VI.

CONCERNING THE TRUE PROPORTIONS OF A CIRCLE TO THE CIRCUMSCRIBED SQUARE, EXPRESSED IN RATIONAL NUMBERS.

Act. Erudit. Lips. Feb. 1682

For a long time geometers have tried to find the proportions of curved lines to rectilinear lines, and yet now too, even after the use of algebra, that matter still has not been settled in a satisfactory manner within the power of the methods published recently, nor indeed can these problems be reduced to algebraic equations, and yet they have the most beautiful uses, especially in reducing the terms in Mechanics to pure geometry, which those who have got to know, but especially the best of the mathematicians, who have examined the matter deeper. In the first place Archimedes, that much is agreed, had found what the ratio shall be between the volumes of cone, sphere, and cylinder of the same height and base, surely the amount is in the ratio of the numbers 1, 2, 3, thus so that [the volume of] the cylinder shall be three times that of the cone, and one and a half that of the sphere ; from which it was ordered to inscribe both a sphere and a cylinder on his tomb: likewise he found the square of the parabola. [Later Cicero was to find the tomb overgrown and neglected at Syracuse in Sicily, and tried to restore it; now lost for ever. However, the mathematics of Archimedes lives on, and is still a wonder to read.] In our own time a way has been found of measuring innumerable curvilinear figures, especially when the ordinates *BC* (fig. 22) are in some multiple or sub-multiple, directly or inversely of the abscissas *AB* or *DC* ; for the figure *ABCA* will be to the circumscribed rectangle *ABCD*, as one to the number expressing the multiple of the ratio, increased by one. For example, because in a parabola with the natural numbers 1, 2, 3 etc. present for the abscissas *AB* or *DC*, the ordinates *BC* are as the squares 1, 4, 9 etc. of these or in the duplicate ratio of the numbers, then the number expressing the multiple of the ratio will be 2; therefore the figure *ABCA* will be to the circumscribed rectangle *ABCD*, as 1 to $2+1$ or as 1 to 3, or the figure will be the third part of the rectangle. If *AB* or *CD* may remain the natural numbers, and *BC* may become the cubes 1, 8, 27 etc. (evidently in the case for the cubic paraboloid), the ratio will become the triplicate of the ratio of the abscissas ; therefore the figure to the rectangle, shall be as 1 to $3+1$ or 4, or the fourth part. But if *DC* shall be the squared, *BC* the cubes, or if the ratio of *BC* themselves DC shall be the triples divided by two, the figure (three on two paraboloid) *ABCA* to the rectangle *ABCD*, shall be as 1 to $\frac{3}{2}+1$ or may constitute two fifths of the rectangle. The number multiplying in the reciprocal of a ratio may be expressed with the sign - or minus in front.

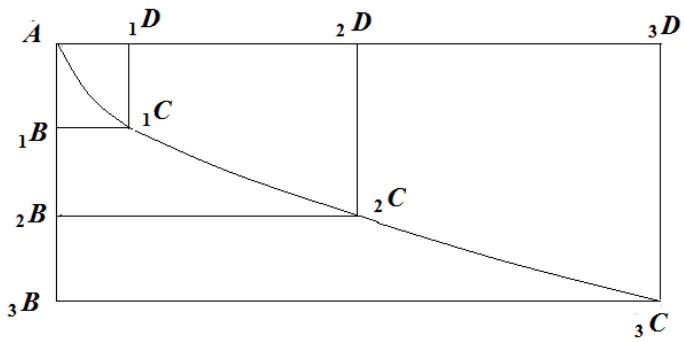


Fig. 22

However at no time up to the present has the [area of the] circle been able to be exercised under laws of this kind, by any geometers at any time. Indeed at no time has a

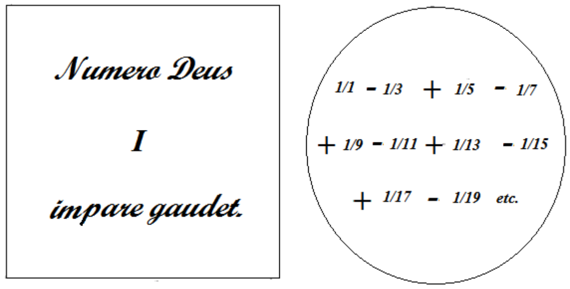


Fig. 23

number been found expressing the ratio of the circle *A* to the square of the circumscribed square *BC* (fig. 23), which is the square of the diameter *DE*. [The inscription may be translated : '*God delights in the odd number* 1.'] Nor was the ratio able to be found of the circumference to the diameter, which is the quadruple of the ratio of the circle to the square. Indeed Archimedes inscribing and circumscribing to the circle, because with the inscribed polygon greater and the circumscribed polygon lesser, a way is shown of presenting the limits, between which the circle must fall, or of showing the approximations: clearly the ratio of the circumference to the diameter to be greater than 3 to 1 or than 21 ad 7, and less than 22 to 7. Others have pursued this method, Ptolemy, Vieta, Metius, but especially Ludolph van Ceulen, who showed the circumference to the diameter to be as 3.14159265358979323846 etc. to 1.00000000000000000000.
[Later he extended this ratio to 35 decimal places, following the method of Archimedes with a 2^{64}-gon.]

