XIII.

A Supplement to the Geometry of Measurements, or the Most General of all Quadratures to be Effected by a Motion : and likewise the various constructions of a curve from a given condition of the tangent.

Act. Erudit. Lips. Sept.1693.

The measurement of curves, surfaces and of most volumes, as well as the finding of centres of gravity, may be reduced to the quadratures of plane figures, and hence a *geometry of measurement* arises, on the whole, as I may say thus, in general different from that *prescribed* [by Euclidian geometry], as there only the magnitudes of right lines may enter and hence the points sought may be determined from points given.
[Recall that at this time the idea of a function had not yet been developed, and thus the rectification of a curve as a measure of its length, if possible, was represented generally by a formula derived from integration, which was in this sense a measure of the first ; the same idea could be extended to surfaces and certain volumes, where the quadrature could be found. Here however, Leibniz will discuss a mechanical method of integration.]
And indeed determinate geometry regularly can be reduced to algebraic equations, in which clearly the unknowns may rise to a certain order. But the geometry of measures by its nature does not depend on algebra, and if sometimes it may arise (clearly in the case of ordinary quadratures) that it may be reduced to algebraic quantities; just as determined geometry does not depend on arithmetic, and if it may arise occasionally (evidently in the case of commensurable quantities) so that it may be resolved in terms of numbers or rational quantities. From which we have *three kinds of magnitudes: rational, algebraic,* and *transcendent.* Moreover the origin of *irrational quantities* is from the algebraic ambiguity of a problem or its *multiplicity;* for indeed neither will it be possible for several values to be expressed satisfying the same problem, except by irrational quantities; truly these cannot be reduced to rationality except in special cases. But the source of *transcendent magnitudes is infinity,* thus so that the corresponding *Analysis* of *transcending Geometry* (of which the dimensions is a part) shall be its very self the *science of the infinite.* Again certainly motion may be used for the construction of any algebraic quantity, in which no material curves are present but only straight edges, or, if rigid curves may be present, they should be taken to cross each other or to intersect each other only in a known manner : thus for constructing transcending quantities at this stage the application or assignment of the measure of curves to right lines, just as shall be in describing the cycloid, either in the unwinding or winding of a string or solid line around a surface. For because if one should wish to describe the spiral of Archimedes or the quadratrix of an ancient geometer (that is with a continued exact motion), this can easily be established by measuring a certain right line against a curve, so that the right motion of a circle may be fitted.
[This is a reference to the quadratrix of Dinostratus or Hippias, a curve used in an attempt to square the circle : this curve is the intersection of two moving right lines, the one being a radius of the circle moving with a constant angular speed, and the other a horizontal line traversing the vertical diameter of the circle, also with a constant speed; both motions

to start from the top of the circle. Thus, it is a trigonometric curve from antiquity. See Wikipedia for details.]
Therefore, I am far from excluding these from geometry, even if Descartes did, since the lines described thus are both exact and have the most useful properties, and may be adapted for transcending quantities. Yet there are other accounts of constructing, which may be seen to have some use pertaining to physics: so that if it may be constructed by rays of light by which the problems of geometry may be determined (because often that may bear fruit), or in some manner we may find the quadrature of the hyperbola: either we may have the logarithms constructed equally by a uniform frictional retardation
[*i.e.* The method used by John Napier, where the hypothetical frictional retardation of a moving point is proportional to the speed at that point, from a given starting point],
or with the aid of a heavy rope or chain (with small links), with the curve of the catenary put in place. And indeed if the reasoning has been put together exactly, the theorem will be received into geometry; and if it shall be easy and useful, then it may be used in practice. For the motion has been treated according to a certain geometrical hypothesis constructed from the example of the centre of gravity. Moreover, there is a certain new kind of motion, which I suppose we have been the first to use for geometrical constructions, on an occasion to be discussed soon, since it may be considered to excel by referring to pure geometry, and it shall be of a relation of the lines described by a thread from a centre or focus, whenever nothing else is required from that, so that as the point describes a curve in a plane, for one end of the thread to be put bound to that in the same plane (or equivalently), while the other end of the thread may be moved by a motion, but only by being pulled, and truly not by a transverse impulse, which must not be expected from the thread on account of its flexibility, but rather may be drawn in the direction of the tension of the string or the pulling direction, which by itself arises if no impediment occurs along the way. Yet because a material thread, since at no time may it have as great a flexibility as geometry supposes, the stylus or describing point (certainly placed freely in a plane) may easily be able to act a little transversely, thus so that the motion of the stylus may not be left exposed from the traction; thus the impediments of the material conveniently is met by remedy of the material, so that clearly the cause shall be, by which the describing point may be made or leaned on a little, either to stick to the point in the plane to which it belongs, by such a reason a weight can be leaning on the describing point, or jointly, by which this point may be pressed onto the horizontal plane, at which to be moving and it must describe the curve. Thus if the resistance from the leaning, from which it arises so that the stylus may not be moved easily from place to place, it may prevail generally for the little remaining resistance to depend on the stiffness of the string, which rather may concede and the string may be extended; and thus it will be acted on by traction, not by an impulse, because that one is required here with respect to the point being described. Hence moreover it comes about, that such a motion shall be adapted wonderfully well to the geometry of transcending curves, because at once it refers back to tangent lines, or the directions of the curves, and thus to elementary quantities, indeed infinite in number, moreover with unassignable or of infinitely small magnitude.
[A reference to Leibniz's idea that a curve is a polygon with an infinite number of sides.]

