RESPONSE TO SOME DIFFICULTIES RAISED BY BERNARD NIEUWENTIJT ABOUT THE DIFFERENTIAL OR INFINITESIMAL METHOD.

G.W. Leibniz : Act. Erudit. Lips. 1695.

I have recently received two new treatises by the distinguished Dutch geometer, *Mr. Bernard Nieuwentijt* [modern spelling], about the differential calculus and the use of the analysis of the infinitely small, send recently the one after the other as it appears, by the most learned geometer *Mr. J. Makreel,* by the order of the author.
[*Analysis Infinitorum*, ….. ; and *Considerationes Circa Analyseos*, …. Amsterdam, 1694-5. See R.H.Vermij, *Bernard Nieuwentijt and the Leibnizian Calculus* : *Studia Leibnitiana*, Bd. 21, H. 1 (1989, pp. 69-86)] And thus since in these in a number of places the solution of certain difficulties may be desired by me most humanely, I have no desire to flee from my duties towards the republic of letters, even if now I may be able only to touch on all the chapters, indeed with so many other distractions. Three matters mainly are addressed :

In the first place, *my method of calculating differentials and of taking sums, labouring with others from a common difficulty, because clearly infinitely small quantities may be discarded, as if they shall be zero;*
secondly, *this method cannot be applied to curves, in the equation of which an undefined exponent may be present*;
thirdly, *even if my differential calculus shall be sustained in the first order, yet in higher order differences of the second, third and of other orders may occur, such as ddx or d^2x, dddx or d^3x, and thus so on, which cannot be reconciled with the principle of the illustrious author* [who accepted first order quantities only], since that still cannot be agreed to be supported by geometry alone. Now I do not touch on several particular problems of the *Marquis L'Hôpital*, the most ingenious *Bernoulli* brothers, to which he objects, as well as some of my own, since they may be able to attend best to so many of their own outstanding discoveries.

Pertaining to the first objection, the most distinguished author asserts in the preface of the *Consideration* [actually to the *Analysis Infinitorum*], as the clearest truth is to be considered: *These quantities alone are equal, the differences of which is null, or equal to nothing.* And in the analysis of curves, under the first axiom l, page. 2: *It is not possible that any given quantity, however small, and however many times it may be taken to be multiplied by some number however great,* (and indeed also may be understood to be infinite)*, that it may come about to be equal to the magnitude of a given quantity, for it is not a quantity, but purely zero in the geometrical context.*

[This translation is taken for Leibniz's re-wording of the assertion ; the original wording of this statement in the *Prefatio* of the *Analysis Infinitorum* of *Nieuwentijt* is as follows : *quicquid per numerum infinitum multiplicatum nullam quantitatem datam , utut exiguam, adaequare valet, entibus annumerandum non est, ac nihilo aequale haberi debet.* This may be translated as follows : *However many times a given null quantity may be* *multiplied by an infinite quantity, however small* [a number] *it may prevail to equalize, it does not exist with the numbers being enumerated, and must have the value zero.*]

Hence because in the equations being investigated for tangents, with *Maxima* and *Minima* (as the author grants to *Barrow*, yet used first by Fermat, lest I am mistaken) the quantities remain infinitely small, but the squares or rectangles of these are to be abandoned; from that it leads to the ratio of this quantity, which quantities are themselves somewhat infinitely small quantities or infinitesimals, because they produce the given quantity multiplied by a given an infinite number (that is, ordinary or assignable); but otherwise the rectangles or squares of these themselves to be had, which hence shall be purely zero from the permitted axiom. Indeed I myself admit to make great use of these, which contend to demonstrate everything accurately as far as to first principles, and with such also always to have put in place much study; still not urged, so that by an excess of carefulness an obstacle may be put in place to the art of discovery, or that we may reject by such a pretext what have been found optimally, and we may be deprived from the fruits of these, just as at one time both Father *Gottigniez* and his students became over-engrossed with insignificant matters about the principles of algebra. Everything else I consider to be equal, not only is the difference of these generally zero, but also the difference of which is incomparably small; and although I have said zero is not allowed generally, yet zero is a quantity comparable with these, of which it is a difference. Just as if you add a point of a line to another line, or a line to a surface, you do not increase the quantity. It is the same, if you add a certain line to a line, but incomparably smaller. Nor by any such construction can the increase be shown. Clearly only these homogeneous quantities are comparable, since I consider by Euclid Book 5, def. 5, of which with the one number, multiplied by a finite number, can exceed another. And which with the quantity of such not being different, I may put to be equal, which also Archimedes accepted, and everybody else after him. And this is the very case, as the difference is said to be given by any small amount whatever. And indeed by Archimedes with a certain process the matter can be confirmed by a reductio ad absurdum proof. Yet because the direct method is quicker to be understood and more useful towards being found, it suffices to know once the way of being reduced after the method being used, in which incomparably smaller amounts are ignored, which certainly carries within itself its own demonstration, following the lemma communicated by me in Feb. 1689 [see *Tentamen de motuum coelestium causis* : translated on this website]. And if anyone rejects such a definition of equality, he disputes in name only. For it suffices to be intelligible and useful for finding results, with those, which can be found in another way (in an example) by a more rigorous method, always by this method it shall be necessary to bring forth results none the less accurate. And thus I assume not only infinitely small lines such as dx, dy, to be for real quantities in their generation, but also the squares or rectangles dxdx, dydy, dxdy, and I think likewise concerning cubes and with other higher orders, especially since I may find these useful for reasoning and discovering. Nor surely do I consider, how the most learned author could be able to consider in his mind, as he has stated, how a line of a length dx can be a quantity, but the square or the rectangle of such lines to be nothing. For these quantities are allowed to be infinitely times infinitely small, with an infinite number of the first order multiplied, they do not produce a given or ordinary quantity, yet by this multiplication they give rise to an infinite number multiplied an infinite number of times, which equally cannot be rejected, if you allow an infinite number; for an infinite number multiplied by itself will be produced. Because moreover in equations of the Fermat kind terms are abandoned, which introduce such squares or rectangles, truly these do not introduce simple infinitesimal lines, of which there is no ratio, because these shall be of another kind, these truly shall be zero, but which are destroyed by ordinary terms themselves, hence the terms remain then, which introduce simple infinitely small lines, which introduce as well the squares or rectangles of these: since truly these terms which shall be incomparably smaller than those are rejected. Because if ordinary terms shall not vanish, also no less ought the terms of infinitesimal lines as with the squares of these be abandoned. Certain of my Lemmas can be added, relating to the fundaments of the differential calculus, from the *Actis Eruditorum Lips.* Febr. 1689, which the distinguished author himself professes to have come across after the publication of the *Considerationes* in the preface of the *Tractatus Analytici*, where now moreover I have given incomparable consideration to these difficulties arising.

