§4. Synopsis of Propositions 18 - 27: Image formation by mirrors and lenses with spheroidal and hyperboloidal surfaces.

Gregory pursues the analogy between reflection and refraction in his investigation of the imaging properties of mirrors and lenses with surfaces derived from conic sections. A sequence of propositions of increasing complexity is established, relating object and image positions for special cases. These follow from the above theorems and also verified from products of the transfer matrices M_1 - M_8 introduced in §0.4.

Prop. 18 considers the focusing or meeting of parallel rays at a point in the same medium by : (a) a concave reflecting paraboloid; (b) a solid lens set in air with a plane and a convex refracting hyperboloidal surface; and (c) a hollow refracting lens set in a dense medium with a plane and a spheroidal refracting surface. The initial rays are parallel to the line joining the vertex to the point of concurrence or focal point of the rays - *i. e.* the optical axis.

Prop. 19 is the converse of Prop. 18, where the rays are reversed in direction.

Prop. 20 considers the divergence of parallel rays from a special point, *i. e.* the focus, in the same medium by: (a) a convex reflecting paraboloid; (b) a solid lens set in air with a plane and refracting convex hyperboloidal surfaces; and (c) a hollow lens set in a dense medium with a plane and a spheroidal refracting surface. The initial rays are again parallel to the optical axis.

Prop. 21 is the converse of Prop. 20, on reversing the rays.

Prop. 22 considers the convergence to a focal point of rays diverging from another focal point in the same medium on the optic axis by means of : (a) a concave reflecting ellipsoid where the foci of the ellipsoid are used; (b) a solid lens set in air with equal convex hyperboloidal refracting surfaces, where the far focal point of each hyperbola is used; and (c) a hollow lens set in a dense medium with equal spheroidal refracting surfaces, where the far focal point of each ellipsoid of cross-section is used .

Prop. 23 is the converse of Prop. 22, on interchanging convex and concave surfaces for both conoidal and spheroidal surfaces.

Prop. 24 considers the convergence to a focal point of rays of the rays converging to another focal point in the same medium on the optic axis by means of : (a) a convex reflecting hyperboloid where the foci of the hyperboloid are used; (b) a solid lens set in air with unequal concave hyperboloidal refracting surfaces, where the unequal far focal point of each hyperbola are used; and (c) a hollow lens set in a dense medium with unequal hyperboloid refracting surfaces, where the far focal point of each hyperboloid of cross-section are used .

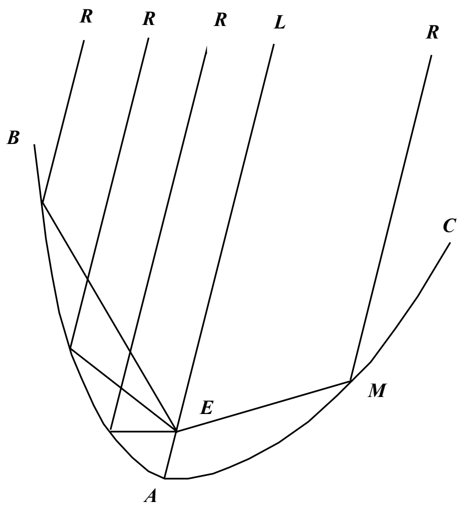
Prop. 25 is the converse of Prop. 22, on exchanging convex spheroidal surfaces for concave conoidal surfaces of unequal focal lengths.

Prop. 26 considers the reflection of a ray incident at some angle to the axis from the vertex of a concave mirror of revolution, together with a ray incident at a different angle to the axis. The rays and the axis are coplanar. The angle formed from the two rays of incidence is equal to the angle formed from the two rays of reflection.

Prop. 27 is a comparable theorem for refraction, forming a vital part of Gregory's theory of image formation by lenses with conoidal or spheroidal surfaces. Two rays at different angles to the axis are incident at the vertex of a lens, one ray on either side of the axis. These rays are refracted into the lens at known angles, and they proceed to the other surface. Two other rays are drawn from the vertex of emergence, each parallel to an incident ray. Within the lens, there are two sets of parallel rays. The theorem contends that the angle between the incident rays is equal to the angle between the emergent refracted rays. We shall discuss this theorem in more depth in the synopsis to Prop. 44.

