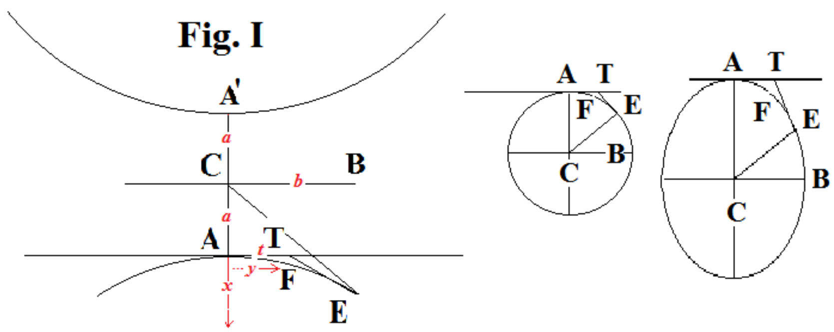
W.W.L. The Arithmetical Quadrature of common Conic Sections which have a centre, with Trigonometric Cannons deduced exactly for any numbers and thence freed from the necessity of Tables: with the special use of curves for the nautical Rhombus, and small approximate planes of the globe adapted for these.

Ac. Erud. Apr. 1691

Now in the year 1675 I was composing a small work on Arithmetical Quadrature read to my friends at that time, but because after other occupation-related tasks intervened, with the material at hand increasing there was no free time to refine the work towards a publication: since now especially setting out the work in more detail in the customary manner would not seem to be worth the effort, which our new work shows with a little analysis. Meanwhile certain conspicuous Mathematicians, by whom the truth of our foremost proposition has been noted, published in these *Acta* some time ago, clearly remember our discovery of such for mankind. Among whom even the most illustrious



Huygens has added something elegantly analogous for the Hyperbola, different from our earlier analogous paper. For indeed we had given the series $\frac{1}{1}t - \frac{1}{3}t^3 + \frac{1}{5}t^5$ etc. for the circle ; thus it had been observed in an analogous manner for the hyperbola, to be shown by the series $\frac{1}{1}t + \frac{1}{3}t^3 + \frac{1}{5}t^5$ etc. itself, which has been added to the paper preceding this one here. And also within reason in our unpublished works nor with that itself having been seen, a single one may be remembered well enough amongst the other propositions on account of its generality, and both those and others may be included :

[In Fig. I, for the circle, we have the equation : $(x-a)^2 + y^2 = a^2$; or $x^2 - 2ax + y^2 = 0$, where A is the origin for each figure, C is the centre of the circle, a the radius, and the + x axis is vertically down, + y axis horizontally to the right; similarly, for ellipse, $\frac{(x-a)^2}{a^2} + \frac{y^2}{b^2} = 1,$

and for the the rectangular hyperbola with centre C at $(-a,0)$:

$$\frac{(x+a)^2}{a^2}-\frac{y^2}{a^2}=1, \text{ or } (x+a)^2-y^2=a^2 \;;$$

i.e. $y=\sqrt{x^2+2ax}$; and $\frac{dy}{dx}=\frac{x+a}{\sqrt{(x^2+2ax)}}$.]

To square a sector arithmetically, taken from a conic curve starting from the vertex, and with the radius drawn from the centre. The part AT of the right line of the tangent at the vertex may be called *t*, taken between the vertex A, and the crossing point T with the tangent at the other extremity, & the conjugate semiaxis CB shall be taken as unity (or the right line, which makes a right angle between both half the transverse and upright sides), the sector CAFEC will be equal to the rectangle taken under AC with the transverse half-side, and by a right line, of which the length shall be $\frac{1}{1}t\pm\frac{1}{3}t^3\pm\frac{1}{5}t^5\pm\frac{1}{7}t^7\ldots$ etc. Not only may the area of a sector of a Circle or of a primary equilateral Hyperbola be expressed when the angle of the asymptotes is right, but also of the sector of any other Ellipses or Hyperbolas whatever.

[For example, the area of the sector CEAC of the hyperbola can be found, starting from:

$$t=y-x\frac{dy}{dx}, \text{ where } \frac{dy}{dx}=\frac{x+a}{\sqrt{(x^2+2ax)}} \text{ and } y=\sqrt{x^2+2ax};$$

$$t=\sqrt{x^2+2ax}-\frac{x^2+ax}{\sqrt{(x^2+2ax)}}=\frac{x^2+2ax-x^2-ax}{\sqrt{(x^2+2ax)}}=\frac{ax}{y}.$$

Now, from $(x+a)^2-y^2=a^2$, $x^2+2ax=y^2$, or $\frac{x}{y}=\frac{y}{x+2a}$, and $t=\frac{ay}{x+2a}$.