Moreover the occasion of the working out of such a construction presented itself to me in Paris. *Claude Perrault*, the eminent Parisian doctor, distinguished both in the studies of

Mechanics and Architecture and for his well-known edition of *Vitruvius*, and likewise while he lived, a prominent member of the French Royal Society of Science, proposed this problem to me and to many others before me, of which he had not yet acknowledged an ingenious solution to have occurred for that [the *Tractrix* or pursuit curve]: to find the curve *BB* (fig. 139) which a weight describes in the horizontal plane with the point *B* [the initial value is $_1B$, with other points to follow, 2, 3, 4, etc.] or an equivalent point for a string or a chain [with small links] *AB* tied [to the weight at the end *B*], while the other end *A* of the chain *AB* is led along a right fixed line *AA*, [the initial value is $_1A$, with the other points 2, 3, 4, etc.] by which we draw the weight itself along a direction in the horizontal plane, on which, or in the equivalent case also of the right line *AA*, the links fall into place during the motion of the string *AB*. Moreover he was using (in order to be understood) his silver watch-case with his watch inside *B*, so that from *A* of the small chain *AB* attached to the case, from the beginning it was led along the table, drawn along the straight line *AA*. Thus the lowest point of the watch-case (which was in the middle on the bottom) described the line *BB* on the table. Considering this line more carefully, (since then I was greatly interested in considering tangents) I suddenly noticed what the problem was about, the string or the right line, was always touching the curve, such as at $_3A_3B$, to be the tangent of the curve *BB* at the point $_3B$. And which I show thus: With centre $_3B$ and with the string $_3A_3B$ as the radius, the arc $_3AF$ of a circle is described however small, thence the string $_3BF$, taken at *F*, directly or along its appropriate remaining direction, is drawn as far as $_4A$, thus so that $_3BF$ may be transferred into $_4B_4A$; and thus as if the process were applied similarly to the point $_3B$, as to the points $_1B$, $_2B$, the point *B* described everywhere the polygon $_1B_2B_3B$ etc., the sides of which always lie on the string, from which an indefinitely small arc could be found, such as $_3AF$, and by vanishing finally, which comes about in the continued motion of pulling, such is the description of our curve, where the lateral displacement of the string is continued, but it is always *unassignable* [*i.e. undetermined*], and it is evident that the polygon becomes a curve, the tangent of which is the string. And thus I saw the matter returned to that problem of the converse tangent: to find the curve *BB* of which kind, so that the part *AB* of the tangent intercepted between the axis *AA* and the curve *BB* shall be equal to a given constant [*i.e.* the length of the string pulling the watch-case]. Nor was it difficult for me to grasp, the description of this line can be related to the quadrature of the hyperbola. Without doubt with the centre *C* or *A* (where the string $_1A_1B$ at the same time is the ordinate and the

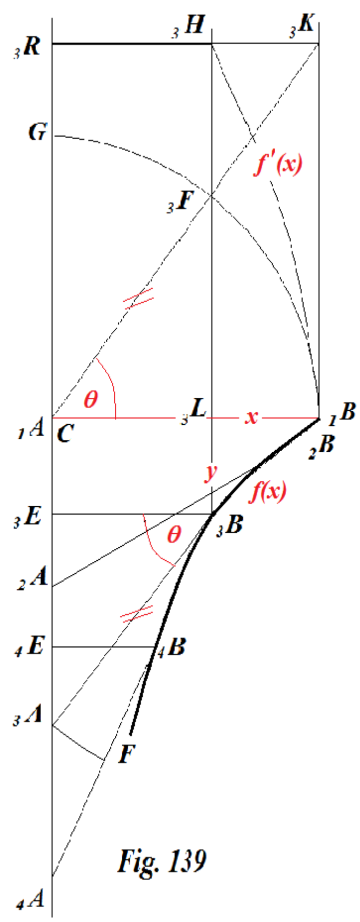

Fig. 139

tangent of the curve), truly with the radius *AB* the circle ${}_1BFG$ is described, crossing the axis *AE* at *G* , and for this ${}_1BK$ shall be parallel to the axis, to which from *C* draw *CF* crossing at *K*, ${}_1BK$ will be the tangent of the arc of the circle ${}_1BF$. Now through *F* there may be drawn *FLB*, parallel to the axis *AE*, crossing ${}_1A_1B$ itself at *L*, and the curve *BB* at *B*, taking on which *LH* equal to ${}_1BK$, and by proceeding in the same manner everywhere, the curve of the tangents ${}_1BHH$ will be produced, and the rectangle ${}_1B_1AE$ will be found to be equal to the figure of the tangents, or to the area of the trilateral form ${}_1BLH_1B$; for example, the product ${}_1B_1A$ into ${}_1A_3E$ equals the trilateral ${}_1B_3L_3H_1B$. Therefore since the area of the figure of the tangent may be shown by the quadrature of the hyperbola or by logarithms, as is well known, it is apparent with its aid also to obtain ${}_1A_3E$ or ${}_3L_3B$, and thus any point ${}_3B$ of the curve *BB*. Hence in turn from a given description of the line *BB* the quadratures of the hyperbola or logarithms may be constructed.

I shall not linger with further explanations, since I may consider especially the same to have been best presented by the most celebrated of men, *Christian Huygens*, who thus not long ago indicated to me by letter to have come upon a singular account of the squaring of the hyperbola, which also most recently was published in the *Histoire des Ouvrages des Savants*, and this itself I gather from these, which recently are presented by the most excellent Bernoulli brothers given in the *Acta Eruditorum*, where from the opportunity afforded from the same work of Huygens, a similar motion was shown to be transferred beautifully to describing a curve, where a portion of the tangent intercepted between the curve and the axis is to a portion of the curve between a fixed point and the crossing of the tangent, or as *AB* to *CA* (in fig. 139*a*) as a constant right line to another constant right line. Which has reminded me too finally to publish my own old material of this kind.

