MATHEMATICA

No. XI

On Finding the Measures of Figures.

[This paper is concerned with an aspect of rectifiable areas.]

It is the sign of a completed analysis, when a problem either can be solved, or its impossibility can be shown : because as hitherto concerning the transformation of curved into rectilinear lines, [*i.e.* the rectification of curves and of areas] no one has excelled, an imperfection is apparent in this part of geometry and of algebra itself, as what hitherto has been discussed has not been extended to such problems. Yet many years ago now I had worked out an analytical aid and shown it to friends, which makes an appearance here : It is observed by those skilled in the inner workings of geometry, for some given curve (*fig.*19) AFC (from a number of these, of which the nature or relation between the ordinate and the abscissa can be expressed by an algebraic equation, or by a certain power, which *Descartes* calls geometrical, which on account of deep reasons I am accustomed rather to call algebraic), it is possible to find another curve AGD also algebraic, whose figure shall be able to squared with the aid of the former : and that can arise in many ways, for example, with the curve AFC given, a curve of such a kind AGD can be found, so that the rectangle under the ordinate FE of the first curve and with a constant right line H, is equal always to the three lines of the latter curve [area bounded by the base BD, the altitude AB, and the curve AGD], or of the figure AEGA ; or as the rectangle under the ordinate FE of the first curve, and the abscissa AF of the second, is equal to the same three lined curve [area]; or with others in an infinitude of ways.

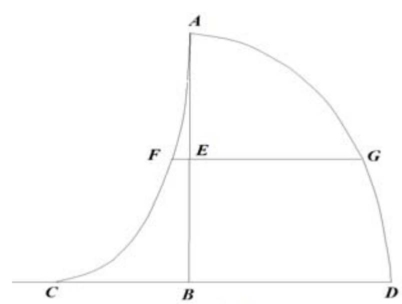
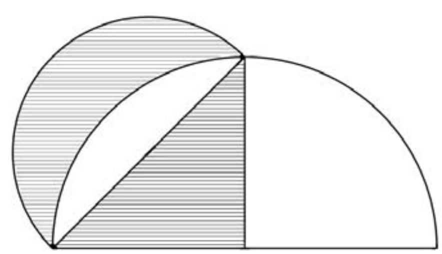


Fig. 19

[For example, following *Naissance…* p. 89, in the first instance, if $y = f(x)$ is the equation of the first curve AFC, and $z = g(x)$ is the equation of the second curve AGD, then in this case $y \times H = \int_0^x g(t)\,dt$; in the second case, $y \times x = \int_0^x g(t)\,dt$; these integrals may be rectifiable for algebraic functions, but no necessarily so for transcendental functions. L. shows by an example that the lunes of Hippocrates are rectifiable, but the analysis cannot be extended to the whole circle ; the diagram here shows such a lune, where the shaded curved area formed from the two circles is equal to the area of the triangle, found from elementary considerations.]