Hence, $t^2=\frac{a^2y^2}{(x+2a)^2}=\frac{a^2\times(x^2+2ax)}{(x+2a)^2}=\frac{a^2x}{(x+2a)}; a^2x=t^2(x+2a)\therefore x=\frac{2at^2}{a^2-t^2}$.

The area of the segment CAEC can then be found, following e.g. the treatment in *Naissance du Calculi* …

If E has the co-ordinates (X, Y), then this consists of the area of the triangle $\frac{1}{2}(a+X)Y$ from which are taken the area of the lesser triangle $\frac{1}{2}XY$, and the area contained by the chord of the hyperbola AE and the arc AFE : $\frac{1}{2}\int_0^X tdx$, giving $\frac{1}{2}aY-\frac{1}{2}\int_0^X tdx$. Now, $\int_0^X tdx=\mathrm{TX}-\int_0^T xdt=\mathrm{TX}-\int_0^T \frac{2at^2}{a^2-t^2}dt$. Finally, the area CAEC is found :

$$\mathrm{A}=a.\mathrm{T}+\int_0^T\frac{at^2}{a^2-t^2}dt=a.\mathrm{T}+\frac{\mathrm{T}^3}{3a}+\frac{\mathrm{T}^5}{5a^3}+\ldots\text{etc.}]$$

The others can be found from infinite series to as great an accuracy as it pleases, as shown by me and others such *Mercator, Newton, Gregory,* and the use of the trigonometric rules can be followed without the use of tables. And neither always is it possible to carry tables by sea or land. Clearly, the radius shall be one, *t* the tangent of the arc *a,* the right sine shall be *s,* the versed sine *v*, the logarithm *l,* the number $1+n$ (for the logarithm of unity or for itself *l* being 0), the series will become :

$$a=\tfrac{1}{1}t-\tfrac{1}{3}t^{3}+\tfrac{1}{5}t^{5}-\tfrac{1}{7}t^{7}+\tfrac{1}{9}t^{9}\text{...etc.} \qquad (1)$$

$$s=a-\frac{a^{3}}{1.2.3}+\frac{a^{5}}{1.2.3.4.5}-\frac{a^{7}}{1.2.3.4.5.6.7}\text{....etc.} \qquad (2)$$

that is, $a-\frac{a^{3}}{6}+\frac{a^{5}}{120}\text{...etc.}$

$$v=\frac{a^{2}}{1.2}-\frac{a^{4}}{1.2.3.4}+\frac{a^{6}}{1.2.3.4.5.6}-\frac{a^{8}}{1.2.3.4.5.6.7.8}\text{...etc.} \qquad (3)$$

$$l=\tfrac{1}{1}n-\tfrac{1}{2}n^{2}+\tfrac{1}{3}n^{3}-\tfrac{1}{4}n^{4}+\tfrac{1}{5}n^{5}\text{...etc.} \qquad (4)$$

$$n=\frac{l}{1}+\frac{l^{2}}{1.2}+\frac{l^{3}}{1.2.3}+\frac{l^{4}}{1.2.3.4}+\frac{l^{5}}{1.2.3.4.5}\text{...etc.} \qquad (5).$$

But the quantity, the powers of which are used in infinite series, always must be less than unity, so that in a progression they may become as small as it pleases. Many series of this kind can be given, and it is appropriate to be effected by series, so that from the arcs the artificial sines and the tangents may be given, or of the logarithms (with these not being supposed natural [*i.e*. the logarithms of the natural sines and tangents]) and in turn the arc may be taken from these : But it pleases only to describe these series, which are of so simple a composition, that they may be retained in memory easily and everywhere free from defects: and shall be able to take the place of tables. And thus I add only one series, on account of its simplicity, and because it provides the occasion for this paper, if the complements of the sines [*s*] shall be *c*, the logarithms of these right sines, or rather (because it returns the same) of the reciprocals from these sines, thus become equal to :

$$\left[\log\tfrac{1}{s}\right]=\frac{c^{2}}{2}+\frac{c^{4}}{4}+\frac{c^{6}}{6}+\frac{c^{8}}{8}\text{...etc.} \qquad (6)$$