§4. Prop.18.1. Prop. 18. Problem.

Parallel rays are to meet at a single point of the same medium, with the vertices of the required lens or mirror also given: nevertheless the line drawn through the vertices of the lens or mirror and the point of concurrence is parallel to the given rays.



Prop. 18 - Figure 1.

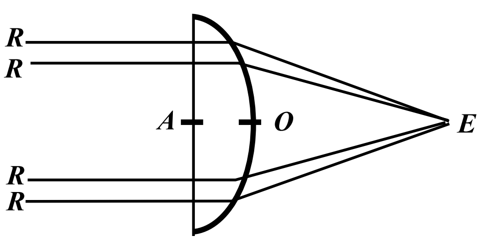
Reflection.

Ler the parallel rays R, R, etc. meet in the single point E: a concave parabolic mirror BAMC is made with vertex A, focus E, and vertex A. I say that all the rays R, R, etc. falling upon this mirror are reflected through its focus E.

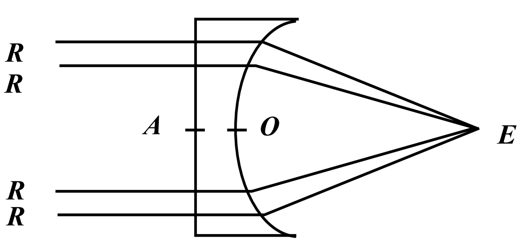
For let M be the point of incidence of the ray RM, and

[28]

since the axis of the mirror AE, and the ray RM by supposition are parallel, then they are in the same plane, and the parabola BAC from which the mirror BAMC has been generated by rotation, will be the reflecting surface. Hence the ray RM is reflected in the focus E. See Prop.10, Cor. 2, and Prop. 11. Q.e.d.



Prop. 18 - Figure 2.



Prop. 18 - Figure 3.

Refraction.

Let the parallel rays R, R, etc. be brought together in a single point E, and let the vertices of incidence and emergence of the lens be A and O. The rays R, R, etc. are led with the help of the plane surface of the lens drawn through A, to which the rays R, R are normal, and within the lens they are sent parallel. [Beyond the lens] they meet at the focal point E, by means of refraction by the [solid] conoid, (Prop. 18 - Fig. 2) or [hollow] spheroid (Prop. 18 - Fig. 3), the vertex of which is O. See also Prop.13 & 14. Q.e.d.

Corollary.

From this it also follows that parallel rays meet near E, even if the rays are not parallel to the line AE.

[29]

§4. Prop.18.2. Prop. 18. Problema.

Radios parallelos, in unicum punctum datum ejusdem diaphani congregare; datis quoque lentis vel speculi verticibus : opertet tamen, ut linea ducta per vertices lentis, vel speculi, & punctum concursi sit radiis datis parallela.

Catoptrice.

Sint radii paralleli R R &c., in unam punctum E congregari, sitque vertex speculi A : foco E, & vertice A, fiat speculum parabolicum concavum BAMC; dico omnes radios R R &c. in hoc speculum incidentes reflecti in ipsius focum E. Sit enim M punctum incidentiae radii RM, &

[28]

quoniam AE axis speculi , & Radius RM, ex suppositione sunt parallelae;erunt in eodem plano ; & parabola BAC ex cujus circumvolutione est genitum speculum BAMC, erit superficies reflectionis ; radius igitur RM reflectetur in focum E, quod erat estendendum.

Dioptrice.

Sint radii R R &c. paralleli in unum punctum E congregandi, sitque Lentis vertex incidentiae A, emersionis O. Radii R R &c. ope planae lentis superficiei per A ductae, cui sunt normales radii RR, intra lentem paralleli mittantur, in E punctum congregentur, ope conoidis, vel sphaeroidis, cujus vertex O; quod erat faciendum.