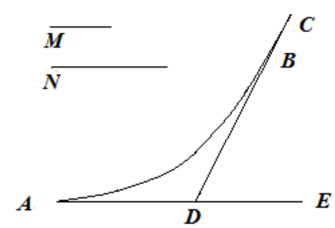

Fig. 140 a

[Reference is made here finally to an article by Johann Bernoulli in the May 1693 *Acta*, to a problem proposed by Descartes, Fig. 140*a* : *Find the curve ABC, so that from any point on the curve B, the tangent crosses the axis AE at D, so that the abscissa at D satisfies* $AD : BD = M : N$ *always*.]

[Previous to this, from Fig. 139, we see that if the tractrix curve is denoted by $y = f(x)$, then in modern terms from the diagram, the upper curve is $\frac{dy}{dx} = f'(x) = \tan\theta$ for the abscissa *x*, or $\tan\theta = \frac{{}_1BK}{{}_1A_1B}$; hence we can write the coordinates of *HH* in the form $x = AB(1-\cos\theta)$ and $y = AB\tan\theta$. The area under the upper curve *HH* is given by :

$$\int ydx = AB^2\left[\int\frac{d\theta}{\cos\theta} - \int\cos\theta d\theta\right] = AB^2\left[\log\left|\tan(\frac{\theta}{2}+\frac{\pi}{4})\right| - \sin\theta\right],$$

which is equal to the value of the corresponding rectangle *xy* for *BB*, using the form adopted earlier from the work of Tschirnhaus. Thus the coordinates for the tractrix are of the form :

$x = a(1-\cos\theta)$ and $y = a\left[\log\left|\tan(\frac{\theta}{2}+\frac{\pi}{4})\right| - \sin\theta\right]$, where θ varies between 0 and $\frac{\pi}{2}$.]

Clearly it was possible to understand at once, from the relation of the motion to the tangents, how innumerable other curves, thus not able to be reduced easily to quadrature, could be constructed by the same method. For even if *AA* may not be a right line, but curved, nevertheless the string itself may be a tangent to *BB*. Furthermore, even if the string *AB* were to increase or decrease between being pulled, thus none the less the tangent shall remain. And so accordingly if some relation may be given between *CA* and *AB* (for example, so that if *AB* were the sines arising, and *CA* the corresponding tangents of the same angle) the motion of the string may be able to be guided by diverse means, and so that the point may advance according to a given rule understood between the curves. Also an infinitude of curves satisfying the same problem can be drawn by this construction, through any point given, if it is wished. But if the describing point may be drawn by several strings at the same time, the direction used can be composite. But even if there shall be only one string, its length can be varied, for the weight *B* may itself be connected to a wheel, or a part rotated of such a kind that it describes a cycloid in the plane. Also the rigid right line may be attached normal to the string always, or given having a constant angle, or an angle varied according to a certain rule, carried around by *B*, in which also it can be understood a moving point to be describing that curve. Also two weights likewise can be drawing [at an angle to each other] in a plane, either maintaining the same distance, or also with that varied during the motion. Also two planes can be considered, one in which the point *C* is moving and with that firmly leaned on, the other in which the stylus moving forwards from *B* with a light attachment (nothing from that is going to disturb *B*'s motion) describing the new curve, and this plane may have its own proper motion, and the tangent of the new curve designating that right direction of motion composed from the motion of the stylus in the fixed plane and of motion of the other plane. From which again the properties of the new tangent curves thus described will be determined. And thus since this kind of motion may be spread out laterally and it may undertake many applications, at one time I filled many pages of paper in considering that matter, and I have learned also from practice about being cautious, especially because I saw a use both for the converse of tangents as well as quadratures in the first place. Therefore when I had found a *construction*, generally extending itself to all quadratures, for which I know not of another thence arising from geometry which may be thought out fuller, than that which I have decided to publish. For even with that part, of just as many parts of the whole work [which I have undertaken], such as for example I might preserve the scientific matter, yet more and more kinds [of problems and ideas] have arisen thereafter, so that with the opportunity finally to discharge some old material, to be better done with that lest matter lost, and indeed after a sufficiently long time, twice as long in fact as the limits imposed by *Horace* [on his fellow poets], expected by *Lucina*.

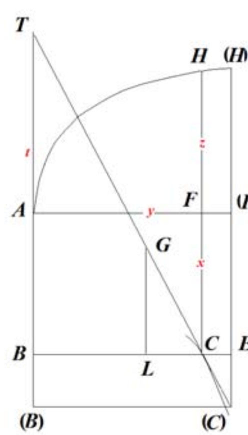

Fig. 140

[*Lucina* was the goddess protecting women in childbirth in Roman mythology, called upon by *Horace* who advised poets to wait nine years before publishing their work. Thus, Leibniz has waited 18 years before publishing this work.]

