frac{\sqrt{3}}{2}$ and $\mathrm{BV} = \frac{\sqrt{3}}{2}.$ Hence the *y*-coordinate of the point L is $\frac{3\sqrt{3}}{4}$ and from $y = c\sec\varphi$ we have $\frac{3\sqrt{3}}{4} = \frac{\sqrt{3}}{2}\sec\varphi$ or $\cos\varphi = \frac{2}{3}$. Thus

$\mathrm{AW}\cos\varphi = \mathrm{AW} \times \frac{2}{3} = AB$ as required.]

[Thus, for the (3,4,5) triangle we can find *c* from $y = c\sec\varphi$ or $c = y\cos\varphi = 3 \times \frac{4}{5} = \frac{12}{5}$, in which case $\mathrm{BV} = \frac{3}{5}$; Hence the *y*-coordinate of the point L is $y = \frac{12}{5} + \frac{3}{10} = \frac{27}{10}$ and from $y = c\sec\varphi$ we have $\frac{27}{10} = \frac{12}{5}\sec\varphi$ or $\cos\varphi = \frac{12}{5} \times \frac{10}{27} = \frac{8}{9}$. Thus

$\mathrm{AW}\cos\varphi = \mathrm{AW} \times \frac{8}{9} = 4,$ giving $\mathrm{AW} = 4\frac{1}{2},$ as required.]

2. Again either a right line can be found equal to the catenary, or to any given part of that. Indeed always with the angle CGA given, the ratio of the axis BV to the curve VA will be given. Just as if the sides GB, BA, AG shall be as 3, 4, 5, the curve VA will be three times the axis VB.

[Here we can use $y^2 = c^2 + s^2$, and $y\cos\varphi = c$; in which case $c = y\cos\varphi = 3\times\frac{4}{5} = \frac{12}{5}$ and $s = \sqrt{y^2 - c^2} = \sqrt{9 - \frac{144}{25}} = \sqrt{\frac{225-144}{25}} = \frac{9}{5}$; and $\mathrm{VB} = 3 - \frac{12}{5} = \frac{3}{5}$ as required.]

3. Likewise the radius of curvature at the vertex V can be defined, as the radius of the greatest circle, which passes through the vertex describing this whole curve. For if the angle CGA shall be 60°, the radius of curvature will be equal to the axis BV itself. Truly if the angle CGA shall be right, the radius of curvature of the curve will be equal to VA.

[The radius of curvature can be shown to be $c\sec^2\varphi = \frac{y^2}{c}$; in this case, $c = \frac{\sqrt{3}}{2}$ and $\mathrm{BV} = \frac{\sqrt{3}}{2}$; hence in this case the radius of curvature is BV itself, etc.]

4. Also a circle of equal area can be found from the surface of the cone formed from the rotation of the catenary about its axis. Thus if the angle CGA shall be 60°, the surface of the cone generated from the catenary CVA equals a circle, of which the radius shall be twice the rectangle BVG.

5. Also some points of the curve KN may be found from the evolute of which, together with the right line KV, with the radius of curvature at the vertex, will be described by the curve VA, and KN the length of the evolute itself. Just as if the angle CGA were $60°$, KN will be three times the axis BV. Indeed if the sides GB, BA, AG were as 3, 4, 5, that will be $\frac{9}{4}$ of the axis BV.

6. In addition the square of the area NKVAN is given. For on putting the angle CGA to be 60°, that area will be equal to the rectangle formed from the axis BV and that which can be three times to square of the same BV. Truly if the sides GB, BA, AG shall be as 3, 4, 5, the same area will be equal to the seven times the square BV with the eighth part.

7. Again any number of catenaries can be found, with the quadrature of one of these curves put in place: $\mathrm{xxyy} = \mathrm{a}^4 - \mathrm{aayy}$ or $\mathrm{xxyy} = 4\mathrm{a}^4 - \mathrm{x}^4$, or also with the distance of the centre of gravity from the axis given, in the parts of the plane, which the right lines cut off parallel to the axis of these in the former curve. But the quadrature of this curve depends on the sum of the secants of the arc increasing equally through the minimum , which sums are found from tables of sines very closely from certain outstanding approximations in use. Hence e.g. the discovery, that if the angle CGA shall be right, and BV may be put of the axis of 10000 parts, BA will be 21279, and not one less. Moreover with the curve indicated above VA is known here to be of 24142 parts, not one less.

In all these problems I have given the solutions to individual cases only, I have striven to avoid prolixity, and since without doubt five universal rules shall have been shown sufficiently to the learned men. But if yet they may require some things from us, these I will send gladly. And now some time ago I have deposited everything with that most distinguished of men *G. G. Leibniz* in a sealed envelope.

Addition to the Problems of Ropes
by *Jacob Bernoulli.* Act. Erudit. 1691.

After my brother had shown the most recent solution of the problem regarding funicular curves, I have continued to move that inquiry further and applied myself to other cases too, which are understood besides these, of which now mention has been made; moreover with several present, which I consider worth the effort to review.

l. If the thickness or weight of the rope or chain shall be uneven and thus adjusted, so that while it is in a state of rest it will be parabolic, the weight of a portion HI (fig.7) shall be in some ratio to a part of the right line LM drawn with the same perpendiculars HL, IM intercepted by the curve AIHB, as formed by the rope or chain thus suspended by its own weight. But if the weight of the part HI shall be in the ratio of the area LOPM with the same perpendiculars HL, IM intercepted, the funicular curve AB will be either parabolic, cubic, biquadratic, or of some fractional power etc., provided the Figure CLO is a triangle, the complement of a common semi-parabola, or of a semi-parabolic cubic etc. Because indeed if the weight of the part HI shall be in the ratio of the areas QRST with the same horizontal right lines HQ, IR of the abscissa, the funicular curve IB will be some curve from a kind of hyperbolas (with the right line present AG with one asymptote), for example the Apollonian, or from cubics, biquadratics, etc., just as it may be seen the Figure AQT either is a triangle, or the complement of a common semi-parabola, or of a cubic, etc.