What in the second place concerns the most learned man, he considers exponential equations (as they may be called by me) able to be treated by his method, but not likewise by mine. And thus with the account of such in Ch. l *Analys*. p. 62 onwards and Ch.8 p. 280 he attempts to show in the account of his calculation, that I can express thus still with the use of my symbols and reasoning. The equation shall be (for the transcending curve)

$$y^{x} = z \quad (1),$$

from which by another rule equally, there becomes :

$$\overline{y+dy}^{\,x+dx} = z+dz \quad (2).$$

And thus with equation (l) requiring to be differentiated, that is with equation (2) being subtracted from equation (l), so that dz or the difference between the two values of z (surely z and of $z+dz$ themselves) may be had (which is the fundamental equation of the differential calculus), everywhere from (2) and (l) the equation becomes :

$$\overline{y+dy}^{\,x+dx} - y^{x} = dz \quad (3), \text{ or } \overline{y+dy}^{\,x+dx} = y^{x+dx} + x.y^{x+dx-1}dy \quad (4),$$

(as because at one time in these *Actis* it has been noted by me generally :

$$\overline{y+a}^{\,m} = y^{m} + \tfrac{m}{1}y^{m-1}a^{1} + \tfrac{m.m-1}{1.2}y^{m-2}a^{2} \text{ etc.};$$

from which in the opinion of the author, with the term $\frac{m.m-1}{1.2}y^{m-2}a^{2}$ vanishing with those that follow, because *a* is infinitely times an infinitely small number, and for *a* by substituting dy, and for the letter *m* by substituting $x+dx$ equation (4) is produced. And thus from equation (3) by equation (4) there becomes :

$$y^{x+dx} + x.y^{x+dx-1}dy - y^{x} = dz \quad (6).$$

In truth expressing this ratio labours with the greatest difficulty, because it is not addressed by the rules of homogeneity of differential calculus [*i.e.* all the differentials are of the same order], and because the heading is not showing what is sought, truly the ratio dx to dy or of the subtangent to the ordinate, expressed in ordinary terms, nor indeed can it be constructed from the drawing of assignable lines. Indeed it reverts to the identical. For placing my beginning near set out above, an incomparably smaller quantity is added to the other greater amount in vain, and, if this does not vanish (actually or virtually), that itself must be abandoned. And thus in equation (6) for dy, dx, dz added to another incomparably greater amount, by writing 0, there becomes $y^{\overline{x+0}} + x.y^{\overline{x+0-1}} 0 - y^x = 0 \quad (7)$, this is equally with 0 discarded 0, as well as with the term multiplied by 0, becoming $y^x - y^x = 0$, which equation indeed is true, but is the identify, from which such a calculation is not useful. Which kind too I have tested, so that if there shall be $b^x = y$, on putting *b* constant, then $b^{\overline{x+dx}} = b^x$, it will be $= dy$; and then by dividing dx by b^x the equation becomes $b^{\overline{dx}} - 1 = dy : b^x$, $b^0 - 1 = 0$, or $b^0 = 1$, as agreed, therefore it becomes $1 - 1 = 0$. But such an identity is to be avoided in my differential calculus. Meanwhile I cannot deny having offered this case myself, where also that ratio being calculated cannot be discarded in a straight-forwards manner. Truly as the most illustrious Niewentijt considered my differential method for equations also, where an unknown or indeterminate exponent is present, (and indeed usefully) to be extended, which I perhaps was the first of all geometers to have proposed for consideration, with my numerical squaring of the circle I gave in the *Actis Eruditorum* in Feb. of the year 1682, I will touch now here on a few matters, which I had discovered many years ago, and which I had described earlier to the greatest of mathematicians *Christian Huygens*, truly a way of differentiating exponential equations, which with my algorithm published some time ago certainly would not be necessary to insert on account of such a rare and unusual expression, which, I admit, is so great, that Huygens himself would scarcely have admitted these. Nor with anything know to me, besides it is the most ingenious *Bernoulli*, who on his own account, without a mention to me, and had himself arrived at this point in the differential calculus and had penetrated this secret, which *Huygens* in jest has called hypertranscendental. Truly let there be $x^v = y$, becoming $v.\log.x = \log.y$; now $\log.x = \int, dx : x$ and $\log.y = \int, dy : y$. Therefore $v.\int, dx : x = \int, dy : y$, which on differentiation shall become $vdx : x + dv \log.x = dy : y$. Again, v ought to be given from x and y, with both together or individually, therefore it is possible to write $dv = mdx + ndy$, and both *m* and *n* equally may be given from x and y, and there will be produced : $vdx : x + \log.x.mdx = dy : y - \log.x.ndy$, and dx to dy (or the subtangent to the ordinate) becomes as :

$$\left[\frac{1}{y} - n\log x\right] \text{ to } \left[\frac{v}{x} + m\log.y\right].$$

