§6. Synopsis of Propositions 37 - 43:

The following six theorems prepare the way for non-axial rays to be used in paraxial ray image formation. Prop. 37 and Prop. 38 deal with parallel rays incident at a small angle to the optical axis of a hyperboloid and a spheroid where they are refracted to a focus at some point a little off the optical axis and nearly in the focal plane, while Prop. 39 deals with the similar case of reflection for a parabolic mirror. Prop. 40 and Prop. 41 are concerned with the almost equality of the angles subtended at the far focus and a point close to it by two nearby points near the vertex of a hyperboloid or of spheroid. Prop. 42 and Prop. 43 are concerned with convex hyperboloidal and concave spheroidal mirrors, where a point in one focal plane near the focus is reflected to a point in the other focal plane.

Until this point, only axial rays have been used in the *Promata* in image formation, in a rather restricted sense. This was the stage in the development, indicated in the preface, where Gregory felt at a loss to know how to proceed further, and he admits to having received some help from his brother David, also a mathematician. The use of paraxial rays was of course a great step forward, when one considers the muted response to lenses adopted by writers on optics at the time. Consider, for example, the writings of Mersenne (The version of his *Cogitata* & *Geometriae* published in 1644 represent his collected works on mathematics and physics apart from the letters), which set out what was known at the time about optics. The only major advance not known to Gregory was the law of refraction in a more general form than he had found himself. This is set out for example by Mersenne in his *Ballistica*, p. 79, following Descartes' *Dioptique*. Mersenne also includes two extra optical works at the end of his *Universae Geometriae,* following more of his own thoughts on optics: the first is a short previously unpublished tract by Warner, who gives the law of refraction using a different kind of diagram; and Hobbes's ideas are presented, which again follow Descartes. Warner had of course been present when Thomas Harriot had made the original deduction of the law of refraction from experiments performed around 1600 at Scion House - which Harriot only circulated amongst his friends and never published. Gregory professes his unfortunate ignorance of Descartes' work in the preface to the *Promoto*. We should also mention the sterling work done by Fermat in deriving the laws of both reflection and refraction from a least time principle at this time.

Prop. 37: If a visible object sends parallel rays through a hyperboloidal surface into a dense medium, then the apices of the pencils of rays from individual paraxial point sources lie very close to a single plane.

We are to understand the visible object to be either an extended distant object, such as the canopy of the stars, or the intermediate image formed by a previous lens. The idea of dividing an extended object up into a number of point sources is used. Gregory does not pursue the obvious question of relating the paraxial angle to the distance of the point of convergence of the paraxial rays from the focal plane, which we consider briefly in §6. Prop. 37.2. .

Prop. 38: If a light source sends parallel rays into a spheroid of a denser medium , then the apices of the pencils of individual point sources are almost coplanar.

Prop. 39: If a light source sends parallel rays to the concave surface of a parabolic mirror, then the apices of the pencils from individual point sources are almost coplanar.

Prop. 40: From one focus of an ellipse, a short line is drawn at right angles to the axis, and two lines are drawn from some point on this line. One line goes to the vertex of the ellipse and the other is truly perpendicular to the circumference of the ellipse. Also, from the preceeding focus of the ellipse, another line is drawn meeting the perpendicular in the circumference. In this case, the angle between the first two lines is almost equal to the angle between the third line and the axis of the ellipse.

Prop. 41: A short line is drawn at right angles to the axis of a hyperbola from one focus. Two lines are drawn from some point on this line: one line goes to the vertex of the hyperbola and the other is truly perpendicular to the circumference of the hyperbola. From the preceeding focus of the hyperbola, another line is drawn meeting the perpendicular in the circumference. In this case, the angle between the first two lines is almost equal to the angle between the third line and the axis of the hyperbola.

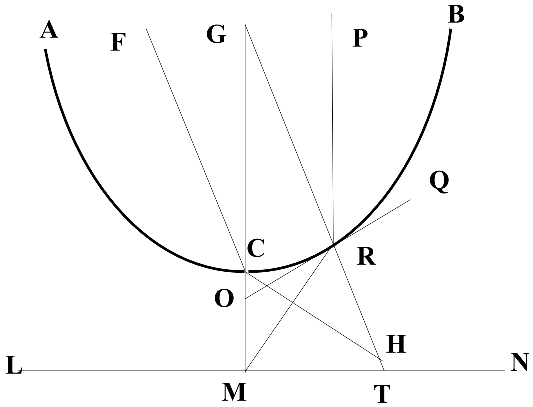
Prop. 42: If rays are sent from a planar object present in the focal plane of a concave elliptic mirror (of which the normal is the axis of the mirror) to the mirror surface, then the apices of the cones formed by the reflected rays from individual points are almost co-planar.

Prop. 43: If rays are sent from a planar object present in the focal plane of a convex hyperboloidal mirror (of which the normal is the axis of the mirror) to the mirror surface, then the apices of the cones formed by the reflected rays from individual points are almost co-planar.

§6. Prop. 37.1. Prop. 37. Theorem.

If a visible object sends parallel rays into a dense medium through a hyperboloidal surface, then the apices of the pencils of rays from individual paraxial point sources are almost coplanar.

