§3. Synopsis of Propositions 7 - 17: The Generation of Spheroidal and Conoidal Surfaces from Conic Sections and the Behaviour of Rays Refracted through them.

The first of these Propositions show how the refracting surfaces are generated from conic sections. We are to regard the conoid either as a paraboloid or the branch of a hyperboloid, while the spheroid is an ellipsoid with an axis of symmetry. The refraction of a set of parallel rays along the optic axis for individual surfaces is then considered. In general, parallel rays are refracted through the far focal point by these surfaces. The situations considered include parallel rays incident on a dense spheroid, parallel rays in a dense hyperboloid incident at a less dense interface, the ray reversed cases of these, and the cases where the less and more dense media are interchanged, and their ray reversed cases. The reflection by the inner surface of a parabola is also considered.

Prop. 7 shows the axial symmetry of spheroidal and conoidal surfaces.

Prop. 8 asserts that all sections perpendicular to the axis of the cone are circles.

Prop. 9 asserts that a plane tangential to a conoid or spheroid is perpendicular to a plane through the point of contact and the axis of the conoid or spheroid.

Prop. 10 asserts that a ray incident on a refracting or reflecting conoidal or spheroidal surface in a plane containing the axis, is refracted or reflected in that plane by the conic section which generated the conoid or spheroid.

Prop. 11 demonstrates the focusing property of a parabola for the reflection of rays parallel to the axis, and locates the position of the focus at one quarter of the length of the latus rectum from the vertex within the parabola.

Prop. 12 asserts that an ellipse or hyperboloid of a given kind, *i. e.* one with a known axis to inter-focal separation ratio, can be constructed from the known positions of a vertex and focus from proportion.

Prop. 13 asserts that equally spaced parallel rays incident at some angle on the plane interface between two media are refracted and sent out parallel and equally spaced at a different angle in the second medium.

Prop. 14 asserts that parallel rays incident along the axis of a spheroid in a less dense medium are refracted through the far focal point of the ellipse of cross-section. Also, parallel rays incident along the axis of a hyperboloid in the denser medium are refracted through the focal point of the far branch of the hyperboloid of cross-section in the rarer medium.

Prop. 15 is the converse of Prop. 14; which follows by reversing the directions of the rays.

Prop. 16 considers the case where the dense medium lies on the outside of the surface. We are to consider refraction by a hollow spheroidal surface and by a hyperboloidal surface with the denser medium filling the space between the branches.
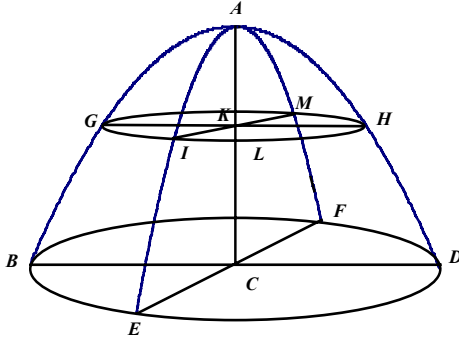
Prop. 17 is the converse of Prop. 16 with the rays reversed.

§3. Prop.7.1.   Prop. 7.   Lemma.

All sections of a conoid or spheroid cut through the axes return the same section of the cone from which they are generated, while the axis of the section is the same as that of the conoid or spheroid.

Prop. 7- Figure 1.

Let ABCD be any conoid or spheroid you wish of which the axis is AC, which is cut by a plane to make the section ABCD. I say that the section EAFC shall be the same conic section, from which the conoid or spheroid ABCD was generated. I also say that AC is the axis of this conic section. The conic section ABCD may be plainly seen, from which the figure has been generated by rotation. Indeed if ABCD itself is rotated, so that it may fall in the place of the section AECF, then either the whole section will agree in every part, or it will be different from AECF itself. If it agrees, then both AECF and ABCD are the same, and as we assert the axis AC of both is the same. If it should differ in some part then it follows that there is something is in the figure which does not agree with the rotation of the conic section ABCD, which is absurd. Therefore, there will be no disagreements, and therefore they are equal; and they have a common axis AC, which had to be shown.

§3. Prop.7.2.   Prop. 7. Lemma.

Omnis conois, vel sphaerois, per axes secta, reddit eandem coni sectionem, ex qua est genita. Sectionis autem axis, idem est cum axe conoidis, vel Sphaeroidis.

Sit Conois, Sphaerois quaelibet ABCD, cujus axis AC, per quem plano secetur, fiatq; sectio EAFC : Dico sectionem EAFC , esse eandem sectionem Conicam, ex qua est generita Conois vel sphaerois ABCD: cujus etiam sectionis conicae dico AC esse axem. Intelligatur ABCD sectio conica, ex cujus revolutione genita est figura: etenim si ipsa ABCD revolvatur, quod incidat in locum sectionis AECF, aut tota toti conveniet, aut ab ipsa AECF discrepabit: si conveniat, eadem est AECF, ac ipsa ABCD, & utriusq; idem est axis AC, ut intendimus : si discrepet aliqua parte, sequetur aliquid esse in figura, quod a sectionis conicae ABCD revolutione, non est constitutum, quod est absurdum. Non ergo discrepabunt, & igitur equales sunt, habentq; axem communem AC, quod demonstrandum erat.

§3. Prop.8.1. Prop. 8. A Lemma.

A plane section perpendicular to the axis of a conoid or spheroid is a circle having its centre on the axis.