Truly approximations of this kind, even if they have practical uses in geometry, yet show nothing, which may satisfy the mind in great need of the truth, unless a progression of such numbers being considered to infinity may be found. Indeed many have declared to have perfected the square, as Cardinal Nicholas of Cusa, Oronce Fine, Joseph Scaliger, Thomas Gephyrander, Thomas Hobbes, but all in error: for they are refuted by the calculations of Archimedes or of Ludolph today. Moreover because I see it must be known, that many people are not perceptive enough in the way required, squaring or the conversion of a circle into an equal square or another rectilinear figure (which depends on the ratio of the area of the circle to the square of the diameter) can be understood in four ways, evidently either by calculation or by line construction, each either accurately or an approximation. The quadrature by accurate calculation I call *Analytic;* that truly which comes about by accurate construction I call *Geometric,* by a near calculation truly an *approximation* may be had, by a near construction truly a *Mechanism.* Ludolph produced the longest approximation; Vieta, Huygens and the others gave outstanding mechanisms.

An accurate *geometrical* construction can be had, from which not only the whole circle, but also any sector or arc may be allowed to be measured by an exact ordered motion, but which may coincide with transcending curves, which otherwise may be considered *mechanical* by mistake, since yet they shall be just as geometric as ordinary curves, although they shall not be algebraic nor shall they be able to be reduced to algebraic curves of a certain order; yet they have their own analytical properties, even if they shall not be algebraic: but these cannot be set out here in the manner they deserve. *Analytical quadrature* or that which is made by an accurate calculation, again can be separated into three parts: Analytical transcending, Algebraic and Arithmetical. *Analytical transcending* quadrature is had by equations of indefinite degree between each other, hitherto not considered by anyone, such as if there shall be $x^x + x$ equals 30, and x may be sought, it shall be found to be 3, because $3^3 + 3$ is $27 + 3$ or 30 : we will have such equations in place for the circle. An *Algebraic* expression arises through common numbers, allowed to be irrational, or through the roots of common equations: which indeed is impossible for the general quadrature of the circle and of sectors. There remains *Arithmetical Quadrature*, which at least arises through series, by showing the exact value of the circle by a progression of terms, especially of rational ones, such as I propose here. Therefore I have discovered (fig. 23) :

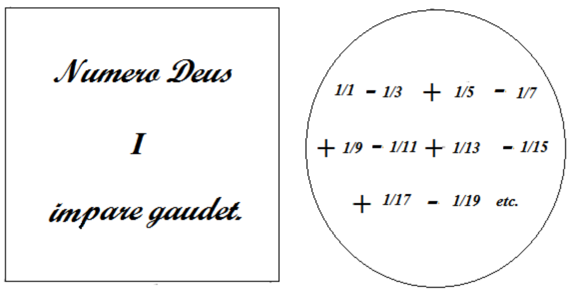


Fig. 23

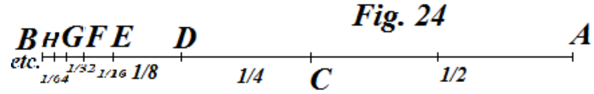
With the square of the diameter present equal to 1, the area of the circle will be : [$\frac{\pi}{4} =$] $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19}$ etc., truly the square of the diameter with the third part removed (lest the value may become excessive), on the contrary with the fifth part added (because we have removed too much), and in turn with the seventh part removed (because we have added on too much again), and thus again, the

value will be just greater than	1,	with the error present below $\frac{1}{4}$
" " " " smaller than	$\frac{1}{1} - \frac{1}{3}$	" " " " " $\frac{1}{5}$
" " " " greater than	$\frac{1}{1} - \frac{1}{3} + \frac{1}{5}$	" " " " " $\frac{1}{7}$
" " " " smaller than	$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$	" " " " " $\frac{1}{9}$
	etc.	etc.