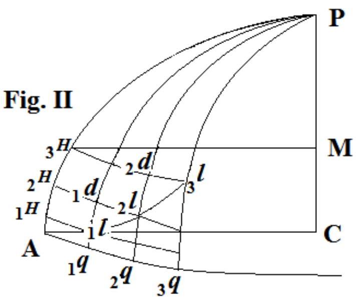
just as it follows from these, which we have indicated in the paper *de Resistentia Medii* Act. Jan. 1689 p. 4 art. 5 prop. 6; from which again it depends on the quadrature of the hyperbola. Nor do they differ from what Nic. Mercator gave, from which according to my squaring of the circle in the second month of the first year of publishing our Acta, I had shown a not inelegant analogy with the hyperbola. Clearly I had arrived at the circle to be to the circumscribed square as $\tfrac{1}{1}-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}\text{...etc.}$ to unity, or the circle to be to the inscribed square as $\frac{1}{4-1}+\frac{1}{36-1}+\frac{1}{100-1}$...etc. where the numbers 4, 36, 100 etc. are the squares from the numbers 2, 6, 10 , etc differing by four. Similarly from what has been said above, when the number is sought whose logarithm $1+x$ is 2, then x is 1, and thus $\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}$ etc. is the hyperbolic logarithm of two. The same series makes $\frac{4}{9-1}+\frac{4}{49-1}+\frac{4}{121-1}$etc. (for $\frac{4}{9-1}$ is equal to $\frac{1}{1}-\frac{1}{2}$, and $\frac{4}{49-1}$ is equal to $\frac{1}{3}-\frac{1}{4}$, and thus again), there the hyperbolic logarithm of two is to one, as $\frac{1}{9-1}+\frac{1}{49-1}$ etc. is to $\frac{1}{4}$, where the numbers 9, 49, 121, etc. are the squares from 3, 7, 11 etc. which are the odd numbers exceeding by one the above mentioned numbers equally different by four : from which the origin is apparent shown by us at one time in these Actis, as has been said. Moreover the log. of $1:\sqrt{1-cc}$ shall be $\frac{1}{2}cc+\frac{1}{4}c^4+\frac{1}{6}c^6$ etc., I may show thus :

log. from $1+c=\frac{1}{1}c-\frac{1}{2}cc+\frac{1}{3}c^3$...etc. and log. from $1-c=-\frac{1}{1}c-\frac{1}{2}cc-\frac{1}{3}c^3$...etc. each by equation 4 ; therefore log. $\overline{1+c}+$ log. $\overline{1-c}$ that is log. $\overline{1-cc}=-\frac{2}{2}cc-\frac{2}{4}c^4-\frac{2}{6}c^6$...etc. and $\frac{1}{2}$log. $\overline{1-cc}$ that is log.$\sqrt{1-cc}$ $=-\frac{1}{2}cc-\frac{1}{4}c^4-\frac{1}{6}c^6$....etc.;

[hence $\log\frac{1}{s}=-\log.\sqrt{1-cc}=\frac{1}{2}cc+\frac{1}{4}c^4+\frac{1}{6}c^6$....etc]

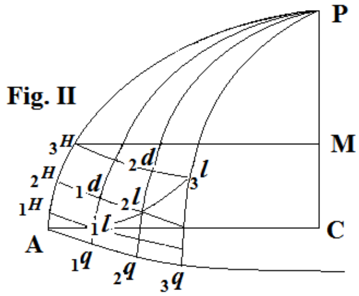
But so that the uses of these may become more apparent, it will be worth the effort to show the same calculation to be advantageous for correctly judging the rhombic curves [now called a rhomb curve or loxodrome : *i.e.* the curve traced out by a vessel sailing which maintains a constant bearing to true North, where a spiral is formed terminating at the pole], described on the celestial sphere used in sailing, and by these being projected onto a plane, which commonly are treated with little accuracy. We will explain the matter most fully by considering a few uses. P shall be the Pole, (fig. II) A*qq* the equator, PA, P*q*, etc. the meridians, $A_1l_2l_3l$ etc. is the rhombic curve described as long as the same course may be held by the direction of the wind. Through the points *l*, the parallels H*l* are drawn, truly

Fig. II



$_1H\,_1l$, $_2H\,_1d\,_2l$, $_3H\,_2d\,_3l$, etc. Because if now the distance between the points *q*, *q* shall be incomparably small, the parts of the arcs indeed will become unassignable and will be taken as right lines, and the triangles with right sides, and the triangles $_1l_1d_2l_2d_3l$, etc. will be similar, on account of the rhombic curve always making the same angle to the meridian of the place. Therefore the magnitude $_1l_3l$ *of the rhombic curve traversed, or of the journey along the same rhombic line, is to* $_1H_3H$ *, the difference of the initial and final latitudes, as the whole sine to the sine of the angle of the rhombus*. [Here Leibniz confuses the issue by taking the angle of the curve to the latitude rather than to the longitude; he uses the correct ratio in what follows, corresponding to $\tan\phi$ as indicated in the extra diagram.] And thus from the given rhombic angle and the difference of the latitudes the magnitude of the path is given. So far the matter here has been common knowledge, but so that from the same differences the longitude may be estimated by calculation, the work belongs to the geometry of transcending quantities, which few people have treated correctly. That therefore can be supplied by our method. The radius or the whole sine shall be unity, and the tangent of the rhombic angle shall be the constant *i.e.* $b = \tan\varphi$, and it is apparent :