The conoid or the spheroid is cut by a plane perpendicular to the axis AC, meeting the axis in the point K, and the section is GLHI, which I say is a circle [Prop.7 - Fig.1] Furthermore, the plane through the axis AKC is extended making the section EAFC, meeting the plane section perpendicular to the axis in the points I and M. The lines KI and KM are drawn to these points from the point K perpendicular to the axis AKC. Hence each section is connected in turn: for indeed the circle BEDF is made from the rotation of the section BAFC in constructing the figure, the radii of which are CB, CE, CD, and CF at right angles to the axis, and hence the axis is connected with each of these in turn. Therefore (*Apol*. 1.21) each of the three equal radii CB, CE and CD has indeed the same ratio to its own parallel line from the three lines KG, KI and KH. Thus it is apparent (as CB, CE and CD are equal) that KG, KI and KH are equal. Therefore the circle is GMHLI, the centre K of which lies on the axes of the conoid or spheroid (*Apol*. 3.9). Q.e.d.

§3. Prop.8.2. Prop. 8. Lemma.
Conoide vel spheroide, secta plano ad axem perpendiculari, sectione fit Circulus, Centrum habens in axe.
   Dividatur conois, vel Sphaerois, plano ad axem AC perpendiculari, occurrente in puncto K, & sit sectio GLHI, quam dico esse circulum: Etenim per axem AKC planum agatur faciens sectionum EAFC, occurrentem secanti plano perpendiculari ad axem, in punctis I, M, ad quae a puncto K ducantur lineae KI, KM, qui ad axem AKC sunt perpendiculares ; & proinde ordinatim applicatae: cum enim ex revolutione sectionis BAFC, in constitutione figurae, factus sit circulus BEDF, cujus radii sunt CB, CE, CD, CF, [*Ap. 1.21*] ad axem recti, & proinde ordinatim quoque applicati; igitur unusquisque trium Radiorum aequalium CB, CE, CD, [*3.9*] habet ad quamlibet sibi parallelam lineam e tribus KG, KI, KH, eandem rationem, unde patet (cum CB, CE, CD, sint aequales) KG, KI, KH esse aequales; igitur circulus est GMHLI, cujus centrum K, in axe conoidis vel sphaeroidis; quod demonstrandum erat.

§3. Prop.9.1. Prop. 9. A Lemma.

If a plane touches a conoid or spheroid, and another plane is drawn through the point of contact and the axis. I say that these two planes mutually cut each other at right angles.
   ABCD is the conoid or spheroid to which the plane HIGF is a tangent at the point E. The plane BEALD is produced through the point E and the axis of the figure, cutting the plane HIGF in the line IF. I say that the planes HIGF and BEALD cut each other mutually at right angles. The plane EMLN is drawn through the point E, perpendicular to the axis AC, and if it is produced then it cuts the plane HIGF in the line GH . Since HG touches the circle EMLN, it is perpendicular to the diameter EL, by Prop. 8. Because the
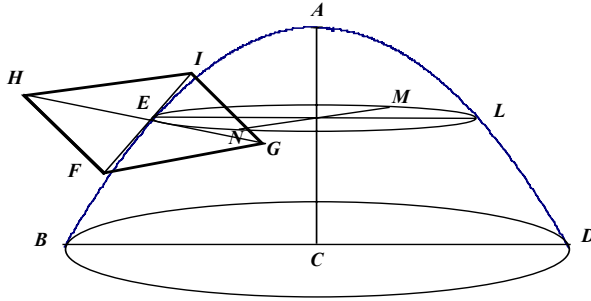
Prop. 9 - Figure 1.

planes BAD and EML cut each other mutually at right angles (*Apol*. 3.18), and HG lying in the plane EMLN is

[18]

perpendicular to the common section EL, then HG will be perpendicular to the plane ABD. Therefore (*Apol*. 11.18) all the planes drawn through HG, (out of which number is the plane HIGF) are normal to the plane BAD.

Q.e.d.

Scholium.

These three Lemmas are explained in almost the same way, generated by rotation in all the solids.

§3. Prop.9.2.   Prop. 9.   Lemma.

Si Conoidem, vel Sphaeroidem, tangat planum, & per punctum contactus, & axem, ducatur aliud planum ; Dico haec duo plana se invicem normaliter secare.

Sit Conois vel spherois ABD, quam tangat planum HIGF in puncto E, perque punctum E, & axem figurae producatur planum BEALD, donec secet planum HIGF in recta IF. Dico plana HIGF, BEALD, se invicem normaliter secare. Per punctum E ducatur planum EMLN, perpendiculare ad axem AC, & producatur, donec secet planum HIGF in recta GH; & quoniam HG tangit circulum EMLN, erit ad ipsius diametrium EL perpendicularis ; & quia plana BAD, EML se invicem secant normaliter, & HG in plano EMLN, est

[18]

 perpendicularis ad communem sectionem EL; erit HG perpendicularis ad plano ABD: ergo & omnia plana per HG ducta, (e quorum numero est planum HIGF) ad planum BAD; erunt normalia ; quod erat demonstrandum.

Scholium.

Haec tria Lemmata, eodem fere modo demonstrantur, in omnibus solidis ex circumvolutione genitis.

§3. Prop.10.1.   Prop. 10. Theorem.

If the surface of a conoid or spheroid is also the surface of a dense medium for refraction, or of a polished medium for reflection, and the line of incidence of a ray lies in the same plane as the axis of the conoid or spheroid, then the conic section from which the conoid or spheroid has been generated will always be the refracting or reflecting surface for the ray.

