No. XVIII.

New Considerations….

…regarding the nature of the Angle of contact and of osculation, and with the practical use of these in Mathematics, towards substituting easier figures in place of more difficult ones.

From Actis Erud. Lips. June 1686.

With regard to the infinitely small parts of each curve, not only the direction can be considered, or the declination or inclination, as has been done hitherto, but also the change of the direction or curvature can be considered; just as the geometers have measured the direction of curves, this is with the right line tangent, the simplest line at the same point having the same direction ; thus I am going to measure the turning of the curve with the simplest curve through the same point, not only going in the same direction but also having the same curvature : this is done by a circle to the proposed curve, not only being a tangent to the curve, but further by osculating, which term I will explain at once. Moreover just as a right line is the most suitable for determining the direction, because its direction is the same everywhere; thus a circle is the most suitable curve for determining curvature, because the curvature of the one circle is the same everywhere. But the circle which makes the smallest angle of contact with that proposed curve in the same plane is said by me to osculate at the point proposed. Indeed from the infinity of circles tangent to a curve, where its concavity for the same sense at a proposed point, one can always determine a single circle which is most similar to the curve there; creeping along the furthest with that as it were, that is, as I may say geometrically, thus to approach that curve, so that between itself and the proposed curve no other circular arc shall be able to be described meeting the curve at the proposed point. And this minimum angle of contact of the circle to the proposed curve, I call the angle of osculation [*osculare* in Latin means 'to kiss' : you may recall Soddy's verses : *The Kiss Precise*, in Coxeter's *Geometry*], just as the smallest angle of the right line to the curve at the point is called the angle of contact. So that indeed just as no other right line can fall between the right line and the curve making the angle of contact ; thus no other arc of a circle can fall between the circle and the curve making the angle of osculation. But so that it may be had, and the way of finding the circle of osculation is known, just as the tangents may be found by equations, or which have two equal roots or two points on a right line crossing the curve at two coinciding points, and the opposite curvature from three equal roots [*i.e.* an inflection point] ; thus the osculation of the circle, or of any other given curve, may be found either by four equal roots, or by two contact points merging into one. And just as two curves which have the same right line tangent, may touch each other ; thus these, which osculate with the same circle, osculate with each other. And thus so that any curve crossing the same curve itself may be considered to make the same common or rectilinear angle, which the right tangents make at the same crossing point, because the difference lies in the angle of contact, which is infinitely small with respect to the rectilinear right lines, and indeed is nothing; thus when two right tangents of two crossing curved lines themselves coincide, or when two curves touch each other, then the curve crossing the other curve may be considered to make the same contact angle at the crossing which the osculating circles make at their crossing point, because the difference lies between the angle of osculation with respect to the angle of contact of the two circles, which is indefinitely small - indeed nothing. From which it can be understood the common angle either of two right lines, the contact angle of two circles, and the angle of osculation (of the first order) are found in a certain way, just as it can be found for a body, a surface, or a line. For not only is a curve lesser than some surface, but neither indeed is it a part of a surface, but only a certain minimum or limiting part. Because if three contact points coincide, or four, or more, (with six roots, or eight, or with even more equalities present,) osculations arise of the second, third, or higher order than hitherto, accordingly so much more perfect than contained in an osculation of the first order, and with a common contact angle. Again it is not possible for a circle touching to a straight line to osculate [for the radius of curvature is infinite] ; and if a circle may osculate with another circle, they cannot be different, but coincide. Concerning the other cases, any curve can have an osculating circle in the same plane, and generally, so that it may be able to establish wherever the point of contact may be, or what the order of the osculation of the curve may be by which it is joined to the curve, it is required to consider in how many points it shall cut that curve. This again has significant use in practice: as indeed from a consideration of what the angle becomes, the curves have the same inclination or direction as the right tangents, which has led to significant consequences in mechanics, catoptrics and dioptrics ; for if a body may be carried by a composite motion, its direction is along the right tangent of the curve as described, and if it may be left to itself the motion shall continue along the tangent, and the incident direction makes the same angle to the receiving surface, as it would make to that tangent plane: thus also from a consideration of the curves of osculation, significant practices can be introduced. For if a certain curve or figure shall be discovered, having a distinctive useful property, but which shall be difficult to fashion into a material, either by turning or by some other means, it will be allowed to substitute for its arc (clearly not exceedingly great, but sufficient in practice) the arc as it were of another curve easier to be described than that, as if the arc coincident with the other described with greater ease were that which is permitted most perfectly of the tangents or osculating circles, but of the largest circle which generally is easier to be described. And hence now it arises, because in practice in catoptrics and dioptrics, a circle is to be substituted for a parabola, hyperbola , or an ellipse, having as it were an imitation of their focus. For a circle whose diameter is equal to the parameter of a conic section, and whose centre is taken on the axis within the curve, moreover the circumference passes through the vertex, osculating at the vertex of the conic section, and thus with the arc assumed to be small, large enough so that not to differ sensibly from that. Which is the reason, why the focus of a circular concave mirror departs a quarter part of the diameter from the mirror, because the focus of the parabola is a quarter part of the parameter from the vertex, and both the focus of the parabola and of the osculating circle coincide.