Therefore the whole series contains all the approximations likewise either with the values just greater or just less : for it is understood that as it is made to continue for a long time, the error will be less than the fraction given, and thence smaller than any given quantity. Whereby the whole series expresses the exact value. Although it shall not be possible to express the sum of this series by a single number, even if the series may be produced

indefinitely, because yet it agrees with a single law of progressing, the whole may be understood well enough. For if indeed the circle is not commensurable with the square, it cannot be expressed by a single number, but it must be shown by necessity by a series in terms of rational numbers, just as with the diagonal of a square, cut both in the extreme and mean section, and which some call the divine ratio [*i.e.* the golden section], and many other quantities, which are irrational [Thus, Leibniz justifies the assumed irrationality of his ratio, compounded from rational numbers only, by referring to analogous irrational quantities such as the golden mean, which also is derived from rational quantities]. And indeed if Ludolph had been able to give a rule, by which the numbers 3.14159 etc may be continued to infinity, he would have given us an exact arithmetical quadrature, as we can ourselves show by fractions.
Moreover lest anyone little experienced in these matters should think, that a series constructed from an infinite number of terms should not to be equal to the circle, which is a finite quantity, it is required to know that for many series with an infinite number of terms, the sum to be in a finite quantity. For to put in place the easiest example, there may be the series of the geometrical progression starting from one decreasing geometrically by two $\frac{1}{2}+\frac{1}{4}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}$ etc. indefinitely, which still does not make more than l. For added to the right line AB (fig. 24) which shall be 1, there will be AC $\frac{1}{2}$, and on bisecting the remainder (CB) at D, you will have CD $\frac{1}{4}$; and on bisecting the remainder (DB) at E, you will have DE $\frac{1}{8}$; and on bisecting at F the remainder (EB), you will have EF $\frac{1}{16}$; and thus by continuing without end, you will never advance to the end B. Likewise with the fractions of the figures of number [*i.e.* triangular, square, pentagonal, etc. numbers] to arise for the harmonic triangle, which has been shown by me elsewhere.

Fig. 24



[Recall Leibniz's harmonic triangle, which resembles Pascal's triangle, but uses fractions instead: for example

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & \frac{1}{2} & & \frac{1}{2} & & \\
 & \frac{1}{3} & & \frac{1}{6} & & \frac{1}{3} & \\
\frac{1}{4} & & \frac{1}{12} & & \frac{1}{12} & & \frac{1}{4}
\end{array}
$$

...............etc...............
The sum of any two adjoining fractions on the same row is equal to the fraction between these on the row just above.]

Much may be observed about this quadrature, but for which now there is no free time to pursue ; yet here it ought not to be disregarded, the terms of our series $\frac{1}{1}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}$ etc. to be of a harmonic progression or in continued harmonic progression, as will be apparent by trial ; indeed by leaping forwards by a term,

$\frac{1}{1}, \frac{1}{5}, \frac{1}{9}, \frac{1}{13}, \frac{1}{17}$ etc. is also a series in harmonic progression, and likewise $\frac{1}{3}, \frac{1}{7}, \frac{1}{11}, \frac{1}{15}, \frac{1}{19}$ etc. is a series with harmonic proportionals. [The terms in such harmonic series are the reciprocals of terms in an arithmetic progression.] Therefore since the circle shall be $\frac{1}{1}+\frac{1}{5}+\frac{1}{9}+\frac{1}{13}+\frac{1}{17}$ etc. $-\frac{1}{3}-\frac{1}{7}-\frac{1}{11}-\frac{1}{13}-\frac{1}{19}$ etc., the latter partial series subtracted from the former, the magnitude of the circle will be the difference of two series of harmonic progressions. And because the sum by adding together whatever number of finite terms of a harmonic progression can be found in some manner, hence short approximations can be introduced (if there is a need for these after Ludolph).

If it should be wished to remove the – sign affecting the terms in our series, these two nearby terms $+\frac{1}{1}-\frac{1}{3}$, likewise $+\frac{1}{5}-\frac{1}{7}$, $+\frac{1}{9}-\frac{1}{11}$, $+\frac{1}{13}-\frac{1}{15}$, and $+\frac{1}{17}-\frac{1}{19}$ thus so on, by being added into one, a new series will be had for the magnitude of the circle, clearly $\frac{2}{3}$ (that is $\frac{1}{1}-\frac{1}{3}$) $+\frac{2}{35}$ $\left(\text{that is } \frac{1}{5}-\frac{1}{7}\right)+\frac{2}{99}$ $\left(\text{that is } \frac{1}{9}-\frac{1}{11}\right)$, and thus with the inscribing square having the area $\frac{1}{4}$, *the area of the circle will be* $\frac{1}{3}+\frac{1}{35}+\frac{1}{99}+\frac{1}{195}+\frac{1}{324}$ etc.

But the numbers 3, 35, 99, 195, 323 etc. to be taken by jumps from the series of square numbers (4, 9, 16, 25 etc.) diminished by one, from which the series becomes 3, 8, 15, 24, 35, 48, 63, 80, 99, 120, 143, 168, 195, 224, 255, 288, 323, 360, 399 etc., each is the fourth term of which series of numbers after our first. Moreover I have found (which is remarkable) the sum of the infinite series $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{35}+\frac{1}{48}+\frac{1}{63}+\frac{1}{80}+\frac{1}{99}$ etc. is $\frac{3}{4}$; and indeed by picking out in a simple jump, truly $\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\frac{1}{63}+\frac{1}{99}$ etc., the sum of which infinite series makes $\frac{2}{4}$ or $\frac{1}{2}$.