Let ABC be the conoid or spheroid, either dense or with a polished surface. The axis AD and the line of the incident ML are coplanar, meeting the dense refracting medium or polished reflecting surface at the point L. I say that the surface for reflection or refraction of the incident ray ML is the same conic section by which the conoid or spheroid was described previously. The plane FEHG is drawn, touching the same conoid or spheroid in the point L, and it
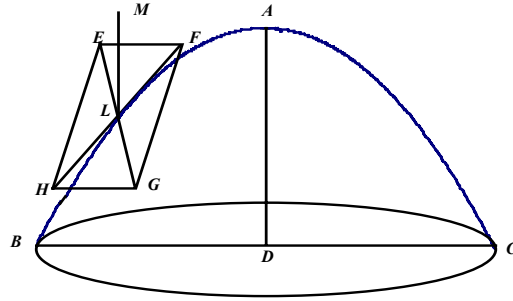
Prop. 10 - Figure 1.

is apparent from the laws of optics that the plane is the surface for reflection or refraction for that incident ray ML cutting the perpendicular plane FEHG in the point L. Thus truly if the incident ray ML is there, and if the plane BAFLHC is drawn through the axis AD and the incident ray, then by Prop. 9 it shall be perpendicular to the plane FEHG, crossing through the point L. Hence the surface is the required surface of reflection or refraction for the line of incidence ML, according to Prop. 7. Q.E.D.

§3. Prop.10.2.   Prop. 10. Theorema.

Si superficies densi, aut politi, fuerit superficies Conoidis, aut Sphaeroidis, fueritque linea incidentiae in eodem plano cum axe Conoidis, vel Spheroidis : sectio Conica ex qua genita est Conois, vel Sphaerois ; semper erit superficies Refractionis, vel Reflectionis.

Sit Conois, vel Sphaerois ABC, densum vel politum; cujus axis AD , sitque linea ML in eodem cum axe plano, L: Dico superficiem reflectionis, vel refractionis, lineae incidentiae ML, esse eandem sectionem conicam ex qua describitur Conois vel Sphaerois. Ducatur planum FEHG, tangens Conoid vel Sphaeroid in puncto L: & patet ex doctrina opticorum, illud planum esse superficiem reflectionis, vel refractionis, lineae incidentiae ML, quod perpendiculariter secat planum FEHG, in puncto L, ita ut linea incidentiae ML, in eo existat: Si vero per axem AD,

[19]

& lineam incidentiae ML, ducatur planum BAFLHC; erit illud perpendiculare ad planum FEHG, & transibit per punctú L; ideoque erit superficiem reflectionis, vel refractionis lineae incidentiae ML; est autem sectio conica, ex qua describitur Conois vel Sphaerois : quod erat demonstrandum.

§3. Prop.11.1.   Prop. 11. Theorem.

If a straight line is tangent to a parabola, and from the contact point a single line is drawn within the parabola parallel to the axis. Another line is drawn from the same contact point, making an angle with the tangent equal to the angle of the first line with the tangent. This second line cuts the axis of the parabola within the parabola such that the distance between the vertex and the point of intersection is always equal to a quarter of the latus rectum. The point of intersection is called the focus of the parabola.
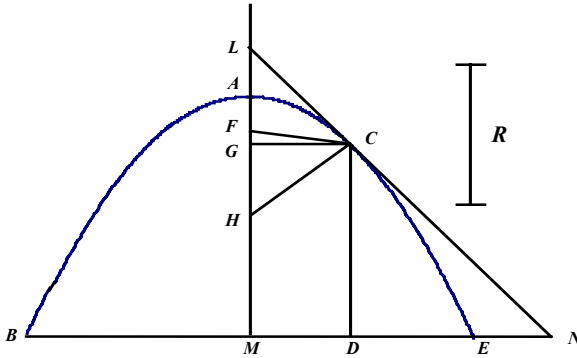
Prop.11-Figure 1.

This is a most beautiful deduction, which I know was first come upon by Witelo, but since he has overlooked the pleasing corollary, we ourselves - and perhaps in an easier manner - shall demonstrate this theorem itself otherwise. Let BACE be the parabola with axis LAM, and the line LCN is a tangent to the parabola at the point C, from which CD is drawn parallel to the axis, and the angle DCN is made equal to the angle LCF. I say that AF is equal to a quarter of the latus rectum or focal chord R [drawn to the right of the figure]. Let GC and AG [the text uses ME] be drawn symmetrically perpendicular and parallel to the axis, with CH perpendicular to the tangent line. These lines will be in the proportion R : GC : : GC : GA (= AL) , and HG : GC :: GC : GL; [as Δ's CGH and LGC are similar] hence GL : GA :: R : HG; but GL is twice the length GA, therefore R is twice the length HG;

[20]

 and since LM, CD are parallel; the angle NLM [the text has ELC] will be equal to the angle DCN, that is FCL; therefore FL and FC are equal: also the angles DCH, CHF, FCH are equal; hence, FH and FC are equal, and consequently FL is equal to FH. If therefore the halves of LH and LG are taken from each other, *i.e.* giving LF and LA , then AF will be the difference of LF and LA. Since FH is half of the latus rectum R, then AF is the fourth part of the latus rectum [*i. e.* half of (R - R/2)]. (*Apol.* 5.15). Q.e.d. The same theorem can also be easily shown for the two remaining cases, which we omit for the sake of brevity.

Corollary 1.

