No. XIX.

CONCERNING A RECONDITE GEOMETRY AND THE ANALYSIS OF THE INDIVISIBLE AND THE INFINITE.

From Acta Eruditorum Lips. June, 1686.

Since becoming aware that several articles which I published in these *Acta*, relating to the progress of geometry, have been approved strongly by certain scholars, why not, [I ask], gradually implement these into use, yet certain ones have not been received so well, by a fault in the writing perhaps, or on account of some other cause; thus I have considered it to be worth the effort for this article to be added here, which may be able to illustrate the former ideas. Without doubt I have accepted *John Craig's* tract on the dimensions of figures [*Methodus figurarum lineis rectis & curvis comprehensarum quadraturas determinandi*], published last year in London [1685], from which certainly it is evident, the author is to be viewed with admiration regarding the advances he has made in higher geometry. Indeed he has approved strongly a distinction forced on me a number of times, between the dimensions [*i.e.* the ability to be rectified or squared] of general figures and those of special ones, as he says in the optimum manner on page1, he presents rightly an observation to have been made recently of geometers, of trying to prove the impossibility of a quadrature, and by ignoring this distinction they have made many paralogisms [*i.e.* flawed arguments. This distinction in the case of lunes had been pointed out by Leibniz in *AE*11.] Also the figures recognised by me to be transcendental, which generally are rejected by geometers, have been published by me in the *Acta* of October 1684 p. 26. ….*Methodum quoque Tangentium* (*AE*13), as geometers for the most part praised pages 27 and 29 as the most outstanding, and its help may have aided greatly the method of dimensions, providing the best cure against irrational thinking. Yet there are some observations, about which I need to advise *John Craig* and other geometers, which I have considered neither to be superfluous nor unpleasant. For I do not know how it happened, nor how the author might be believed, who wrote an article in an off-hand manner [schediasma] in the *Acta* of May 1684. p. 233, to have changed his mind, from the start of the *Acta*. of October 1683, where he had proposed to give the impossibility of all sorts of demonstrations of squaring the circle, after considering in May of the following year that the impossibility of squaring the circle had not yet been fully established in particular cases. However since the schediasma of October, 1683 came from E. T. [*i.e.* E.W. Tschirnhaus].Truly the schediasma of May1884 came from me [*AE*11 in this series of translations]: a part of the same which I claim myself, lest at some time I may be accused of usurping the property of others, and a part of which the use may be attributed to *E.T.*, and I was dissenting to that part in a friendly manner. For that author was thinking it followed from the impossibility of indefinite squaring [for all cases] entailed that of all determined quadratures and of their definite impossibility [for certain cases] : my true constant belief was not-prevailing (now moreover vindicated, by the squaring arithmetic I had produced in the second month of the first year of the *Acta*, surely 1682.) from that to this consequence. In order that I might prove which, I have brought an example of a certain figure into the *Acta* of May 1684, which undertakes a special squaring, (because I am able to show) it is not truly general, as from the theorem itself of *E.T.* I had undertaken to show there ; though quickly, and I may have strayed to some extent of establishing the matter in the sure manner, which I will explain and correct later. To which *E.T.* responded in private, that his method itself had not been drawn from mine, but that he had come upon it from his own investigation, and because there might be concern about some objection, that logical consequence from indefinite to definite squaring itself could be demonstrated, by which his method would stand out chiefly; in truth my counter example depended on a faulty calculation. I was willing to admit (in *Acta.* December 1684. p. 587.) if he were able to demonstrate that consequence, it would be making something available which hitherto no one had done; yet I was always uncertain, and with the calculation corrected after I had strengthened my counter example, more about which soon. But nevertheless I have had this method for ten years or more, since we were together in Paris, and we were discussing geometrical matters most often, in which time that clearly it fell out in different ways, for me truly then and now to be most familiar with the general equations to be used for expressing the nature of the line sought, being determined in the progress of the calculation, which stands at the heart of the method, such as I had come upon nowhere else: yet I have never seen the equal of its beauty, and by attributing so much to its nature, that I may believe easily either this may have fallen into place by itself, or perhaps not to have remembered more [by me], by what previous occasion the seeds of such a contemplation were planted : especially since I know it has excelled in solving even more difficult problems, and many outstanding and great instances can be expected from its ingenuity.

[Here L. is commenting rather regretfully on the transformation now called after Tschirnhaus, by which the coefficients of certain leading terms in a polynomial can be reduced to zero; clearly he is unsure who invented it, either himself or his friend, or perhaps together. These matters have little to do with the title of this article, but have been included as clearly they have been a worry to Leibniz.]