[Thus,

$$\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{35}+\frac{1}{48}+\frac{1}{63}+\frac{1}{80}+\frac{1}{99}\ldots+\frac{1}{(n+1)(n+3)}+\ldots$$

$$=\sum\frac{1}{(n+1)(n+3)}=\frac{1}{2}\sum\left(\frac{1}{(n+1)}-\frac{1}{(n+3)}\right)=\frac{1}{2}\left(1-\cancel{\tfrac{1}{3}}+\frac{1}{2}-\cancel{\tfrac{1}{4}}+\cancel{\tfrac{1}{3}}-\cancel{\tfrac{1}{5}}+\cancel{\tfrac{1}{4}}-\frac{1}{6}+\cancel{\tfrac{1}{5}}\ldots\right)=\frac{3}{4};$$

$$\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\frac{1}{63}+\frac{1}{99}+\ldots+\frac{1}{(2n+1)(2n+3)}+\ldots=\frac{1}{2}\sum\frac{1}{(2n+1)}-\frac{1}{(2n+3)}$$

$$=\frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\frac{1}{5}-\frac{1}{7}+\ldots\right)=\frac{1}{2}.]$$

But if from these again truly we may select the terms, $\frac{1}{3}+\frac{1}{35}+\frac{1}{99}$ etc., the sum of this infinite series will be [an approximation for] a semicircle, with the square of the diameter put to be l.

[It is noted here in the *Naissance du Calculi* that re-arrangements of convergent series are also convergent, but that sub-groupings of terms may not converge fast enough to be useful. We Leibniz found these sums above as shown.]

Moreover because there is a need for arithmetical quadrature of the hyperbola the whole of the harmonic series may be set out :

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
1 4 9 16 25 36 49 64 81100 121 144 169 196 225 256 289 324 361 400
0 3 8 15 24 35 48 63 80 99 120 143 168 195 224 255 288 323 360 399

$$\frac{1}{3}\,\frac{1}{8}\,\frac{1}{15}\,\frac{1}{24}\,\frac{1}{35}\,\frac{1}{48}\,\frac{1}{63}\,\frac{1}{80}\,\frac{1}{99}\,\frac{1}{120}\,\frac{1}{143}\,\frac{1}{168}\,\frac{1}{195}\,\frac{1}{224}\,\frac{1}{255}\,\frac{1}{288}\,\frac{1}{323}\,\frac{1}{360}\,\frac{1}{399}\ \text{etc.}$$

etc. being equal to $\frac{3}{4}$

$$\frac{1}{3}\cdot\frac{1}{15}\cdot\frac{1}{35}\cdot\frac{1}{63}\cdot\frac{1}{99}\cdot\frac{1}{143}\cdot\frac{1}{195}\cdot\frac{1}{255}\cdot\frac{1}{323}\cdot\frac{1}{399}$$

etc. being equal to $\frac{2}{4}$

$$\frac{1}{8}\cdot\frac{1}{24}\cdot\frac{1}{48}\cdot\frac{1}{80}\cdot\frac{1}{120}\cdot\frac{1}{168}\cdot\frac{1}{224}\cdot\frac{1}{288}\cdot\frac{1}{360}$$

etc. being equal to $\frac{1}{4}$

$$\frac{1}{3}\cdot\frac{1}{35}\cdot\frac{1}{99}\cdot\frac{1}{195}\cdot\frac{1}{323}\cdot\ \text{etc.}$$

being equal to the circle *ABCD*, the inscribed power of which is $\frac{1}{4}$

$$\cdot\frac{1}{8}\cdot\frac{1}{48}\cdot\frac{1}{120}\cdot\frac{1}{224}\cdot\frac{1}{360}\cdot\text{etc.}$$

is equal to the Hyperbola *CBEHC*, for which the square *ABCD* is $\frac{1}{4}$.

In fig. 25 the curve of the hyperbola *GCH* shall be described with the normals themselves *AF*, *AE* for the asymptotes, the vertex of which is *C*, truly the power of the inscribed hyperbola, which always is equal to the rectangle under some ordinate *EH* multiplied by its abscissa *AE*, the square shall be *ABCD*; around this square the circle may be described, and the hyperbola may be put thus continued from *C* as far as to H, thus so that *AE* shall be the double of AB. Then on putting *AE* to be one, there will be *AB* $\frac{1}{2}$, and its square *ABCE* will be $\frac{1}{4}$, and the circle, of which the inscribed power is *ABCD*, will be $\frac{1}{3}+\frac{1}{35}+\frac{1}{99}$ etc.; truly the part of the hyperbola (of which the inscribed potential is the same square $\frac{1}{4}$) the part *CBEHC*, which represents the logarithm of the ratio of *AE* ad *AB* itself (or of two), will be $\frac{1}{8}+\frac{1}{48}+\frac{1}{120}$ etc.