It follows from this Theorem that: DC + CF is equal to MA + AF; indeed DC + CF is equal to MG + FH, that is MF + GH; but GH is double FA itself; hence DC + CF = MF + 2FA = MA + AF, which is the proposition. From this corollary the easiest way of describing the parabola in the plane is given, that Kepler touched on *in Ast. Opt.*
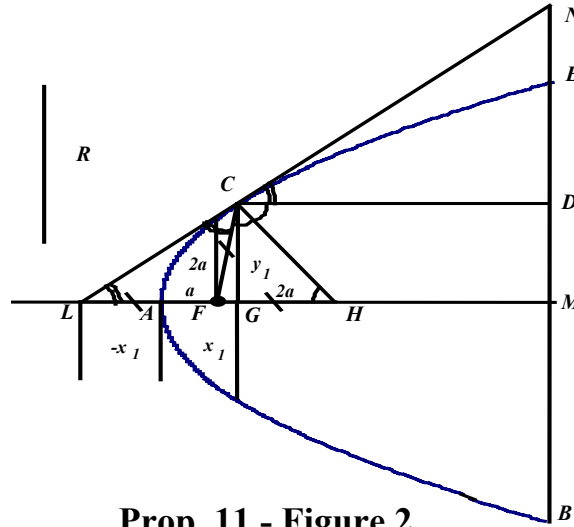
Corollary 2.

From this Theorem the second [corollary] follows, all the rays parallel to the axis are reflected in the focus of the parabola, if the mode of reflection should be the concave surface of a parabola.

[20]

§3. Prop.11.2.
Notes on Theorem 11.

The first proportionality written in the form $GC^2 = R.GA$ resembles the modern standard equation for the parabola be $y^2 = 4ax$ with origin at A (0, 0), and C is the point with co-ordinates (x_1, y_1) as in Prop. 10 - Fig.2, where we have marked in the various equal angles and lengths. If we let the point G move to F, then the equation becomes $(R/2)^2 = R.FA$, giving FA = R/4 = a.

Prop. 11 - Figure 2.

We may note the confusion of lettering in the text and diagram: it would appear that one had been changed but not the other. Note that the first corollary translates straight into the language of equal path lengths of all the rays: hence the focusing property of the parabola.

§3. Prop.11.3.   Prop. 11. Theorema.

Si parabolam recta linea tangat, & a tactu ducatur una recta intra parabolam, axi parallelae, alia vero ad easdem partes, faciens angulum cum contingente, equalem prioris angulo cum contingente ; dico hanc lineam secundam, secare axem parabolae intra parabolam, ita ut linea inter verticem & intersectionem; semper aequalis sit quadranti lateris Recti. Dicitur autem intersectionis punctum, focus Parabolae.

Hanc Pulcherrimam conclusionem, primus quod sciam invenit Vitellio, sed quoniam jucundam omisit Corollarum, nos aliter, & forsan facilius, hanc ipsam demonstrabimus. Sit Parabola BACE, cujus axis LAM, tangatque eam linea LCN, in puncto C, a quo ducatur axi parallela CD, fiatque angulo DCN, aequalis LCF, dico AF esse aequalem quadranti lateris recti R. Sint ordinatim applicatae, GC, ME perpendicularis ad contingentem CH, eruntque hae lineae proportiones R:GC :: GC:GA = AL, & HG:GC::GC:GL; ergo GL:GA::R:HG; sed Gl est dupla lineae GA, ergo R est dupla linea HG;

[20]

& quoniam LM, CD sunt parallelae; erit angulus ELC , aequalis angulo DCN, hoc est FCL; ergo FL, FC sunt aequales: sunt etiam anguli DCH, CHF, FCH aequales; ergo, FH & FC sunt aequales, & consequenter FL & FH: si igitur duplorum LH, LG sumantur

dimedia LF, LA; erit AF differentia dimediorum, semissis GH differentiae duplorum ; est autem GH semissis lateris recti R; ergo & AF est quanta pars lateris recti, quod erat demonstrandum. Eadem etiam facilitate demonstrabitur hoc Theorema in duobus Reliquis casibus, quos brevitatis gratia praetermittimus.

Corollary 1.

Sequiter ex hoc Theoremate: DC + CF esse aequales MA + AF; sunt enim DC + CF aequales MG + FH, hoc est MF + GH; est autem GH dupla ipsius FA; ergo DC + CF = MF + 2FA = MA + AF, quod est propositum. Ex hoc corollario, datur facillimus modus describendi parabolam in plano, quem attigit *Keplerus in Ast. Opt.*

Corollarium 2.

Ex hoc Theoremate sequiter secundo, omnes radios axi parallelos reflecti in focum parabolae, si modo superficies reflectionis fuerit concavitas surface parabolae.

[21]

§3. Prop.12.1.   Prop. 12. Problem.

Given the position of a single focus and vertex for a given kind of ellipse or hyperbola [i. e., one with a known axis to inter-focal separation ratio], the ellipse or hyperbola can be found.