The former curve AFC I call the *Quadratrix* [*i.e.* the rectifying curve], the latter AGD the *rectifiable* curve. [*i.e.* for a given curve AGD of which the quadrature or area is sought, AFC represents that area in the form of a rectangle, or a curve with the ordinate related to the area or quadrature of the other]. But here there is a need, for a given figure requiring to be squared, to find the quadratrix of that other ; especially since some quadratrix may be impossible to find (for it is required to be expressed algebraically). Therefore in order that I might perform everything possible in this manner, I had thought out such a method, which I knew before, and had not usurped [from others]; which moreover is of the greatest use, as well as for other matters. I use the general equations of curves, each of which expresses all the curves of the same order. And of such a general curve, such as of the quadrature considered, the general curve for which the quadrature is required, following any considered from the above ways, which I always keep the same. Because I can show, if the quadratix is not given following the one way, nor is the same to be given following the other. Now an equation is required to be prepared of a particular quadrature provided by some general formula, expressing the nature of the quadrature ; but if there shall be no agreement, it is evident that itself cannot be undertaken, and thus clearly no algebraic quadrature can be had. By the same method I am able to find, so that I may have quadrature, if not algebraic then at least transcendent, that is of a circle, a hyperbola, or the quadrature of another supposed figure, so that at any rate of course we may reduce the remaining dimensions to these simpler. I have done much work of this kind, by which geometry may be advanced an immense distance beyond the boundaries set by *Vieta* and *Descartes*. For the ancients were unwilling to use lines of higher orders, and the solutions, which were done mechanically with the aid of these, *Descartes* held that back, and he kept back all the geometrical curves, the nature of which could be expressed by some algebraic equation, or certainly of some order. Indeed rightly, but in that he sinned no less than the ancients, because other infinite curves, which also again can be described accurately, he excluded from geometry, and he called mechanical, because these certainly can be returned as equations, and will not be able to be handled following these rules. Truly it is to be understood that also I can calculate these too, such as the cycloid, the logarithmic curve, and others of that kind, which have the greatest use, and which can be expressed even by finite but not algebraic equations (of a certain order), but of an indefinite, or transcendental order. And thus are able to be subjected to the same calculation as the rest : although this calculation shall be of another kind than what is commonly taken. I shared my thoughts of this kind, which I had not observed elsewhere, with a most talented friend, who also increased these by his own discoveries, and in his time will become famous [*i.e.* Tschirnhaus]; he gave the same algebraic calculation requiring to be found based on the said method, introduced by some theorems he published. Yet the love of the truth alone forces me [to say] this, my very own method of finding quadrature, indeed has significant uses, but not to be sufficient for finding any quadrature, nor from that to be able to say that the quadrature of the circle or of the hyperbola to be impossible. Indeed it can happen, that somehow a certain part of the quadrant of a circle, or even the whole quadrant ABDGA shall be able to be squared, although the indefinite quadrature may not be given, or any of a general portion given, following some one common rule, or the general algebraic calculation, which may express a relation between the area AEGA, and the rectangle AEF; from which neither will a certain algebraic equation always be given expressing a relation between AE and EF, the abscissa and ordinate of the squared curves AFC; and hence the quadratrix will not be Algebraic, or of a certain order, but transcendent. And indeed it is easy to show by many ways that the circle is capable of definite quadrature, but no demonstration stands out at this time free from fallacy, which may show the impossibility of a special quadrature of the whole circle [*i.e.* squaring the circle, or representing its area by that of a square of known dimensions, which was shown eventually by Lindemann to be impossible]. But it pleases to consider the example of a figure, where a special quadrature can be found without the general found. Let there be in the square AEBZ (*fig.* 20.), an orthogonal three-line AENMA, now truly the opposite sides of the square AE, ZB may be cut at the points *G*, R, and the curve may be cut at the point M by the right line GR drawn parallel to the remaining sides of the square AZ, EB. The abscissa BR may be called *v*, and the ordinate RM may be called *y*, and the side of the square *h* and the equation expressing the nature of the curve shall be

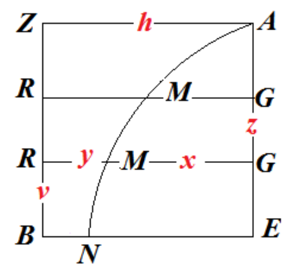


Fig. 20

$$y^4 - 6hhyy + 4yyvv + h^4 = 0.$$

[This example is actually the lune of Hippocrates from antiquity, known better under the form $y = \sqrt{2hh - vv} - \sqrt{hh - vv}$, the quadrature of which is known to be $\frac{1}{2}h^2$.]
GM may be called x, and AG may be called *z*, there becomes $y = h - x$ & $v = h - z$, which values on being substituted into the previous equation, gives :

$$h^4 - 4h^3x + 6hhxx - 4hx^3 + x^4 - 6h^4 + 12h^3x - 6hhxx + 4h^4$$
$$- 8h^3z + 4hhzz - 8h^3x + 16hhxz - 8hxzz + 4hhxx - 8hxxz + 4xxzz + h^4 = 0$$

or with the terms cancelling :

$$4hhzz - 8hxzz + 4xxzz - 8h^3z + 16hhxz - 8hxxz + 4hhxx - 4hx^3 + x^4 = 0;$$

which equation on dividing by $hh - 2hx + xx$ gives :

$$4zz - 8hz + \frac{xx(4hh - 4hx + xx)}{hh - 2hx + xx} = 0.$$

And thus if our figure is capable of being squared by the above-mentioned method, here the second equation must be able agree with that proposed elsewhere [*i.e.* the general rule proposed a little earlier by Tschirnhaus] :

$$0 = bzz + caz + eaa + 2dxz + 2fax + 4gxx$$
$$+ \frac{aaxx(dde + ccg + bff - cdf - 4beg)}{4beaa + 4bfax + 4bgxx - ccaa - 2cdax - ddxx}$$

But it is evident, the combination cannot proceed, even if only the numerator of the fraction present may be compared everywhere, $4hh-4hx+xx$ must coincide with $aa\left(dde+ccg+bff-cdf-4beg\right)$; since there shall be an indeterminate term with a determined term: so that I may remain silent the letters *d, e, f, g* become equal to zero, from the rest of the comparison : from which in the equation sought for the quadrature of the curve, which is

$$\begin{aligned} byy+cay+eaa& \\ +dxy+fax& \\ +gxx=0,& \end{aligned}$$

only $byy+cay=0$ may remain , which is equivalent not to a line but to a point. And thus the quadrature for the curved line cannot be had in this way. And we know from elsewhere, the three-lines proposed is to be squared : and thus this method, although it shall be of great benefit, still does not suffice for all quadrature requiring to be found, moreover it is a help to have used other means hitherto, which indeed I may explain elsewhere, for the matter is entirely within our power.