And thus a way may be had of deducing such a curve from the supposed quadrature of the hyperbola, or from logarithms; but in the general case for the differential of the exponential by my algorithm, it suffices to be ascribed to my rule:

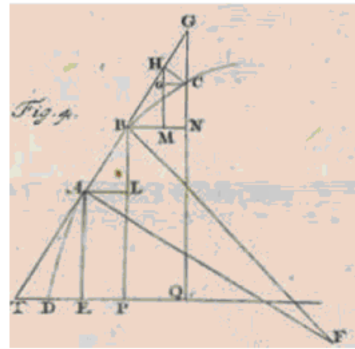
$$d\left(x^v\right) = x^v\left(\frac{v}{x}.dx + dv.\log.x\right).$$

From which if v shall be a constant number such as e, it produces $d,x^e = x^e \frac{e}{x} dx$, that is $e.x^{\frac{e-1}{\cdot}}.dx$, which is the theorem of our algorithm for the differentiation of powers or roots, treated before.

It remains, that I may resolve briefly the third difficulty of our illustrious man, evidently opposing successive differentials or the differentiation of differentials. And thus neither he does not consider allowing ddx themselves to be quantities, because they shall not give an ordinary quantity on being multiplied by an infinite number. But it is known explicitly to produce that, as I have advised now concerning the first difficulty, if the multiplying number shall be an infinity of a higher order. And the matter whence can certainly be performed in many ways. For whenever the terms do not increase uniformly, it is necessary that the increments of these again have differentials, which certainly are the differences of differences. Then the illustrious author concedes that dx is a quantity ; now with two quantities the third proportional certainly is also a quantity ; but the quantity ddx is of such a kind, with respect to the quantities x and dx, which I show thus. Let x be in a geometric progression, and y in an arithmetic progression, [*i.e.* we may be looking at the function $x = a^y$, in which case *y* is the abscissa axis and *x* the ordinate axis ; if this is the case then $\log x = y \log a$ and $\frac{1}{x}\frac{dx}{dy} = \log a$; and $dx = xdy \log a$; and $ddx = dxdy \log a$; hence the argument does not apply in this example quite as stated;] then dx will be to the constant dy, as x is to the constant *a*, or :

$$dx = xdy : a \text{ ; therefore } ddx = dxdy : a .$$

From which by taking $dy : a$ by the first equation there becomes $xddx = dxdx$, from which it is apparent that x to dx, is as dx to ddx. And from the continued geometric progression also the remaining differences of higher order will be produced. And generally in a geometric progression not only is the series of differences of the same order, but also the series of transitions or of differentials is a geometric progression. But also the truth and use of these successive differentiations may be confirmed from these matters themselves. Certainly, as now I remember noting in other places, ordinary quantities, first order differential quantities or differentials, and differential of the differential quantities or second order infinitesimals, themselves can be had as motions or speeds and by with a force acting, which is the first element of the speed. A line is described by the motion, with the velocity an element of the line, with a force acting as the element of an element (just as descending initially from gravity, or the motion from a centrifugal attempt). And in geometry itself the ordinary quantities are those of common algebra, the differentials of the first order refer to the tangents or the directions of lines, but the differentials of higher order refer to the radii of osculation or the curvature of the lines, which I recall also to note. I will finish, when I add this one remark, for me to wonder, how the most learned Niewentijt was able to believe that this absurdity followed from our principles: that in any curve the subtangent shall be equal to the ordinate [*i.e.* an isosceles rt. angle is formed and $TE = AE$], *Considerationes*, Fig. 4, p. 19. An element of the curve shall be dc, there will be $dxdx + dydy = dcdc$, as it is agreed ; therefore by differentiating $dxddx + dyddy = dcddc$. If now dc is constant there becomes $ddc = 0$, and there arises $dxddx + dyddy = 0$, but this differential I say can again be turned into a sum to produce : $\frac{1}{2}dxdx = \frac{1}{2}dydy$, and thus $dx = dy$, which certainly is absurd. If we were to use such a calculation, how many truths might we uncover with its aid ? But I respond by requiring the summation, or by turning the differential into a summation, is going to produce $\frac{1}{2}dxdx + \frac{1}{2}dydy - \beta dc = 0$, or a constant increment of area is required to be subtracted, in any case indeed there cannot become $dxdx = dydy$, but rather $-dxdx = dydy$, or $dy = dx\sqrt{-1}$, which is an impossible equation, which indicates β cannot be equal to 0, but to have a − sign, and to be a constant quantity, which is none other than $\frac{1}{2}dc$, because we have put dc itself to be constant. From which the initial equation put in place is returned $dxdx + dydy = dcdc$, as required. And he is troubled with a similar abuse of the differential calculus in *Consid*. on p. 21 ; it is indeed a miracle that the calculus is safe and has not descended into absurdity in this manner. Thus also in the greater tract itself or *Analys. Inf*. Ch.8 p. 283, he put in place the characteristic triangle [*i.e.* the rt. angled triangle formed from the above elements] of the same curve, only they shall be with finite numbers and they follow each other in an infinite series, to be similar amongst themselves ; from which it can be inferred readily, with the elements of the abscissas equal, also the elements of the ordinates, etc. will be equal. But since certainly the curve changes the inclination of its direction (otherwise it becomes a right line and not a curve) also the angles change continuously, although insensibly or by incomparably small divisions. I recall putting in place a line of reasoning about this matter a long time ago. The difficulty objected to also, *Consid.* p. 20 against a triangle, of which the base is incomparably smaller than the height, is part of the same comment: Indeed that state of affairs is had for an isosceles triangle, because in actuality the difference between the height and the hypotenuse is incomparably small, and therefore the differences between the radius and the secant of infinitely small angles. But this I judge to be sufficient, and I hope it will give satisfaction to Mr. Niewentijt, who may wish to overturn these studies and to undertake them anew, if he wishes to add to his ingenuity and learning, without doubt he will be able to produce outstanding results, just as one can judge from his examples.