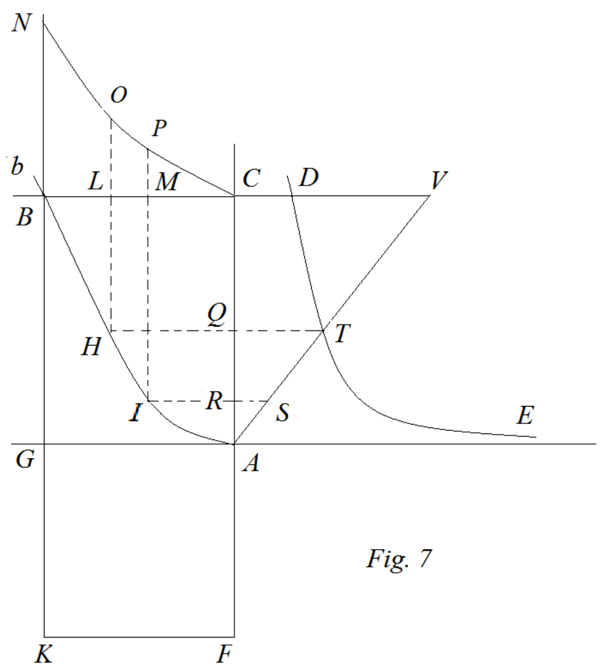


Fig. 7

2. If the rope shall be of uniform thickness, but extendable by its weight, there is a need for a special mathematical device (fig.7). A part of the rope may be said not to be extended, of which the weight is equivalent to the force extending one point a of the rope, and the excess of the length, by which this part extended by the said force may not exceed *b,* and it may be taken from the perpendicular $FA = a$, and restricted to $FC = x$: then a curve DE arises of this nature, so that the applied line shall be

$$CD = \frac{ab}{\sqrt{2aa+2bx-2a\sqrt{aa+bb+2bx}}}, \text{ or from } \sqrt{\frac{aa+bx-2a\sqrt{aa+bb+2bx}}{2xx-2aa}},$$

and indeed likewise ACDE may be put in place with the area equal to the rectangle FG, and the right lines KG, DC may be produced to cross each other at B; thus the point B will be for the required funicular AB. Moreover I suppose, the extensions from the stretching forces to be in proportion, even if I may have some doubt, or since that hypothesis may be satisfied well enough from reasoning and experiments. But it may be allowed for us to retain that, while we are ignorant of the truth.

3. From the occasion of the funicular problem soon we have fallen on another one not less illustrious, concerning the bending or curvature of beams, of stretched bows or of any kind of sideways bending from its own weight, or by hanging on a weight, or made from some other pressing force ; with regard to which the most celebrated *Leibniz* in private letters, with which at the same time he has honoured me, I see that he intends to get involved also. Moreover it may be seen this problem, since on account of the uncertain hypothesis, while there is a multiple variety of causes, at first to involve a little more difficulty, thought this without an involved calculation, but only a need for diligence. I myself through the solution of the simplest case (at least of the extension in the afore mentioned hypothesis) happily I have opened up the heart of the problem ; truly so that following the method of the most excellent man and from others I may concede the extent of his grasp of analysis, for now I will suppress the solution, and that for the present I will hide by a code, the key being communicated with a demonstration in the autumn. If an elastic lamina shall be free from weight AB (fig. 8), with a uniform width everywhere and of a length, with the lower end A made firm somewhere and a weight B may be attached to the top, as much as it suffices for the lamina at that point to be curved, so that the line of the direction of the weight BC shall be perpendicular to the curved lamina at B, the curvature of the lamina will be of the following nature:

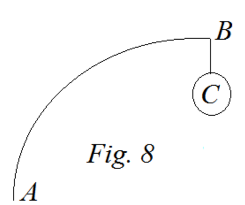


Fig. 8

Qrzumu bapt dxqopddhbp poyl fy bbqnfqbfp lty ge mutds utlthh tubs tmixy yxdksdhxp gqsrkfgudl bg ipqandtt tcpgkbp aqdbkzs. [This becomes : The proportion between the applied line of the axis and the tangent is to the tangent itself thus as the square of the applied line to a certain constant area.]

4. Truly much more uplifted is the consideration about the figure of a sail inflated by the wind, although so far it has a relation with the funicular problem, in as much as the impulses of the wind continually approaching the sail can be considered as the weight of the rope. Anyone who has understood the nature of fluid pressure, indeed may grasp the problem without difficulty, because the part of the sail BC (fig.9), which has stretched out perpendicular to the direction of the wind DE, ought to be curved in the arc of a circle. But such a curvature may be adopted by the remaining part of the