Corollarium.

Ex hoc etiam sequitur, radios parallelos congregati in unam punctum circiter E, etiamsi radii non sint geometrice paralleli rectae AE.

[29]

§4. Prop.19.1. Prop. 19. Problem.

To show that rays diverging from a single given point can be made parallel in the same medium. The vertex of the required lens or mirror is also given, and it is necessary that the vertices of the lens and the point of divergence are in the same straight line.

This problem is the converse of the preceding, and it is brought about by the same method; as it appears from *Prop. 9, book 10* of Witelo. [*Note:* The proposition referred to in Witelo does not discuss lenses but gives the plane refraction case, which is applicable at the vertex of the lens.]

Corollary.

From this it also follows that rays diverging from some point near E can be made parallel with the help of the lens, or of the mirror mentioned above.

§4. Prop.19.2. Prop. 19. Problema.

Radios ex unico puncto dato divergentes, ad parallelismum reducere, in eodem diaphano; datis quoque lentis vel speculi verticibus : Oportet autem, ut vertices lentis, & punctum divergentia, sint in eadem recta linea.

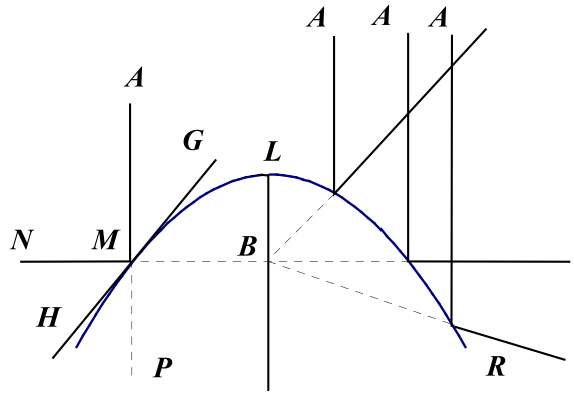
Hoc Problema est conversum antecedentis, eodemque modo perficitur ; ut patet ex *Prop. 9, lib. 10,* Vitellionis.

Corollarium.

Ex hoc etiam sequitur, radios divergentes ab aliquo puncto circiter E, fieri parallelos ope lentis, vel speculi supradicti.

§4. Prop.20.1. Prop. 20. Problem.

Parallel rays are made to diverge from some given point in the same medium, where the vertices of the required lens or mirror are given too. It is necessary that the line drawn through the vertex and the point of divergence is parallel to the given rays.



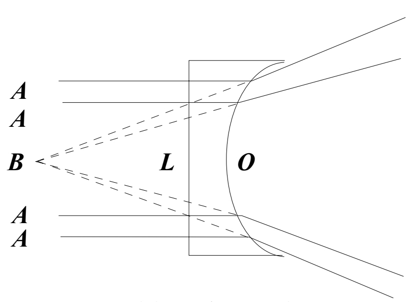
Prop. 20 - Figure 1.

Reflection.

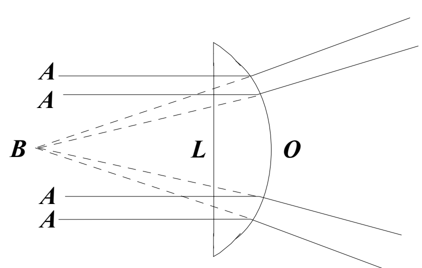
The parallel rays A, A, etc. are thus to be reflected, in order that they appear to be diverging from the point B, and the vertex L of the mirror is given. From the vertex L and the focus B the parabolic convex mirror MLR is described, intersecting the parallel rays A,A, etc. The rays which are falling on the surface MLR, I say, are reflected by the surface,

[30]

and diverge from the point B. Also a single ray AM incident on the point M, as it is parallel to the axis of the conoid, has the parabolic surface MLR for reflection (from which the conoid is described). (See Prop 10).The line HMG is drawn through M tangent to the parabola MLR, and the ray AM is produced in P, from B BN is drawn through the point M. The angle BMG is equal to the angle PMH, that is angle NMH is equal to angle AMG, and therefore AM is reflected into N. (See Prop 11). Q.e.d.