Addition to this Tract.

It pleases to add one thing at this point, the dispute may remove everything regarding the reality of differentials of any kind, as these can be expressed always by ordinary right lines in proportion. Truly let there be any line, the ordinates of which increase or decrease, the ordinates of the curve of the second order can be applied at the ends of the new curve at the same points of the same axis, proportional to the differences of the first order or ordinates of the first line. Because if now the same become the ordinates to be handled, what was done for the first order can be done to the second, and the ordinates of a third curve, proportionals of the first order differentio-differentials order differentials, or, what is the same, of the second order differentials from the first. And from that in the same manner also, the differences of the third and of any assignable quantity can be put in place. Moreover since now I have set out the manner showing the proportional right lines from the differences of the first order, with the first of these elements of the calculus I treated in the *Actis* of October 1684. Truly there the

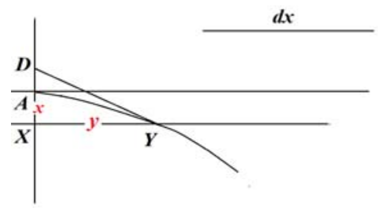


diagram may be shown, dx is found, the element of the abscissa AX or x, represented by the assignable right line put in a separate figure, and then dy, the element of the ordinate XY or y, to be represented by the right line which shall be to the said right line dx now assigned, as the ordinate XY is to the intercept XD on the axis between the tangent and the ordinate. And because the same work shall have a way of setting out the differentials of the second order by proportionals from these differentials of the first order, and in general the latter by the closest preceding, it is apparent there is no grade of differentials however distant, which finally cannot be shown by assignable right lines. Because if the first differences alone may be given, all the orders increasing uniformly shall follow, or every line is right. But meanwhile, by continuing the other differentiations, finally coming to an end, since without doubt the curve corresponding to the differential shall be a right line, either the second or third or some higher order. Certainly if the first ordinates shall be as the abscissas, then the first curve is right and without second differences. If the first ordinates shall be for a parabola (surely squared) or if they shall be as the square of the abscissas, then the curve of the second order shall be right, and the first curve (a parabola of course) will be without third order differences. If the ordinates of the first curve shall be as a cubic parabola, or they shall be as the cubes of the abscissas, then the curves of the third order will be right, and the first curve (clearly the cubic parabola) will be without the fourth order differences, and thus so on. It is the same if the ordinates (of the first curve) may be composed from the said parabolic ordinates, either by addition or subtraction; then indeed finally the differences will be finished with the ordinates of the greatest paraboloid entering. But in the remaining curves with all the differences proceeding to infinity, evidently as often as the abscissa is found in the value of the ordinate named or in a chain of abscissas. Now from these it is understood, the differential calculus can be considered as if it were only with ordinary quantities, even if the origin shall be required from unassignable quantities, in order that a ratio may be returned from the rejection or annulment of quantities. And thus either if Mr. Niewentijt had thought enough about these initial calculations published by me, he would have seen easily, there can be no more doubt about the further differences than about the first, or thus mention of unassignable quantities would be avoided by me, with matters concerning the orders treated, as such scruples would be removed; otherwise if something was worthy of consideration, from that it would seem to me to be part of its nature, so that willingly I would let the truth lead me on, just as now with the matter considered more carefully, I approve of what the celebrated *Jacob Bernoulli* has advised about the number of roots of osculation, with which I might have agreed less before, there being no other reason, than how many different tasks and thoughts had been accomplished, so that I would agree later to the whole matter being considered well enough. While I write this, I hear the sad news of the death of that incomparable man, *Christian Huygens* [Huygens died after a painful illness on the 8th July, 1695]. No greater loss could be allowed than these sublime letters, which admit the human mind into the arcane workings of nature. I put *Huygens* alone after the time of *Galileo* and *Descartes*. Since he gave so much, people expected nothing less. And I hope that amongst his papers some thesis of his is going to be found, which may give us some consolation. Therefore the greater his brother is beseeched, a man of merit in the illustrious state − indeed I would wish to consult with him, so that a complete edition of all his works be published for common use, and equally to the glory of his brother. I have forgotten about these matters brought forth by me regarding concavity or convexity, in the case of the parabola, about which Mr. Niewentijt objected. But that is not worth wondering about, just an error either in writing or in the type-setting words, a concavity required to be put for convexity and vice versa. And thus not only the case of the parabola must be changed (when in all curves the opposite shall be meant of what the words insinuate) as generally the inverse is to be noted. And thus the rule being carried out is this : if, with the ordinates increasing the differences themselves also increase, the curve will be turned convex to the axis, otherwise it will be concave, of course with equal differences of the abscissas between themselves.

RESPONSIO AD NONNULLAS DIFFICULTATES A DN. BERNARDO NIEUWENTYT CIRCA METHODUM DIFFERENTIALEM SEU INFINITESIMALEM MOTAS.