Let ACB be the medium with the hyperboloidal or conoidal surface, with the axis GC produced to the exterior focus M, and through M is drawn the focal plane LMN normal to GM. I say that the apices of all the pencils coming from the converging parallel rays in the conoid lie very close to the focal plane LMN. For let the axis of one such pencil GRH be incident normally on the surface of the conoid at the point R. A plane is drawn through the axis of the conoid and the axis of the pencil (which lie in the same plane), giving a common hyperbolic section ACRB with the conoid, from which the conoid itself is generated. Now in the plane containing the line LMN and the point R, a line QRO is drawn tangent to the hyperbola, and it crosses to meet the axis in O. Another ray PR is drawn parallel to the axis of the hyperbola, and this ray is refracted in the exterior focus of the hyperbola M, which is the apex of the pencil of rays parallel to the axis. A line FC is drawn to meet the vertex of the hyperbola,



Prop. 37 - Figure 1.

[48]

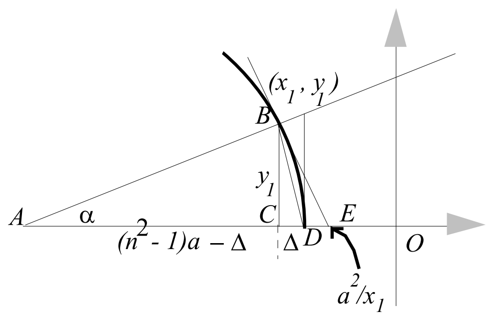
which is parallel to GH itself, which is refracted to the apex H of the pencil of rays which are all parallel to GH. Since the angles of incidence GRP and GCF are equal, then both the angles of refraction RMG and GHC are also equal. The triangles GCH and GRM are similar, hence:
GR : GM : : GC : GH.
Again, from the similar triangles GRO and GMT:
GR : GM : : GO : GT.
Therefore GC : GO : : GH : GT.

If the point R is near the vertex of the hyperbola C, (concerning the pencils of which we have so much to say) GC will be nearly equal to GO: (as those who have been well versed in conics are fully aware) and therefore GT will be itself approximately equal to GH; therefore the apex of the pencil H approaches as near as possible to the plane surface LMTN. Q.e.d.

§6. Prop. 37.2.

Notes.

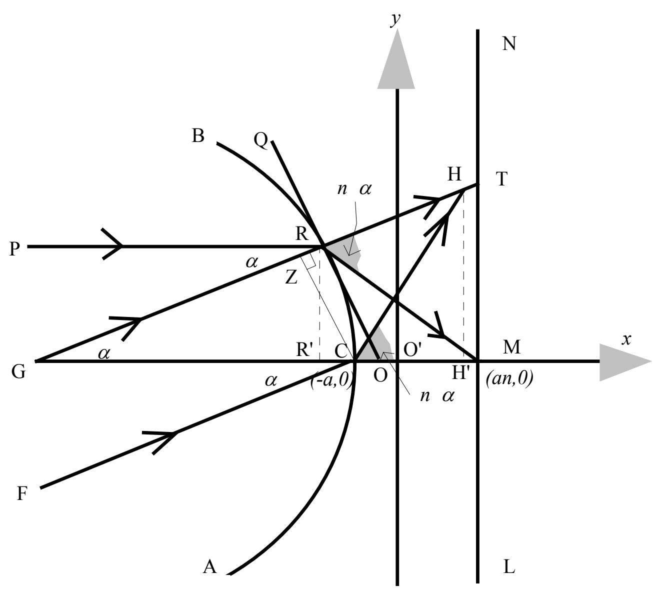
I. Initially, we show that for points on the left- hand branch of the hyperbola, close to the vertex D in Fig. 2, the y co-ordinate can be found in terms of some sagittal distance $\Delta > 0$ associated with the point B, which has co-ordinates (x_1, y_1). BE is the tangent at B, while A has the x co-ordinate n^2x_1, where $x_1 = -a - \Delta$. Now, from the right-angled similar triangles ABC & BCE, $BC^2 = AC.CE$, and on inserting the various lengths in terms of co-ordinates, keeping only first order terms in Δ, we obtain : $y_1^2 = (n^2 - 1).2a\Delta$.



Prop. 37 - Figure 2.

We may observe that $y_1 \sim \Delta^{1/2}$, which is an order of magnitude greater than Δ, a result giving the extent of the focusing action of the surface for paraxial rays . This result can also be obtained by direct substitution into the equation for the hyperbola.

II. If the point R with co-ordinates (x_1, y_1) is located near the x axis with origin at O' in fig. 3 on the left-hand branch of the regular hyperbola $x^2/a^2 - y^2/b^2 = 1$, (reverting to Gregory's notation), then the equation of the tangent QRO is given by: $xx_1/a^2 - yy_1/b^2 = 1$. This line cuts the x-axis at the point O (a^2/x_1, 0), and the length OC = a (1 + a/x_1), where $x_1 < 0$. Also, the normal GH to the hyperbola through R (x_1, y_1) has the equation: $y - y_1 = \alpha.(x - x_1)$, or $a^2\, x/x_1 + b^2 y/y_1 = a^2 + b^2$ in general. From the latter equation, when $y = 0$ we have $x_G = (1 + b^2/a^2).\, x_1 = n^2.x_1$, the x co-ordinate of G.



Prop. 37 - Figure 3.

According to §0.2 Fig. 1(c), there is a constant path difference between any ray RM parallel to the axis and the equivalent ray CM from the same wave front through the apex C: a requisite for focusing. We now consider the paraxial ray case, and in particular the path difference (p.d.) for the rays arriving at H that start from the wave front ZC in fig. 3. One ray GH lies along the auxiliary axis, while another lies along CH : the p.d. = (n.ZR + RH) - CH. We now proceed to evaluate these quantities.