Fig. II



${}_1d_2l$ shall be to ${}_1l_1d$ or ${}_1\mathrm{H}_2\mathrm{H}$, or ${}_2d_3l$ shall be to ${}_2l_2d$ or to ${}_2\mathrm{H}_3\mathrm{H}$, as b to 1

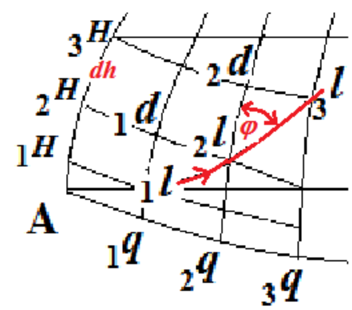
$$\left[i.e.\ \frac{{}_1d_2l}{{}_1l_1d} = \frac{{}_1d_2l}{{}_1\mathrm{H}_2\mathrm{H}} = \frac{b}{1} = \tan\varphi = \frac{{}_2d_3l}{{}_2l_2d} = \frac{{}_2d_3l}{{}_2\mathrm{H}_3\mathrm{H}}. \right]$$

But ${}_2q_3q$ is to ${}_2d_3l$, as AC (the whole sine or the radius of the sphere) to ${}_3$HM, [i.e. the radius of cross-section at this latitude]; $\left[i.e.\ \frac{{}_2q_3q}{{}_2d_3l} = \frac{\mathrm{AC}}{{}_3\mathrm{HM}}; \right]$

and the sine of the angle ${}_3\mathrm{HCP}[=\varphi]$ of which the arc ${}_3$HP is the complement of the latitude A${}_3$H,

or, ${}_2q_3q$ is to ${}_2d_3l$, as C${}_3$H is to ${}_3$HM;

$\left[i.e.\ \frac{{}_2q_3q}{{}_2d_3l} = \frac{\mathrm{C}_3\mathrm{H}}{{}_3\mathrm{HM}} = \frac{\mathrm{CE}}{\mathrm{AC}} = \frac{n}{1} = \frac{\sec h}{1}; \right]$see Fig.III.



or as CE the secant of the angle of latitude to the whole sine AC [, from triangle ACE]. The latitude or the arc of the meridian AH shall be h, and ${}_2$H${}_3$H will be dh. Now the secant CE shall be n, and ${}_2d_3l$ will be bdh, and ${}_2q_3q$ will be $bndh$, and the total part of the equator A${}_3q$ will be equal to $b\int ndh$, and $\int ndh$ is the area bounded by the secant applied to the arc. Now with the right angle CEN drawn, EN itself shall cross CA at N, and with the elementary part ${}_3$HQ taken of ${}_3$HM, and QF drawn normally from Q to the circle. Then on account of similar triangles, truly the ordinary triangle NEC and the unassignable characteristic triangle ${}_3$HQF [are similar], and the rectangle CE by ${}_3$HF or ndh will be equal to the rectangle CN by QF.

[$\frac{\mathrm{CN}}{\mathrm{CE}} = \frac{{}_3\mathrm{HF}}{\mathrm{QF}} = \frac{ndh}{\mathrm{QF}}$] If now the sine of the latitude CM shall become e, QF will be de, & CN or MV (taken at M${}_3$H continued) will be found to be $1 : 1 - ee$:

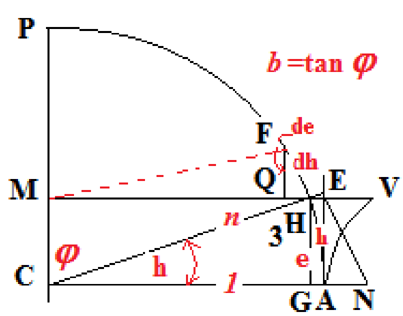


Fig. III

[for $\frac{\mathrm{CE}}{\mathrm{CN}} = \cos h = \frac{\mathrm{CA}}{\mathrm{CE}} \therefore \frac{\mathrm{CE}}{\mathrm{CN}} \times \frac{\mathrm{CA}}{\mathrm{CE}} = \cos^2 h; \therefore \mathrm{CN} = \frac{1}{1-e^2}.$]