[The focal point for the equivalent circular concave mirror with the same curvature as the parabola at the vertex is half the radius for small angles of incidence ; from the standard equation of the parabola $y = 4ax$, the radius of curvature at the point (x, y) is found from

$$y^2 = 4ax; 2yy' = 4a; y' = \frac{2a}{y}; y'' = -\frac{2a}{y^2}y' = -\frac{2a}{y^2}\cdot\frac{2a}{y} = -\frac{4a^2}{y^3};$$

$$\rho = \frac{\left(1+y'^2\right)^{\frac{3}{2}}}{y''} = -\frac{\left(y^2+4a^2\right)^{\frac{3}{2}}}{\frac{4a^2}{y^3}\cdot y^3} = -\frac{\left(y^2+4a^2\right)^{\frac{3}{2}}}{4a^2} = -\frac{\left(4ax+4a^2\right)^{\frac{3}{2}}}{4a^2}; \text{when } x = 0 \text{ , } \rho = 2a.$$

As a is the focal length of the parabola, then this disagrees with Leibniz's statement, as stated in the *Naissance.*]

Likewise for all of the other curves, for the kinds of properties used, there is a place for the method arising. Anyone of intelligence cannot fail to see, how much life the geometrical subtleties bring together on being transferred into use. Truly for us to have uncovered the approach, lest perhaps it be lost, and now indeed this contemplation gave satisfaction. Nor will it be unpleasant to consider how thus at last the controversy of the geometers about the contact angle, which has considered by some to be void of interest, will have become solid truths, and to be useful in the future.

No. XVIII.

MEDlTATIO NOVA

De natura Anguli contactus & osculi, horumque usu in practica Mathesi, ad figuras faciliores succedaneas difficilioribus substituendas.

Ex Actis Erud. Lips. 1686.