Prop. 20 - Figure 2.



Prop. 20 - Figure 3.

Refraction.

Let the parallel rays A, A, etc. be refracted thus in order that they diverge from the point B by the lens, of which the incident vertex of is L and of emergence O. The rays A, A, etc., with the help of the plane surface of the lens, are drawn through L (to which the rays are themselves the normals). They are sent parallel into the lens, and travel parallel within the lens, and finally are refracted to diverge from point B, with the aid of the solid hyperboloid (Fig. 2) or the hollow spheroid (Fig. 3), the vertex of which is O. (Following Prop's 17&16). Q.e.d.

Corollary.

From this it follows also that parallel rays diverge from some point around B, even if they shall not be geometrically parallel to BL themselves.

§4. Prop.20.2. Prop. 20. Problema.

Radios parallelos, ad divergentiam in eodem diaphano a quocunque puncto dato reducere, datis quoque lentis, vel speculi verticibus : Oportet tamen, ut recta per vertices, & divergentiae punctum ducta, sit radiis datis parallela.

Catoptrice.

Sint radii paralleli AA &c. ita reflectendi, ut appareant divergi e puncto B, sitque data speculi vertex L. Vertice L, & foco B, describatur speculum parabolicum convexum MLR, intercipiens radios parallelos AA &c., quos radios in superficiem MLR incidentes, dico in ipsa reflecti, &

[30]

divergi a puncto B. Sit enim ex illis, unus AM incidens in punctum M, qui, quoniam axi Conoidis est parallelus, habebit parabolam MLR (ex qua describitur Conois) superficiem reflectionis : ducatur per M linea HMG tangens parabolam MLR in M, & producutur radius AM in P, & ex B ducatur BN, per punctum M; & erit angulus BMG, aequalis angulo PMH, hoc est NMH, aequalis angulo AMG; reflectitur igitur AM in N: quod erat ostendendum.

Dioptrice.

Sint radii paralleli AA &c. ita refringendi, ut divergant ex puncto B, lente cujus vertex incidentiae L, emersionis autem O. Radii AA &c., ope planae lentis superficiei, per L ductae (cui normales sunt ipsi radii) intra lentem paralleli, mittantur, & intra lentem paralleli, & B puncto divergantur, ope Conoidis, vel Sphaeroidis, cujus vertex O: quod erat faciendum.

Corollarium.

Ex hoc etiam sequitur, radios parallelos divergi ex aliquo puncto circiter B, etiamsi non sint geometrice paralleli ipsi BL.

[31]

§4. Prop.21.1. Prop. 21. Problem.

The rays converging to one given point can be made parallel in the same medium, the vertices of the required lens or mirror are given too. It is necessary that the vertices of the lens and the point of convergence are in the same straight line.

This problem is the converse of the preceding, and is solved by the same method, as is clear from *Prop. 9. book 10*, Witello.

Corollary.

From this it follows too, that the rays converging to some point around B will also be made parallel with the help of the above mentioned lens or mirror.

[31]

§4. Prop.21.1. Prop. 21. Problema.

Radios ad unicum punctum datum convergentes, ad parallelismum in eodem diaphano reducere; datis quoque lentis, vel speculi, verticibus : oportet tamen ut vertices lentis, & punctum convergentiae sint in eadem recta linea.

Hoc Problema est conversum antecedentis, eodemque modo solvitur, ut patet ex *Prop. 9. lib. 10*, Vitellionis.

Corollarium.

Ex hoc quoq sequitur radios convergentes ad aliquod punctum circiter B, fieri etiam parallelos, ope lentis, vel speculi supradicti.

§4. Prop.22.1. Prop. 22. Problem.