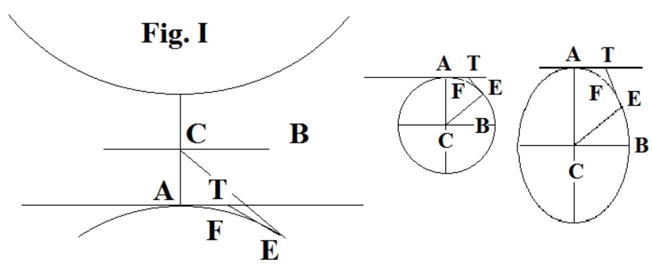
and with the curve drawn through AVV, the area ACMVA will become $\int ndh$, or equal to $\int de:\overline{1-ee}$ and $b\int de:\overline{1-ee}$, or A_3q will be $\frac{b.e}{1}+\frac{b.e^3}{3}+\frac{b.e^5}{5}$ etc., the arc of the equator between A (the starting point of the rhombic curve A_3l at the equator) and the meridian P_3l_3q to which it arrives, the intercept *e* put to be the sine of the final latitude $_3l$ and *b* shall be the number which shall be to one, as the tangent of the constant angle of the rhombus to the meridian is to the total sine. From which if $_1q_3q$ may be sought, the difference of the length of the two points $_1l$ & $_3l$ is taken from the rhombic curve $_1l_2l_3l$ on account of the difference of latitude $_IH_3H$ of the same; it is necessary only to find A_1q and A_3q and the difference will be $_1q_3q$; and thus if the sine of the latitude of the point $_3l$ shall become *e*, and the sine of the point $_1l$ shall be (*e*), it is needed only to multiply $\frac{e-(e)}{1}+\frac{e^3-(e)^3}{3}+\frac{e^5-(e)^5}{5}$ etc. by *b*, the tangent of the angle of the rhombus to the meridian, on putting the whole sine to be one: and the product will be the difference of the longitudes sought. [At this time there was no way of indicating the limits of an integral.] Finally, from the above matter concerning logarithms, it is reduced according to the manner of *art*. 5, *prop*. 4 of our paper *de Resistantia Media*, the differences of the lengths of the points $_3l$ and $_1l$ will be as the logarithms of the ratio $\overline{1+e}:\overline{1-e}$ to $\overline{1+(e)}:\overline{1-(e)}$. With the radius of the sphere put to be one, and the sine of the latitude of the said points to be *e* and (*e*) respectively. Now the experienced practitioner will be led by these rules. Just as if you seek the rhombic line from a given difference of longitude and latitude of the places or the angle of the rhombic curve leading from one place to the other: For the tangent of the angle which the rhombic sought makes with the meridian, is to the whole sine, as the arc of the difference of the longitude is to the hyperbolic logarithm of the said ratio, or to $\frac{e-(e)}{1}+\frac{e^3-(e)^3}{3}+\frac{e^5-(e)^5}{5}$ etc. Because if the meridians may be projected by parallel right lines onto the tangent plane of the sphere, because with due caution used conveniently generally it can be satisfied with reasonable exactness, then also the rhombic curves will be straight lines. If now we may project steps of longitude, and parts of these with equal intervals, it will be required t assume unequal steps of latitude ; and thus indeed to a map being constructed geometrically, so that with all the meridians drawn obliquely freely with the secant, the latitudes may have points of intersection, so that from what has been said it is apparent the numbers, of such a kind as $\overline{1+e}:\overline{1-e}$ intersect in a geometric progression; that if indeed one line may be put in place, all will be able to be established. From which by comparison with the numbers of the scale of the latitude, it will facilitate measurements on the map from some true right line I may draw in that, or the magnitude of the given rhombus. If others may be joined to these maps, where the parts of the sphere are projected from the centre onto tangent planes and all the arcs of the great circles, thus the shortest paths may be shown by right lines, generally true enough in practice to be completely satisfactory.

O.V.E. QUADRATURA ARITHMETICA
communis Sectionum Conicarum quae centrum habent, indeque ducta Trigonometria canonica ad quantumcunque in numeris exactitudinem a Tabularum necessitate liberata: cum usu speciali ad lineam Rhomborum nauticam, aptatumque illi planisphaerium.