Applying the sine rule to triangle ACH, we find that $CH = \frac{n^2 x_1 - a}{n-1} = (n+1)a + \frac{n^2}{n-1}\Delta$

Now, ZR = GR - GZ ~ (GO - GC) where cos α ~ 1.

$GO = -n^2 x_1 + a^2 / x_1 = n^2(a+\Delta) - a(1-\Delta/a) = (n^2-1)a + \Delta(n^2+1)$.

GC = $-n^2.x_1 - a = (n^2-1)a + n^2\Delta$.

Hence ZR = $a^2/x_1 + a \sim \Delta$, a result we might have guessed.

Again, RH = GH - GR ~ GH - GO. From the sine rule applied to triangle GHO:

GH/n = GC/(n - 1); hence, $GH = \frac{n}{n-1}\left[n^2(a+\Delta) - a\right] = n(n+1)a + \frac{n^3}{n-1}\Delta$,

where GM = $an(n + 1)$.

Hence, RH = GH - GO

$= n(n+1)a + \frac{n^3}{n-1}\Delta - (n^2-1)a - \Delta(n^2+1) = (n+1)a + \frac{\Delta}{n-1}(n^2-n+1)$.

The above p.d. becomes:

$n.ZR + RH - CH = -(n+1)a - \frac{n^2}{n-1}\Delta + n\Delta + (n+1)a + \frac{\Delta}{n-1}(n^2-n+1) = (n-1)\Delta.$

Hence, to the first order appriximation considered, the path difference is proportional to Δ, and it follows that for a point x_1, with say $\Delta < \lambda/4$ of C, for a given wavelength λ, reasonable focusing of the beam will occur at H. Following part I of this note, an equivalent variation for y can be established. The distance HT of the paraxial focal point H from the focal plane NM is given by HT = CO.GT/GO from Prop. 37 ; CO = Δ ; GT = $an(n + 1) + O(\Delta)$; GO = $(n^2 - 1)a + O(\Delta)$: giving HT = $\frac{an(n+1)\Delta}{(n^2-1)a+(n^2+1)\Delta} \sim \frac{n}{n-1}\Delta$,

placing further constraints on the magnitude of Δ for focusing of paraxial rays in the original plane.

Finally, we note that the observable quantity α is related to Δ according to :

$$\alpha = \frac{y_1}{(n^2-1)x_1} \sim \sqrt{\frac{2\Delta}{a(n^2-1)}}$$

Similar arguments can be given for the two following theorems.

§6. Prop. 37.3. Prop. 37. Theorema.

Si visibile mittat radios parallelos, per Conoidem densitatis ; erunt apices pencilllorum, singulorum visibilis punctorum, in uno quam proxime plano.

Sit Conois densitatis ACB, cujus axis GC producetur ad focum exteriorem M; & per M ducatur planum LMN, cui normalis sit GM; dico omnes apices pencillorum, ex radiis in Conoide parallelis, genitorum, cadere quam proxime in planum LMN. Sit enim unius pencili axis GRH, normaliter cadens in conoidis superficiem in puncto R, & per axem conoidis, & axem pencilli (qui sunt in eodem plano) ducatur planum, faciens cum conoide, communem sectionem hyperbolam ACRB, ex qua generatur, cum plano vero LMN, rectam LMN; & per punctum R, ducatur una linea tangens hyperbolam, nempe

QRO, axi occurrens in O, alia vero PR, axi hyperbolae parallela, quae refringitur in focum hyperbolae exteriorem M, seu apicem pencilli radiorum

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axi parallelorum ; ducaturque ad vertice hyperbolae linea FC, ipsi GH parallela, qui ideo refringetur H, apicem pencilli radiorum, ipsi GH parallelorum: & quoniam angulus incidentiae GRP, est aequalis angulo incidentiae GCF, erit & angulus refractionis RMG angulo refractionis GHC aequalis {cor. 14. Hujus}: eruntq; triangla GCH, GRM similia, & ut

GR : GM : : GC : GH;

Et ob similitudinem triangulorum GRO, GMT, ut

GR : GM : : GO : GT; ergo ut GC : GO : : GH : GT;

Si vero punctum R non multum recedat a vertice Hyperbolae C, (de quibus pencilllis nos tantum loquimur) GC ferme aequalis erit ipsi GO : (ut satis norunt qui in conicis versati sunt) ergo & GT, quam proxime aequalis erit ipsi GH; apex igitur pencilli H, quam proxime incidet in superficiem planum LMTN; quod demonstrare oportuit.

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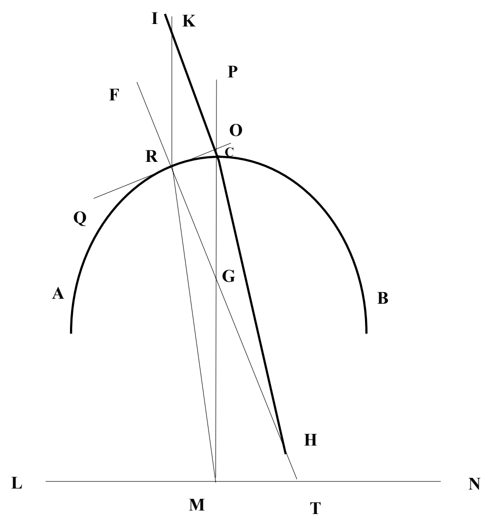
§6. Prop. 38.1. Prop. 38. Theorem.