Act. Erudit. Lips. an. 1695.

Egregii Geometrae Batavi, *Domini Bernardi Niewentijt,* tractatus duos novas circa calculum differentialem et Analysin infinite parvis utentem, nuper missu alterius, ut apparet, doctissimi Geometrae *Dn. J. Makreel,* autoris jussu accepi. Itaque cum a me pluribus in locis difficultatum quarundam solutio humanissime petatur, operam reipublicae literariae debitam defugere nolui, tametsi summa tantum capita attingere tot aliis distractus nunc quidem possim. Ad tria potissimum res redit:
methodum meam calculi differentialis et summatorii laborare communi cum aliis difficultate, quod scilicet quantitates infinite parvae abjiciantur·, quasi essent nihil;
secundo, *hanc methodum non posse applicari ad curvas, in quarum aequatione indeterminata ingreditur exponentem;*
tertio, *tametsi meus calculus differentialis primi gradus sustineri possit, differentias tamen inferiores, secundi, tertii et aliorum graduum, ut ddx seu d^2x, dddx sive d^3x, et ita porro, non posse conciliari cum principio clarissimi Autoris,* quo tamen solo Geometriam hanc statuminari posse arbitratur. Specialia nonnulla, quae Hospitalianis, Bernoullianis et meis objecit, nunc non attingo, cum illustrissimus *Marchio Hospitalius* et ingeniosissimi *Fratres Bernoullii* tot praeclara inventa sua optime tueri possint.

Quod ad primam objectionem attinet, clarissimus Autor hanc in praefatione Considerationum ponit enunciationem, quam liquidissimae veritatis esse autumat: *Solae eae quantitates aequales sunt, quarum differentia nulla est seu nihilo aequalis.* Et in Analysi curvilineorum, sub initium axiom. § l pa. 10: *Quicquid toties sumi, hoc est per tantum numerum* (etiam infinitum, sic enim intelligit) *multiplicari non potest, ut datam ullam quantitatem, utut exiguam, magnitudine sua aequare valeat, quantitas non est, sed in re Geometrica merum nihil.* Hinc quia in aequationibus pro tangentibus investigandis, Maximisque et Minimis (quam Dn. Autor Barrovio tribuit, primus tamen, ni fallor, Fermatius usurpavit) remanent quantitates infinite parvae, abjiciuntur autem earum quadrata vel rectangula; hujus rei rationem ex eo ducit, quod quantitates ipsae infinite parvae seu infinitesimae sunt aliquid, quoniam per numerum infinitum multiplicatae quantitatem datam (id est, ordinariam val assignabilem) efficiunt; secus autem se habere earum rectangula vel quadrata, quae proinde ex axiomate praemisso sint merum nihil. Ego quidem fateor magni me eorum diligentiam facere, qui accurate omnia ad prima principia usque demonstrare contendunt et in talibus quoque studium non raro posuisse; non tamen suadere, ut nimia scrupulositate arti inveniendi obex ponatur, aut tali praetextu optime inventa rejiciamus, nosque ipsos eorum fructu privemus, quod et olim Patri *Gottignies* et discipulis ejus circa Algebrae principia scrupulosis inculcavi. Caeterum aequalia esse puto, non tantum quorum differentia est omnino nulla, sed et quorum differentia est incomparabiliter parva; et licet ea Nihil omnino dici non debeat, non tamen est quantitas comparabilis cum ipsis, quorum est differentia. Quemadmodum si lineae punctum alterius lineae addas, vel superficiei lineam, quantitatem non auges. Idem est, si lineam quidem lineae addas, sed incomparabiliter minorem. Nec ulla constructione tale augmentum exhiberi potest. Scilicet eas tantum homogeneas quantitates comparabiles esse, cum Euclide lib. 5 defin. 5 censeo, quarum una numero, sed finito multiplicata, alteram superare potest. Et quae tali quantitate non differunt, aequalia esse statuo, quod etiam Archimedes sumsit, aliique post ipsum omnes. Et hoc ipsum est, quod dicitur differentiam esse data quavis minorem. Et Archimedeo quidem processu res semper deductione ad absurdum confirmari potest. Quoniam tamen methodus directa brevior est ad intelligendum et utilior ad inveniendum, sufficit cognita semel reducendi via postea methodum adhiberi, in qua incomparabiliter minora negliguntur, quae sane et ipsa secum fert demonstrationem suam secundum lemmata a me Febr. 1689 communicata. Et si quis talem aequalitatis definitionem rejicit, de nomine disputat. Sufficit enim intelligibilem esse et ad inveniendum utilem, cum ea, quae alia magis (in speciem) rigorosa methodo inveniri possunt, Hac methodo semper non minus accurate prodire sit necesse. Itaque non tantum lineas infinite parvas, ut dx, dy, pro quantitatibus veris in suo genere assumo, sed et earum quadrata vel rectangula dxdx, dydy, dxdy, idemque de cubis aliisque altioribus sentio, praesertim cum eas ad ratiocinandum inveniendumque utiles reperiam. Nec profecto video, quomodo doctissimus Autor in animum suum inducere potuerit, ut statueret, lineam seu latus dx esse quantitatem, at quadratum vel rectangulum talium linearum esse nihil. Licet enim hae quantitates infinities infinite parvae, numero infinito primi gradus multiplicatae, non producant quantitatem datam seu ordinariam, faciunt tamen hoc multiplicatae per numerum infinities infinitum, quem rejicere par non est, si numerum infinitum admittas; prodibit enim numero infinito primi gradus ducto in se. Quod autem in aequationibus Fermatianis abjiciuntur termini, quos ingrediuntur talia quadrata vel rectangula, non vero illi quos ingrediuntur simplices lineae infinitesimae, ejus ratio non est, quod hae sint aliquid, illae vero sint nihil, sed quod termini ordinarii per se destruuntur, hinc restant tum termini, quos ingrediuntur lineae simplices infinite parvae, tum quos ingrediuntur harum quadrata vel rectangula: cum vero hi termini sint illis incomparabiliter minores, abjiciuntur. Quod si termini ordinarii non evanuissent, etiam termini infinitesimarum linearum non minus, quam ab his quadratorum abjici debuissent. Adjungi possunt Lemmata quaedam mea, calculi differentialis fundamentis inservientia, ex Actis Eruditorum Lipsiensibus Febr. 1689, quae Cl. Autor non nisi post editas Considerationes in praefatione Tractatus Analytici sibi occurrisse profitetur, ubi jam tum incomparabilium considerationem adhibui ad has difficultates praeveniendas.