Let A be the vertex and B the focus of the ellipse ALD, for which the ratio of the separation of the foci to the axis is given as E to GH. The axis length and the focal separation of the ellipse ALD is sought. From GH, MN equal to E itself is taken away [i. e. the ratios 1 - E/GH, 2NH/GH, NH/GH, and also 1 - NG/GH or GN/GH, are also known]. Thus, as GM is equal to NH, and the ratio GN to GH is as AB to AD, and AC is equal to BD, I say that [the unknown] AD is the axis, and C and B are the foci of the ellipse ALD.
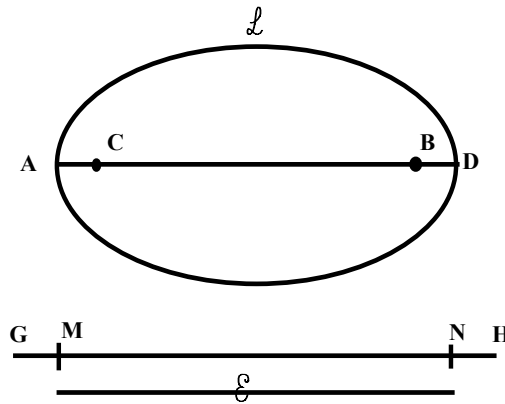
Prop. 12 - Figure 1.

Conversely:
If GN:GH::AB:AD, then
GN:AB::GH:AD ; & as
GH:AD::NH:BD; [for GN/GH - 1 = AB/AD - 1, so NH/GH = BD/AD]
GH:AD::2NH:2BD; [hence 1 - 2NH/GH = 1 - 2BD/AD, etc, leading to....]
GH:AD::MN = E:CB; *i. e.*
 E:GH::CB:AD; (*Apol.* 5.19). Q.e.d.

 Also one can proceed in the same way, if A is the focus of a hyperbola, B the vertex; E to GH, the given ratio of the axes to the separation of the foci, and AD is the separation of the foci, and CB the axis. [This is the case of a conjugate ellipse and hyperbola. In this theorem, everything follows by proportion from the given ellipse or hyperbola.]

[21]

§3. Prop.12.2.   Prop. 12. Problem.

Ex datis positione, uno foco, una vertice, cum ellipseos, vel hyperbolae specie ; Ellipsem aut Hyperbolam invenire.

Sit A vertex ellipseos ALD, B focus , sitque ut E ad GH; ita distantia focorum ellipseos ALD, ad ipsius axem; quaeritur illius axis, & focorum distantia. Ex GH auferatur MN, aequalis ipsae E ; ita, ut GM sit aequalis NH; fiatque ut GN ad GH, ita AB, ad AD; sitq; AC aequalis ipsi BD: Dico AD esse axem ellipseos ALD, & C, B, illius focos . Quoniam enim ut GN:GH::AB:AD; erit ut
GN:AB::GH:AD; & ut
GH:AD::NH:BD;
GH:AD::2NH:2BD;
GH:AD::MN = E:CB; hoc est
 E:GH::CB:AD;
quod erat ostendendum. Eodem etiam modo esset praecedendum, si A esset focus Hyperbolae, B vertex; E ad GH, ratio axeos ad distantium focorum, essetque AD distantia focorum, & CB axis.

[22]

§3. Prop.13.1.   Prop. 13. Problem.

Equally spaced parallel rays in one medium,[on refraction] *by another medium of differing density are to be sent out equally spaced..*
Let the common surfaces of the mediums of different density be plane. Therefore the parallel rays incident on the plane surface are chosen with equal angles of incidence everywhere; therefore all of the angles of refraction of these are equal, which results in parallel rays of refraction too since they are made parallel by the surface of refraction, as may be shown by the converse of *Prop. 10, Book 11. Elements.* Q. E. D.
Scholium
    Even though this problem is demonstrated most generally here, in the following however we shall make use of only one case ; indeed we always draw such a plane through any given point, to which the parallel rays are normal [i.e. lie in a perpendicular plane] ; this special case is used more for the sake of convenience than necessity, as will become apparent to the knowing Reader with what follows.

[22]

James Gregory's Optica Promota

§3. Prop.13.2.   Prop. 13. Problema.

Radios parallelos in uno Diaphano, per aliud diversae densitatis, aequi distantes mittere.

Sit communis superficies Diaphanorum diverarum densitatum plana, radii igitur paralleli, in superficiem planam incidentes, undique sortiuntur aequales incidentiae angulos, ac proinde aequales refractiones, omnes ergo eorum anguli refracti sunt aequales, qui cum fiant in superficiebus refractionum parallelis efficiunt quoque radios refractos parallelos, ut patet per conversum *Prop. 10, libri 11. Elementi.* Quod erat ostendendum.

Scholium

Hoc Problema etiamsi hic generalissime demonstretur, in sequentibus tamen unum tantum illius casum usurpamus; semper enim Ducimus tale planum per punctum aliquod datum, cui normales sunt radii paralleli ; quo casu magis ob commoditatem, quem necessitatem utimur, ut intelligenti Lectori ex sequentibus patebit.

## §3. Prop.14.1.   Prop. 14. Problem.

The parallel rays in one medium are gathered together in a given single point by a conoidal lens composed of a medium of different density; with the vertex of the conoid or spheroid given too, it is moreover in order that the line drawn through the point of convergence and the given vertex shall be parallel to the given rays.

Prop. 14 - Figure 1.

In the first place the rays R, R, .. etc. are parallel in the rarer medium, and they are to converge to the point E of the denser medium. The vertex of the spheroid is L with E the further focus, by Prop. 12. An ellipse LMA is made from the denser transparent medium, and with the axis LEA continuing parallel to the given rays, the spheroid LMA is made from the ellipse by revolution, by Prop. 10. I say all the rays parallel to the axis of the Spheroid (from which a number are RR etc.) which are incident on the spheroid, are

[23]

refracted at the surface of the spheroid and concur at the focus E.