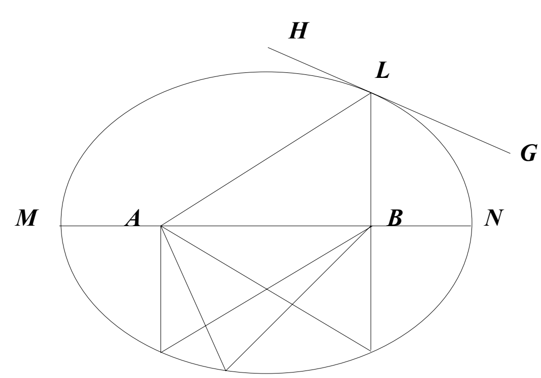
The rays diverging from one given point converge to some other point of the same medium ; with the vertices of the required lens or mirror given too : nevertheless it is necessary that the points of divergence and concurrence, and the vertices of the lens or mirror are in the same straight line.

Reflection.

Let the rays diverging from point A converge in point B, with the mirror of which the vertex is M. From the foci A, B

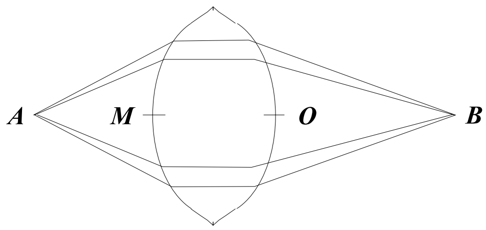
[32]

and the vertex M, the concave elliptical mirror MLN is drawn,

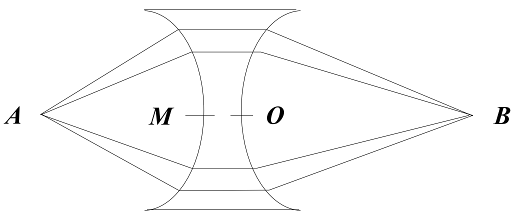


Prop. 22 - Figure 1.

in the surface of which the ray AL is incident in the point L: I say that the ray AL is reflected in the point L and passes through the focus B. For since AL and the axis of the spheroid are in the same plane; the ellipse LMN (from which the spheroid has been generated) will be the surface of reflection of the ray AL, and the angle of the ray AL with the contact will be equal to the angle BL from the same contact with the ellipse MLN; therefore the ray AL is reflected in B [Prop. 10]. Q.e.d.



Prop. 22 - Figure 2.



Prop. 22 - Figure 3.

Refraction.

Let the rays diverging from the [focal] point A be made to converge to the [other focal] point B, by means of a lens for which the vertices of incidence and emergence are M and O. The rays diverging from A are sent parallel within the lens with the help of the conoid [hyperboloidal surface: Prop. 22 - Fig. 2] or the spheroid [spheroidal surface : Prop. 22 - Fig. 3], of which the vertex is M. Subsequently they are made to converge in the point B with the help of the conoid or spheroid with vertex O. [Prop's 15 & 14.] Q.e.d.

Corollary.

From this result it appears that if a radiating point is near A then the rays from that point converge to some point around B by means of the lens or mirror mentioned above.

§4. Prop.22.2. Note.

We may note the use of the matrix methods introduced initially in the introductory section to verify these theorems independently of the geometrical presentation. Thus, for example, the lens in Prop. 22 - Fig. 2 can be represented by the matrices M_6 and M_5 for a thin lens, giving the transfer matrix :

$$M_5 M_6 = \begin{pmatrix} 1 & 0 \\ \frac{-1}{a(n+1)} & n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-1}{an(n+1)} & \frac{1}{n} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{-2}{a(n+1)} & 1 \end{pmatrix}$$

. The focal length of the lens is thus $a(n + 1)/2$, while the focal lengths A & B associated with the hyperboloidal surfaces are at a distance $a(n + 1)$ from their associated vertex. The effect of this transfer matrix is to change the sign of the slope of an incident ray from A that passes through B. Similarly, the lens in Prop. 22 - Fig. 3 can be represented by the matrices M_1 and M_2, resulting in a transfer matrix:

$$M_1 M_2 = \begin{pmatrix} 1 & 0 \\ \frac{-n}{a(n+1)} & \frac{1}{n} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-n^2}{a(n+1)} & n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{-2n}{a(n+1)} & 1 \end{pmatrix}$$

with the same effect on the rays. The focal length associated with this lens is $a(n + 1)/2n$.