Ac. Erud. Apr. 1691

Iam anno 1675 compositum habebam Opusculum Quadraturae Arithmeticae amicis ab illo tempore lectum, sed quod materia sub manibus crescente limare ad editionem non vacavit, postquam aliae occupationes supervenere : praesertim cum nunc prolixius exponere vulgari more, quae Analytis nostra nova paucis exhibet, non satis pretium opera: videatur. Interim insignes quidam Mathematici, quibus veritas primariae nostrae propositionis dudum in his Actis publicatae innotuit, pro humanitate sua nostri qualiscunque inventi candide meminere. Quos inter *Ill. Hugenius* etiam analogum aliquid in Hyperbola elegnater adjecit, a nostri olim schediasmatis analogia diversum. Ut enim nos dederamus seriem $\frac{1}{1}t-\frac{1}{3}t^{3}+\frac{1}{5}t^{5}-\text{\&c.}$ per circulum; ita ipse $\frac{1}{1}t+\frac{1}{3}t^{3}+\frac{1}{5}t^{5}+\text{\&c.}$ per hyperbola primariam exhiberi notavit, de quo adde dicta ad schediasma hic praecedens. Et sane etiam in Opusculo nostra inedito nec ipsi viso, inter alias propositiones una continebatur satis memorabilis ob generalitatem, ambasque illas & plura complexa: *Sectorem, curva conica a vertice incipiente, & radiis ex centro eductis, comprehensum, arithmetice quadrare.* AT, portio rectae in vertice tangentis, comprehensa inter verticem A, & T occursum tangentis alterius extremi vocetur *t,* & CB semiaxis conjugatus (seu recta, quae potest rectangulum sub dimidiis lateribus recto & transverso) sit unitas, erit sector CAFEC aequalis rectangulo comprehenso sub AC semilatere transverso, & recta, cujus longitudo sit $\frac{1}{1}t+\frac{1}{3}t^{3}+\frac{1}{5}t^{5}+\text{\&c.}$ Ita exprimitur non solum area sectoris Circularis aut sectoris Hyperbolae primariae aequilaterae cum angulus Asymptotarum est rectus, sed & alterius sectoris Elliptici aut Hyperbolici cujuscunque. Caeterum ex seriebus infinitis a me aliisque ut *Mercatore, Newtono, Gregorio* exhibitis, sequitur Trigonometriae Canonicae sine Tabulis praxis quantum libet exacta. Neque enim semper Tabulas per maria & tertas circumferre in potestate est. Nempe sit radius unitas, arcus *A* tangens *t,* sinus rectus *s,* sinus versus *v,* logarithmus *l,* numerus $1+n$ (logaritho ipsius unitatis seu *l* existente 0), fiet,

Fig. I



(1)

$a = \frac{1}{1}t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \frac{1}{9}t^9$etc.

(2)

$s = a - \frac{a^3}{1.2.3} + \frac{a^5}{1.2.3.4.5} - \frac{a^7}{1.2.3.4.5.6.7}$ etc.

id est, $a - \frac{a^3}{6} + \frac{a^5}{120}$ etc.

(3)

$v = \frac{a^2}{1.2} - \frac{a^4}{1.2.3.4} + \frac{a^6}{1.2.3.4.5.6} - \frac{a^8}{1.2.3.4.5.6.7.8}$ etc.

(4)

$l = \frac{1}{1}n - \frac{1}{2}n^2 + \frac{1}{3}n^3 - \frac{1}{4}n^4 + \frac{1}{5}n^5$etc.

(5)

$n = \frac{l}{1} + \frac{l^2}{1.2} + \frac{l^3}{1.2.3} + \frac{l^4}{1.2.3.4} + \frac{l^5}{1.2.3.4.5}$ etc.

Semper autem quantitas; cujus potentiae in serie infinita adhibentur, debet esse minor unitate, ut in progressu fiant quantumvis parvae. Hujusmodi series dari possunt plures, & efficere etiam per series licet, ut ex arca dentur sinus et tangentes artificiales, seu logarithmici (nonsuppositis naturalibus) & vicissim arcus ex ipsis : Sed placuit eas tantum adscribere series, quae tam simplicis sunt compositionis, ut facillime memoria retineri et ubivis defectum librorum: ac tabularum supplere possint. Itaque unam tantum ob suam simplicitatem et quia hujus schediasmatis occasionem praebuit, addo, si sinus complementi sint c, logarithmos sinuum rectorum, vel potius (quod eodem redit) reciprocorum ab his sinubus,

(6)

fore = ut $\frac{c^2}{2} + \frac{c^4}{4} + \frac{c^6}{6} + \frac{c^8}{8}$ etc.