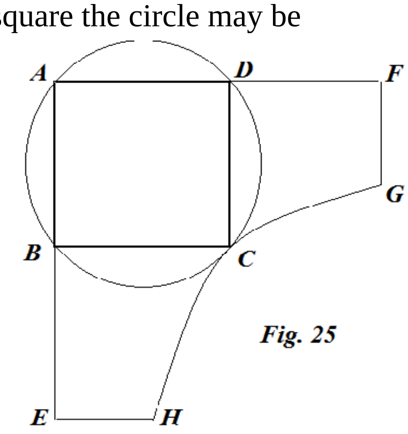


Fig. 25

[Recall that $\ln 2 = 0.693...$; thus this series multiplied by 4 would give an approximation for ln2.]

VI.

DE VERA PROPORTlONE CIRCULI AD QUADRATUM CIRCUMSCRIPTUM IN NUMERIS RATIONALIBUS EXPRESSA.

Act. Erudit. Lips. an. 1682

Proportiones curvilineorum ad Rectilinea investigare Geometrae semper sunt conati, et tamen nunc quoque, etiam post Algebram adhibitam, nondum ea res satis in potestate est secundum methodos quidem hactenus publicatas : neque enim haec problemata ad aequationes Algebraicas revocari possunt, et usum tamen pulcherrimum habent, inprimis in Mechanica ad purae Geometriae terminos reducenda, quod norunt, qui talia profundius inspexere, pauci quidem, sed maxime eximii Mathematicorum.

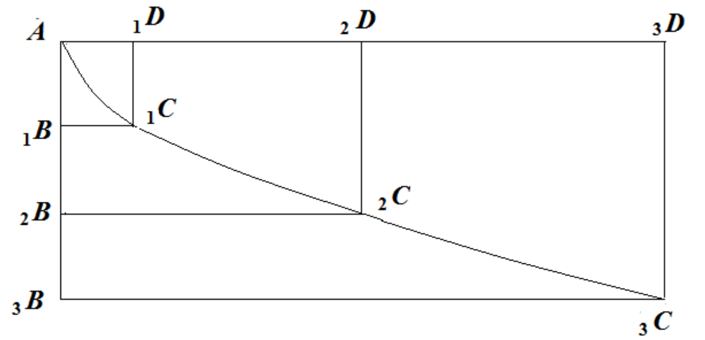


Fig. 22

Primus Archimedes, quantum constat, invenit, quae sit ratio inter conum, sphaeram et cylindum ijusdem altitudinis et basis, nempe qualis est numerorum 1, 2, 3, ita ut cylindrum sit triplus coni et sesquialter sphaerae ; unde sphaeram et cylindrum etiam sepulcro suo insculpi jussit: idem invenit quadraturam Parabolae. Nostro seculo repertus est modus metiendi figuras curvilineas innumerabiles, inprimis quando ordinatae BC (fig. 22) sunt in ratione utcunque multiplicata aut submultiplicata, directa aut reciproca abscissarum AB vel DC ; erit enim figura ABCA ad rectangulum circumscriptum ABCD, ut unitas ad numerum rationis multiplicationem exprimentem, unitate auctum. Exempli gratia, quia in Parabola abscissis AB sive DC existentibus at numeris naturalibus l, 2, 3 etc. ordinatae BC sunt ut eorum quadrata 1, 4, 9 etc. seu in duplicata ratione numerorum, tunc numerus rationis multiplicationem exprimens erit 2; ergo erit figura ABCA ad rectangulum circumscriptum ABCD, ut 1 ad $2+1$ seu ut l ad 3, sive figura erit tertia pars rectanguli. Si AB vel CD maneant numeri naturales, et BC fiant cubi 1, 8, 27 etc. (nempe in Paraboloide cubica), foret ratio ordinatarum triplicata rationis abscissarum ; ergo figura ad rectangulum, ut 1 ad $3+1$ sive 4, seu pars quarta. Sin DC sint quadrata, BC cubi, sive ratio ipsarum BC rationis ipsarum DC triplicata subduplicata, erit figura (Paraboloides cubico-subquadratica) ABCA ad rectangulum ABCD, ut l ad $\frac{3}{2}+1$ seu duas quintas rectanguli constituet. In reciprocis numero rationis multiplicationem exprimenti praefigitur signum - sive minus.