Moreover I will show, how *a general problem of quadratures can be reduced to the finding of a given curve having a given rule of the gradient*, or in which the sides of the characteristic triangle may have a given relation between each other, then I will show that *this curve can be described by a motion considered by us.* Without doubt (fig. 140) in the whole curve *C*(*C*) I consider a *two-fold characteristic triangle*: with *TBC* assignable, and *GLC* unassignable, similar to each other. And indeed the unassignable may be taken with the elements themselves *GL, LC* from the legs of the coordinates *CB, CF*, and *GC* for the element of the arc, as the base or hypotenuse. But the assignable *TBC* is taken between the axis, the ordinate and the tangent, and thus expresses the angle, which the direction of the curve (or its tangent) makes to the axis or base, and this is the slope of the curve at the proposed point *C*. Now the region requiring to be squared shall be *F*(*H*), taken between the curve *H*(*H*), the two parallel right lines *FH* and (*F*)(*H*) and the axis *F*(*F*) ; on this axis with the point *A* assumed fixed, *AB* may be drawn through *A* normal to *AF* as the conjugate axis, and on some *HF* (produced as needed) the point *C* may be taken: or a new line comes about *C*(*C*), which shall of be this kind, so that with both the conjugate ordinate *CB* (equal to *AF*) as well as the tangent *CT* drawn from the point *C* to the conjugate axis *AB* (if there is a need for it to be produced), the part of this axis taken *TB*, the ratio between this shall be to *BC*, as *HF* to a constant *a*, or *a* by *BT* is equal to the rectangle *AFH* (circumscribed around the trilateral *AFHA*). With these in place, I say the rectangle under *a* and *E*(*C*) (with the ordinates of the curve *FC* and (*F*)(*C*) discriminated between each other) to equal the region *F*(*H*); and thus if the curve *H*(*H*) produced may be incident at *A*, the trilateral *AFHA* of the figure squared to be equal to the rectangle under the constant *a* and the ordinate of the figure of the quadratrix *FC*. Our calculus shows the matter at once. For there shall be *AF*,*y*; and *FH*,*z*; *BT*,*t*; and *FC*,*x*; there will be $t = zy : a$ from the hypothesis; again $t = ydx : dy$ the nature of the tangents expressed from our calculus ; therefore $adx = zdy$, and thus $ax = \int zdy = AFHA$. Therefore the curve *C*(*C*) is the quadratix with respect of the line *H*(*H*), since the ordinate *FC* of *C*(*C*), multiplied by the constant *a*, shall make a rectangle of equal area, or the sums of the ordinates of *H*(*H*) owed to the abscissas of the applied *AF*. Hence since *BT* shall be to *AF*, as *FH* to *a* (from the hypothesis) and the relation of *FH* to *AF* is given (showing the nature of the squared figure), and therefore the relation *BT* to *FH* as well as to *BC* is given, and thus also the relation *BT* to *TC*, that is the relation between the sides of the triangle *TBC*. Thus for all the quadratures and thus also for the dimensions being effected there is such a need for a given relation of the sides of the assignable characteristic triangle *TBC*, or with a given rule of the gradient of the curve, to be able to describe the curve *C*(*C*), which was shown to be squared.

In fig.141 the right angle *TAH* shall remain fixed and placed in the horizontal plane, in the leg of which *AT* a hollow vertical cylinder *TG* moves forward, projecting below the above mentioned horizontal plane, within which there shall be a solid cylinder *FE* moving freely both upwards and downwards, having tied a string *FTC* at the end *F*, thus so that the part *FT* shall be within the hollow cylinder, the part *TC* in the said horizontal plane. Again to the end *C* of the string *TC* the point *C* shall be pressed onto the plane by its weight, and there describing the curve *C*(*C*), moreover the start of the motion will be in the hollow cylinder *TG*, while it is drawn receding along *AT* from *A*, attracting *C*. Truly the describing point or the stylus *C* before it may advance along *HR*, proceeds towards *A* along the right line *AH* in the same horizontal plane (the other arm of the fixed right angle *TAH*), the forwards motion of which is not impeded, so that with less forwards force the point *C* may be moved by the traction of the string only and thus it may move preserving its direction in the motion. Truly there shall be a certain table *RLM*, meeting the straight edge *HR* normally at its same point *R*, the other table being pushed [sideways] continually by the hollow cylinder [rolling along with a uniform speed], thus so that *ATHR* shall be a rectangle. And then on this table the rigid

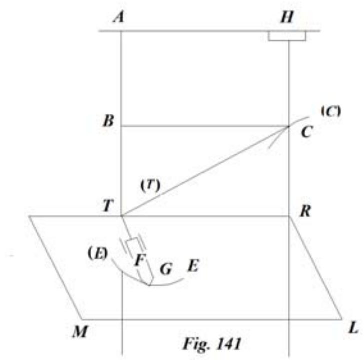

Fig. 141

curve *EE* shall be described (on a thin sheet present, if needed), which the solid cylinder *FE* is tracing out [moving along without turning around with negligible friction inside the hollow cylinder], so that the end *E* is understood always to be leaving marks, thus so that the cylinder *FE* may be dropping as *R* approaches *T*,. Therefore as the quantity $ET + TC$ shall be given (evidently composed from the solid cylinder *EF* and from the whole string *FTC*) and there shall be a given relation between *TC* and *TR* or *BC* (from the given rule of the gradient of the curve), and the relation between *ET* and *TR* will be had, the ordinate and the abscissa of the curve *EE*, of which hence the nature and description can be had from ordinary geometry on the table *LRM* ; therefore also the description of the curve *C*(*C*) may be had by the machine presentation. Moreover *TC* always is a tangent to the curve *C*(*C*) from the nature of our motion, and thus the line *C*(*C*) has been described, where the condition of the gradient or the relation of the sides of the assignable characteristic triangle *TRC* or *TBC* has been given. Which curve since it shall be the quadratrix of the given figure squared, as has been shown a little before, the quadrature or the dimension sought will be obtained.

Likewise in various ways the problems for the converse method of the tangent can be adapted, just as if the point *T* were in motion along the curve *TT* (in place of the right line *AT*), also the coordinate *BC* (or the abscissa *AB*) would be entering into the calculation. And for any reasonable problem for the converse of the tangent it can be reduced to a relation between three lines, surely the two coordinates *CB*, *CH* and the tangent *CT*, or other functions in place of these. But often the thing can be made into a much simpler motion. Just as if the relation were given between *AT* and *TC* (which is to find the curve *C*(*C*) intersecting circles at right angles, with the positions of the circles given in order), a smaller apparatus would suffice. For with these ending which fall at *H* and *R*, it will be sufficient to describe a rigid right line directrix *EE* in the vertical plane at rest passing through *AT*. Thus with the point *T* moving forwards along the fixed line *AT* or through the hollow cylinder *TG*, and with the solid cylinder *TE* descending, just as the given directrix line *EE* orders, as the cylinder may remain, everywhere on account of the constant sum $ET + TC$ (as before) and the relation between *AT* and *TC* given, the relation between *AT* and *TE* ought to be found easily, or the nature of the curve *EE*, with the aid of which the curve described *C*(*C*) will be sought.