In lineae cujusque partibus infinite exiguis, considerari potest non tantum directio, sive declivitas, aut inclinatio, ut hactenus factum est, sed & mutatio directionis , sive flexura ; & quemadmodum linearum directionem mensi sunt Geometrae, simplicissima linea in eodem puncto eandem directionem habente, hoc est recta tangente; ita ego flexuram lineae metior simplicissima linea in eodem puncto, non tantum directionem eamdem , sed & eamdem flexuram habente, hoc est circulo curvam propositam non tantum tangente, sed &, quod amplius est ; osculante , quod mox explicabo. Est autem ut recta linea aptissima ad determinandam directionem, quia eadem ubique ejus directio est ; ita circulus aptissimus ad determinandam flexuram, quia ubique eadem unius circuli est flexura. Circulus autem ille lineam propositam ejusdem plani in puncto proposito osculari a me dicitur, qui minimum cum ea facit angulum contactus. Ex infinitis enim circulis lineam , ubi ad easdem partes cava est, tangentibus in proposito puncto , semper determinari potest unus, qui maxime ibi lineae assimilatur, & cum ea longissime quasi repit , hoc est, ut Geometrice loquar, ita ad eam accedit, ut inter ipsum, & curvam propositam nullus alius arcus circuli , curvae in puncto proposito occurrens describi potest. Et hunc minimum angulum contactus circuli ad lineam propositam, voco angulum osculi, uti minimus angulus rectae ad lineam vocatur angulus contactus. Ut enim inter rectam , & lineam, angulum contactus facientes ; nulla cadere potest recta: ita inter circulum, & lineam, angulum osculi facientes, nullus cadere potest arcus circuli. Ut autem habeatur & modus inveniendi circulum osculantem, sciendum est, quemadmodum tangentes inveniuntur per aequationes , quae habent duas radices aequales , seu duos occursus coincidentes, & flexus contrarii per tres radices aequales ; ita circuli vel aliae quavis lineae datam osculantes inveniuntur per quatuor radices aequales, seu per duos contactus per unum coincidentes. Et quemadmodum duae lineae quae eamdem habent rectam tangentem , se tangunt ; ita eae, quas idem osculatur circulus, se osculantur. Itaque ut linea quaevis eumdem ad lineam sibi occurrentem censetur facere angulum communem , seu rectilineum , quem faciunt in puncto occursus earum tangentes rectae , quia differentia consistit in angulo contactus , qui respectu anguli rectilinei est infinite exiguus, imo nullus; ita quando duae rectae tangentes duarum linearum curvarum sibi occurrentium coincidunt , seu quando duae lineae se tangunt , tunc linea ad lineam occurrentem eumdem censetur facere angulum, contactus, quem faciunt in puncto occursus earum osculantes circuli, quia differentia consistit in angulo osculi, qui respectu anguli contactus duorum circulorum est nfinite parvus, imo nullus. Ex quo intelligi potest angulum communem seu duarum rectarum, angulum contactus duorum circulorum, & angulum osculi (primi gradus) quodammodo se habere, ut corpus, superficiem, & lineam. Non tantum enim linea est minor quavis superficie, sed & ne quidem pars est superficiei, sed tantummodo minimum quoddam , sive extremum. Quod si tres contactus coincidant , aut quatuor, aut plures, (radicibus sex, aut octo, aut pluribus existentibus aequalibus,) oriuntur osculationes secundi, tertiive gradûs , aut adhuc altiores, in tantum perfectiores osculo primi gradus, in quantum prima osculatio perfectiorem contactum continet, quam contactus communis. Porro circulus rectam tangere potest, osculari non potest ; si circulus circulum osculetur , non erunt diversi , sed coincident. De caetero omnem lineam ejusdem plani osculari poterit circulus, & in genere, ut sciri possit , quonam contactus , vel osculi gradu linea lineae conjungi possit , considerandum est, in quot punctis possit eam secare. Haec Porro insignem habent usum in praxi. Ut enim ex consideratione, quod idem fit angulus, eadem inclinatio, vel directio linearum, quae est rectarum tangentium, insignes consequentiae in mechanicis, catoptricis & dioptricts ductae sunt ; nam si corpus motu composito feratur, directio ejus est in recta tangente, lineae , quam describit , & si sibi relinquatur continuat motum in tangente, & radius incidens eumdem angulum facit ad superficiem excipientem, quem faceret ad planum eam tangens: ita ex consideratione quoque linearum osculantium insignes praxes duci possunt. Si enim linea, aut figura egregiam quamdam, atque utilem habens proprietatem inventa sit, sed quam sive torno, sive alia ratione in materiam introducere sit difficile, licebit pro arcu ejus (scilicet non nimis magno, tamen ad praxin suffecturo) substituere arcum quasi coincidentem lineae alterius descriptu facilioris eam quam perfectissime licet, tangentis, sive osculantis, maxime autem circuli, qui omnium descriptu est facillimus. Et hinc jam oritur, quod in praxi catoptrica , & dioptrica circulus esse succedaneum parabolae , hyperbolae , aut ellipseos , suosque ad earum imitationem habet quasi focos. Nam circulus cujus diameter aequatur parametro sectionis conicae, & cujus centrum in axe intra curvam sumitur, circumferentia autem per verticem transit, sectionem conicam in vertice osculatur, adeoque assumpto arcu, quantum satis esse parvo, ab ea non differe ad sensum. Quae causa est, cur focus speculi concavi circularis absit a speculo quarta parte diametri, quia focus parabolae a vertice abest quarta parte parametri, & focus parabolae, atque circuli osculantis, coincidunt. Eadem in omni alio linearum, & utilium proprietatum genere , pro re nata locum habent. Quae quantum conferant ad subtilitates Geometricas in usum vitae transferendas, nemo talium intelligens non videt. Nobis vero aditum aperuisse, ne forte perit et haec meditatio, nunc quidem satis fuit. Nec injucundum erit considerare, quomodo ita tandem controversia Geometrarum de angulo contactus, quae plerisque inanis visa est, in veritates desierit solidas, & profuturas.