Sed circulus nondum hactenus cogi potuit sub hujusmodi leges, quamvis ab omni retro memoria a Geometris exercitus. Nondum enim inveniri potuit numerus exprimens rationem circuli A ad quadratum circumscriptum BC (fig. 23), quod est quadratum diametri DE. Nec inveniri potuit ratio circumferentiae ad diametrum, quae est quadrupla

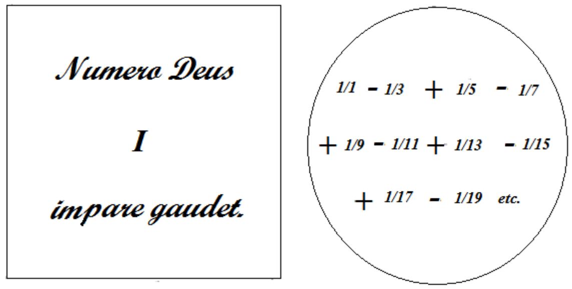


Fig. 23

rationis circuli ad quadratum. Archimedes quidem polygona circulo inscribens et circumscribens, quoniam major est inscriptis et minor circumscriptis, modum ostendit exhibendi limites, inter quos circulus cadat, sive exhibendi appropinquationes: esse scilicet rationem circumferentiae ad diametrum majorem quam 3 ad 1 seu quam 21 ad 7, et minorem quam 22 ad 7. Hanc methodum alii sunt prosecuti, Ptolemaeus, Vieta, Metius, sed maxime Ludolphus Coloniensis, qui ostendit esse circumferentiam ad diametrum ut 3.14159265358979323846 etc. ad 1.00000000000000000000.

Verum hujusmodi appropinquationes, etsi in Geometria practica utiles, nihil tamen exhibent, quod menti satisfaciat avidae veritatis, nisi progressio talium numerorum in infinitum continuandorum reperiatur. Multi quidem perfectum Tetragonismum professi sunt, ut Cardinalis Cusanus, Orontius Finaeus, Josephus Scaliger, Thomas Gephyrander, Thomas Hobbes, sed omnes falso: calculis enim Archimedis vel hodie Ludolphi refellebantur. Sed quoniam video, multos non satis percepisse, quid desideretur, sciendum est, Tetragonismum sive conversionem circuli in aequale quadratum aliamve rectilineam figuram (quae pendet a ratione circuli ad quadratum diametri, vel circumferentiae ad diametrum) posse intelligi quadruplicem, nempe vel per calculum, vel per constructionem linearem, utrumque vel accurate vel propemodum. Quadraturam per calculum accuratum voco *Analyticam;* eam vero quae per constructionem accuratam fit, voco *Geometricam*, per calculum prope verum habetur *appropinquatio*, per constructionem prope veram *Mechanismus.* Appropinquationem longissime produxit Ludolphus; Mechanismos egregios Vieta, Hugenius aliique dedere.

Constructio Geometrica accurata haberi potest, qua non tantum circulum integrum, sed et quemlibet sectorem sive arcum metiri liceat motu exacto atque ordinato, sed qui curvis transcendentibus competat, quae per errorem alioqui Mechanicae censentur, cum tamen aeque sint Geometricae ac vulgares, licet Algebraicae non sint nec ad aequationes Algebraicas seu certi gradus reduci queant; suas enim proprias, etsi non-algebraicas, tamen analyticas habent. Sed ista hic pro dignitate exponi non possunt. *Quadratura Analytica* seu quae per calculum accuratum fit, iterum in tres potest dispesci: in Analyticam transcendentem, Algebraicam et Arithmeticam. Analytica *transcendens* inter alia habetur per aequationes gradus indefiniti, hactenus a nemine consideratas, ut si sit $x^x + x$ aeq. 30, et quaeratur *x*, reperietur esse 3, quia $3^3 + 3$ est $27 + 3$ sive 30 : quales aequationes pro circulo dabimus suo loco. *Algebraica* expressio fit per numeros, licet irrationales, vulgares seu per radices aequationum communium: quae quidem pro quadratura generali circuli sectorisque impossibilis est. Superest *Quadratura Arithmetica,* quae saltem per series fit, exhibendo valorem circuli exactum progressione terminorum, inprimis rationalium, qualem hoc loco proponam. Inveni igitur (fig. 23)

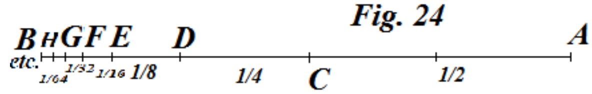
Quadrato Diametri existente 1,

Circuli aream fore $\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\frac{1}{17}-\frac{1}{19}$ etc., nempe quadratum diametri integrum demta (ne nimius fiat valor) ejus tertia parte, addita rursus (quia nimium demsimus) quinta, demtaque iterum (quia nimium re-adjecimus) septima, et ita porro, eritque

valor justo	major	1	errore tamen existente infra $\frac{1}{4}$
	minor	$\frac{1}{1}-\frac{1}{3}$	. . . . $\frac{1}{5}$
	major	$\frac{1}{1}-\frac{1}{3}+\frac{1}{5}$	. . . . $\frac{1}{7}$
	minor	$\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}$	. . . . $\frac{1}{9}$
		etc.	etc.