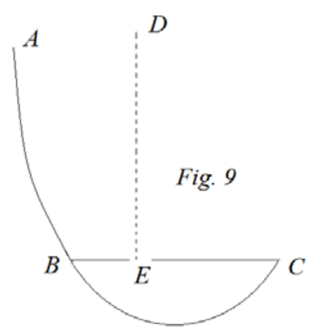


Fig. 9

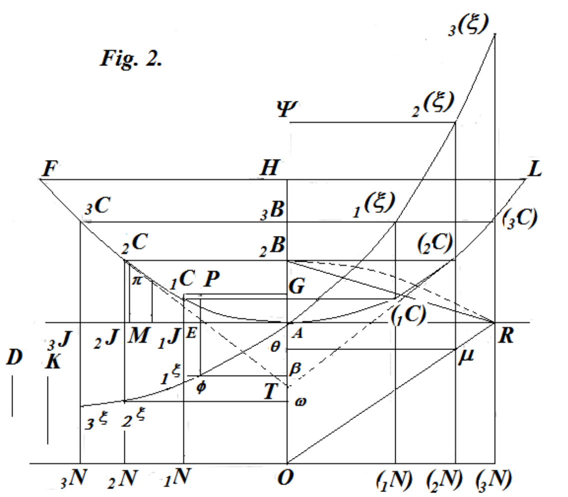
sail AB, so that the enquiry is difficult, thus in nautical matters in short it is going to be of the most outstanding use, so that it may be seen to be deserved to be a task for the most outstanding as well as the most subtle geometers. Finally in all these problems, which anyone tries in vain by another method, plainly the particular use of the *Leibnizian calculus* I have found to be outstanding, so that therefore itself I judge to be required to be considered among the greatest of our secular discoveries. Although indeed, as I have intimated recently, I believe the opportunity for this [to be developed] must be given to *Barrow's* calculus, such as I call it, who in his time prevailed almost everywhere to be amongst the most outstanding geometers, even now each I see to be compared with the most noble *Tschirnhaus* : yet from that it is not to be understood, so that I may wish to detract and assign to others the dignity of the most useful discovery in any respect, or make light of any of the deserved praise whatsoever due to the most celebrated man ; and if on being brought together each presents to me a likeness that has been seen to intercede between these, that likeness cannot be greater, than what may be made out of it, so that with one line of reasoning understood another may be understood more easily, while also the one used which is superfluous is required to be abandoned soon, which on being omitted shortens the working of the other: and concerning the other method in as much as there the saving is such, that the nature of the matter may change completely, and in order that infinitely many things may be done better through this change, which are unable to be done by the other: just as this refinement found certainly itself was not of just any kind, but arose from the most sublime ingenuity, and because it shows the method adopted by the author in the best light.

DE LINEA, IN QUAM FLEXILE SE PONDERE PROPRIO CURVAT, ElUSQUE USU INSJGNI AD INVENIENDAS QUOTCUNQUE MEDIAS PROPORTIONALES ET LOGARITHMOS.

Act. Erudit. Lips. an. 1691.

Problema *Lineae Catenariae* vel *Funicularis* duplicem usum habet, unum ut augeatur ars inveniendi seu Analysis, quae hactenus ad talia non satis pertingebat, alterum ut praxis construendi promoveatur. Reperi enim hanc lineam ut facillimam factu, ita utilissimam effectu esse, nec ulli Transcendentium secundam. Nam suspensione fili vel potius *catenulae* (quae extensionem non mutat) nullo negotio parari et describi potest *physico* quodam *constructionis genere.* Et ope ejus ubi semel descripta est, exhiberi possunt quotcunque mediae proportionales, et Logarithmi, et Quadratura Hyperbolae. Primus *Galilaeus* de ea cogitavit, sed naturam ejus assecutus non est: neque enim Parabola est, ut ipse erat suspicatus. *Joachimins Jungius,* eximius nostri saeculi Philosophus et Mathematicus, qui multa ante *Cartesium* praeclara cogitata habuerat circa scientiarum emendationem, calculis initis et experimentis factis parabolam exclusit, veram lineam non substituit. Ex eo tempore a multis tentata quaestio est, a nemine soluta, donec nuper mihi ab eruditissimo Mathematico praebita ejus tractandae occasio est. Nam *Cl. Bernoullius,* cum meam quandam Analysin infinitorum, calculo differentiali, me suadente, introducto expressam, feliciter applicuisset ad quaedam problemata, a me publice petivit Actorum anni superioris mense Majo p. 218 seq., ut tentarem, an nostrum calculi genus etiam ad hujusmodi problemata, quale est lineae catenariae inventio, porrigeretur. Re in gratiam ejus tentata, non tantum successum habui, primusque, ni fallor, illustre hoc problema solvi, sed et lineam egregios usus habere deprehendi, quae res fecit, ut exemplo Blasii Paschalii aliorumque ad eandem inquisitionem invitaverim Mathematicos certo tempore praestituto, experiundarum Methodorum causa, ut appareret, quid illi daturi essent, qui fortasse alias adhiberent ab ea, qua *Bernoullius* mecum utitur. Tempore nondum elapso, duo tantum significarunt rem se consecutos, *Christianus Hugenius,* cujus magna in rem literariam merita nemo ignorat, et ipse cum fratre ingenioso juvene et pererudito *Bernoullius,* qui his, quae dedit, effecit, ut praeclara quaeque porro ab iis speremus. Cum igitur reapse expertum puto quod significaveram, huc quoque porrigi nostram calculandi rationem, et quae antea diffieillima hahebantur jam aditum admittere. Sed placet exponere, quae a me sunt inventa ; quid alii praestiterint, collatio ostendet.

Fig. 2.



Linea sic construitur Geometrice, sine auxilio fili aut catenae, et sine suppositione quadraturarum, eo constructionis genere, quo pro Transcendentibus nullum perfectius et

magis Analysi consentaneum mea sententia haberi potest. Sint duae quaecunque lineae rectae, determinatam quandam et invariabilem inter se habentes rationem, eam scilicet quam D et K [א & ב] (fig. 2) hic expositae, qua ratione semel cognita caetera omnia per Geometriam ordinariam procedunt. Sit recta indefinita ON horizonti parallela, eique perpendicularis OA, aequalis ipsi O_3N, et super ${}_3N$ verticalis ${}_3N_3\xi$, quae sit ad OA, ut D ad K. Inter OA et ${}_3N_3\xi$ quaeratur media proportionalis ${}_1N_1\xi$; et inter ${}_1N_1\xi$ et ${}_3N_3\xi$, itemque inter ${}_1N_1\xi$ et OA quaeratur rursus media proportionalis, et ita porro quaerendo medias et inventis tertias proportionales, describatur continueturque linea $\xi\xi A(\xi)(\xi)$, quae erit talis naturae, ut ipsis intervallis, verbi gr.
${}_3N_1N,\ {}_1NO,\ O_1(N),\ {}_1(N)_3(N)$ etc sumtis aequalibus, sint ordinatae
${}_3N_3\xi,\ {}_1N_1\xi,\ OA,\ {}_1(N)_1(\xi),\ {}_3(N)_3(\xi)$ in continua progressione Geometrica, qualem lineam *Logarithmicam* appellare soleo. Iam sumtis ON, O(N) aequalibus, super N vel (N) erigatur NC vel (N)(C) aequales dimidiae summae ipsarum $N\xi$, $(N)(\xi)$, et C vel (C) erit *punctum lineae catenarae* FCA(C)L, cujus ita puncta quotcunque assignari Geometrice possunt.