XIII.

SUPPLEMENTUM GEOMETRIAE DIMENSORIAE, SEU GENERALISSIMA OMNIUM TETRAGONISMORUM EFFECTIO PER MOTUM: SIMILITERQUE MULTIPLEX CONSTRUCTIO LINEAE EX DATA TANGENTIUM CONDITIONE.

Act. Erudit. Lips. Sept.1693.

Dimensiones linearum, superficierum et solidorum plerorumque, ut et inventiones centrorum gravitatis reducuntur ad tetragonismos figurarum planarum, et hinc nascitur *Geometria dimensoria,* toto, ut sic dicam, genere diversa a *determinatrice,* quam rectarum tantum magnitudines ingrediuntur atque hinc quaesita puncta ex punctis datis determinantur. Et Geometria quidem determinatrix reduci potest regulariter ad aequationes Algebraicas, in quibus scilicet incognita ad certum assurgit gradum. Sed dimensoria sua natura ab Algebra non pendet, etsi aliquando eveniat (in casu scilicet quadraturarum ordinariarum) ut ad Algebraicas quantitates revocetur; uti Geometria determinatrix ab Arithmetica non pendet, etsi aliquando eveniat (in casu scilicet commensurabilitatis) ut ad numeros seu rationales quantitates revocetur. Unde *triplices* habemus *quantitates: rationales, Algebraicas* et *transcendentes.* Est autem *fons irrationalium* Algebraicarum *ambiguitas* problematis seu *multiplicitas;* neque enim possibile foret, plures valores eidem problemati satisfacientes eodem calculo exprimere, nisi per quantitates radicales; eae vero non nisi in casibus specialibus ad rationalitates revocari possunt. Sed *fons transcendentium* quantitatum est *infinitudo,* ita ut *Geometriae transcendentium* (cujus pars dimensoria est) respondens *Analysis* sit ipsissima *scientia infiniti.* Porro quemadmodum ad construendas quantitates Algebraicas certi adhihibentur motus, in quibus aut non intersunt curvae materiales, sed tantum regulae rectilineae, aut , si curvae rigidae interveniunt, non tamen nisi ratione occursuum seu intersectionum usurpari debent: ita ad construendas quantitates transcendentes hactenus adhihita est applicatio seu admensuratio curvarum ad rectas, uti sit in descriptione cycloeidis, aut evolutione fili vel solii lineae vel superficiei circumligati. Quin et si quis spiralem Archimedis aut Quadraticem Veterum Geometrice (hoc est motu continuo exacto) describere velit, hoc facile praestabit quadam rectae ad curvam admensuratione, ut motus rectus circulari attemperetur. Minime igitur haec excludo ex Geometria, etsi id fecerit *Cartesius,* cum lineae sic descriptae et exactae sint et utilissimas habeant proprietates, et transcendentibus quantitatibus sint accommodatae. Sunt tamen et aliae construendi rationes, quae aliquid Physici videntur habere admistum: ut si quis problemata Geometriae determinatricis construeret per radios lucis (quod saepe cum fructu fieri posset) aut quemadmodum nos aream Hyperbolae quadravimus, vel logarithmos construximus motu composito ex aequabili et per frictionem uniformem retardato, vel ope chordae sive catenae pondere praeditae lineam catenariam vel funicularem (la chainette) constituentis. Et quidem si exacta sit construendi ratio, recipitur in Geometriae theoriam; si facilis sit utilisque, potest recipi in praxin. Nam et motus secundum certas hypotheses factus Geometricae est tractationis, exemplo centri gravitatis. Est autem novum quoddam motus genus, quem nos opinor primi ad constructiones Geometricas adhibuimus, occasione mox dicenda, cum prae caeteris videatur referri posse ad puram

Geometriam, affinisque sit descriptioni linearum per fila ex umbilicis sive focis, quandoquidem in eo nihil aliud requiritur, quam ut punctum, lineam in plano describens , ad unam extremitatem fili in eodem plano (vel aequivalente) positi alligatum, moveatur altera extremitate fili mota, sed non nisi per tractionem, non vero per impulsum in transversum, qui nec a filo ob flexibilitatem debet expectari, trahatur autem in ipsius fili tensi seu trahentis directione, quod per se evenit, si nullum in itinere occurrat impedimentum. Quoniam tamen filum materiale, cum nunquam habeat summam flexibilitatem, quam Geometria supponit, facile stylum seu punctum describens (quippe in plano libere positum) nonnihil in transversum agere posset, ita ut motus styli non esset nuda tractatio; ideo impedimenta materiali commode occurritur remedio materiali , ut scilicet causa sit, quae punctum describens nonnihil faciat vel apprimi, vel adhaerere loco plani cui inest, qualis causa esse potest pondus puncto describenti incumbens, seu conjunctum, quo ipsum hoc punctum apprimetur plano horizontali, in quo moveri lineamque describere debet. Ita si resistentia incumbentis, qua fit, ut non facillime loco moveatur, praevaleat omnino exiguae illi residuae in filo rigiditati, potius cedet filum atque intendetur; atque ita aget tractione, non impulsu, quod unum hoc loco requiritur respectu puncti describentis. Hinc autem fit, ut talis motus mire sit accommodatus ad Geometriam transcendentem, quia immediate refertur ad lineae tangentes, vel directiones, adeoque ad quantitates elementares, numero quidem infinitas, magnitudine autem inassignabiles seu infinite parvas.