Tota ergo series continet omnes appropinquationes simul sive valores justo majores et justo minores : prout enim longe continuata , intelligitur, erit error minor fractione data, ac proinde et minor data quavis quantitate. Quare tota series exactum exprimit valorem. Et licet uno numero summa ejus seriei exprimi non possit, et series in infinitum producatur, quoniam tamen una lege progressionis constat, tota satis mente percipitur. Nam siquidem circulus non est quadrato commensurabilis, non potest uno numero exprimi, sed in rationalibus necessario per seriem exhiberi debet, quemadmodum et diagonalis quadrati, et sectio extrema et media ratione facta, quam aliqui divinam vocant, aliaeque multae quantitates, quae sunt irrationales. Et quidem si Ludolphus potuisset regulam dare, qua in infinitum continuarentur numeri 3. 14159 etc. dedisset nobis quadraturam Arithmeticam exactam in integris, quam nos exhibemus in fractis.
Ne quis autem in his parum versatus putet, seriem ex infinitis terminis constantem non posse aequari circulo, qui est quantitas finita, sciendum est, multas series numero terminorum infinitas esse in summa quantitates finitas. Exempli facillimi loco sit series ab unitate decrescens progressionis geometricae duplae $\frac{1}{2}+\frac{1}{4}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}$ etc. in infinitum, quae tamen non facit plus quam l. Nam in adjecta linea recta AB (fig. 24) quae sit 1, erit AC $\frac{1}{2}$, et residuum (CB) bisecando in D, habebis CD $\frac{1}{4}$; et residuum (DB) bisecando in E, habebis DE $\frac{1}{8}$; et residuum (EB) bisecando in F, habebis EF $\frac{1}{16}$; et ita continuando sine fine, nunquam egredieris terminum B. Idem in fractionibus numerorum figuratorum seu triangulo Harmonico fieri a me alibi ostensum est.

Fig. 24



Multa notari possent circa hanc Quadraturam, sed quae nunc persequi non vacat; hoc tamen praeteriri non oportet, terminos seriei nostrae $\frac{1}{1}$, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$ etc. esse progressionis harmonicae sive in continua proportione harmonica, ut experienti patebit;

quin et per saltum $\frac{1}{1}, \frac{1}{5}, \frac{1}{9}, \frac{1}{13}, \frac{1}{17}$ etc. est etiam series progressionis harmonicae, et $\frac{1}{3}, \frac{1}{7}, \frac{1}{11}, \frac{1}{15}, \frac{1}{19}$ etc. est itidem series harmonice proportionalium. Itaque cum circulus sit $\frac{1}{1}+\frac{1}{5}+\frac{1}{9}+\frac{1}{13}+\frac{1}{17}$ etc. $-\frac{1}{3}-\frac{1}{7}-\frac{1}{11}-\frac{1}{13}-\frac{1}{19}$ etc., posteriorem seriem partialem a priori suhtrahendo, erit magnitudo circuli differentia duarum serierum progressionis harmonicae. Et quoniam quotcunque terminorum numero finitorum progressionis harmonicae summa compendio aliquo iniri potest, hinc appropinquationes compendiosae (si post Ludolphinam illis esset opus) duci possent.

Si quis in serie nostra terminos signo - affectos tollere volet, is duos proximos $+\frac{1}{1}-\frac{1}{3}$, item $+\frac{1}{5}-\frac{1}{7}$, item $+\frac{1}{9}-\frac{1}{11}$, et $+\frac{1}{13}-\frac{1}{15}$, et $+\frac{1}{17}-\frac{1}{19}$ ita porro, in unum addendo, habebit seriem novam pro magnitudine circuli, nempe
$\frac{2}{3}$ (id est $\frac{1}{1}-\frac{1}{3}$) $+\frac{2}{35}\left(\text{id est } \frac{1}{5}-\frac{1}{7}\right)+\frac{2}{99}\left(\text{id est } \frac{1}{9}-\frac{1}{11}\right)$, itaque