Truly because the error in the counter-example has been admitted, as I have said, in the instance above, because *John Craig* has placed that before *E.T.* (to whom that article was attributed) as a proof, I suppose, in order that he could refute the indefinite [i.e. general] method, thus I must correct the calculation. P. 239 may be examined of the *Acta* for the year1684, whereby bringing together the equation $4zz-8hz$, &c. with the equation $bzz+caz$ &c. where *z* must be absent from the final terms in the equation, put in place outside the fraction to be multiplied by the numerator of the fraction, before a comparison can be established, so that in each fraction all the terms are without the letter *z,* may be taken together as a single fraction. And $b=1$ always can be put in place, and because in the first equation the term *xz* plainly is missing, in the latter there becomes $d=0$, and the first equation may be divided, either by 4 given, and in the latter, or by the equation substituted, both the numerator from the fraction, as well as the denominator, may be divided by *g* : thus both the term *zz* in both places, as well as the term *zz* in the numerator of the fraction will agree in both places. With everything being prepared, on account of the term *z,* there becomes $c=2h:a$; from x^4 there becomes $g=1:16$, or $\frac{1}{16}$ *;* on account of x^3 there becomes $f=-1:6a$; for *x* in the numerator there will be

$f = -h:8a$. Therefore there shall be $h = 8:6,$ or $\frac{4}{3}$ which is absurd, as *h* is a given quantity. From other continued absurdities arise from the comparison, for there shall be either *c* or $f = 0$, now the opposite concluded.

Moreover it pleases here, that we may be going to say more useful things, to *open the source of transcendental quantities*, on account of which without doubt certain problems neither shall be planes, solids, or hyper-solids, or of any certain order, but which may transcend every algebraic equation. By the same work we will show the manner, how without any calculation it shall be able to demonstrate, the algebraic square of the circle and of the hyperbola to be impossible. For if that may be given, it may follow by its help, to be able to cut either an angle or a ratio if a logarithm in a given proportion, right line to right line, and that by a single general construction, and therefore the problem of the section of an angle or of the discovery of however many mean proportionals there may be of a certain order, since still for another number of parts of an angle, or of mean proportions, other and still other orders of an algebraic equation may be required, [*e.g.* an equation of the second order is required for a bisection, of the third order for a trisection, etc.] and thus the problem is understood in terms of the number of parts generated, or of just as many means, shall be of an indefinite order, and transcends all algebraic equations. Yet because nevertheless such problems are able to be proposed in geometry, indeed they must be considered amongst the most fundamental, and are to be determined ; and thus it is necessary everywhere, these curves be received into geometry, by which alone they can be constructed ; and by that they can be described exactly by a continuous motion, as is apparent for the cycloid and with similar curves, actually by agreeing not to be of a mechanical but geometrical nature; they leave especially the usefulness of their common geometrical lines (if you remove the right line and the circle) many miles behind, and they may have the greatest outstanding properties, which are capable of geometrical demonstrations. *Therefore none the less with Descartes' Geometry excluding these, he was at fault just as the ancients, who rejected certain three-dimensional or linear places as being less geometrical*. [Parmentier, in *Naissance du Calcul*… p.135 disagrees with this assessment by Leibniz.]

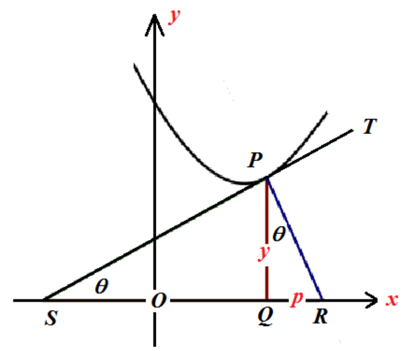
Because also the method of investigating indefinite squaring, or of the impossibilities of these, with me is only a special case (and indeed easier) of a much greater problem, which I call the *Method of the inverse tangent,* in which the greater part of all of transcendental geometry is contained, and which if it can be solved algebraically always, all that might be had is found, and truly hitherto I see nothing satisfying to stand out from that, therefore I may show how none the less it may be resolved, just as indefinite squaring itself. Therefore since before the algebraists assumed letters, or general numbers of the quantities sought, in such transcending problems I have assumed general equations, or indefinites for the lines sought, e.g. the abscissa and ordinates present for *x* and *y*, the equation for the line sought by me is :

$$0 = a + bx + cy + exy + fxx + gyy \ \ \&c. \ ;$$

with the aid of this indefinite equation proposed, actually finite (for it can be determine always, to what extent it may need to rise) I seek the tangent of the line, and what I find, that agrees with the property of the given tangent, I find the value of the assumed letters *a*, *b, c*, &c. and thus I define the equation of the line sought, where still meanwhile a certain arbitrary quantities remain; in which case also innumerable lines can be found, satisfying the question, in which case it happens, that many problems may be considered not to have been defined well enough by hindsight, nor to be in a condition to be solved. The same is evident also for series. Moreover, I have understood much about their calculation, about which at another time. Because if a comparison may not proceed, I proclaim that the line sought is not algebraic, but transcendental.