Quod ad secundum attinet, doctissimus Vir aequationes exponentiales (ut a me appellantur) sua methodo tractari posse putat, mea non item. Itaque tali ratione cap. 1 Analys. pag. 62 seqq. et cap. 8 pag. 280 per suam calculandi rationem ostendere conatur, quam tamen usitatis mihi symbolis ratiociniisque sic exprimo. Sit aequatio (ad curvam transcendentem) $y^x = z \quad (1)$, unde alia pari jure, fiet $\overline{y+dy}^{\,x+dx} = z + dz \quad (2)$. Itaque differentiando aequationem (1), id est aequationem (1) ab aequ. (2) subtrahendo, ut dz seu differentia inter duorum z valores (ipsius nempe z et ipsius $z+dz$) habeatur (quod calculi differentialis fundamentum est), utique ex (2) et (1) fiet $\overline{y+dy}^{\,x+dx} - y^x = dz \quad (3)$, sed $\overline{y+dy}^{\,x+dx} = y^{x+dx} + x.y^{x+dx-1}dy \quad (4)$, (quia ut olim in his Actis a me generaliter notatatum est $\overline{y+a}^{m} = y^{m} + \frac{m}{1} y^{\frac{m-1}{\cdot}} a^{1} + \frac{m.m-1}{1.2} y^{\frac{m-2}{\cdot}} a^{2}$ etc.; unde ex sententia Autoris, evanescente termino $\frac{m.m-1}{1.2} y^{\frac{m-2}{\cdot}} a^{2}$ et sequentibus, quia *a* est infinities infinite parva, et pro *a* substituendo dy, et pro litera *m* substituenda $x+dx$ prodit aequ. (4). Itaque ex aequ. (3) per aequ. (4) fit $y^{\frac{x+dx}{\cdot}} + x.y^{\frac{x+dx-1}{\cdot}} dy - y^{x} = dz \quad (6)$. Verum haec ratio exprimendi maximis laborat difficultatiqus, quia non servat leges homogeneorum calculi differentialis, et quod caput est, non exhibet quaesitum, nempe rationem dx ad dy seu subtangentialis ad ordinatam, in terminis ordinariis expressam, neque adeo ductu linearum assignabilium construi potest. Imo redit ad identicum. Nam juxta principium meum supra expositum, quantitas incomparabiliter minor alteri majori frustra additur, et, si haec non evanescat (actu vel virtualiter), ipsamet abjici debet. Itaque in aequ. (6) pro dy, dx, dz additis ad alia incomparabiliter majora scribendo 0, fiet $y^{\frac{x+0}{\cdot}} + x.y^{\frac{x+0-1}{\cdot}} 0 - y^{x} = 0 \quad (7)$, hoc est abjecto 0 pariter, et termino per 0 multiplicato, fiet $y^{x} - y^{x} = 0$, quae aequatio vera quidem, sed identica est, unde talis calculus non prodest. Quale quid ego quoque expertus sum, ut si sit $b^{x} = y$, posita *b* constante, tunc $b^{\frac{x+dx}{\cdot}} = b^{x}$ erit $= dy$; et hanc dividendo dx per b^{x} fit $b^{\frac{dx}{\cdot}} - 1 = dy : b^{x}$ seu $b^{0} - 1 = 0$, seu $b^{0} = 1$, ut constat, ergo fit $1-1=0$. Sed talis identicismus in meo calculo differentiali evitatur. Interim non diffiteor obtulisse se mihi casus, ubi ista quoque calculandi ratio non prorsus negligenda sit. Verum ut videat Cl. Niewentijt meam methodum differentialem ad aequationes quoque, ubi incognita vel indeterminata ingreditur exponentem, (et quidem utiliter) porrigi, quas ego fortasse omnium primus considerandas Geometris proposui, cum meum Tetragonismum Circuli Numericum darem in Actis Eruditorum anni 1682 mens. Febr., attingam hoc loco paucis, quod jam a multis annis habui, et ad summum Geometram Ch*ristianum Hugenium* dudum perscripsi, nempe modum differentiandi aequationes exponentiales, quem Algorithmo meo olim publicato inserere non admodum necesse erat ob talium expressionum raritatem et insolentiam, quae, fateor, tanta est, ut ipse *Hugenius* eas aegre admiserit. Nec quisquam mihi notus est praeter ingeniosissimum *Bernoullium,* qui proprio Marte, me non monente, et ipse in calculo differentiali huc pervenerit atque ad haec penetrarit, quae *Hugenius* per jocum hypertranscendentia appellabat. Nempe sit $x^{v} = y$, fiet $v.\log.x = \log.y$; jam $\log.x = \int, dx : x$ et $\log.y = \int, dy : y$. Ergo $v.\int, dx : x = \int, dy : y$, quam differentiando fit $vdx : x + dv \log.x = dy : y$. Porro v debet dari ex x et y, ambobus vel singulis, ergo scribi potest $dv = mdx + ndy$, et *m* pariter atque *n* dabuntur ex x et y et prodibit: $vdx : x + \underline{\log.x.mdx} = dy : y - \log.x.ndy$, et fiet dx ad dy (seu subtang. ad ordinatam) ut y ad $\overline{\frac{v}{x} + m\log.y}$. Itaque habetur modus ducendi tangentem talis curvae ex supposita hyperbolae quadratura vel Logaritbmis; pro generali autem differentiatione exponentialium sufficit Algorithmo meo hunc canonem ascribi:

$d, x^v = x^v, \frac{v}{x}.dx + dv + dv.\log.x$. Unde si v sit constans numerus ut e, prodit $d, x^e = x^e \frac{e}{x} dx$, id est $e.x^{\frac{e-1}{\cdot}}.dx$, quod est theorema nostri Algorithmi pro differentiatione potentiarum vel radicum dudum traditum.

Superest, ut tertiam Viri Cl. difficultatem paucis absolvam, contra differentiationes scilicet successivas seu quantitates differentio-differentiales. Itaque ipsas ddx non putat admittendas, nec esse quantitates, quia per infinitum numerum multiplicatae non dent quantitatem ordinariam. Sed sciendum est omnino eam prodire, ut ad primam difficultatem jam monui, si numerus multiplicans sit infinitus altioris gradus. Et res sane etiam aliunde multis modis confici potest. Nam quotiens termini non crescunt uniformiter, necesse est incrementa eorum rursus differentias habere, quae sunt utique differentiae differentiarum. Deinde concedit Cl. Autor, dx esse quantitatem; jam duabus quantitatibus tertia proportionalis utique est etiam quantitas; talis autem, respectu quantitatum x et dx, est quantitas ddx, quod sic ostendo. Sint x progressionis Geometricae, et y arithmeticae, erit dx ad constantem dy, ut x ad constantem *a,* seu $dx = xdy : a$; ergo $ddx = dxdy : a$. Unde tollendo $dy : a$ per aequationem priorem fit $xddx = dxdx$, unde patet esse x ad dx, ut dx ad ddx. Et continuata progressione Geometrica etiam reliquae differentiae ulteriores ordine prodeunt. Et generaliter in progressione Geometrica non tantum series differentiarum ejusdem gradus, sed et series transitus seu differentiationum, Geometricae est progressionis. Sed et harum differentiationum successivarum veritas ususque rebus ipsis confirmatur. Nempe, ut jam alias notare memini, quantitas ordinaria, quantitas infinitesima prima seu differentialis, et quantitas differentio-differentialis vel infinitesima secunda, sese habent ut motus et celeritas et sollicitatio, quae est elementum celeritatis. Motu describitur linea, velocitate elementum lineae, sollicitatione (velut initio descensus a gravitate, vel motus a conatu centrifugo) elementum elementi. Et in ipsa Geometria quantitates ordinariae sunt pro vulgari Algebra, differentiales primi gradus referuntur ad tangentes seu linearum directiones, sed differentiales ulterioris gradus ad oscula seu linearum curvedines, quod etiam jam notare memini. Finiam, ubi hoc unum adjecero, mirari me, quomodo doctissimus Niewentijt credere potuerit, ex nostris principiis sequi hoc absurdum, quod in omni curva subtangentialis sit ordinatae aequalis, Consid. p. 19. Sit curvae elementum dc, erit $dxdx + dydy = dcdc$, ut constat; ergo differentiando $dxddx + dyddy = dcddc$. Si jam dc constans fit $ddc = 0$, et fit $dxddx + dyddy = 0$, sed hac differentiali in summatricem rursus versa ait prodire $\frac{1}{2}dxdx = \frac{1}{2}dydy$, adeoque $dx = dy$, quod utique absurdum est. Si talibus uteremur calculis, quomodo eorum ope tot veritates detexissemus? Sed respondeo summando seu versa differentiali in summatricem, prodíturum $\frac{1}{2}dxdx + \frac{1}{2}dydy - \beta dc = 0$, seu constantem areolam esse subtrahendam, alioqui fieret non quidem $dxdx = dydy$, sed potius $-dxdx = dydy$, seu $dy = dx\sqrt{-1}$, quae est aequatio impossibilis, quod indicat β non debere esse 0, sed habere signum −, et esse quantitatem constantem, quae non alia est, quam $\frac{1}{2}dc$, quia ipsam dc posuimus constantem. Unde redit aequatio initio posita $dxdx + dydy = dcdc$, prout oportet. Et simili abusu calculi differentialis laboratur Consid. p. 21 ; nec mirum est hoc modo calculum non esse tutum aut incidere in absurda. Sic et in ipso Tractatu majore seu Analys. inf. c.8 p. 283 ponit triangula characteristica ejusdem curvae, modo numero sint finita et serie non interrupta sese consequantur, esse similia inter se; unde facile infert, positis elementis abscissarum aequalibus, etiam elementa ordinatarum etc. fore aequalia. Sed cum ubique curva directionis suae inclinationem mutet (alioqui non curva, sed recta foret) etiam anguli continue, licet insensibiliter seu per discrimina incomparabiliter parva mutantur. Qua de re me quoque olim ratiocinationes instituere memini. Difficultas quoque objecta Consid. p. 20 contra triangulum, cujus basis est altitudine incomparabiliter minor, ejusdem est commatis: id enim pro isoscele habetur, quia differentia inter altitudinem et hypotenusam incomparabiliter parva est, perinde ac differentia inter radium et secantem anguli infinite parvi. Sed haec sufficere judico, et ipsi Cl. Niewentijt satisfactura spero, qui si ingenium et doctrinam magis ad augenda, quam retractanda haec studia vertere volet, haud dubie praeclara dare poterit, quemadmodum ex his ipsis speciminibus judicare licet.