If a light source sends parallel rays into a spheroid of a denser medium , then the apices of the pencils of individual point sources are almost coplanar.



Prop. 38 - Figure 1.

Let the dense spheroid be ARCB, the axis GC of which is produced to the further focus M; through M is drawn the plane LMTN which is normal to the axis CGM. I say that all the apices of the pencils arising from the parallel rays incident on the spheroid fall close to the plane LMTN. Let the axis of one pencil be FR, incident normally on the surface of the spheroid at the point R, and a plane is drawn through both the main axis of the spheroid and the axis of the pencil, making a common elliptic section ACRB with the spheroid, from which the spheroid is generated. With the normal plane LMTN there is associated the line LMTN. Through the point R a line QRO is drawn tangent to the ellipse cutting the axis in O. Another line KR, which is parallel to the axis of the ellipse, is refracted through the focus M: the apex of the pencil of rays parallel to the axis. A line IC is drawn through the vertex of the ellipse C parallel to the ray FR itself, which therefore is refracted through H, the apex of the pencil of rays parallel to FR. Since the angles of incidence ICP and FRK are equal, it follows that both the angles of refraction RHC and RMC are equal [Note: we would now call these latter angles the angles of deviation rather than refraction]. The triangles GCH, GRM are similar:

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hence GR : GM : : GO : GH;
And from the similitude of the triangles GRO & GMT:
GR : GM : : GO : GT;
therefore: GC : GO : : GH : GT.
If indeed the point R is near the vertex of the ellipse C, (concerning the pencils of which we have so much to say), then GC will be nearly equal to GH itself, and therefore GT will be itself approximately equal to GH. Therefore the apex of the pencil H approaches as near as possible to the plane surface LMTN. Qed.

[49]

§6. Prop. 38.2. Prop. 38. Theorema.

Si visibile mittat radios parallelos, in sphaeroidem densitatis ; erunt apices pencillorum, singulorum visibilis punctorum, in uno quam proxime plano.

Sit Sphaerois densitatis ARCB, cujus axis GC producetur ad focum remotiorem M; & per M ducatur planum LMTN, cui normalis sit axis CGM; dico omnes apices pencillorum, ex radiis in sphaeroidem parallele incidentibus, ortorum, cadere quam proxime proxime in planum LMTN. Sit enim unius pencili axis FR, normaliter incidens in sphaeroidis superficiem in puncto R, & per axem sphaeroidis & pencilli ducatur planum, faciens cum sphaeroide communem sectionem ellipsen ACRB, ex qua generatur, cum plano vero LMTN, rectam LMTN; & per punctum R, ducatur una linea QRO tangens ellipsem, & axi occurrens in O, alia vero KR, axi ellipseos parallela, quae igitur refringitur in focum M, seu apicem pencilli radiorum
axi parallelorum ; ducaturq; per verticem ellipseos C, recta IC, ipsi FR, parallela, quae ideo refringetur in H, apicem pencilli, Radiorum ipsi FR parallelorum: et quoniam angulus incidentiae ICP, est aequalis angulo incidentiae FRK, erit & angulus refractionis RHC angulo refractionis RMC aequalis {cor. 14. Hujus}: eruntq; triangla GCH, GRM similia, & ut

[50]

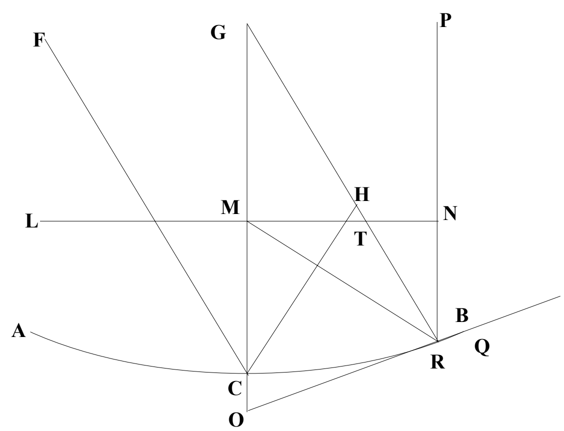
GR : GM : : GO : GT
Et ob similitudenem triangulorum GRO, GMT, erit ut
GR : GM : : GO : GT;
ergo ut GC : GO : : GH : GT;
Si vero punctum R non recedat multum a vertice ellipseos C, (de quibus pencillis nos tantum loquimur) GC ferme aequalis erit ipsi GH : apex igitur pencilli H, quam proxime incidit in superficiem planum LMTN; quod demonstrandum erat.

§6. Prop. 39.1. Prop. 39. Theorem.

If a light source sends parallel rays to the concave surface of a parabolic mirror, then the apices of the pencils from individual point sources are almost coplanar.

Let ACB be a concave parabolic mirror with axis GC and focus M. Through the focus M a plane is drawn normal to the axis GMC. I say that all apices of pencils arising from the incident parallel rays fall quite near the plane LMTN drawn through the focus. For let GR be the axis of one pencil, incident normally on the concavity of the mirror surface at the point R, and through the axis of the mirror a plane is drawn, making a common parabolic section ACRB with the mirror, from the rotation of which section the mirror is generated. Indeed with the plane LMTN, the line LMTN is also generated; & through the point R a line QRO is drawn tangent to the parabola, meeting the axis in O. Also, to the same point R, another line PR is drawn parallel to the axis GC, which therefore is reflected in the focus M of the parabola, or the apex of the pencil of the rays parallel to the axis. The line FC is drawn through the vertex C of the mirror, parallel to GR, which therefore is reflected in H, which is the apex of the pencil of the rays which are parallel to GR. Thus, since the angles of incidence FCG and GRP are equal, both the angles of reflection GRM and GCH are also equal. The triangles GCH and GRM are similar, and

[51]



Prop. 39 - Figure 1.