Contra si linea catenaria physice construatur ope fili vel catenae pendentis, ejus ope exhiberi possunt quotcunque mediae proportionales, et Logarithmi inveniri datorum numerorum vel numeri datorum Logarithmorum. Sic si quaeratur Logarithmus numeri $O\omega$, posito ipsius OA (tanquam Unitatis, quam et *parametrum* vocabo) Logarithmum esse nihilo aequalem; seu, quod eodem redit, si quaeratur Logarithmus rationis inter OA et $O\omega$, sumatur ipsius $O\omega$ et OA tertia proportionalis $O\psi$, et ipsarum $O\omega$ et $O\psi$ summae dimidiae OB, tamquam abscissae, respondans Lineae Catenariae ordinata BC vel ON erit *Logarithmus quaesitus numeri dati.* Contra, dato Logarithmo ON, inde ductae ad Curvam Catenariam verticalis NC duplam oportet secare in duas partes tales, ut media proportionalis inter segmenta sit aequalis datae (unitati) OA (quod facillimum est) et duo segmenta erunt *respondentes dato Logarithmo Numeri quaesiti,* unus major, alter minor unitate. Aliter: Inventa, ut dictum est, NC seu OR (sumto ita puncto R in horizontali AR, ut habeamus OR aequalem OB vel NC) erunt summa ac differentia rectarum OR et AR duo respondentes Logarithmo dato Numeri, unus major, alter minor unitate. Nam differentia ipsarum OR et AR est $N\xi$, et summa earum est $(N)(\xi)$; ut vicissim OR est semisumma, et AR semidifferentia ipsarum $(N)(\xi)$ et $N\xi$.

Sequuntur *solutiones Problematum primariorum,* quae circa lineas proponi solent. *Tangentem ducere ad punctum lineae datum C.* In AR horizontali per verticem A sumatur R, ut fiat OR aequalis OB datae, et ipsi OR ducta antiparallela CT (occurrens axi AO in T) erit tangens quaesita. *Antiparallelas* compendii causa *r* hic voco ipsas OR et TC, si ad parallelas AR et BC faciant non quidem eosdem angulos, sed tamen complemento sibi existentes ad rectum, ARO et BCT. Et Triangula rectangula OAR et CBT sunt sim illa.

Rectam invenire arcui catenae aequalem. Centro O radio OB describendo Circulum, qui horizontalem per A secet in R, erit AR aequalis arcui dato AC. Patet etiam ex dictis fore $\psi\omega$ aequalem catenae CA(C). Si catena CA(C) aequalis esset duplae parametro, seu si AC vel AR aequalis OA, foret catenae in C inclinatio ad horizontem seu angulus BCT 45 graduum, adeoque angulus CT(C) rectus.

Quadrare spatium linea catenaria et recta vel rectis comprehensum. Scilicet invento puncto R, ut ante, erit rectangulum OAR aequale Quadrilineo AONCA. Unde alias quasvis portiones quadrare in proclivi est. Patet etiam, arcus esse arcis quadrilineis proportionales.

Invenire centrum gravitatis catenae, aut partis ejus cujuscunque. Arcui AC vel AR, ordinatae BC, parametro OA inventa quarta proportionalis $O\Theta$ addatur abscissae OB, et summae dimidia OG dabit G centrum gravitatis catenae CA(C). Porro Tangens CT secet horizontalem per A in E, compleatur rectangulum GAEP, erit P centrum gravitatis arcus AC. Cujuscunque arcus alterius ut C_1C distantia centri gravitatis ab axe est AM, posito πM esse perpendicularem in horizontem verticis, demissam ex π consursu tangentium $C\pi$, ${}_1C\pi$, quanquam et centrum ejus ex centris arcuum AC, A_1C facile habeatur. Hinc et habetur BG, maximus descensus possibilis centri funiculi seu catenae aut lineae flexilis non intendibilis cujuscunque, duabus extremitatibus C et (C) suspensae, longitudinem habentis datam $\psi\omega$; quamcunque enim figuram aliam assumat, minus descendet centrum gravitatis quam si in nostram CA(C) curvetur.

Invenire centrum gravitatis figurae, linea catenaria et recta vel rectis comprehensae. Sumatur Oß dimidia ipsius OG, et compleatur rectangulum ßAEQ. erit Q centrum gravitatis quadrilinei AONCA. Unde et cujuscunque alterius spatii linea catenaria et recta vel rectis terminati centrum facile habetur. Hinc porro sequitur illud memorabile, non tantum quadrilinea ut AONCA arcubus AC proportionalia esse, ut jam notavimus, sed et amborum centrorum gravitatis distantias ab horizontali per O, nempe OG et Oß, esse proportionales, cum illa sit semper hujus dupla; et distantias ab axe OB, nempe PG, Qß adeo esse proportionales, ut sint plane aequales.

Invenire contenta et superficies solidorum, rotatione figurarum linea catenaria et recta vel rectis comprehensarum, circa rectam immotam quamcunque genitorum. Habetur ex duobus problematibus praecedentibus, ut notum est. Sic si catena CA(C) rotetur circa axem AB, generata superficies aequabitur circulo, cujus radius posit duplum rectangulum EAR. Nec minus aliae superficies vel etiam solida dicto modo genita mensurari possunt. Multa Theoremata ac Problemata praetereo, quae vel in his continentur, quae diximus; vel non magno negotio inde derivantur cum brevitati consulere visum sit. Sic sumtis duobus catenae punctis, ut C et ${}_1C$, quorum tangentes sibi occurrant in π, ex punctis ${}_1C$, π, C in ipsam AEE horizontalem verticis demittantur perpendiculares, ${}_1C_1I$, πM, CI : fiet, ${}_1I$ in AC minus ${}_1CC$ in ${}_1IM$ aequale ${}_1BB$ in OA.