With which in place so that I may find the *kind of Transcendence* itself (for other transcendental quantities depend on the general division of a ratio, or on logarithms, others on the general division of an angle, or on the arc of a circle, or on some other indefinite questions of greater complexity) thus besides the letters *x* and *y* I take besides a third such as *v*, which indicates a transcendent quantity, and from these three I form a general equation for the curve sought, from which I seek the curve of the tangent, following my method of tangents I published in the October1684 *Acta*, which can handle transcending quantities. Then comparing that which I find with the given property of the tangent curve I find not only the assumed letters *a, b, c,* &c. but also the special nature of the transcending curve. But though it may happen sometimes, that several curves used shall be transcending, and when they shall be of diverse kinds, and the transcendences evident, and with all such proceeding to infinity, yet we are able to content ourselves with much easer and more useful of these ; and it is allowed to use several unusual tricks to understand the calculation, and the problem, as long as it is permitted, is required to be returned in simpler terms, which are not presented here. But with this method applied to squaring, or to finding the quadrature of curves, (in which certainly the property of the tangent always is given) it is not yet apparent, how that may be found, or the indefinite quadrature shall be algebraically impossible, but and how with this impossibility seized upon, the transcending quadrature [Leibniz calls this curve the *quadratrix* in *AE*11, and later by Johann Bernoulli it appears finally to be called the *integral*, which really means the sum, total, or wholeness of a quantity.] shall be able to be found, which hitherto has not been handled. Indeed so that it may not seem to be asserted idly, geometry by this method is to be moved forwards an immense amount beyond the limits set by *Vieta* and *Descartes*. When by this account analysis may be extended to certain and general problems, which are of no certain order, and thus cannot be understood in terms of algebraic equations.

Again because for transcendental problems, wherever dimensions and tangents occur, required to be treated by calculation, scarcely anything more useful, shorter, or more universal can be composed than by my differential calculus or the analysis of the *indivisible* and of the *infinitudes*, of which only as if a small sample or corollary may be contained into that method of Tangents produced by me in the *Acta* of October 1681, and approved so much by *John Craig* ; and *Craig* mistrusted somewhat the more profound matters, and hence on page 29 of his little book has tried to derive Barrow's theorem (so that the sum of the intervals, between the ordinates of a curve taken perpendicular to the axis, and to the axis of the application, is equal to half the square of the final ordinate) in the execution of this he is moved away a little from his aim, which does not surprise me with the new method, and thus I consider myself to be most grateful to him and to others, *if here I may put in place the beginning of a matter of which so wide a use may be apparent.* For thence all the theorems of this kind, and problems, which deservedly were to be admired, these flow with ease, so that now no more do these need to be learned, and shall it be necessary to retain the rest, as for most theorems of common geometry those are required to be learned by heart, which hold some attraction. Therefore I proceed in the aforementioned case. The ordinate shall be *x* and the abscissa *y*, the interval between the perpendicular, and the ordinate, which shall be said to be *p*, at once appears by my method to be $pdy = xdx$, and which *Craig* had observed from that ; with which equation turned into a sum, shall become $\int pdy = \int xdx$. But from these matters, which I have set out in the method of tangents, it is apparent that $d,\frac{1}{2}xx = xdx$, therefore counter wise $\frac{1}{2}xx = \int xdx$ (as indeed the powers, and the roots in common calculations, thus the sum and the difference for us, or $\int$ and d, are reciprocals). Therefore we have $\int pdy = \frac{1}{2}xx$. Q.e.d.



[In the extra diagram, *ST* or *SP* is the tangent to the curve $y = f(x)$ at the point *P*(*x, y*), *PR* is the normal and *QR* the subnormal of length *p*, while *SQ* is the subtangent. From the figure, it follows that $\tan\theta = \frac{dy}{dx} = \frac{p}{y}$; hence $pdx = ydy$ and $\int_0^x pdx = \int_0^y ydy = \frac{1}{2}y^2$].

But I prefer *dx* and the like to be used, as the letters for these, because that *dx* is the modification sought of *x*, and thus it comes about with its help, so that happens only when there is a need for the letter *x*, clearly with its powers, and may enter the calculation with differentials, and the transcending relations may be expressed between *x* and another quantity. By which account also, just as the transcending lines for an equation are set out. For example, let the arc be *a*, the versed sine *x*, there becomes

$a = \int \overline{dx : \sqrt{2x - xx}}$, and if the ordinate of a cycloid shall be *y*, there becomes :
$y = \sqrt{2x - xx} + \int \overline{dx : \sqrt{2x - xx}}$, which equation expresses perfectly the relation between the ordinate *y*, and the abscissa *x*, and from that all the properties of the cycloid can be demonstrated ; and in this manner the analytical calculus can be extended to these curves, which no longer have to be excluded for some reason or another, as which it may be believed incapable of handling: also the interpolations of Wallis, and innumerable others hence may be derived.