[Thus Leibniz refutes the theorem of Tschirnhaus, by presenting a curve able to be squared, but not satisfying the theorem of the latter.]

MATHEMATICA

No. XI

DE DIMENSIONIBUS FIGURARUM

INVENENDIS.

Ex Actis Erudit. Lips. ann. 1684.

Signum est perfectae Analyseos, quando aut solvi problema potest, aut ostendi ejus impossibilitas : quod cum nemo hactenus praestiterit circa transmutationes curvilineorum in rectilinea, patet in hac parte imperfectio Geometriae, & ipsius Algebrae, quae uti hactenus tractata est, ad talia problemata non porrigitur. Excogitavi tamen jam a multis annis subsidium Analyticum, & amicis ostendi, quod huc redit : notum est interioris Geometriae peritis data (*fig.* 19) qualibet curva AFC (ex illarum numero, quarum natura, seu relatio inter ordinatam, & abscissam per aequationem Algebraicam, seu certi gradus exprimi potest, quas *Cartesius* appellat Geometricas, ego ob graves rationes potius Algebraicas appellare soleo)

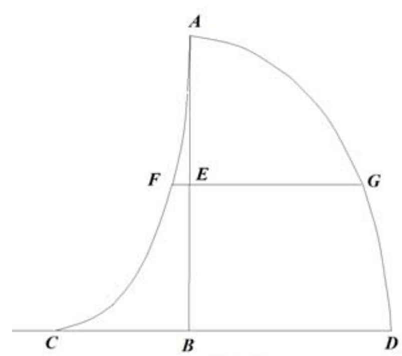


Fig. 19

posse aliam inveniri curvam AGD etiam Algebraicam, cujus figura ope prioris possit quadrari : idque fieri potest multis modis, exempli causa data curva AFC, potest inveniri curva AGD talis naturae, ut rectangulum sub FE ordinata prioris curvae, & recta constante H, semper aequetur trilineo curvae posterioris, seu figurae AEGA; vel ut rectangulum sub FE ordinata curvae prioris, & abscissa ejus AF, aequetur eidem trilineo; vel aliis modis infinitis. Priorem curvam AFC voco *Quadratricem,* posteriorem AGD *Quadrandam.* Sed hoc opus hic labor est, data Quadranda Figura, invenire Quadratricem ejus aliquam ; praesertim cum aliquando quadratricem invenire (Algebraice quidem exprimendam) sit impossible. Ut ergo praestarem quicquid in hoc genere fieri potest, talem methodum excogitavi, antea quod sciam, non usurpatam ; sed quae maximum & in aliis usum habere potest. Adhibeo aequationes Curvarum generales, quarum unaquaeque omnes curvas ejusdem gradus exprimit. Et talis curvae generalis, consideratae tanquam quadraticis, quaesto quadrandam generalem, secundum aliquem ex modis supra ductis, quem semper eundem servo. Quia demonstrare possum, si non datur quadratrix secundum unum modum, nec eam secundum alium dari. Oblatae jam quadrandae specialis aequatio comparanda est cum aliqua ex formulis generalibus, quadrandarum naturam exprimentibus; sed si nulli comparari possit, manifestium est, eam sub ipsis non comprehendi, adeoque nullam habere quadratricem, scilicet Algebraicam. Eadem methodo invenire possum, quam habeat quadratricem, si non Algebraicam, saltem transcendentem, hoc est Circuli, aut Hyperbolae, aut alterius figurae quadraturam supponentem, ut scilicet saltem dimensiones reliquas ad has simpliciores reducamus. MuIta hujusmodi habeo, quibus Geometria in immensum ultra Terminos a *Vieta,* & *Cartesio* positos promovetur. Nam veteres nolebant uti lineis altiorum graduum, & solutiones, quae earum ope fiebant, Mechanicas. *Cartesius* id reprehendit, & omnes curvas in Geometriam recipit, quarum natura aequatione aliqua Algebraica, seu certi alicujus gradus, exprimi possit. Recte quidem, sed in eo peccavit non minus quam veteres, quod alias infinitas, quae tamen etiam accurate describi possunt, ex Geometria exclusit, & Mechanicas vocavit, quia scilicet eas ad aequationes revocare, & secundum suas regulas tractare non poterat. Verum sciendum est, istas ipsas quoque, ut Cycloidem, Logarithmicam, aliasque id genus, quae maximos habent usus, posse calculo, & aequationibus etiam finitis exprimi, at non Algebraicis, seu certi gradus, sed gradus indefiniti, sive transcendentis. Et ita eodem modo posse calculo subjici as reliquas : licet ille calculus sit alterius naturae quam qui vulgo usurpatur. Hujusmodi cogitationum mearum, quae alibi non observavi, participem feci amicum ingeniosissimum, qui etiam eas multis de suo inventis auxit, & suo tempore praeclara dabit ; idem calculum inveniendi quadratrices algebraicas supra dicta methodo aggressus aliquot theoremata dedit. Unum tamen cogit me monere amor veritatis, hanc ipsam methodum meam quaerendi quadratrices, insignes quidem usus habere, sed non sufficere ad inveniendas quadraturas quascunque, neque ex ea probari posse impossibilitatem Quadratur Circuli, aut Hyperbolae. Fieri enim potest, ut aliqua certa portio quadrantis circuli, vel etiam totus quadrans ABDGA, quadrari possit, licet non detur quadratura indefinita, & generalis cujuslibet portionis datae, secundum unam aliquam legem communem, seu calculum algebraicum generalem, qui exprimat relationem inter spatium AEGA, & rectang. AEF; unde nec dari poterit semper aequatio quaedam algebraica exprimens relatione inter AE, & EF, abscissam, & ordinatam quadratricis AFC; ac proinde quadratrix non erit Algebraica, seu certi gradus, sed transcendens. Et quidem circulum esse incapacem quadraturae indefinitae facile multis modis demonstrari potest, sed nulla hactenus extat demonstratio a paralogismo libera, quae ostendat impossibilitatem specialis quadraturae circuli totius. Placet autem ascribere exemplum figurae, ubi succedit quadratura specialis sine generali. Sit in quadrato AEBZ (*figur*.20.) trilineum orthogonium AENMA, jam secentur latera quadrati opposita AE, ZB in punctis *G*, R, curva vero in puncto M, per rectas G R, reliquis quadrati lateribus AZ, EB parallelas. Abscissa BR appelletur *v*, & ordinata RM, appellatur *y*, & latus quadrati *h* & aequatio naturam curvae exprimens sit $y^4 - 6hhyy + 4yyvv + h^4 = 0$. GM appelletur *z*, & AG appelletur *x*, fiet $y = h - x$ & $v = h - z$, quos valores substituendo in aequatione praecedenti fiet:

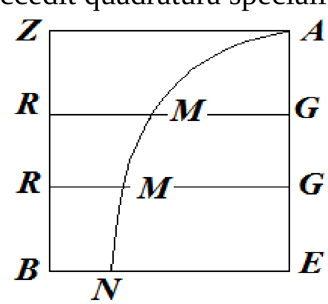


Fig. 20

$$h^4 - 4h^3x + 6hhxx - 4hx^3 + x^4 - 6h^4 + 12h^3x - 6hhxx + 4h^4$$
$$- 8h^3z + 4hhzz - 8h^3x + 16hhxz - 8hxzz + 4hhxx - 8hxxz + 4xxzz + h^4 = 0$$

seu destructis destruendis:

$$4hhzz - 8hxzz + 4xxzz - 8h^3z + 16hhxz - 8hxxz + 4hhxx - 4hx^3 + x^4 = 0$$

quam aequationem dividendo per $hh - 2hx + xx$ habebitur.

$$4zz - 8hz + \frac{4hh - 4hx + xx \text{ multiplicatum in } xx}{hh - 2hx + xx} = 0.$$

Itaque si figura nostra est quadrabilis methodo supradicta, deberet hac aequatio secundum alibi proposita conferri posse cum ista.

$$0 = bzz + caz + eaa + 2dxz + 2\,fax + 4gxx$$
$$+ \frac{dde + ccg + bff - cdf - 4beg \text{ multiplic. in } aaxx}{4beaa + 4bfax + 4bgxx - ccaa - 2cdax - ddxx}$$

Sed manifestum est, collationem non procedere, si vel solus numerator fractionis utrobique existens comparetur, deberet $4hh - 4hx + xx$ coincidere cum aa, in $dde + ccg + bff - cdf - 4beg$; indeterminatum cum determinato: ut taceam ex reliquis comparationibus literas *d*, *e*, *f*, *g* fieri aequales nihilo : unde in aequatione quaesita ad curvam quadratricem quae est

$$\begin{aligned} &byy + cay + eaa \\ &\quad + dxy + fax \\ &\quad + gxx = 0 \end{aligned}$$

restaret tantum $byy + cay = 0$, quae est aequivocatio non ad lineam, sed ad punctum. Itaque linea curva Quadratrex haberi hoc modo non potest. Et tamen aliunde scimus, trilineum propositum esse quadrabile: itaque ista methodus, licet maximi sit momenti, tamen ad omnes quadraturas inveniendas non sufficit, sed opus est alias adhuc artes adhaberi, quas quidem alias exponam, res enim omnino in potestate est.