For let one of the parallel rays RM be incident at the point M in the same plane as the axis, as it is parallel to that plane, it follows that the ellipse LMA will be the surface of refraction of the ray RM (which has been generated by rotation of the spheroid LMA); RM therefore will be refracted in the point E, by Prop. 5. Q.E.D.
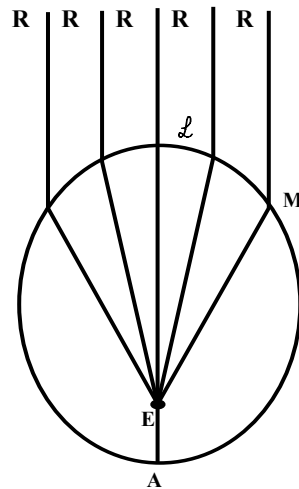
Prop. 14 - Figure 2.
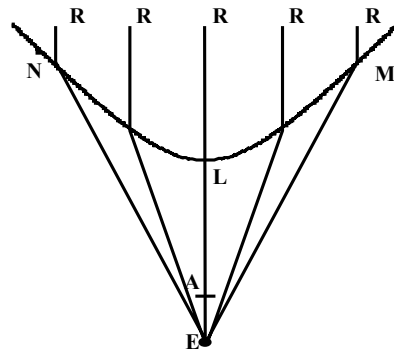
In the second case, let R, R,...etc. be parallel rays in the denser medium, and they converge to the given point E of the rarer medium. L shall be the vertex of the conoid, and it is to be made from the hyperbola NLM of the dense transparent medium with vertex L and further focus F, by Prop. 12. From the axis EAL continuing parallel to the given rays, the hyperbolic conoid is made by rotation of the hyperbola NEM. If the dense material is as before, I say that all the rays parallel among themselves, are refracted in the surface itself, and emerge into the rarer medium to concur at the focus E.

[24]

The ray RM is one of the parallel rays incident in the point M, which therefore is in the same plane as the axis. Hence the hyperbola NLM, (from which the conoid has generated by rotation, by Prop. 10) will be the surface of refraction for the ray RM; which will be refracted in the point E; because of that, the ray coming from the point E and incident in M is refracted in MN, (Prop. 5); as is shown by *Prop. 9, Book 10* of Witelo. Q.E.D.

Corollary

From this it appears to be sufficient, according to visual perception, that parallel rays RR, etc., are to be gathered together in one point near E, : even if LE may be different geometrically from the length of the ray; since if there shall be no sensible difference among the causes, neither will there be any sensible differences among the effects.

§3. Prop.14.2.   Prop. 14. Problema.

Radios parallelos in uno Diaphano, in unicum punctum datum Diaphani diversae densatatis congregare ; Data quoque Conoidis, vel Sphaeroidis vertice ; oportet autem, ut linea, ducta per punctum congregationis, & verticem datam, sit radiis datis parallela.

Sint primo radii in Diaphano rariore paralleli, R, R, .. &c., congregandi in Diapham densioris punctum E ; sitque Sphaeroidis vertex L; foco remotiore E, & vertice L,fiat horum diaphanorum ellipsis densitatis LMA ; & manente axe LEA, radiis datis parallelo, ex ellipseos circumvolutione fiat Sphaerois LMA, cujus materia sit ex praedicto diaphano denso : dico omnes radios axi Sphaeroidis parallelos, (e quorum numero sunt R R &c.) in Sphaeroidem incidentes,

[23]

in superficie Sphaeroidis refringi, & in focum E concurrere : Sit enim Radius RM, unus e parallelis, incidens in punctum M, qui erit in eodem plano cum axe, ex eo quod sit illi parallelus; erit igitur Ellipsis LMA (ex cujus circumvolutione genita est sphaerois LMA) superficies refractionis radii RM; refringetur igitur RM in focum E; quod ostendendum erat.

Secundo; Sint radii in diaphano densiore paralleli RR &c. congregandi in diaphani rarioris punctum datum E : Sitq Conoidis vertex L; foco remotiore E, & vertice L, fiat horum diaphanorum Hyperbole densitatis NLM ; & manente axe EAL radiis datis Parallelo, ex circumductione hyperbolae NEM, fiat Conois hyperbolica, cujus materia si fuerit ex praedicto denso; dico omnes radios in ipsa parallelos, in ipsius superficie refringi, & in focum E, in diaphano Rariore existentem, concurrere : Sit enim radius RM,

[24]

unus e parallelis incidens in punctum M, qui propterea erit in eodem cum axe plano ; erit igitur hyperbola NLM, (ex cujus circumductione genita est Conois) superficies refractionis Radii RM; qui refringetur in punctum E, ex eo quod radius puncto E egrediens, & incidens in M refringatur in MR; ut patet per *Prop. 9, lib. 10* Vitellionis. quod erat ostendendum.

Corollarium

Ex hoc satis apparet, Radios Parallelos RR, &c., congragari in unum punctum prope E, quo ad sensum: etiamsi LE non aequi distet radii &c. geometrice; Quoniam si non sit sensibilis differentia inter causas, nec erit sensibilis differentia inter effectus.

[24 cont'd]

§3. Prop.15.1.   Prop. 15. Problem.

The rays diverging from a single point of one medium are restored to parallelism in another medium of different density. The vertex of the required conoid or spheroid is also given.

This problem is the converse of the antecedent, and is solved in the same way; as shown by *Prep. 9, book 10* of Witelo.

Corollary

From this too it appears to be sufficient that the rays coming from any point near E, (on account of the reasoning reported above) are perceptibly restored to parallelism by the conoid or spheroid NLM.

[24 cont'd]

§3. Prop.15.2.   Prop. 15. Problema.