quemadmodum sequitur ex his, quae innuimus in schediasmate de Resistentia Medii Act. Januarii 1689 pag. 4 artic. 5 prop. 6; unde rursus etiam pendere a quadratura Hyperbolae. Nec abludunt quae dederat Nic. Mercator, unde ad meum Circuli Tetragonismum secundo mensae primi anni horum Actorum editum duxeram Analogiam cum Hyperbola non inelegantem. lnveneram scilicet circulum esse ad quadratum circumscriptum ut $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$etc. ad unitatem, seu circulum esse ad quadratum inscriptum ut $\frac{1}{4-1} + \frac{1}{36-1} + \frac{1}{100-1}$etc. ubi numeri 4, 36, 100 etc. sunt quadrati a paribus quaternario differentibus 2, 6, 10 , etc. Similiter ex supradictis, cum Numerus cujus logarithmus quaeritur $1+$ x est 2, tunc x est 1, adeoque $\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$ etc. est Logarithmus Hyperbolicus binarii. Eadem series facit $\frac{4}{9-1} + \frac{4}{49-1} + \frac{4}{121-1}$etc. (nam $\frac{4}{9-1}$ est aeq. $\frac{1}{1} - \frac{1}{2}$, et $\frac{4}{49-1}$ est aeq. $\frac{1}{3} - \frac{1}{4}$, et ita porro), ergo logarithmus Hyperbolicus binarii est ad unitatum, ut $\frac{1}{9-1} + \frac{1}{49-1}$ etc. est ad $\frac{1}{4}$, ubi numeri 9, 49, 121, etc. sunt quadrati a 3, 7, 11 etc. qui sunt impares unitate excedentes supra dictos pares quaternario differentes : unde origo patet analogia olim a nobis exhibitae in his Actis, ut dictum est. Esse autem

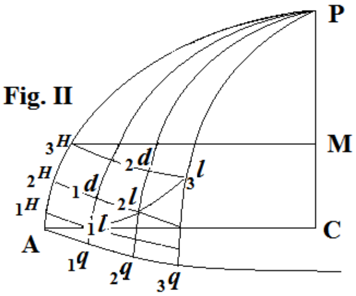
$\frac{1}{2}cc+\frac{1}{4}c^4+\frac{1}{6}c^6$ etc. log. de 1: $\sqrt{1-cc}$, sic demonstrator :

log. de $1+c=\frac{1}{1}c-\frac{1}{2}cc+\frac{1}{3}c^3$ etc. et log. de $1-c=-\frac{1}{1}c-\frac{1}{2}cc-\frac{1}{3}c^3$ etc. utrumque per aeq. 4 ; ergo log. $\overline{1+c}+$ log. $\overline{1-c}$ id est log. $\overline{1-cc}=-\frac{2}{2}cc-\frac{2}{4}cc-\frac{2}{6}c^3$ etc. et proinde

$\frac{1}{2}$log. $\overline{1-cc}$ id est log.$\sqrt{1-cc}$ $=-\frac{1}{2}cc-\frac{1}{4}cc-\frac{1}{6}c^3$ etc.

Sed quo magis horum usus appareat, ostendere operae pretium erit, eundem calculum prodesse ad lineam Rhombicam in superficie sphaerica a navigantibus descriptam recte aestimandam atque in plano projiciendam, quae vulgo parum accurate tractantur. Rem usu amplissimam paucis explicemus. Sit Polus P, (fig. 29) Aequator A*qq*, Meridiani PA, P*q*, etc., Linea Rhombica, $A_1l_2l_3l$ etc. quae describitur quamdiu eadem plaga seu venti rhombus tenetur. Per puncta *l*, *l* ducantur paralleli H*l*, nempe ${}_1H\,{}_1l$, ${}_2H\,{}_1d\,{}_2l$, ${}_3H\,{}_2d\,{}_3l$, etc. Quam si jam punctorum *q*, *q* intervalla sint incomparabiliter parva, portiones arcuum quippe inassignabiles erunt pro rectis, & triangula rectis, & triangula ${}_1l_1d_2l_2d_3l$, etc. erunt similia, ob angulum linea Rhombicae semper eundem ad loci meridianum. Ergo ${}_1l_3l$ *quantitas Rhombicae percursae seu itineris in eodem rombo, est ad* ${}_1H_3H$ *, differentiam latitudinis extremorum, ut sinus totus ad anguli rhombici sinum*. Itaque ex dato angulo rhombico & differentia latitudinum datur quantitas itineris, vel contra. Huc usque res pervulgata est, sed ut ex iisdem differentia longitudinum calculo aestimetur, negotium est Geometriae transcendentis, quam pauci recte tractaverunt. Id ergo supplere nostrae methodi est. Radius seu sinus totus sit unitas, & tangens anguli rhombici constantis sit *b* : patet esse