Possunt et series infinitae utiliter adhiberi. Sic si parameter OA sit unitas, et Arcus AC vel recta AH dicatur a, et ordinata BC vocetur y, fiet $y=\frac{1}{1}a-\frac{1}{6}a^3+\frac{3}{40}a^5-\frac{5}{112}a^7$ etc., quae series facili regula continuari potest. Datis quoque lineam determinantibus, haberi possunt reliqua ex dictis. Sic dato vertice A, et alio puncto C, et AH longitudine catenae interceptae AC, haberi potest lineae parameter AO vel punctum O: quoniam enim datur et B, jungatur BR, et ex R educatur recta $R\mu$ ita ut angulus $BR\mu$ sit aequalis angulo RBA, et ipsa $R\mu$ (producta) occurret Axi BA (producto) in puncto O quaesito.

Atque his quidem potissima contineri arbitror, unde caetera circa hanc lineam, ubi opus, facile duci poterunt. Demonstrationes adjicere supersedeo, prolixitatis vitandae gratia, praesertim cum novae nostrae Analyseos calculos in his Actis explicatos intelligenti sponte nascantur.

Beilagen.

Solutio Problematis Funicularis, exhibita a Johanne Bernoulli, Basil. Med. Cand.

Act. Erud. Lips. an. 1691.

Annus fere est, cum inter sermocinandum cum Cl. Fratre mentio forte incidisset de Natura Curvae, quam funis inter duo puncta fixa libere suspensus format. Mirabamur rem omnium oculis et manibus quotidie expositam nullius hucusque attentionem in se concitasse. Problema videbatur eximium et utile, at tum ob praevisam difficultatem tangere voluimus; statuimus itaque illud publice Eruditis proponere, visuri num qui vadum tentare auderent: nesciebamus enim, quod jam inde a Galilaei temporibus inter Geometras agitatum fuisset. Interea dignum censuit nodum hunc, cui solvendo se accingeret summus Geometra Leibnitius, significavitque non multo post [Vid. Act. Erudit. Lips. an. 1690, pag. 360.] se clave sua aditus problematis feliciter reserasse, concesso tamen et ullis tempore, intra quod si nemo solveret, ipse solutionem suam publicaturus esset. Id animum addidit, ut problema denuo aggrederer, quod eo quidem cum successu factum, ut brevi et ante termini a Viro Cl. positi exitum ejus solutionem omnimodam et plenariam, qualem antea ne sperare quidem ausus fuissem, invenerim. Reperi autem Curvam nostram Funiculariam non esse Geometricam, sed ex earum censu, quae Mechanicae dicuntur, utpote cujus natura determinata aequatione Algebraica exprimi nequit, nec nisi per relationem curvae ad rectam, vel spatii curvilinei ad rectilineum habetur, sic ut ad illam describendam alterius curvae rectificatio vel curvilinei quadratura supponatur, ut ex sequentibus Constructionibus liquet.

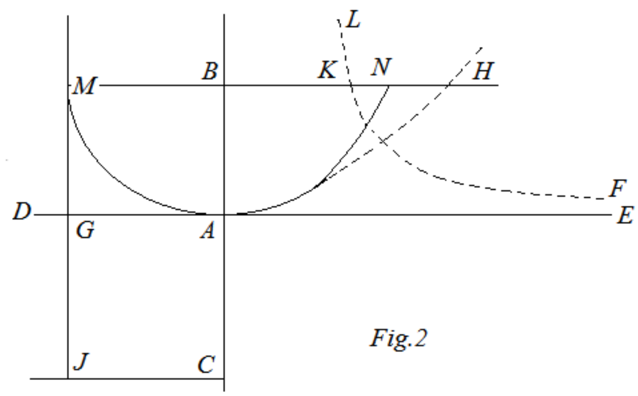


Fig.2

Constr. I. Ductis normalibus CB, DE (fig. 122) sese secantibus in A, centroque C ubivis sumpto in axe CB, et vertice A descripta Hyperbola aequilatera AH, construatur curva LKF, quae talis sit, ut ubique CA sit media proportionalis inter BH et BK. fiat rectangulum CG aequale spatio EABKF, erit productis JG, HB punctum concursus M in Curva Funicularia MAN.

Constr. 11. Descripta ut prius ad axem BH (fig. 3). Hyperbola aequilatera BG, construatur ad eundem axem Parabola BH, cujus latus rectum aequetur quadruplo lateris recti vel transversi Hyperbolae, ordinatimque applicata HA producatur ad E, ita ut recta GE sit aequalis lineae Parabolicae BH; dico punctum E esse in Curva Funicularia EBF.

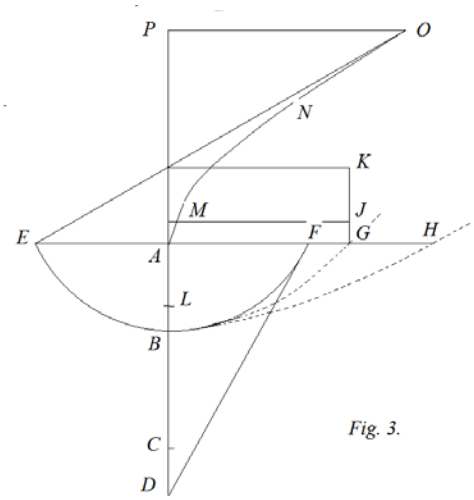


Fig. 3.

Ex his patet, Curvae hujus EBF naturam per aequationem Geometricam haberi non posse, nisi simul rectificatio lineae Parabolicae detur. Hujus autem et praecedentis Constructionis demonstrationem lubens omitto, ne Celeberrimo Viro primae inventionis palmam vel praeripiam, vel inventa sua super hac materia plane supprimendi ansam praebeam: sufficiet hic, si notabiliores hujus Curvae proprietates addidero:

I. Ducta tangente FD (fig. 3), erit AF.AD :: BC.BF curvam.
2. AE vel AF aequatur curvae Parabolicae BH, dempta recta AG.
3. Curva BE vel BF aequalis est rectae AG, i. e. portiones curvae funiculariae ad axem applicatae conficiunt Hyperbolam aequilateram: insignis est hujus Curvae proprietas.
4. Spatium Funicularium BAE vel BAF est aequale rectangulo sub BA et AF, diminuto rectangulo sub CB et FG.
5. Curva MNO, ex cujus evolutione describitur Funicularia BE, est tertia proportionalis ad CB et AG.
6. Recta vero evolvens EO est tertia proportionalis ad CB et CA.
7. Recta BM usque ad principium curvae MNO sumta aequatur ipsi CB.
8. MP est dupla ipsius BA.
9. Rectangulum sub CB et PO duplum est spatii hyperbolici ABG.
10. Recta CP bisecta est in puncto A.
11. Curva EB est ad curvam MNO, ut recta CB ad rectam AG.
12. Si ad AG applicentur duo Rectangula AI, AK, quorum unum AI ei quod sub semilatere transverso CB et recta FG comprehenditur rectangulo, alterum AK quod ipsi spatio Hyperbolico BGA aequatur, et differentiae latitudinum KI sumatur in axe a vertice B aequalis BL, erit punctum L centrum gravitatis curvae Funiculariae EBF.
13. Si super EF infinitae intelligantur descriptae curvae ipsi Funiculariae EBF aequales, illaeque in rectas extendantur, et in singulis singulae extensae punctis applicentur rectae ipsis respective distantiis a linea EF aequales, erit omnium spatiorum quae sic efficiuntur illud quod a Funicularia gignitur maximum.

Coepit Hon. Frater speculationem hanc extendere etiam ad funes inaequaliter crassos, quorum crassities ad longitudinem relationem obtinet aequatione algebraica exprimibilem, notatque unum casum, quo problema per Curvam simplicem Mechanicam solvi possit, nempe si supponatur Figura Curvilinea ABDEG (fig. 4), cujus applicata GE sit reciproce in dimidiata ratione abscissae AG, eaque sit in omnibus suis applicatis flexilis, hoc est, si concipiatur funis AG gravatus in singulis suis punctis respectivis rectis GE, vel (quod tantundem est) differentiis applicatarum GH in Parabola AHI, aut denique portiunculis curvae cycloidalis AHI (cujus vertex A) isque sic gravatus suspendi intelligatur, ita ut punctum A sit omnium infimum (quod sit, ubi connexum habuerit a parte A alium funem ejusdem longitudinis et in aequalibus a puncto A distantiis

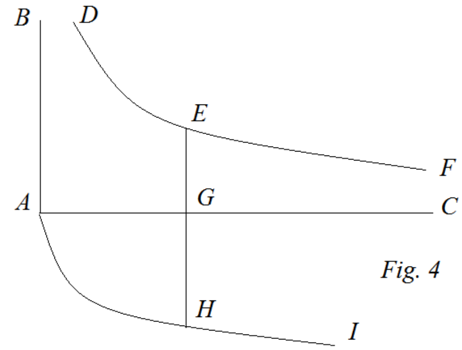


Fig. 4

aequaliter gravatum): tum jubet ad axem AG (fig. 5) construere Hyperbolam aequilateram ABC cujus vertex A, applicatamque BD producere ad E, ita ut rectangulum sub semilatere recto vel transverso et linea DE sit aequale spatio ADB, ostenditque punctum E esse ad curvam quaesitam AEF, quam funis dicta ratione gravatus format, ipsam vero curvam AE esse tertiam proportionalem ad rectum vel transversum latus Hyperbolae et applicatam ejus DB; tangentem EH haberi sumpta III quarta proportionali ad semilatus rectum, abscissam AD et applicatam DB etc. Reperi autem, quod memorabile est, curvam hanc AEF illam ipsam esse, ex cuius evolutione altera BE, quam uniformis crassitiei funis format, describitur, adeoque eandem cum curva MNO.

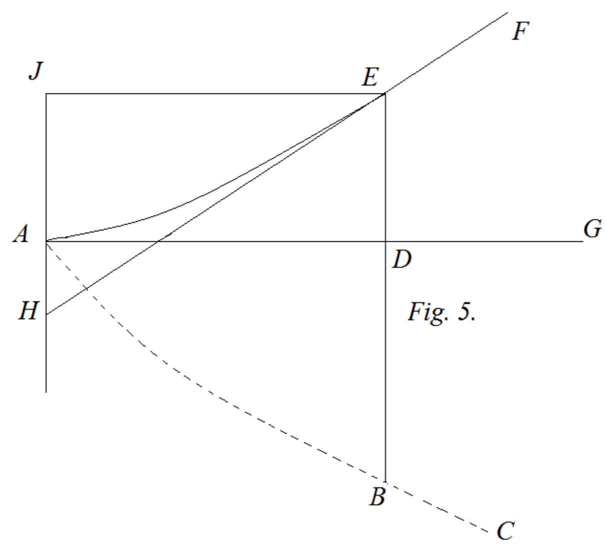


Fig. 5.

Notare convenit, quod si quis experimentis haec examinare instituat, catenulam prae fune seligere debeat, quem ob nimiam cum levitatem tum rigiditatem ad id ineptum deprehendimus. Caeterum qui materiam hanc perficere et ampliare volet, poterit investigare naturam curvae, quam refert funis in hypothesi a Terrae centro distantiae finitae, vel si supponatur insuper a proprio pondere extensibilis, aut quocunque alio modo gravatus: vel etiam vice versa qualiter illum gravare conveniat, ut referat lineam Parabolicam, Hyperbolicam, Circularem aliamve quamcunque datam curvam; res enim omnino in potestate est.

Christiani Hugenii, Dynastae in Zeelhem, solutio ejusdem Problematis.