Radios ex unico puncto unius diaphani; provenientes, ad Parallelismum in alio diaphano diversae densitatis reducere ;data quoque Conoidis vel Sphaeroidis vertice.

Problema hoc, conversum est antecedentis, eodemque modo solvitur; ut Patet *Prop. 9, lib.10,* Vitellionis.

Corollarium.

Ex hoc quoque satis apparet radios, e puncto aliquo prope E provenientes, (propter rationem superius allatam) sensibiliter ad parallelismum Reduci Conoide vel Spheroide NLM.

**[25]**

## §3. Prop.16.1.   Prop. 16. Problem.

Parallel rays in one medium shall diverge in a medium of another density. Such rays can be traced back to some given point, for which the vertex of the required conoid or spheroid has also been given. It is also the case that the line drawn through the vertex and the given point is parallel to the given rays.

The parallel rays in the rarer medium A, A, etc., are to be refracted thus so that they appear to diverge from the point N: with the vertex M of the spheroid given too {prop. 12}. The dense ellipse LMR [*i.e.* on the outside] is made of this transparent medium, N is the more removed focus, and M is the vertex. The axis NM of this ellipse remaining fixed with respect to the parallel rays A, A, etc. The spheroid LMR is

Prop. 16 - Figure 1.

described by the rotation of this ellipse, composed of the rarer medium; indeed, the space which is situated around the spheroid is composed of the denser medium. I say that all the rays A, A, etc, incident on the surface of the spheroid, are refracted by the surface itself, and diverge from the focus N. Indeed from the rays A, A, etc., let one ray AL be incident on the spheroid at the point L, and this ray is produced to P, and the line HLG is drawn touching the spheroid in the point L, crossing the plane of the ellipse through the axis of the spheroid and the point L. The ellipse is the surface of refraction RML. The line NLB is drawn from the point N through the point of contact. Therefore, from prop. 10, if the spheroid LMR is made from the denser medium and is enclosed by the rarer medium, then the angle of incidence

[26]
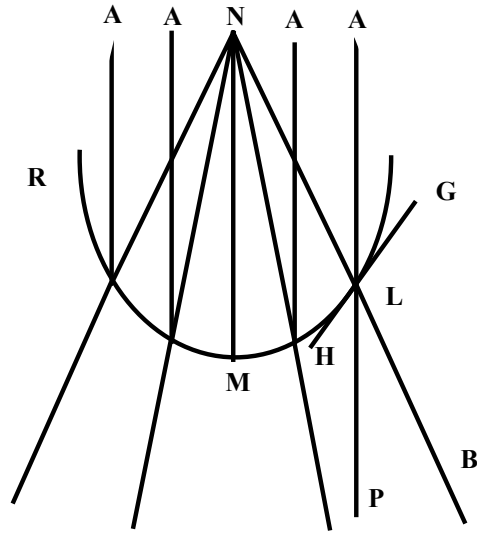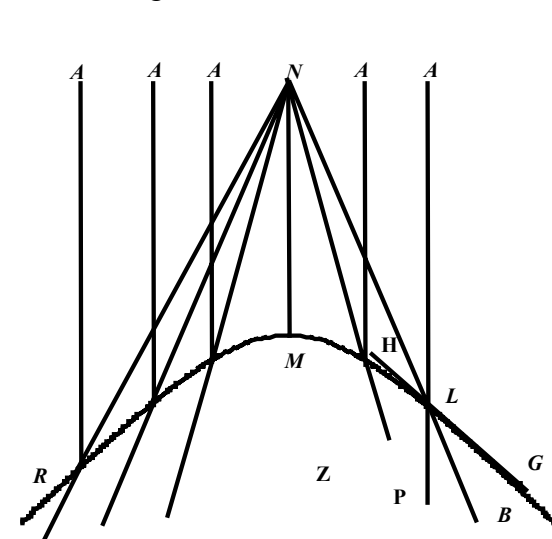
Prop. 16 - Figure 2.

HLP agree with the angle of refraction NLA from prop. 9 [*i.e.* the ray PL is refracted along LN]. Hence, in the present case for the equivalent angle of incidence ALG the equivalent angle of refraction PLB agrees [*i.e.* in the sense that the ray has bent through this angle]. Therefore the ray AL is refracted at B, and appears to diverge from the point N. Q.e.d.

Secondly the parallel rays A A A, etc. in the denser medium can be refracted thus, in order that they appear to diverge from the point N. Given too the vertex M of the conoid, N the more distant focus, and M the vertex. The dense hyperbola LMR is

made of this transparent medium, and by the rotation of which, with the axis NM remaining parallel to the given rays, the hyperbolic conoid RML is generated. Thus as the space outside the conoid (where assuredly the parallel rays A, A, etc are present) is made from the denser medium, truly the cone itself consists of the rarer medium. I say that all the rays A, A, etc. are refracted in the surface of the conoid RML, and diverge from the point N. Indeed the ray AL is one of these, for which the surface of refraction is the hyperbola RML itself, from which the conoid is described by Prop. 10. Through the point of incidence L, the line HLG is drawn touching the hyperbola, and AL is produced to P, and from the point N, through L the line NLB is drawn. If therefore the conoid RML is composed of the denser medium, and is present in the rarer medium, from which it is now composed by Prop. 6, then the line PL is refracted in N. Therefore for the equivalent situation with the angle of incidence PLG, the refraction NLA is in agreement, and the line AL is refracted in B. Q.e.d.

Corollary.