§4. Prop. 22.3. Prop. 22. Problema.

Radios ab unico puncto dato divergentes, in aliud punctum ejusdem diaphani congregare ; datis quoque lentis vel speculi verticibus: oportet tamen, ut punctum divergentiae, punctum concursus, & vertices lentis vel speculi, sint in eadem rectae linea.

Catoptrice.

Sint radii e puncto A divergentes, in punctum B congregati, speculo cujus vertex M. Focis A, B

[32]

& vertex M, describatur speculum ellipticum Concavum MLN, in cujus superficiem, incidat radius AL, in puncto L: Dico radium AL in puncto L reflecti , & in focum B tendere. Quoniam enim AL, & axis Sphaeroidis sunt in eodem plano; erit ellipsis LMN (ex qua genita est Sphaerois) superficies reflectionis radii AL, & angulus radii AL cum contingente, aequalis erit angulo BL cum eadem contingente ellipsem MLN; reflectetur igitur radius AL in B ; quod erat ostendendum.

Dioptrice.

Sint radii e puncto A divergentes, in punctum B congregandi, lente, cujus vertex incidentiae M, emersionis O, radii ex A divergentes intra lentem paralleli mittantur, ope Conoidis, vel Sphaeroidis, cujus vertex M, & intra lentem paralleli, in punctum B congregentur, ope Conoidis vel Sphaeroidis cujus vertex O; quod erat faciendum.

Corollarium.

Ex hoc patet, si punctum radians fuerit prope A, ipsius radios congregari, in aliquod punctum circiter B, ope lentis vel speculi supradicti.

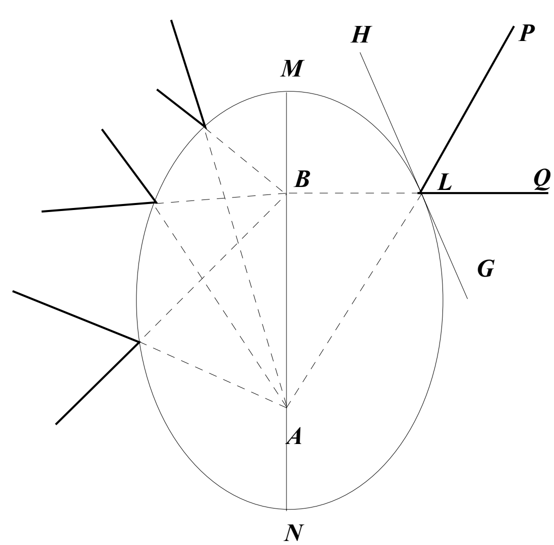
[33]

§4. Prop. 23.1. Prop. 23. Problem.

The rays that converge to one given point are made to diverge from another point in the same medium. The vertices of the required lens or mirror are given too. It is also necessary for the points of convergence and divergence to be collinear with the vertices.

Reflection.

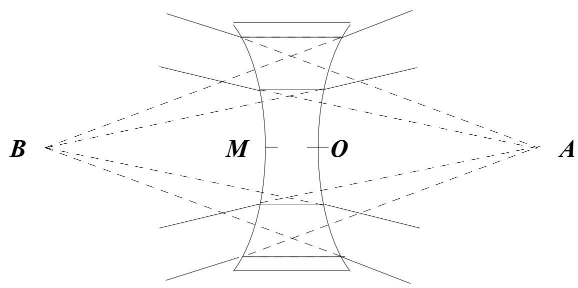
The elliptic convex mirror MLN is described with vertex M, foci A & B, and vertex M. Let the rays that converge to the point A diverge from the point B . I say that all the rays incident on the surface of the mirror converge to the same focus A, and are reflected by it to diverge from the other focus B. For let the ray P converge to the focus A, which therefore will be in the same plane as the axis of the mirror, and the surface of reflection of the ray PL will be the ellipse