What remains, lest I may write excessively or I may seem to be detracting from others, I may say a little to express my opinion about what may be owed chiefly in this kind of geometry, to the most conspicuous mathematicians of our century. In the first place *Galileo* and *Cavalleri* had begun to uncover the most complicated works of *Conon* and *Archimedes*. But the Cavallerian geometry of indivisibles was but the infancy of the rebirth of the science. Great advances were brought forth by a celebrated triumvirate of men : *Fermat* discovered a method of maxima and minima, *Descartes* by showing how the common curves of geometry can be expressed by equations (for he excluded transcendent curves), & *Father Gregorius a St. Vincentio* with many outstanding discoveries. To which I add *Guldin's* outstanding rule concerning the motion of the centre of gravity [of the generating curve; also known as the theorem of *Pappus*]. But these men certainly were confined within certain limits, which the celebrated mathematicians *Huygens*, and *Wallis* had passed over with the appearance of a new opening. For it is probable enough that *Van Heuraet* [drew inspiration] from the works of *Huygens* [The MacTutor website considers this interaction to have been the other way round!], *Neil* from *Wallis,* and *Wrenn,* who equally were the first to show the rectification of curves, the occasion of the most handsome of discoveries to have been given. Because yet nothing can be detracted from the most deserving of praise of the discoveries. Following these are the Scot *James Gregory* and the Englishman *Isaac Barrow*, who have enriched our knowledge of theorems of this kind in an amazing way. Meanwhile *Nicolas Mercator,* of Holstein, a most outstanding mathematician, and the first, as far as I know, who gave a certain quadrature in terms of an infinite series. But the same discovery not only has been pursued with his own strict discipline and resolved by the reasoning of that most profound and talented geometer, *Isaac Newton*, who if he had published his own deliberations, those which I understand to be the first at that time, without doubt would have revealed for us a new entry point into a great increase and compendium of knowledge.

It happened for me at this time to be a novice in these studies, so that from a single study of a certain demonstration concerning the magnitude of the area of a sphere, suddenly a great light arose. For I was considering generally the figure drawn by the normals to a curve, applied to an axis (for a circle it depends on the radii) to be proportional to the area of the volume itself, to be generated by the rotation of the figure about the axis. So that I was transported by joy with the first theorem (since I was unaware that such had become known to others), at once I was devising a triangle, which I may call the characteristic for any curve, the sides of which should be indivisibles (or more accurately by being called infinitely small) or differential quantities ; from which at once I had no difficulty in establishing innumerable theorems, the latter part of which could be found in the works of *Gregorius* and *Barrow*. Not even in truth while I was using that for an algebraic calculation with that added , soon I was finding my arithmetical quadrature and many other things. But in some manner my algebraic calculations were not satisfying me at this stage in this work, and many things which I wanted to know from analysis, hitherto outstanding I was gathering together of spiral figures, then at last I found the true algebraic supplement for transcending quantities, evidently my indefinite calculus of the very small, and which differential, either of summations, or of quadratures, and unless I am deceived, I call aptly enough, the *analysis of the indivisibles and of the infinitudes*, with which once uncovered, whatever I was viewing before of that kind with wonder, is seen now as a game or a joke. So that not only is it a significant shortening, but it has allowed the most general method I have set out before to be established, from which either quadratics, or any other kind of algebraic lines sought, even transcendental ones, may be determined, exactly as it is possible. Before I may finish, I may advise besides that one cannot in differential equations rashly discard *dx* itself, just as a little before there was $a = \int dx : \sqrt{1-xx}$, because in that case it can be ignored, in which the increases of *x* are assumed to be uniform : for in this most people are mistaken, and by themselves have closed the path to greater things, because indivisibles of this kind, such as *dx*, cannot relinquish their general nature (as evidently a progression of values of *x* of some kind may be assumed); since again from this alone innumerable transformations of figures may arise of equal value.

[Evidently this integral can be regarded as the sum of the increments of an angle, equal to the angle.]

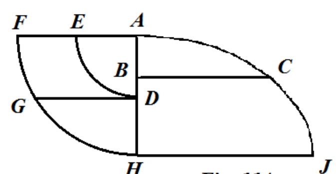


Fig. 114

Now this complete small work has just come to hand, which *D.T.* has published in the March of this year *Acta* page 176. Where he has proposed some elegant questions, and worth solving. I can see that the line ACJ (Fig. 114 here) is a certain one of the sine curves, and always the rectangle AH by GD to be equal to the area ABCA.

[Solutions are provided in *Naissance du Calculi*…. note 44, p. 120].

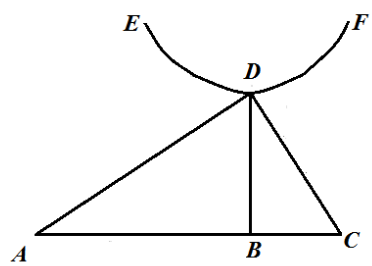


Fig. 115

And [in the other problem] in Fig.115 if the square BC by BD or *x* must always be equal to the cube given by *a*, to satisfy the paraboloid, the equation of which is $4a^3yy = 25x^5$. Similarly just as to determine the matter for other powers. But if AD, DB, BC = to the given cube, the matter is restored to finding the quadrature of the figure of the square, of which the ordinate is ax^3 divided by $\sqrt{a^6 - x^6}$; but in general the problem is, for some relation given between the right lines AB, BC, CD, AD, DB given in Fig. 115 to find the curve, because it coincides with the finding of the quadratures. But if in the right line AC a fixed point L may be assumed, new relations of another kind are found, so that if the relation shall be given between LC and CD, because the problem still receives a solution in the same manner.