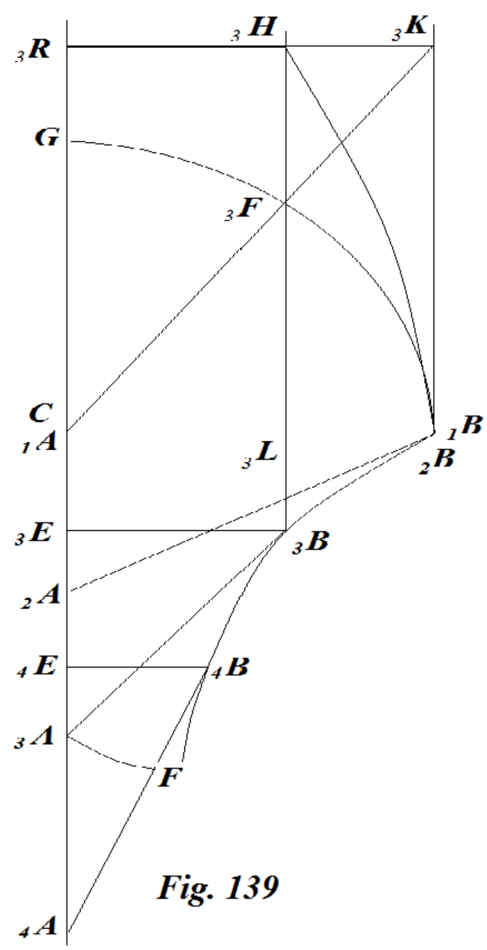

Fig. 139

Hujus autem Constructionis excogitandae talis mihi olim occasio Lutetiae praebita est. *Claudius Perraltus* , Medicus Parisinus insignis, tum et Mechanicis atque Architectonicis studiis egregius et Vitruvii editione notus, idemque in Regia Scientiarum Societate Gallica, dum viveret, non postremus, mihi et aliis ante me multis proposuit hoc problema, cujus nondum sibi occurrisse solutionem ingenue fatebatur: invenire lineam *BB* (fig. 139), quam pondus fili vel catenulae *AB* extremitati *B* annexum, puncto *B* vel aequivalente describat in plano horizontali, dum alteram fili *AB* extremitatem *A* ducendo per rectam immotam *AA*, eo ipso pondus *B* trahimus per directum planum horizontale , in quod vel aequivalens etiam recta *AA* et durante motu filum *AB* cadunt. Utebatur autem (intelligentiae causa) horologio portatili suae thecae argenteae incluso *B*, quod catenulae *AB* ad thecam alligatae principio *A*, secundum regulam *AA* ducto, per tabulam trahebat. Ita imum thecae punctum (quod in fundi medio est) in tabula describebat lineam *BB*. Hanc lineam ego attentius considerans (cum tunc maxime in tangentium contemplatione versarer) statim animadverti, quod res est, filum perpetuo lineam tangere, seu rectam, ut $_3A_3B$, esse tangentem lineae *BB* in puncto $_3B$. Quod et sic demonstrator: Centro $_3B$ et filo $_3A_3B$ tanquam radio describatur arcus circuli utcunque parvus $_3AF$, inde filum $_3BF$, apprehensum in *F*, directe seu per sua propria vestigia trahatur usque ad $_4A$, ita ut ex $_3BF$ transferatur in $_4B_4A$; itaque si ponatur similiter fuisse processum ad puncta $_1B$, $_2B$, ut ad punctum $_3B$, utique punctum *B* descripsisset polygonum $_1B_2B_3B$ etc. cujus latera semper incident in filum, unde imminuto indefinite arcu, qualis erat $_3AF$, ac tandem evanescente, quod fit in motu tractionis continuae, qualis est nostrae descriptionis, ubi continua, sed semper *inassignabilis* fit circumactio fili, manifestum est, polygonum abire in curvam, cujus tangens est filum. Itaque videbam rem redire ad hoc problema conversae tangentium: invenire lineam *BB* ejus naturae, ut *AB* portio tangentis inter axem *AA* et curvam *BB* intercepta sit constanti datae aequalis. Nec difficile mihi fuit deprehendere, hujus lineae descriptionem ad quadraturam Hyperbolae revocari posse. Nimirum centro *C* vel *A* (ubi filum $_1A_1B$ simul est ordinata et tangens curvae), radio vero *AB* describatur circulus $_1BFG$, axi AE occurrens in *G* , et huic axi parallela sit $_1BK$, cui ex *C* educta *CF* occurrat in *K*, erit $_1BK$ tangens arcus circularis $_1BF$. Jam per *F* ducatur *FLB*, parallela axi *AE*, occurrens ipsi$_1A_1B$ in *L*, et curvae *BB* in *B*, in qua producta sumatur *LH* aequalis ipsi $_1BK$, idemque ubique faciendo, prodibit linea tangentium $_1BHH$, et rectangulum $_1B_1AE$ reperietur aequari figurae tangentium, seu areae trilineae $_1BLH_1B$; verbi gratia $_1B_1A$ in $_1A_3E$ producet aequale trilineo $_1B_3L_3H_1B$. Cum igitur figurae tangentium area exhiberi possit per quadraturam Hyperholae vel Logarithmos, ut notum est, patet ejus ope etiam haberi $_1A_3E$ seu $_3L_1B$, adeoque punctum curvae ut *BB*. Vicissim hinc data descriptione lineae *BB* quadratura Hyperbolae vel Logarithmi construentur. Quibus ulterius explicandis non immoror, cum praesertim arbitrer idem optime praestitisse *Christianum* Hugenium, Virum celeberrimum, qui mihi non ita pridem per literas significaverat incidisse sibi singularem Hyperbolae quadrandae rationem, quam etiam in Historia Operum Eruditorum publicatam nuperrime, et hanc ipsam esse colligo ex iis, quae nuper a praestantissimis fratribus Bernoulliis data exhibentur in Actis Eruditorum, ubi Hugenianorum istorum occasione, motum similem apparet pulchre transferri ad describendam lineam, ubi portio tangentis intercepta inter curvam et axem est ad portionem axis inter punctum fixum et occursum tangentis, seu *AB* ad *CA* (in fig. 139) ut recta constans ad aliam rectam constantem. Quae me quoque veterum in hoc genere meorum tandem edendorum admonuere.