Fig. II



${}_1d_2l$ ad ${}_1l_2l$ seu ad ${}_1H_2H$, vel ${}_2d_3l$ ad ${}_2l_2d$ seu ad ${}_2H_3H$ ut *b* ad 1 Sed ${}_2q_3q$ est ad ${}_2d_3l$, ut AC (sinus totus seu sphaerae radius) ad ${}_3$HM, sinum anguli ${}_3$HCP cuius arcus ${}_3$HP est latitudinis A_3H complementum, seu ${}_2q_3q$ ad ${}_2d_3l$, ut C_3H ad ${}_3$HM, seu ut CE secans anguli latitudinis ad AC sinum totum. Latitudo seu arcus meridiani AH sit *h*, & ${}_2H_3H$ erit *dh*. Jam CE secans sit *n*, & ${}_1d_3l$ erit *bdh*, & ${}_2q_3q$ erit *bndh*, & portio tota aequa totis A_3q erit $b\int ndh$, et $\int ndh$ est area secantium arcui applicatorum. Jam angulo CEN recto educta EN ipsi CA occurrat in N, sumtaque ${}_3$HQ, particula ipsius ${}_3$HM & normaliter ex Q educta ad circulum QF ob triangula similia, nempe ordinarium NEC & characteristicum inassignabile 3HQF erit rectangulum CE in ${}_3$HF seu *ndh* aequale rectangulo CN in QF. Si

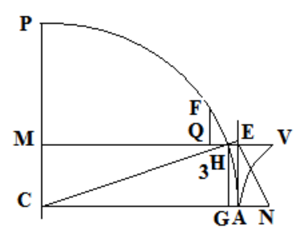


Fig. III

jam CM sinus latitudinis fit *e*, QF erit *de*, & CN vel MV (sumta in M_3H continuata) reperietur esse $1:1-ee$ ductaque linea per AVV, erit $\int ndh$ seu area ACMVA aequ. $\int de:\overline{1-ee}$ & $b\int de:\overline{1-ee}$, seu $\frac{b.e}{1}+\frac{b.e^3}{3}+\frac{b.e^5}{5}$ etc. erit A_3q, arcus aequatoris inter A (initium lineae rhombicae A_3l in aequatore) & meridianum P_3l_3q ad quem pervenit, interceptus *e* posito esse sinum latitudinis extremi $_3l$ & *b* esse numerum qui sit ad unitatem, ut tangens Constantis anguli Rhombicae cum meridiano est ad sinum totum. Unde si quaeratur $_1q_3q$, differentia longitudinis duorum rhombicae linea: $_1l_2l_3l$ punctorum $_1l$ & $_3l$, ex data, $_IH_3H$, differentia latitudinis eorundem; oportet tantum invenire A_1q, & A_3q eritque differentia $_1q_3q$; adeoque si sinus latitudinis puncti $_3l$ fit *e*, & puncti $_1l$ sit (*e*) tantum opus $\frac{e-(e)}{1}+\frac{e^3-(e)^3}{3}+\frac{e^5-(e)^5}{5}$ etc. multiplicare per *b* tangentem anguli Rhombici ad meridianum, posito sinum totum esse unitatem: & productum erit differentia longitudinis quaesita. Denique ex superioribus re ad logarithmos redacta ad modum *artic. 5, prop. 4 nostri schediasmatis de Resistantia Media*, erunt differentia Iongitudinum, punctorum $_3l$ & $_1l$, ut logarithmi rationis $\overline{1+e}:\overline{1-e}$ ad $\overline{1+(e)}:\overline{1-(e)}$. Posito radium sphaerae esse unitatem, & sinus latitudinum dictorum punctorum respective esse *e*, & (*e*). Ex his jam canones practicos facile ducet peritus. Veluti si data differentia longitunus & latitudinis locorum quaeras rhombum seu angulum rhombicae lineae ab uno ad alium ducentis: Nam tangens anguli quem Rhombus quasitus facit ad meridianum, est ad sinum totum, ut arcus differentiae longitudinem est as Logarithmum hyperbolicum dictae rationis, seu ad $\frac{e}{1}+\frac{e^3}{3}+\frac{e^5}{5}$ etc. Quod si meridiani in planisphario projiciantur rectis parallelis, quod cautionibus debitis adhibitis plerumque commode satis fieri potest salva exactitudine, tunc etiam lineae Rhombicae erunt rectae. Si jam gradus longitudinis horumque partes projiciamus aequalibus intervallis, oportet gradus latitudinis assumi inaequales ; & sic quidem ad mappam Geometrice construendam, ut ducta ad libitum recta omnes meridianos oblique secante, latitudines punctorum intersectionis habeant, ut ex dictis patet, numeros, qualis est $\overline{1+e}:\overline{1-e}$; geometrica progressione incedentes; id enim si una recta praestet, praestabunt omnes. Unde comparando cum numeris scalae latitudinis facillimum erit in ipsa mappa mensurare ex vero rectam quamvis in ea ducibilem, seu quantitatem Rhombicae data. His mappis si alias jungas, ubi sphaericae superficiei partes projiciuntur ex centro in plana tangentia omnesque arcus circulorum magnorum, adeoque viae brevissimae exhibentur rectis, pleraque in praxi probe satis praestari possunt.