No. XIX.
DE GEOMETRIA RECONDITA

ET ANALYSI INDIVISIBILIUM ATQUE INFlNITORUM.

Ex Actis Erudis. Lips. ann. 1686.

Cum intelligam nonnulla, quae in his Actis ad Geometrae prosectum publicavi, non mediocriter a viris quibusdam doctis probari, quin & paulatim in usum transferri, quaedam tamen, sive scribentis vitio, sive aliam ob causam, ab aliquibus non satis fuisse percepta, ideo pretium operae putavi, hoc loco adjicere, quae illustrare priora possint. Accepi nimirum tractum Dn. *Craigii* de dimensione figurarum, Londini anno superiore editum, ex quo sane apparet, autorem non contemnendos in Geometria interiore progressus fecisse. Is quidem valde approbat distinctionem a me aliquoties, inculcatam inter dimensiones figurarum generales, & speciales, quam pag. 1 ait optime nuper a Geometris fuisse observatum, & neglectioni hujus distinctionis paralogismos complures tetragonismi impossibilitatem probare conantium, recte tribuit. Mecum etiam figuras, quas vulgo e Geometria rejiciunt, agnoscit esse Transcendentes pag. 26. Methodum quoque Tangentium a me in Actis Octobr. 1684. publicatam, pro humanitate sua plurimum laudat pag. 27. & 29. tanquam praestantissimam, & cujus ope methodus dimensionum valde juvetur, optima contra irrationalitates remedio suppeditato. Sunt tamen nonnulla, de quibus monere eum, aliosque, nec inutile, nec ipsi ingratum fore putavi. Nescio enim quomodo factum sit, ut crediderit, eum qui schediasma Act. Maii 1684. p. 233. scripsit, retractasse sententiam, & cum initio Act. Octobr. 1683. proposuisset omnimodam dare demonstrationem impossibilitatis tetragonismi circuluris, postea agnovisse Majo anni sequentis, nondum satis demonstratam esse impossibilitatem tetragonismi specialis. Cum tamen schediasma Octobr, 1683. sit a Dn. D. T. Schediasma vero Maii 1884. a me sit prosectum *:* qui partim eamdem methodum & mihi asserebam, ne aliquando rei alienae usurpatae accusarer, partim ab usu quem ei tribuebat Dn. D. T. amice dissentiebam. Nam putabat ille ex indefiniti tetragonismi impossibilitate, sequi & cujusque definiti impossibilitatem : meum vera constans dogma fuerat (jam tum indicatum, cum tetragonisinum arithmeticum ederem, mense secundo anni primi Actorum, nempe 1682.) ab illa ad hanc non valere consequentiam. Quod ut probarem, instantiam cujusdam figurae attuli in Actis Maii 1684, quae tetragonismum specialem recipit, (quod possum demonstrare) non vera generalem, ut ex ipsis Dn. D. T. theorematis ibi ostendere susceperam ; quamquam festinus, & rei certus in modo probandi per calculum nonnihil aberraverim, quod postea explicabo, & corrigam. Ad haec Dn. D. T. privatim respondit, se methodum istam non ex meis hausisse, sed in eam proprio Marte devenisse, & quod ad objecitionem attineret, se consequentiam illam a tetragonismis indefinitis ad definitos posse demonstrare, inque eo potissimum methodum suam eminere; instantiam vera meam pravo calculo niti. Ego vera lubens fassus sum (in Actis. Decembr. 1684. p. 587.) si eam consequentiam demonstrare possit, facturum quod hactenus nemo; semper tamen subdubitavi & correcto calculo postea instantiam meam roboravi, de quo mox. Quamquam autem ego hanc methodum jam habuerim ante decennium, & amplius, cum una essemus Parisiis, et de rebus Geometricis creberrime loqueremur, quo tempore ipse per alias plane vias incedebat, mihi vera jam tum famillarissimum erat aequationes generales adhibere pro exprimenda natura lineae quasitae, progressu calculi determinandas, in quo methodi nervus consistit, quale quid alibi nusquam animadverteram: attamen candori ejus pariter, & ingenio tantum tribuo, ut facile credam vel ipsum per se in haec incidisse, vel saltem non amplius meminisse, qua olim occasione talium meditationum semina fuerint jacta : praesertim cum sciam, etiam difficiliora ipsum per se praestitisse, & multa praeclara maximique momenti ab ejus ingenio posse expectari.