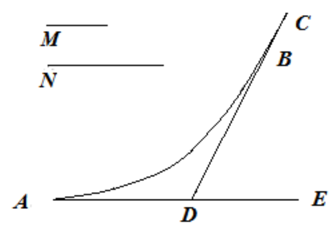

Fig. 140

Pronum scilicet statim fuerat intelligere, percepta semel relatione motus ad tangentes, innumeras alias lineas, non ita facile ad Quadraturam revocabiles, hac eadem arte construi posse. Nam etsi *AA* non recta esset, sed curva, non ideo minus filum ipsam *BB* tangeret. Quin amplius, etsi filum *AB* inter trahendum cresceret aut decresceret, non ideo minus tangens maneret. Itaque si data utcunque esset relatio inter *CA* et *AB* (verbi gratia ut *AB* existentibus sinubus, cssent *CA* tangentes ejusdem anguli) variis machinationibus moderari motum fili licet et, ut data lege inter contrahendum promoveretur. Infinitae etiam lineae eidem problemati satisfacientes hac construendi ratione duci possunt, quaelibet per punctum, si lubet, datum. Quodsi punctum describens a pluribus filis simul trahatur, composita directio poterit adhiberi. Sed etsi unum tantum sit filum, poterit ejus longitudo variari, ipsi ponderi *B* annexa existente rota vel figura per modum describendae cycloeidis in plano voluta. Recta etiam rigida ad filum semper normalis, vel datum aut certa lege variablem angulum habens, cum *B* ferri potest, in quo etiam intelligi potest moveri punctum describens aliud. Possunt etiam duo pondera plano innitentia simul trahi, sive eandem semper distantiam servantia, sive etiam durante motu eam variantia. Possunt etiam duo plana intelligi, unum in quo movebitur punctum *C* eique firmiter innitetur, alterum in quo stylus ex *B* egrediens levissimo attactu (nihil ad eo motum ipsius *B* turbaturo) describat lineam novam, et hoc planum suum habeat motum proprium, eritque lineae novae tangens ipsa recta designans directionem motus compositi ex motu styli in plano immoto et motu plani alterius. Unde rursus tangentium lineae novae sic descriptae determinabuntur proprietates. Itaque cum hoc motuum genus latissime diffundatur et innumeras applicationes recipiat, multa olim chartae folia meditando in eam rem implevi, ac de cautionibus etiam practicis cogitavi, praesertim quia usum tam insignem ad tangentium conversam et inprimis ad Tetragonismos videbam. Cum ergo *constructionem* repererim, generaliter sese extendentem ad omnes quadraturas, qua nescio an alia amplior inde a nata Geometria excogitata sit, eam publicare tandem constitui. Tametsi enim ista hactenus in justi operis integraeque velut scientiae materiam servaverim, tam multa tamen alia alleriusve generis subnascuntur, ut veteribus quacunque occasione defungi tandem praestet, ne intercidant, et satis diu ista, ultra Horatiani limitis duplum pressa, Lucinam expectarunt.

Ostendam autem, *problema generale Quadraturarum reduci ad inventionem lineae datam habentis legem declivitatum*, sive in qua latera trianguli characteristici assignabilis datam inter se habeant relationem, deinde ostendam, *hanc lineam per motum a nobis excogitatum describi posse.* Nimirum (fig. 140) in omni curva *C*(*C*) intelligo *triangulum characteristicum duplex:* assignabile *TBC*, et inassignabile *GLC*, similia inter se. Et quidem inassignabile comprehenditur ipsis *GL*, *LC*, elementis coordinatarum *CB*, *CF* tanquam cruribus, et *GC*, elemento arcus, tanquam basi seu hypotenusa. Sed assignabile *TBC* comprebenditur inter axem, ordinatam et tangentem, exprimitque adeo angulum, quem directio curvae (seu ejus tangens) ad axem vel basin facit, hoc est curvae declivitatem in proposito puncto *C*. Sit jam zona quadranda *F*(*H*), comprehensa inter curvam *H*(*H*), duas rectas parallelas *FH* et (*F*)(*H*) et axem *F*(*F*) ; in hoc axe sumto puncto fixo *A*, per *A* ducatur ad *AF* normalis *AB* tanquam axis conjugatus, et in quavis *HF* (producta prout opus) sumatur punctum *C*: seu fiat linea nova *C*(*C*), cujus haec sit natura, ut ex puncto *C* ducta ad axem conjugatum *AB* (si opus productum) tam ordinata conjugata *CB* (aequali *AF*) quam tangente *CT*, sit portio hujus axis inter eas comprehensa *TB* ad *BC*, ut *HF* ad constantem *a*, seu *a* in *BT* aequetur rectangulo *AFH* (circumscripto circa trilineum *AFHA*). His positis, aio rectangulum sub *a* et sub *E*(*C*) (discrimine inter *FC* et (*F*)(*C*) ordinatas curvae) aequari zonae *F*(*H*); adeoque si linea *H*(*H*) producta incidat in *A*, trilineum *AFHA* figurae quadrandae aequari rectangulo sub *a* constante et *FC* ordinata figurae quadratricis. Rem noster calculus