GR : GM : : GC : GH;

From the simililarity of the triangles GRO & GMT:

GR : GM : : GO : GT;

therefore GC : GO : : GH : GT.

If the point R in near the vertex of the parabola C, (concerning the pencils of which we have so much to say) GC will be nearly equal to GO: and therefore GH is approximately equal to GT. Therefore the apex of the pencil H approaches as near as possible to the plane surface LMTN. Which was to be show.

§6. Prop. 39.2. Prop. 39. Theorema.

Si visibile mittat radios parallelos, in speculum conum parabolicum, erunt apices pencillorum, singulorum visibilis punctorum, in uno quam proxime plano.

Sit speculum cavum parabolicum ARCB, cujus axis GC, focus M; & per focum M ducatur planum cui perpendicularis sit axis GMC; dico apices omnium pencillorum, ex radiis parallele incidentibus ortorum, cadere quam proxime proxime in planum LMTN, per focum ductum. Sit enim unius pencili axis GR, perpendiculariter incidens in cavem speculi superficiem in puncto R, per quem, & axem speculi ducatur planum, faciens communem sectionem cum speculo parabolam ACRB, cujus revolutione generatur speculum; cum plano vero LMTN, rectam LMTN; & per punctum R, ducatur recta tangens parabolam, & axi occurrens in O, nempe QRO; & ad idem punctum R, ducatur alia recta PR, axi GC parallela, quae igitur reflectetur in focum parabolam M, seu apicem pencilli radiorum axi parallelorum ; ducaturq; ad verticem speculi C, recta FC, ipsi GR parallela, quae ideo reflectetur in H, apicem pencilli radiorum, ipsi GR parallelorum: quoniam itaq; angulus incidentiae FCG, est aequalis angulo incidentiae GRP, erit quoque angulus reflexus GRM aequalis angulo reflexo GCH {cor. 18. Hujus}. Triangla igitur GCH, GRM sunt similia, & ut

[51]

GR : GM : : GC : GH;

Et ob similitudenem triangulorum GRO, GMT, erit ut

GR : GM : : GO : GT;

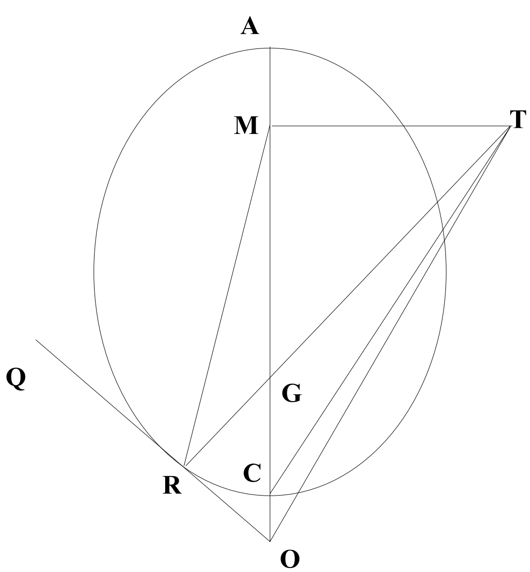
ergo ut GC : GO : : GH : GT;

Si vero punctum R non multum recedat a vertice parabola C, (de quibus pencillis nos tantum loquimur) GC propendicum aequalis erit ipsi GO : erit ergo & GH fere aequalis ipsi GT, apex igitur pencilli H, quam proxime incidit in planum LMTN; quod demonstrandum erat.

[52]

§6. Prop. 40.1. Prop. 40. Lemma.

A short line is drawn at right angles to the axis of an ellipse from one focus. Two lines are drawn from some point on this line: one line goes to the far vertex of the ellipse and the other is truly perpendicular to the circumference of the ellipse. Also, from the preceeding focus of the ellipse , another line is drawn meeting the perpendicular in the circumference. In this case, the angle between the first two lines is almost equal to the angle between the third line and the axis of the ellipse.



Prop. 40 - Figure 1.

ARC is the ellipse, and from the focus M a line MT is drawn perpendicular to the axis AC. From a point T, the line TC is drawn to the vertex C. Also from T to the circumference of the ellipse, the perpendicular line TR is drawn to the tangent QRO with point of contact R. The line MR is drawn from the focus M to the point R. I say that the angles RMC & RTC are almost equal. The axis AC is produced so that it meets the tangent QRO. If therefore from the diameter OT, a circle is described then it will pass through the points R & M, as the angles TMO & RTO are right. Therefore, the angles RTO and RMO are equal, lying in the same arc of the circle. But the angle

RTC does not differ much from the angle RTO, for if the point R of the contact line RO lies near the vertex C, (which always happen on account of the shortness of the line MT), it meets the axis almost at C. And with the line OC being short, then the difference between the angles RMC & RTC, *i.e.* the angle CTO, is diminished. Therefore, if the line MT is made very short, then the angles RTC and RMC will be almost equal. Qed.

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§6. Prop. 40.2. Prop. 40. Lemma.