Quoniam vero instantiae supradictae calculo erratum a me, ut dixi, admissum est, quod Dn. Craigius Dno *D.T.* (cui id tribuerat) tamquam argumentum, opinor, ad hominem objecit, ut ipsam methodum indefinitam refutaret, ideo corrigere calculum debeo. Inspiciatur Actorum anni 1684. pag. 239, ubi aequationem $4zz-8hz$, &c. conferendo cum aequatione $bzz+caz$ &c. debent in aequatione posteriore termini ubi abest z, extra fractionis positi multiplicari per fractionis nominatorem, antequam comparatio instituatur, ut in utraque fractione omnes termini carentes litera *z*, una fractione comprehendantur. Ponatur ; & $b=1$ quod semper fieri potest, & quia in aequatione priore terminus *xz* plane abest, fiat in posteriore $d=0$, dividatur *&* aequatio priore, seu data per 4, & in posterioris, seu supposititiae aequatione, fractione tam numerator, quam numerator dividatur per *g* : ita tam tcrminus *zz* utrobique, quam terminus *zz* in nominatore fractionis utrobique consentient. Caetera comparando, ob terminum *z*, fiet $c=2h:a$; ob x^4 fiet $g=1:16$, seu $\frac{1}{16}$ *;* ob x^3 fiet $f=-1:6a$; ob *x* in nomeratore fiet $f=-h:8a$. Ergo fit $h=8:6,$ seu $\frac{4}{3}$ quod absurdum, nam *h* est quantitas data. Oriuntur & alia ex comparatione continuata absurditates, nam fit vel *c* vel $f=0$, contra jam conclusa.

Caeterum placet hoc loco, ut magis profutura dicamus, *fontem aperire Transcendentium Quantitatum*, cur nimirum quaedam problemata neque sint plana, neque solida, neque sursolida, aut ullius certi gradus, sed omnem aequationem Algebraicam transendant. Eademque opera modum ostendemus, quomodo sine calculo demonstrari possit, lineam quadratricem Algebraicam circuli, & hyperbolae esse impossibilem. Si enim ista daretur, sequeretur ejus ope angulum, aut rationem sive logarithmum secari posse in data ratione rectae ad rectam, idque una generali constructione, & proinde problema sectionis anguli vel, inventionis quotcunque mediarum proportionalum foret certi gradus, cum tamen pro alia numero partium anguli, aut mediarum proportionalium, alius atque alius gradus aequationis Algebraicae requiratur, & ideo problema intellectum in genere de numero partium, aut mediarum quocunque, sit gradus indefiniti, & omnem Algebraicam aequationem transcendat. Quoniam tamen nihilominus talia problemata revera in Geometria proponi possunt, imo inter primaria haberi debent, & determinata sunt ; ideo necesse utique est, eas quoque lineas recipi in Geometriam, per quales solas construi possunt ; & cum ea exacte, continuoque motu describi possunt, ut de cycloide, & similibus patet, revera censendas esse non Mechanicas, sed Geometricas ; praesertim cum utilitate sua lineas communis Geometriae (si rectam circulumque exceperis) multis parasangis post se relinquant, & maximi momenti proprietates habeant, quae prorsus Geometricarum demonstrationum sunt capaces. *Non minor ergo* Cartesii *Geometria eas* *excludentis, quam veterum lapsus fuit,* qui loca solida, aut linearia tamquam minus Geometrica rejiciebant.

Quoniam etiam methodus investigandi Tetragonismos indefinitos, aut eorum impossibilitates, apud me casus tantum specialis est (& quidem facilior) problamatis multo majoris, quod appello *Methodum Tangentium inversam*, in quo maxima pars totius Geometriae transendentis continetur, & quod si Algebraice semper posset solvi, omnia reperta haberentur, & vero nihil adhuc de eo extare video satisfaciens, ideo ostendam quomodo non minus absolvi possit, quam Tetragonismus ipse indefinitus. Cum igitur antea Algebraistae assumerent literas, seu numeros generales pro quantitatibus quaesitis, ego in talibus problematibus transcedentibus assumi aequationes generales, seu indefinitas pro lineis quaesitis, v. g. abscissa, & ordinata existentibus x & y, aequatio pro linea quaesita mihi est, $0 = a + bx + cy + exy + fxx + gyy$ &c. ope hujus aquaequationis indefinite propositae, revera finitae (semper enim determinari potest, quousque assurgi opus sit) quaero lineae tangentem, & quod invenio, id cum proprietate tangentium data conferens, reperio valorem literarum assumtitiarum, *a, b, c,* &c. atque adeo aequationem lineae quaesitae definio, ubi tamen interdum quaedam manent arbitrariae; quo casu etiam innumerae lineae reperiri possunt, quaesito satisfacientes, quod in causa fuit, ut multi problema non satis definitum a posteriori videntes, putarent, nec in potestate esse. Eadem per series quoque praestantur. Ad calculum autem contrahendum multa habeo, de quibus alias. Quod si comparatio non procedat, pronuntio lineam quaesitam non esse Algebraicam, sed transendentem.