Quadrato inscripto existente $\frac{1}{4}$, *erit Area Circuli* $\frac{1}{3}+\frac{1}{35}+\frac{1}{99}+\frac{1}{195}+\frac{1}{324}$ etc. Sunt autem numeri 3, 35, 99, 195, 323 etc. excerpti per saltum ex serie numerorum quadratorum (4, 9, 16. 25 etc.) unitate minutorum, unde fit series 3, 8, 15, 24, 35, 48, 63, 80, 99, 120, 143, 168, 195, 224, 255, 288, 323, 360, 399 etc., ex cujus seriei numeris quartus quisque post primum noster est. Inveni autem (quod memorabile est) seriei infinitae $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{35}+\frac{1}{48}+\frac{1}{63}+\frac{1}{80}+\frac{1}{99}$ etc. summam esse $\frac{3}{4}$; quin et simplici saltu excerpendo, nempe $\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\frac{1}{63}+\frac{1}{99}$ etc., ejus seriei infinitae summa facit $\frac{2}{4}$ seu $\frac{1}{2}$. Sed si ex hac iterum simplici saltu terminos excerpamus, nempe $\frac{1}{3}+\frac{1}{35}+\frac{1}{99}$ etc., , ejus seriei infinitae summa erit Semicirculus, posito quadratum diametri esse l.

Quoniam autem eadem opera quadratura Hyperbolae arithmetica habetur placet totam harmonium oculis subjicere :

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
1 4 9 16 25 36 49 64 81100 121 144 169 196 225 256 289 324 361 400
0 3 8 15 24 35 48 63 80 99 120 143 168 195 224 255 288 323 360 399

$$\frac{1}{3}\,\frac{1}{8}\,\frac{1}{15}\,\frac{1}{24}\,\frac{1}{35}\,\frac{1}{48}\,\frac{1}{63}\,\frac{1}{80}\,\frac{1}{99}\,\frac{1}{120}\,\frac{1}{143}\,\frac{1}{168}\,\frac{1}{195}\,\frac{1}{224}\,\frac{1}{255}\,\frac{1}{288}\,\frac{1}{323}\,\frac{1}{360}\,\frac{1}{399}\text{ etc.}$$

etc. aequatur $\frac{3}{4}$

$$\frac{1}{3}\cdot\frac{1}{15}\cdot\frac{1}{35}\cdot\frac{1}{63}\cdot\frac{1}{99}\cdot\frac{1}{143}\cdot\frac{1}{195}\cdot\frac{1}{255}\cdot\frac{1}{323}\cdot\frac{1}{399}$$

etc. aequatur $\frac{2}{4}$

$$\frac{1}{8}\cdot\frac{1}{24}\cdot\frac{1}{48}\cdot\frac{1}{80}\cdot\frac{1}{120}\cdot\frac{1}{168}\cdot\frac{1}{224}\cdot\frac{1}{288}\cdot\frac{1}{360}$$

etc. aequatur $\frac{1}{4}$

$$\frac{1}{3}\cdot\frac{1}{35}\cdot\frac{1}{99}\cdot\frac{1}{195}\cdot\frac{1}{323}\cdot\text{ etc.}$$

aequatur circulo ABCD, cujus potentia inscripta est $\frac{1}{4}$

$$\cdot\frac{1}{8}\cdot\frac{1}{48}\cdot\frac{1}{120}\cdot\frac{1}{224}\cdot\frac{1}{360}\cdot\text{ etc.}$$

aequatur circulo ABCD, cujus potentia inscripta Hyperbolae CBEHC, cujus quadratum ABCD est $\frac{1}{4}$.

Sit in fig. 25 Asymptotis AF, AE sibi normalibus Hyperbolae curva descripta GCH, cuis vertex C, potentia vero Hyperbolae inscripta sive quadratum, quod rectangulo sub ordinate quacunque EH lin suam abscissam AE simper aequale est, sit ABCD; circa hoc quadratum describatur circulus, et ponatur Hyperbola ita continuata a C usque ad H, ut sit AE dupla ipsius AB. Tunc posito AE esse unitatem, erit AB $\frac{1}{2}$, et ejus quadratum ABCD erit $\frac{1}{4}$, et Circulus, cujus potential inscripta est ABCD, erit $\frac{1}{3}+\frac{1}{35}+\frac{1}{99}$ etc., Hyperbolae vero (cuius potentia inscripta est idem quadratum $\frac{1}{4}$) portio CBEHC, quae logarithmum rationis ipsius AE ad AB (sive binarii) repraesentat, erit $\frac{1}{8}+\frac{1}{48}+\frac{1}{120}$ etc.

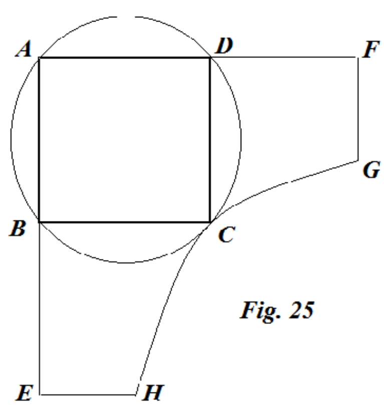


Fig. 25