Si Catena CVA (fig. 6) suspensa sit ex filis FC, EA utrinque annexis ac gravitate carentibus, ita ut capita C et A sint pari altitudine, deturque angulus inclinationis filorum productorum CGA et catenae totius positus, cujus vertex sit V, axis VB,

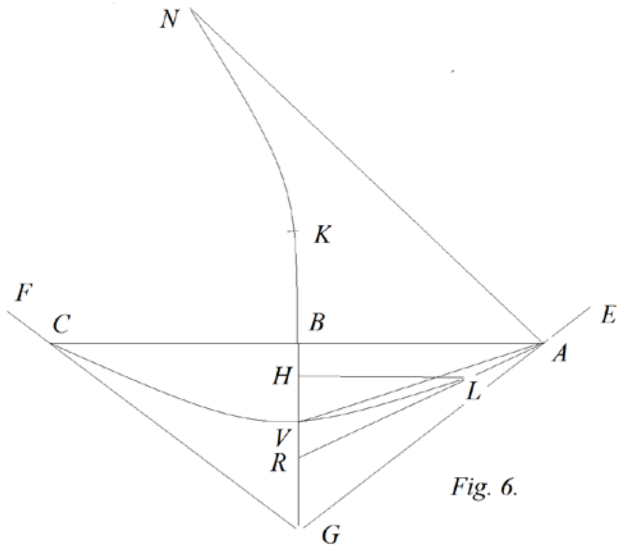


Fig. 6.

l. licebit hinc invenire tangentem in dato quovis catenae puncto. Velut si punctum datum sit L, unde ducta applicata LH dividat aequaliter axem BV. Jam si angulus CGA sit 60°, erit inclinanda a puncto A ad axem recta AIV aequalis $\frac{3}{2}AB$, cui ducta parallela LR tanget curvam in puncto L. Item si latera GB, BA, AG sint partium 3, 4, 5, erit AIV ponenda partium $4\frac{1}{2}$.

2. Invenitur porro et recta linea catenae aequalis, vel datae cuilibet ejus portioni. Semper enim dato angulo CGA, data erit ratio axis BV ad curvam VA. Velut si latera GB, BA, AG sint ut 3, 4, 5, erit curva VA tripla axis VB.

3. Item definitur radius curvitatis in vertice V, hoc est semidiameter circuli maximi, qui per verticem hunc descriptus totus intra curvam cadat. Nam si angulus CGA sit 60°, erit radius curvitatis ipsi axi BV aequalis. Si vero angulus CGA sit rectus, erit radius curvitatis aequalis curvae VA.

4. Poterit et circulus aequalis inveniri superficiei conoidis ex revolutione catenae circa axem suum. Ita si angulus CGA sit 60°, erit superficies conoidis ex catena CVA genita aequalis circulo, cujus radius possit duplum rectangulum BVG.

5. Inveniuntur etiam puncta quotlibet curvae KN, cujus evolutione, una cum recta KV, radio curvitatis in vertice, curva VA describitur, atque evolutae ipsius KN longitudo. Veluti si angulus CGA fuerit 60°, erit KN tripla axis BV. Si vero latera GB, BA, AG sint ut 3, 4, 5, erit illa $\frac{9}{4}$ axis BV.

6. Praeterea spatii NKVAN quadratura datur. Posito enim angulo CGA 60°, erit spatium illud aequale rectangulo ex axe BV et ea quae potest triplum quadratum ejusdem BV. Si vero latera GB, BA, AG sint ut 3, 4, 5, erit idem spatium aequale septuplo quadrato BV cum parte octava.

7. Porro puncta quotlibet catenae inveniri possunt, posita quadratura curvae alterius harum: $xxyy = a^4 - aayy$ vel $xxyy = 4a^4 - x^4$, vel etiam data distantia centri gravitatis ab axe, in portionibus planis, quas abscindunt rectae axi parallelae in curva harum

priore. Quadratura autem hujus curvae pendet a summis secantium arcuum per minima aequaliter crescentium, quae summae ex Tabulis sinuum egregio quodam adhibito compendio inveniuntur quamlibet proxime. Hinc ex. gr. inventum, quod si angulus CGA sit rectus, et ponatur axis BV partium 10000, erit BA 21279, non una minus. Curva autem VA per superius indicata cognoscitur hic esse partium 24142, non una minus.

In his omnibus non nisi ad casus singulares solutiones problematum dedi, vitandae prolixitatis studio, et quoniam non dubito quin regulas universales Viri docti affatim sint exhibituri. Quod si tamen aliquae ex nostris requirentur, eas lubenter mittam. Ac jam pridem omnes apud Clarissimum Virum *G. G. Leibnitium* involucra quodam obtectas deposui.

Additamentum ad Problema Funicularium
von *Jacob Bernoulli.* Act. Erudit. 1691.

Postquam Problematis de Curva Funicularia solutionem nuperrime exhibuisset Frater, speculationem istam continuo promovi ulterius et ad alios quoque casus applicui, quo pacto praeter ea, quorum tum mentio facta est, nonnulla sese obtulerunt, quae recensere operae pretium existimo.

l. Si crassities vel gravamina funis aut catenae inaequalia sint et sic attemperata, ut dum est in statu quietis, gravamen portionis HI (fig.7) sit in ratione portionis rectae utcunque ductae LM iisdem perpendiculis HL, IM interceptae, curva AIHB, quam funis vel catena sic suspensa proprio pondere format, erit Parabolica. Sin gravamen portionis HI sit in ratione spatii LOTM iisdem perpendiculis GL, IM intercepti, erit Funicularia AB curva Parabolae vel Cubicalis, vel Biquadraticae, vel Surdesolidalis etc., prout Figura CLO est vel Triangulum, vel Complementum semiparabolae communis, aut semiparabolae Cubicalis etc. Quod si vero gravamen portionis HI sit in ratione spatii QRST iisdem rectis horizontalibus HQ, IR abscissi, erit Funicularia IB curva aliqua ex genere Hyperbolicarum (recta AG existente una ex asymptotis), puta vel Apolloniana, vel Cubicalis, vel Biquadratica etc , prout videlicet Figura AQT est vel Triangulum, vel Complementum semiparabolae communis aut cubicalis etc.