Quo posito ut ipsam *Transcendentiae speciem* reperiam (aliae enim transcendentes pendent a sectione generali rationis, seu a Logarithmis, aliae a sectione generali anguli, seu ab arcubus circuli, aliae ab aliis indefinitis quaestionibus magis compositis) ideo praeter literas x & y assumo adhuc tertiam ut v, quae transendentem quantitatem significat, & ex his tribus formo aequationem generalem ad lineam quaesitam, ex qua lineas tangentem quaero, secundum meam methodum tangentium in Actis Octobr. 1684. publicatam, quae nec transcendentes moratur. Deinde id quod invenio comparans cum data proprietate tangentium curvae reperio non tantum literas assumtitias *a, b, c,* &c. sed & specialem transendentia naturam. Quamquam autem aliquando fieri possit, ut plures adhibendae sint transcendentes, naturae quandoque inter se diversae, & dentur transendentes transcendentium, & omnino talia procedant in infinitum, tamen facilioribus, & utilioribus contenti esse possumus; & plerumque peculiaribus artificiis uti licet ad calculum contrahendum, problemaque, quoad licet, ad terminos simplices revocandum, quae non sunt hujus loci. Hac autem methodo ad Tetragonismos applicata, seu ad inventionem linearum quadratricium, (in quibus utique semper tangentium proprietas data est) patet non tantum, quomodo inveniatur, an quadratura indefinita sit Algebraice impossibilis, sed & quomodo impossibilitate hac deprehensa reperiri possit quadratrix transcendens, quod hactenus traditum non fuit. Adeo ut videar non vane asseruisse, Geometriam hac methodo ultra terminos a *Vieta,* & *Cartesio* positos in immensum promoveri. Cum hac ratione Analysis certa & generalis ad ea porrigatur problemata, quae nullius sunt certi gradus, atque adeo Algebraicis aequationibus non comprehenduntur.

Porro quoniam ad problemata transcendentia, ubicunque dimensiones, tangentesque occurrunt, calculo tractanda, vix quicquam utilius, brevius, universalius fingi potest *calculo* meo *differentiali* seu *analysi indivisibilium, atque infinitorum,* cujus exiguum tantum velut specimen, sive corollarium continetur in methodo illa mea Tangentium in Actis. Octobr. 1681. edita, & Dn. *Craigio* tantopere probate ; & ipse Dn. Craigius suspicatus est aliquid altius in ea latere, ac proinde pag. 29. sui libelli inde derivare conatus est theorema Barrovianum (quod summa intervalorum, inter ordinatas, & curvae perpendiculares in axe sumtorum, & ad axem applicatorum, aequetur semiquadrato ordinatae ultimae) in cuius executione tamen nonnihil a scopo deflexit, quod in nova methodo non miror, ideo gratissimum ipsi, aliisque fore arbitror, *si hoc loco aditum rei cujus tam late patet utilitas, patifecero.* Nam inde ommia hujusmode theoremata, ac problemata, quae admirationi merito fuere, ea facilitate fluunt, ut jam non magis ea disci, tenerique necesse sit, quam plurima vulgaris Geometriae theoremata illi ediscenda sunt, qui speciosam tenet. Sic ergo in casu praedicto procedo. Sit ordinata x abscissa y, intervallum inter perpendicularem, & ordinatam, quod dixi sit p, patet statim methodo mea fore $pdy = xdx$, quod & Dn. *Craigius* ex ea observavit ; qua aequatione differentiali versa in summatricem, fit $\int pdy = \int xdx$. Sed ex iis, quae in methodo tangentium exposui, patet esse $d, \frac{1}{2}xx = xdx$, ergo contra $\frac{1}{2}xx = \int xdx$ (ut enim potestates, & radices m vulgaribus calculis, sic nobis summae, & differentiae seu $\int$ & d, reciprocae sunt.) Habemus ergo $\int pdy = \frac{1}{2}xx$. Quod erat dem. Malo autem dx & similia adhibere, quam literas pro illis, quia istud dx est modificatio quaedam ipsius x, & ita ope ejus fit, ut sola quando id fieri opus est litera x cum suis scilicet potestatibus, & differentialibus calculum ingrediatur, & relationes transscendentes inter x & aliud exprimantur. Qua ratione etiam lineas transcendentes aequatione explicare licet, verbi grat. Sit arcus a, sinus versus x, fiet $a = \int \overline{dx : \sqrt{2x - xx}}$, & si cycloidis ordinata sit y, fiet : $y = \sqrt{2x - xx} + \int \overline{dx : \sqrt{2x - xx}}$, quae aequatio perfecte exprimit relationem inter ordinatam y, & abscissam x, & ex ea omnes cycloidis proprietates demonstrari possunt ; promotusque est hoc modo calculus analyticus ad eas lineas, quae non aliam magis ob causam hactenus exclusae sunt, quam quod ejus incapaces crederentur : interpolationes quoque Wallisianae, & alia innumera hinc derivantur.