Si ex uno ellipseos foco, ducatur rectula, normalis axi ; & ab aliquo rectulae puncto, ducantur duae rectae, una ad elllipseos verticem, altera vero ellipseos circumferentiae perpendicularis ; a foco etiam ellipseos praedicto, ducantur alia, occurens perpendiculari, in circumferentia : erit angulus comprehensus a duabus primis rectis, ferme aequalis angulo comprehenso, a tertia, & axe ellipseos.

Sit ellipsis ARC, ex cujus foco M, ducatur recta MT, axi AC perpendicularis; & a puncto T ad ellipseos verticem C, ducatur recta TC ; & ab eodem puncto T, ad ellipseos circumferentiam, ducatur contingenti QRO, perpendicularis in puncto contactus TR; & a foco M, ad punctum R, ducatur recta MR. Dico angulos RMC, RTC fere aequales esse. Producatur axis AC, ut concurrat cum tangente QRO in O;si igitur diametro OT, describatur circulus, transibit per puncta R, M; quoniam anguli TMO, RTO sunt recti; anguli igitur RTO, RMO in eodem circuli portionis sunt aequales; angulus autem RTC non multum differt ab angulo RTO, quoniam si R punctum, cadat prope C, (quod semper evenit propter brevitatem lineae MT), linea contactus RO, concurret cum axe, fere in C. Et pro parvitate rectae OC, diminuitur angulus CTO, differentia inter angulos RMC & RTC: quare si recta MT fuerit brevis; erunt anguli RTC, RMC ferme aequales; quod demonstrandum erat.

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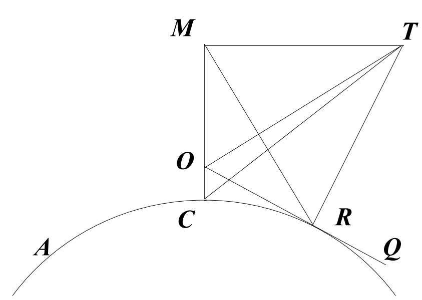
§6. Prop. 41.1. Prop. 41. Lemma.

A short line is drawn at right angles to the axis of a hyperbola from one focus. Two lines are drawn from some point on this line: one line goes to the vertex of the hyperbola and the other is truly perpendicular to the circumference of the hyperbola. From the preceeding focus of the hyperbola, another line is drawn meeting the perpendicular in the circumference. In this case, the angle between the first two lines is almost equal to the angle between the third line and the axis of the hyperbola.

Let ARC be the hyperbola, from the far focus M of which the short line MT is drawn perpendicular to the axis AC. The line TC is drawn From the point T to the vertex C of the hyperbola, and also from T another is drawn to the point of contact R perpendicular to the tangent QRO. The line MR is drawn from the focus M to the point R. I say that the angles CMR & CTR are almost equal. The tangent QRO is produced



Prop. 41 - Figure 1.

so that it meets the axis in O. A circle can be described on the diameter TO passing through the points R & M, since the angles TMO & RTO are right. Angles OMR & OTR are equal, lying on the same arc of the circle. However the angle RTC does not differ much from the angle RTO, for if the point R of the contact line RO lies near the vertex C, (which always happens on account of the shortness of the line MT), it meets the axis almost at C. And if the line OC is made small in length, then the difference between the angles RMC & RTC, *i.e*, the angle CTO , is diminished. Therefore, if the line MT is made very short, then the angles RTC & RMC are almost equal. Qed.

[53]

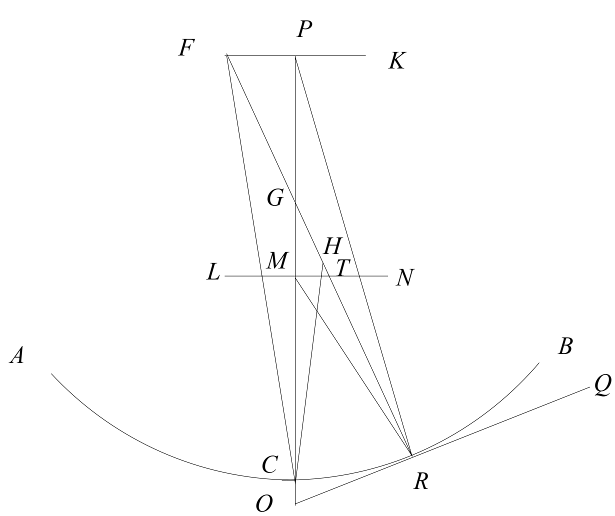
§6. Prop. 41.2. Prop. 41. Lemma.

Si ex uno hyperbolae foco, ducatur rectula, normalis axi ; & ab aliquo rectulae puncto, ducantur duae rectae, una ad hyperbolae verticem, altera vero circumferentiae perpendicularis ; a foco etiam hyperbolae praedicto, ducantur alia, occurens perpendiculari, in circumferentia : erit angulus comprehensus a duabus primis rectis, ferme aequalis angulo comprehenso, a tertia, & axe hyperbolae.