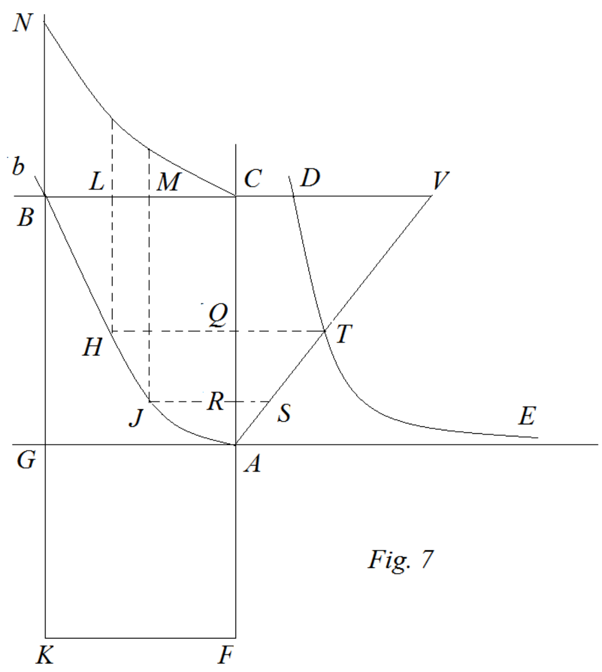


Fig. 7

2. Si funis sit uniformis crassitiei, at a pondere suo extensibilis, peculiari opus est artificio. Vocetur portio funis non extensi, cujus ponderi aequipollet, vis tendens unum funis punctum *a,* et excessus longitudinis, quo portio haec a dicta vi extensa non

extensam superat, *b*, sumaturque in perpendiculo $FA = a$, et in definita $FC = x$: tum fiat curva DE ejus naturae, ut sit applicata

$$CD = \frac{ab}{\sqrt{2aa+2bx-2a\sqrt{aa+bb+2bx}}} \text{ sive } a\sqrt{\frac{aa+bx-2a\sqrt{aa+bb+2bx}}{2xx-2aa}},$$

perinde enim est ac spatio curvilineo ACDE constituatur aequale rectangulum FG, producanturque rectae KG, DC ad mutuum occurum in B; sic erit punctum B ad requisitam funiculariam AB. Suppono autem, extensiones viribus tendentibus proportionales esse, tametsi dubium mihi sit, an cum ratione et experientia hypothesis illa satis congruat. Retinere autem istam nobis liceat, dum veriorem ignoramus.

3. Occasione Problematis funicularii mox in aliud non minus illustre delapsi sumus, concernens flexiones seu curvaturas trabium, arcuum tensorum aut elaterum quorumvis a propria gravitate vel appenso pondere aut alia quacunque vi comprimente factas; quorsum etiam Celeberrimum *Leibnitium* in privatis, quibus sub idem me tempus honoravit, literis digitum opportune intendere video. Videtur autem hoc Problema, cum ob hypotheseos incertitudinem, tum casuum multiplicem varietatem, plus aliquanto difficultatis involvere priori, quanquam hic non prolixo calculo, sed industria tantum opus est. Ego per solutionem casus simplicissimi (saltem in praememorata hypothesi extensionis) adyta Problematis feliciter reseravi; verum ut ad imitationem Viri Excellentissimi et aliis spatium concedam suam tentandi Analysin, premam pro nunc solutionem, eamque tantisper Logogripho occultabo, clavem cum demonstratione in nundinis autumnalibus communicaturus. Si lamina elastica gravitatis expers AB (fig. 8), uniformis ubique crassitiei et latitudinis, inferiore extremitate A alicubi firmetur et superiori B pondus appendatur, quantum sufficit ad laminam eousque incurvandam, ut linea directionis ponderis BC curvatae laminae in B sit perpendicularis, erit curvatura laminae sequentis naturae:

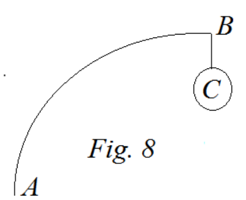


Fig. 8

Qrzumu bapt dxqopddhbp poyl fy bbqnfqbfp lty ge mutds utlthh tubs tmixy yxdksdhxp gqsrkfgudl bg ipqandtt tcpgkbp aqdbkzs. [Dies bedeutet: Portio axis applicatam inter et tangentem est ad ipsam tangentem sicut quadratum applicatae ad constans quoddam spatium.]

4. Istis vero omnibus multa sublimior est speculatio de *Figura veli vento inflati*, quanquam cum Problemate Funiculario eatenus affinitatem habet, quatenus venti continuo ad velum adlabentis impulsus ceu funis alicujus gravamina spectari possunt. Qui naturam pressionis fluidorum intellexerit, haud difficulter quidem capiet, quod portio veli BC (fig.9), quae subtensam habet directioni venti DE perpendicularem, curvari debeat in arcum circuli. At qualem curvaturam induat reliqua portio AB, ut difficilis est perquisitio, sic in re nautica eximii prorsus usus futura est, ut praestantissimorum Geometrarum occupationem juxta cum subtilissimis mereri videatur. Caeterum in his Problematibus omnibus, quae quis nequicquam alia tentet methodo, *calculi Leibnitiani* eximium et

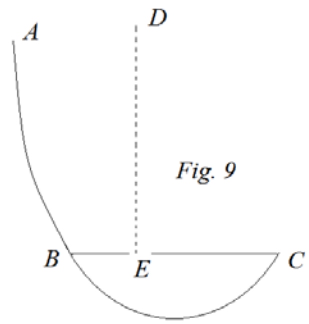


Fig. 9

singularem plane usum esse comperi, ut ipsum propterea inter primaria seculi nostri inventa censendum esse aestimem. Quanquam enim, ut nuper innui, ansam huic dedisse credam calculum *Barrovii,* qualem appello, qui ab hujus viri tempore passim fere apud Geometras praestantiores invaluit, quemque etiamnum Nobilissimo *Tschirnhausio* solemnem esse video : hoc tamen non eo intelligendum est, quasi utilissimi inventi dignitatem ullatenus elevare aut Celeberrimi Viri laudi meritae quicquam detrahere et aliis ascribere cupiam; et si quae conferenti mihi utrum que intercedere inter illos visa est affinitas, ea major non est, quam quae faciat, ut uno intellecto ratio alterius facilius comprehendatur, dum unus superfluas et mox delendas quantitates adhibet, quas alter compendia omittit: de caetero namque compendium isthoc tale est, quod naturam rei prorsus mutat, facitque ut infinita per hunc praestari possint, quae per alterum nequeunt: praeterquam etiam quod ipsum hoc compendium reperisse utique non erat cujusvis, sed sublimis ingenii et quod Autorem quam maxime commendat.