Additio ad hoc Schediasma.

Unum adhuc addere placet, ut omnis de realitate differentiarum cujuscunque gradus tollatur disputatio, posse eas semper exprimi rectis ordinariis proportionalibus. Nempe sit linea quaecunque, cujus ordinatae crescunt vel decrescunt, poterunt ad eundem axem in iisdem punctis applicari ordinatae secundae ad novam lineam terminatae, proportionales differentiis primi gradus seu elementis ordinatarum lineae primae. Quod si jam idem fiat prosecundis ordinatis, quod factum est pro primis, habebuntur ordinatae ad lineam tertiam, proportionales primarum ordinatarum differentio-differentialibus seu differentiis secundis, seu, quod idem est, secundarum ordinatarum differentiis primis. Et eodem modo etiam differentiae tertiae et aliae quaecunque per quantitates assignabiles exponi possunt. Modum autem differentiis primi gradus proportionales exhibendi rectas ordinarias jam tum explicui, cum primum hujus calculi elementa traderem in Actis Octobris 1684. Nempe inspiciatur ibi fig.III, reperietur dx, elementum abscissae AX vel x, repraesentari per rectam assignabilem in figura separatim positam, et deinde dy, elementum ordinatae XY seu y, repraesentari per rectam quae sit ad dictam dx jam assignatam, ut XY ordinata est ad XD interceptam in axe inter tangentem et ordinatam. Et quoniam eadem opera habetur modus exponendi differentias gradus secundi per proportionales illis differentias gradus primi, et in universum posteriores per praecedentes proximas, patet nullum esse gradum differentialium utcunque remotum, qui non per rectas assignabiles exhiberi tandem queat. Quod si solae darentur differentiae primae, sequeretur omnes ordinatas crescere uniformiter, seu omnem lineam esse rectam. Interdum autem, continuando aliquousque differentiationes, tandem finiendum est, cum nimirum linea differentiarum repraesentatrix, secunda vel tertia vel alia ulterior, fit recta. Nempe si ordinatae primae sint ut abscissae, tunc linea prima est recta et caret differentiis secundis. Si ordinatae primae sint ad parabolam (nempe quadraticam) seu si sint ut quadrata abscissarum, tunc linea secunda erit recta, et linea prima (parabola scilicet) carebit differentiis tertiis. Si ordinatae primae sint ad paraboloeidem cubicam, seu sint ut cubi abscissarum, tunc linea tertia erit recta, et linea prima (paraboloeides scilicet cubica) carebit differentiis quartis, et ita porro. Idem est si ordinatae (primae scilicet) componantur ex ordinatis paraboloeidum dictis, sive per additionem sive per subtractionem; tunc enim finientur tandem differentiae cum altissimae paraboloeidis ingredientibus ordinatis. Sed in caeteris lineis omnibus differentiationes procedunt in infinitum, quoties scilicit in valore ordinatae abscissa in nominatore vel vinculo reperitur. Ex his jam intelligitur, calculum differentialem posse concipi tamquam si fieret non nisi in quantitatibus ordinariis, tametsi origo ex inassignabilibus petenda sit, ut abjectionum seu destructionum ratio reddatur. Itaque si vel ipsa initia calculi a me publicata satis meditatus fuisset Cl. Niewentijt, facile vidisset, non magis de ulterioribus quam de primis differentiis dubitari posse, et vel ideo evitatam tunc a me fuisse mentionem inassignabilium, re ad ordinarias traducta, ut tales scrupuli tollerentur; caeterum si quid notasset animadversione dignum, sensisset me eo esse ingenio, ut libenter dem veritati manus , quemadmodum nunc re accuratius considerata, ea quae Celeberrimus *Jacobus Bernoullius* de numero radicum osculi monuerat probo, quibus quo minus assentirer antea, non alia causa fuit, quam quod diversae occupationes cogitationesque effecerant, ut tardius accederem ad rem de integro satis considerandam. Dum haec scribo, tristem nuntium mortis Viri incomparabilis, *Christiani Hugenii,* accipio. Non poterant majorem jacturam pati literae illae sublimiores, quae humanae menti aditum faciunt in arcana naturae. Ego *Hugenium* solo tempore *Galilaeo* et *Cartesio* postpono. Cum maxima dederit, expectabantur non minora. Et spero inter schedas ejus thesaurum quendam repertum iri, qui nos utcunque soletur. Eoque magis ordunus est Frater ejus, vir meritis in rempublicam illustris, ut maturata editione communi utilitati pariter ac fraternae gloriae, imo suae consulere velit. Oblitus eram eorum quae Dn. Niewentijt contra notam concavitatis vel convexitatis a me allatam objicit, instantia parabolae producta. Sed mirum est ipsum nonanimadvertisse, tantum errore sive scribentis sive typothetae transposita esse verba, et pro concavitate ponendam esse convexitatem, ac vice versa. Itaque non tam afferri debuerat instantia parabolae (quando in omnibus curvis contrarium fit ejus quod verba insinuabant) quam generaliter notari inversio. Adeoque regula sic efferenda est: si crescentibus ordinatis crescant etiam ipsarum differentiae, curva axi obvertet convexitatem, alias concavitatem, posito scilicet aequales inter se esse differentias abscissarum.