Sit hyperbola ARC, ex cujus foco M, ducatur recta MT, axi AC perpendicularis; & a puncto T ad hyperbolae verticem C, ducatur recta TC ; & ab eodem puncto T, ducatur altera ad punctum contactus R; tangenti ORQ perpendicularis TR ; & a foco M, ad punctum R, ducatur MR. Dico angulos CMR, CTR fere aequales esse. Producatur tangens QRO ut concurrat cum axe in O; & diametro TO, ducatur circulus, qui transibit per puncta R, & M; quoniam anguli TMO, RTO sunt recti; eruntque anguli OMR, OTR, in eodem circuli portion ; inter se aequales ; angulus autem RTC non differt multum ab angulo RTO, quoniam si R punctum, cadat prope verticem C, (quod semper evenit propter brevitatem lineae MT), tangens QRO, concurret cum axe, ferme in vertice C. Et pro brevitate rectae OC, diminuitur angulus CTO, differentia inter angulos RMC & RTC: quare si MT fuerit breviuscula; erunt anguli RTC, RMC aequales fere; quod demonstrandum erat.

[54]

§6. Prop. 42.1. Prop. 42. Theorem.



Prop. 42 - Figure 1.

If rays are sent from a planar object present in the focal plane of a concave elliptic mirror (of which the normal is the axis of the mirror) to the mirror surface, then the apices of the cones formed by the reflected rays from the individual points are almost co-planar.

Let ACB be the surface of the elliptic concave mirror, PC is the axis and the foci are P and M. Let the plane surface object FPK send rays from the focal plane through P, to which the axis PC is perpendicular.

[55]

Another plane LMT is drawn through the other focal plane M N. I declare that the apices of the pencils of the single points on the radiating surface lie approximately in the plane LMTN. In order to show this, let the axis of one of the pencils be FR, incident normally on the mirror at the point R. A plane is drawn through this line and the axis of the mirror giving the ellipse ACRB as a common section with the mirror, from which the mirror can be generated. From the corresponding planes the lines FPK and LMTN are taken, and the line QRO is drawn which is a tangent to the ellipse at the point R, which meets the axis at O. The line PR is drawn from the focus P to the same point R, which is reflected through the other focus M, which is the apex [i.e. meeting point] of the pencil of rays diverging from P , by Prop. 22. The line FC is drawn from the point F to the vertex of the mirror, where it is reflected through H, the apex of the pencil of the diverging rays from the point F. And so the angles CFR and CPR are equal; (for they shall not be very different, as we have shown above), according to the cor. of Prop. 22. Therefore the angles of incidence PRF and PCF are equal; and also the angles of reflection GRM, GCH are equal. Therefore the triangles GRM and GCH are similar.
And as

$$GR : GM :: GC : GH;$$

And from the similitude of the triangles GRO and GMT,

$$GR : GM :: GO : GT;$$

therefore as GC : GO : : GH : GT. Now if the point R is actually very near the vertex of the ellipse C (about the pencils of have so much to say at present) then GC will be nearly equal to GO itself, and therefore GH will be itself approximately equal to GT. Hence the apex or image point H of the pencil of rays diverging from F approaches closely to the plane surface LMTN. Qed.

§6. Prop. 42.2. Prop. 42. Theorem.

Si superficies plana, in foco speculi cavi elliptici (cui normalis sit axis speculi) mittat radios in speculum; erunt apices pencillorum, singulorum superficiei puncturom, in uno quam proxime plano.

Sit speculum ellipticum concavum, ACB cujus axis PC, foci P, M; sitq; in foco P, superficis plana radians, FPK

[55]

cui perpendicularis sit axis PC ; per alterem focum M, ducatur aliud planum priori parallelum, LMTN. Dico apices pencillorum singulorum superficieis radiantis punctorum, cadere quam proxime in planum LNTN. Sit enim unius pencilli axis FR, perpendiculariter incidens in speculum, in puncto R; per quem, & axem speculi, ducatur planum, faciens commonem sectionem cum speculo, ellipsem ACRB, ex qua generatur speculum;cum planis vero, rectas lineas FPK, LMTN; & per punctum R, ducatur recta QRO, tangens ellipsem, & occurens axi in O; & ad idem punctum R, ducatur a foco P,

recta PR, quae ideo reflectetur in alterum focum M, seu apicem pencilli radiorum , ex foco P divergentium ; ducaturq; ad verticem speculi, ex puncto F, recta FC, quae reflectetur in H, apicem pencilli, radiorum ex puncto F divergentium. Sint itaque anguli CFR, CPR aequales; (parum enim differunt ut demonstrimus) ; eruntq; igitur anguli incidentiae PRF, PCF aequales : & ideo anguli reflexi FRV, PCS, seu illis ad verticem MRG, HCG, aequales erunt ; quare triangula GRM, GCH sunt similia.
Et ut

GR : GM : : GC : GH;

Et ob similitudinem triangulorum GRO, GMT, ut

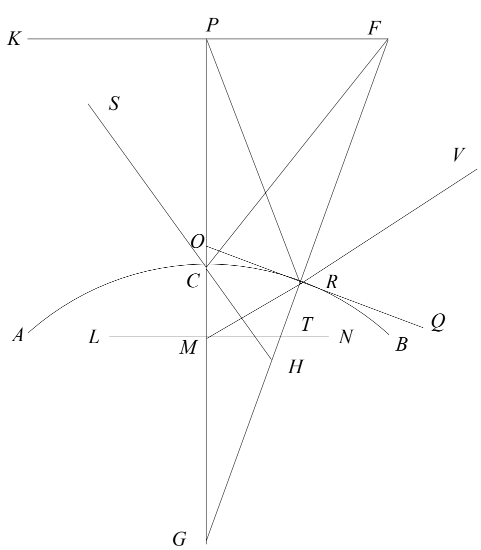
GR : GM : : GO : GT;

ergo ut GC : GO : : GH : GT; Si vero punctum R, non recedat multum a vertice ellipseos C ; (de quibus pencillis nos tantum loquimur) GC ferme aequalis erit rectae GO ; erit ergo & GH , quam proxime aequalis ipsi GT; Incidit igitur apex pencilli radiorum, ex F divergentium, nempe H, in superficiem planam, LMTN sere; quod demonstrantium erat.