Quod superest, ne nimium mihi adscribere, aut detrahere aliis videar, paucis dicam quid potissimum insignibus nostri saeculi Mathematicis in hoc Geometriae genere mea sententia debeatur. Primi *Galilaeus* & *Cavallerius* involutissimas *Cononis* & *Archimedis* artes detegere coeperunt. Sed Geometria indivisibilium Cavalleriana, scientiae renascentis non nisi infantia fuit. Majora subsidia attulerunt triumviri celebres, *Fermatius* inventa methodo de maximis & minimis, *Cartesius* ostensa ratione lineas Geometriae communis (transendentes enim exclusit) exprimendi per aequationes,& P. *Gregorius a S. Vincentio* multis praeclaris inventis. Quibus egregiam *Guldini* regulam de motu centri gravitatis addo. Sed & hi certos limites constitere, quos transgressi sunt novo aditu aperto, *Hugenius,* & *Wallisius,* Geometrae inclyti. Satis enim probabile est, Hugeniana *Heuratio,* Wallisiana *Neilio,* & *Wrennio,* qui primi curvis aequales rectas demonstravere, pulcherrimorum inventorum occasionem dedisse. Quod tamen meritissimae laudi inventionum nil detrahit. Secuti has sunt *Jacobus Gregorius* Scotus, & *Isaacus Barrovius* Anglus, qui praeclaris in hoc genere theorematibus scientiam mire locupletarunt. Interia

Nicolaus Mercator, Holsatus, Mathematicus & ipse praestantissimus, primus, quod sciam, quadraturam aliquam dedit per seriem infinitam. At idem inventum non suo tantum Marte assecutus est, sed & universali quadam ratione absolvit profundissimi ingenii Geometra, *lsaacus Newtonus,* qui si sua cogitata ederet, quae illum adhuc premere intelligo, haud dubie nobis novos aditus ad magna scientiae incrementa, compendiaque aperiret.

Mihi contigit adhuc tironi in his studiis, ut ex uno aspectu cuiusdam demonstrationis de magnitudine superficiei sphaericae, subito magna lux oboriretur. Videbam enim generaliter figuram factam ex perpendicularibus ad curvam, axi ordinatim applicatis (in circulo radiis) esse proportionalem superficiei ipsius solidi, rotatione figurae circa axem geniti. Quo primo theoremate (cum aliis tale quid innotuisse ignorarem) mirifice delectatus, statim comminiscebar triangulum, quod in omni curva vocabam characteristicum, cujus latera essent indivisibilia (vel accuratius loquendo infinite parva) seu quantitates differentiales ; unde statim innumera theoremata nullo negotio condebam, quorum partem postea apud *Gregorios,* & *Barrovium* deprehendi. Nec dum vera Algebraico calculo utebar, quem cum adjecissem, mox quadraturam meam Arithmeticam, aliaque multa inveni. Sed nescio quomodo non satisfaciebat mihi calculus Algebraicus in hoc negotio, multaque quae analysi voluissem, praestare adhuc cogebar figurarum ambagibus, donec tandem verum Algebrae supplementum pro transendentibus inveni, scilicet meum calculum indefinite parvorum, quem & differentialem, aut summatorium, aut tetragonisticum, & ni fallor, satis apte *analysim indivisibilium, & infinitorum* voco, quo semel detecto, jam ludus jocusque visum est quicquid in hoc genere ipse antea fueram admiratus. Unde non tantum insignia compendia, sed & methodum generalissimam paulo ante expositam condere licuit, qua sive quadratrices, sive aliae quaesitae linae Algebraicae, vel transcendentes, prout possibile est, determinantur. Antequam finiam, illud adhuc admoneo, ne quis in aequationibus differentialibus, qualis paulo ante erat $a=\int dx:\sqrt{1-xx}$ ipsam dx temere negligat, quia in casu illo quo ipsae x uniformiter crescentes assumuntur, negligi potest : nam in hoc ipso peccarunt plerique, & sibi viam ad ulteriora praeclusere, quod indivisibilibus istiusmodi, velut *dx*, universalitatem suam (ut scilicet progressio ipsarum x assumi posset qualiscunque) non reliquerunt; cum tamen ex hoc uno innumerabiles figurarum transfigurationes, & aequipotentiae oriantur.

Scriptiuncula hac jam absoluta, venere in manus meas, quae Dn. *D. T.* in Martio hujus anni Acta pag. 176. communicavit. Ubi nonnullas quaestiones elegantes proposuit, & solvi dignas. Video autem lineam ACI *(fig. VIII.* ibi) esse quamdam ex lineis sinuum, semperque rectangulum AH in GD esse aequale spatio ABCA. Et in *fig. IX.* si quadratum BC in BD seu X semper aequale debeat esse dato cubo ab a, satisfacere paraboloeidem, cujus aequatio est $4a^3yy=25x^5$. Similiter rem determinare licet pro aliis potentiis. Sin AD, DB, BC$=$ cubo dato, res redit ad quadratricem figurae, cujus ordinata: valor est a x^3 divis. per $\sqrt{a^6-x^6}$ in genere autem data relatione quacunque inter rectas AB, BC, CD, AD, DB in dicta *fig. IX.* invenire lineam, problema est, quod coincidit cum inventione quadraturarum. Sed si in recta AC assumatur punctum fixum L, novae oriuntur alterius naturae relationes, ut si data sit relatio inter LC, & CD, quod problema tamen itidem solutionem recipit.