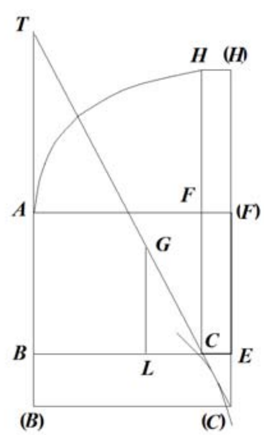

Fig. 140

statim ostendit. Sit enim *AF,y*; et *FH,z*; et *BT,t*; et *FC,x*; erit $t = zy : a$ ex hypothesi; rursus $t = ydx : dy$ ex natura tangentium nostro calculo expressa; ergo $adx = zdy$, adeoque $ax = \int zdy = AFHA$. Linea igitur *C*(*C*) est quadratix respectu lineae *H*(*H*), cum ipsius *C*(*C*) ordinata *FC*, ducta in *a* constantem, faciat rectangulum aequale areae, seu summae ordinatarum ipsius *H*(*H*) ad abscissas debitas *AF* applicatarum. Hinc cum *BT* sit ad *AF*, ut *FH* ad *a* (ex hypothesi) deturque relatio ipsius *FH* ad *AF* (naturam exhibens figurae quadrandae), dabitur ergo et relatio *BT* ad *FH* seu ad *BC*, adeoque et relatio *BT* ad *TC*, id est relatio inter latera trianguli *TBC*. Itaque ad omnes quadraturas adeoque et ad dimensiones efficiendas tantum opus data relatione laterum trianguli characteristici assignabilis *TBC*, seu data lege declivitatum curvae, posse describere curvam *C*(*C*), quam ostensum est esse quadratricem.

Haec descriptio ita fiet: In figur. 141 sit angulus rectus *TAH* immotus et in plano horizontali positus, in cujus crure *AT* precedat cylinder cavus verticalis *TG*, infra dictum planum horizontale prominens, in quo sit sursum deorsumque mobilis cylinder solidus *FE*, in summitate *F* alligatum habens filum *FTC*, ita ut pars *FT* sit intra cylindrum cavum, pars *TC* in dicto plano horizontali. Porro ad fili *TC* extremitatem *C* sit punctum pondere sibi incumbente eidem plano innitens, atque in eo describens lineam *C*(*C*), initium autem motus erit in cylindro cavo *TG*, qui dum ducitur per *AT* recedens ab *A*, attrahet *C*. Punctum vero describens seu stylus *C* ante se protrudat *HR*, regulam in eodem plano horizontali normaliter ad *AH* (alterum crus anguli recti immobilis *TAH*) incedentem versus *A*, quae protrusio non impedit, quo minus protrudens punctum *C* sola tractionc fili moveatur adeoque ejus directionem in motu servet. Sit vero et tabula quaedam *RLM*, eodem sui puncto *R* normaliter incidens ad regulam *HR*, caeterum propulsa continue a cylindro cavo, ita ut *ATHR* sit rectangulum. Denique in hac tabula sit descripta (per laminam extantem, si placet) linea rigida *EE*, quam cylinder solidus *FE* incisura, quam in extremitate *E* habere intelligitur, semper mordeat, ita prout *R* accedet ad *T*, cylinder *FE* descendet. Cum igitur quantitas $ET + TC$ sit data (nempe composita ex cylindro solido *EF* et toto filo *FTC*) sitque data relatio inter *TC* et *TR* vel *BC* (ex lege declivitatum curvae data), habebitur et relatio inter *ET* et *TR*, ordinatam et abscissam curvae *EE*, cujus proinde natura et descriptio haberi potest in tabula *LRM* per geometriam ordinariam; habetur ergo etiam descriptio lineae *C*(*C*) per machinationem praesentem. Est autem *TC* semper tangens curvae *C*(*C*) ex natura nostri motus, itaque descripta est linea *C*(*C*), ubi lex declivitatum seu relatio laterum trianguli characteristici assignabilis *TRC* vel *TBC* est data. Quae linea cum sit quadratrix figurae datae quadrandae, ut paulo ante ostensum est, habebitur quadratura vel dimensio quaesita.

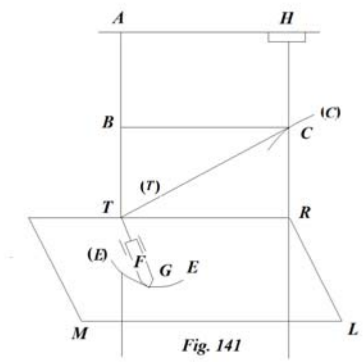

Fig. 141

Similia variis modis ad conversae tangentium methodi problemata accommodari possunt, velut si punctum *T* fuisset motum in curva *TT* (loco rectae *AT)*, etiam *BC* coordinata (seu abscissa *AB*) calculum fuisset ingressa. Et sane omne problema conversae tangentium reduci potest ad relationem inter tres rectas, nempe duas coordinatas *CB*, *CH* et tangentem *CT*, aut alias functiones harum loco. Sed saepe res multo simpliciore motu confici potest. Velut si data fuisset relatio inter *AT* et *TC* (quod est circulis lineam ad angulos rectos secantibus, ordinatim positione datis, invenire Lineam *C*(*C*)), suffecisset minor apparatus. Cessantibus enim iis quae incedunt in *H* et *R*, satis erit *EE* lineam rigidam directricem describere in plano verticali immobili transeunte per *AT*. Ita promoto in recta immota *AT* puncto *T* seu cylindro cavo *TG*, descendenteque cylindro solido *TE*, prout jubet linea data directrix *EE*, quam cylinder mordet, utique ob summam $ET+TF$ constantem (ut ante) et relationem inter *AT* et *TC* datam facile invenietur relatio debita inter *AT* et *TC*, seu natura lineae *EE*, cujus ope descripta *C*(*C*) erit quaesita.