§6. Prop. 43.1. Prop. 43. Theorem.

If rays are sent from a planar object present in the focal plane of a convex hyperbolic mirror (of which the normal shall be the axis of the mirror) to the mirror, then the apices of pencils of the reflected rays of the individual points are almost coplanar.

[56, 57]



Prop. 43 - Figure 1.

Let ACB be the surface of the convex hyperbolic mirror, the axis is PCG, and the foci are P and M. Let the planar object FPK sent rays from the focus P, which is perpendicular to the axis PC. Through the other focus M another plane LMTN is drawn parallel to the first. I declare that the apices or image points of the pencils of the individual object points on the surface are near the plane surface LMTN. For let the axis of one of the cones of rays FR be incident normally on the mirror at the point R. Through R and the axis of the mirror, a plane is drawn making as a common section with the mirror the hyperbola ACRB, from which the mirror is generated. Indeed, the intersection of this plane with the planes KPF and LMN gives the lines KPF and LMN. The line QRO is drawn through the point R tangent to the hyperbola, meeting the axis in O. The line PR is drawn from the focus P to the same point R, which is thus reflected

from the other focus M, or from the apex or image point of the cone of rays diverging from the point P, by Prop. 24. The line FC is next drawn from the point F to the vertex of the mirror which is then reflected from H, the apex or image point of the cone of diverging rays from the point F, by cor. Prop. 24. Therefore the angles CFR and CPR are equal; (which indeed differ little, as we have explained); and therefore the angles of incidence PRF and PCF are equal: and also the angles of reflection FRV and PCS are equal. Therefore the triangles GRM, GCH are similar.
And as

GR : GM : : GC : GH;

And from the similitude of the triangles GRO, GMT:

GR : GM : : GO : GT;

therefore GC : GO : : GH : GT.

If the point R is near the vertex of the Hyperbola C (about which cones of rays we say so much) then GC will be nearly equal to GO, and therefore GH will itself be approximately equal to GT. Therefore the apex H of the pencil of rays of reflection from the point F is not far from the plane surface LMTN. Qed.

Scholium.

In the corollories to our catoptic and dioptic problems (which have been resolved geometrically to a large extent by the use of conic sections through the foci), we have said these same problem can be adequately resolved according to our visual sense for certain other points a little distance from the foci. But from these theorems just demonstrated, it is evident that all these points almost lie in a plane perpendicular to the axis of the lens or mirror.

§6. Prop. 43.2. Prop. 43. Theorema.

Si superficies plana, in foco speculi hyperbolici convexi (cui normalis sit axis speculi) mittat radios in speculum; erunt apices pencillorum, singulorum superficiei puncturom, in uno quam proxime plano.

[56, 57]

Sit speculum hyperbolicum convexum; cujus axis PCG, foci P, M; sitq; in foco P, superficis plana radians, FPK cui perpendicularis sit axis PC ; per alterem focum M, ducatur aliud planum, priori parallelum LMTN. Dico apices pencillorum singulorum superficieis radiantis punctorum, non procul abesse a superficie plana LNTN. Sit enim unius Coni radiosi axis FR, perpendiculariter incidens in specului superficiem in puncto R; per quem, & axem speculi, ducatur planum, faciens commonem sectionem cum speculo, hyperbolam ACRB, ex qua generatur speculum;cum planis vero, rectas lineas FPK, LMN & per punctum R, ducatur recta QRO, tangens Hyperbolam, & axi occurens in O; ducaturq; a foco P, ad idem punctum R, recta PR, quae ideo reflectetur a foco M, seu apice coni radiorum puncti P reflexorum, ducatur deinde ad verticem speculi, ex puncto F, recta FC, quae reflectetur ab H, apice coni radiorum reflexorum puncto F. Sint itaque anguli CFR, CPR aequales; (qui parum enim differunt ut demonstravimus) {22 & Cor.22. Hijus}; eruntq; igitur anguli incidentiae PRF, PCF aequales : & ideo anguli reflexi GRM, GCH, aequales erunt ;quare triangula GRM, GCM sunt similia.
Et ut

GR : GM : : GC : GH;

Et ob similitudinem triangulorum GRO, GMT, ut

GR : GM : : GO : GT;

ergo ut GC : GO : : GH : GT; Si vero punctum R, non recedat multum a vertice Hyperbolae ; (de quibus conis radiosis nos tantum loquimur) rectae GC ferme aequalis erit rectae GO ; erit ergo & GH , quam proxime aequalis ipsi GT; apex igitur coni radiorum reflexorum puncti F, nempe H, non procul distantia plano LMTN; quod demonstrantium erat.

Scholium.

In Corollariis ad problemata nostra catoptrica, & dioptrica, diximus eadem problema, (quae geometrice tantum resolvuntur per focos sectionum conicarum) quo ad sensum etiam resolvi, per alia quaedam puncta, a focis paululum distantia. Ex his autem Theorematibus, nuper demonstratis, paret omnia illa puncta esse in uno quam proxime plano, cui perpendicularis est axis lentis, vel speculi.