Hence too it is clear enough to the senses that if parallel rays A, A, etc., diverge from some point near N, then the line NM shall not be geometrically parallel to the rays A A etc.

<div style="text-align:center">

[25]

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§3. Prop.16.2.   Prop. 16. Problema.

Radios *parallelos in uno diaphano, ad divergentiam, in alio diaphano aliterius densitatis, ab aliquo puncto dato reducere;data quoque Conoidis vel Sphaeroidis vertice: Oportet autem, ut linea per verticem, & punctum datum ducta, sit radiis datis parallela.*

Sint radii Paralleli in Diaphano rariore A, A etc., ita refringendi, ut appareant divergere ex puncto N; data quoque Sphaeroidis vertice M, foco remotiore N, & vertice M, fiat ellipsis densatitis horum diaphanorum LMR, & ejus axe NM immobili manente, radiis A, A, &c. parallelo, ex ejus circumvolutione describatur Sphaerois LMR constans ex diaphano rariore; spatium vero in quo est Sphaerois, constet ex diaphano densiore: Dico omnes radios A, A, &c., in Sphaeroidis superficiem incidentes, in ipsius superficies refringi, & a foco N divergere: Sit enim e radius A, A, &c.,unus AL, incidens in Spheroidis superficiem in punctum L, & producatur in P, ducaturque contingens Sphaeroidem in puncto L, linea HLG, in plano ellipseos, per axem Sphaeroidis, & punctum L, transeuntis, quae ellipsis est superficies refractionis, sitque RML; & ducatur ex puncto N, per punctum contactus L, linea NLB: Si igitur Sphaerois LMR, esset diaphanum densius, & includeretur diaphano rariore, ex quo nunc constat, tunc angulo incidentia.

<div style="text-align:center">

[26]

</div>

HLP, competeret refractio NLA; ergo & aequali angulo incidentiae ALG competit aequalis refractio PLB; refringitur ergo radius AL in B, divergens a puncto N; quod ostendendum erat.

Sint secundo radii paralleli in diaphano densiore A A &c. ita refringendi, ut appareant divergere e puncto N; Data quoque Conoidis vertice M: foco remotiore N, & vertice M, fiat hyperbola densitatis horum diaphanorum LMR, ex cujus circumvolutione, axe NM

radiis datis parallelo manente, fiat Conois Hyperbolica RML ; ita ut spatium extra Conoidem, (ubi nimirum sunt radii paralleli A A &c.) constet ex diaphano densiore, ipsa vero Conois ex diaphano rariore. Dico omnes radios A A &c. refringi in superficie Conoidis RML, & divergi a puncto N. Sit enim ex illis unus, radius AL, cujus superficies refractionis est ipsa hyperbole RML, ex qua describitur Conois ; & per punctum incidentiae L, ducatur tangens hyperbolam Recta HLG , & producatur AL in P, & a puncto N, per L ducatur recta NLB; Si igitur Conois RML constatet ex diaphano densiore, & existeret in diaphano rariore, ex quo nunc constat ; tunc refrangeretur radius PL, in N; & angulo incidentiae PLG,competeret refractio NLA; aequali igitur angulo incidentiae ALH, competit aequalis refractio PLB ; refringitur igitur radius AL in B; quoderat ostendendum..

<div align="center">*Corollarium.*</div>

Hinc quoque satis patet radios A A &c. Parallelos, divergi ab aliquo puncto circiter N; quo ad sensum; etiamsi linea NM nonsit radiis A A &c., geometrice parallela.

<div align="center">[27]</div>

§3. Prop.17.1.   Prop. 17. Problem.

The rays in one medium converging to a single given point become parallel in another transparent medium of different density; with the vertex of the required Conoid or Spheroid given too.

This problem is the converse of the preceding too, and is solved in the same way; as is clear from *Prop. 9, Book 10,* Witelo.

<div align="center">*Corollary.*</div>

From this also it is apparent, that rays converging to some other point near N are sensibly reduced to being parallel for the Conoid or Spheroid LMR.

<div align="center">*Scholium.*</div>

   Up to this point we have talked only about a single refraction, which happens at the surface of a conoid or spheroid; indeed now we are to talk about the two-fold refraction of lenses (one refraction happens in the incidence of the rays, the other in the emergence), which are composed of conoidal or spheroidal frustrums. The same conclusions are to be shown always, for both mirrors and lenses - in order that the wondrous harmony may appear between Catoptrics and Dioptrics.

<div align="center">[27]</div>

§3. Prop.17.2.   Prop. 17. Problema.

Radios in uno diaphano,ad unicum punctum datum convergentes, ad parallelismium in alio diaphano diversae densatatis reducere ; data quoq; Conoidis, vel Sphaeroidis vertice.

Hoc problema est conversum quoque antecedentis, eodemque modo solvitur ; ut patet ex *Prop. 9, lib. 10,* Vitellionis.

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Corollarium.

Ex hoc etiam evidens est, radios ad punctum aliquod prope N convergentes, ad parallelismum reduci quo ad sensum, Conoide, vel Sphaeroide LMR.

Scholium.

Huc usque loquati sumus de unica tantum refractione, quae fit in superficie Conoidis, Sphaeroidis; nunc vero loquimur de duplice refractione lentium (una fit in radiorum incidentia, altera in radiorum emersione), quae ex frustis conoideon, vel sphaeroideon componuntur, easdem semper conclusiones demonstrando, & in speculis, & in lentibus ; ut appareat admiranda Harmonia, inter Catoptricam, & Dioptricam.