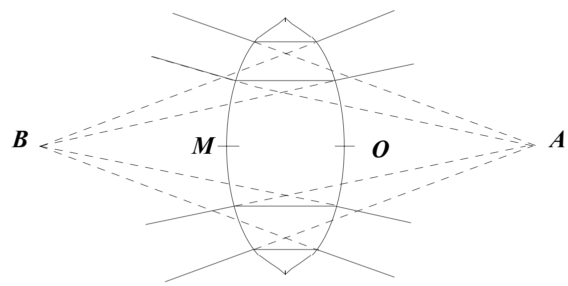
Prop. 23 - Figure 1.

MLN (from which the spheroid has been generated), by Prop. 10. This ray is produced to A, and BQ is drawn through L from B, and the straight line HLG is a tangent to the ellipse at the point L. Therefore it is clear the angle ALG is equal to the angle BLH: from the vertex L the former will equal PLH and the latter QLG. The ray PL is therefore reflected into LQ. Q.e.d.

[34]



Prop. 23 - Figure 2.



Prop. 23 - Figure 3.

Refraction.

The rays that converge to the point A are made to diverge from the point B. This is achieved by a lens with vertex M for the incident rays and vertex O for the rays that emerge. The rays that converge to the [far focal point of the first surface] A are rendered parallel by the conoidal [Prop. 23 - Fig. 2] or spheroidal surface [Prop. 23 - Fig. 3] with vertex M, and sent parallel within the lens, subsequently the rays diverge from the point B [the far focal point associated with the second surface], by means of the conoidal or spheroidal surface with vertex O. (Prop's 17 & 16.) Q.e.d.

Corollary.

Thus it also follows, that rays converging to some point near A will also diverge from some point around B, with the help of the lens or mirror mentioned above.

§4. Prop. 23.2. Note.

The diverging lens in Prop. 23 - Fig. 2 can be represented by the matrices M_7 and M_8 for a thin lens, giving the transfer matrix :

$$M_8 M_7 = \begin{pmatrix} 1 & 0 \\ \frac{1}{a(n+1)} & n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{an(n+1)} & \frac{1}{n} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{2}{a(n+1)} & 1 \end{pmatrix}$$

. The focal length of the <u>lens</u> is thus - $a(n + 1)/2$, while the focal lengths A & B associated with the hyperboloidal surfaces are at a distance $a(n + 1)$ from their associated vertex. The effect of this transfer matrix is to change the sign of the slope of an incident ray going towards A that appears to come from B. Similarly, the lens in Prop. 23 - Fig. 3 can be represented by the matrices M_3 and M_4, resulting in a transfer matrix:

$$M_4 M_3 = \begin{pmatrix} 1 & 0 \\ \frac{n}{a(n+1)} & \frac{1}{n} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{n^2}{a(n+1)} & n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{2n}{a(n+1)} & 1 \end{pmatrix}$$

with the same change of direction of the rays. The focal length associated with this lens is - $a(n + 1)/2n$.

[33]

§4. Prop. 23.2. Prop. 23. Problema.

Radios ad unicum punctum datum convergentes, in eodem diaphano ab alio puncto dato divergere ; datis ; datis quoque lentis, vel speculi verticibus : Opertet tamen, ut punctum convergentiae, punctum divergentiae, & vertices lentis, vel speculi, sint in eadem recta linea.

Catoptrice.

Sint radii ad punctum A convergentes, a puncto B divergendi, speculi cujus vertex M: focis A, B, & vertice M, describatur speculum ellipticum convexum MLN. Dico omnes Radios ad focum convergentes, & in speculi superficiem incidentes, in ipsa reflecti, & a foco B divergere. Sit enim ad focum A convergens radius PL qui igitur erit in eo plano cum axe speculi, eritque Ellipsis MLN (ex qua genita est Sphaerois), superficies Reflectionis radii PL, qui producetur in A; & a B, per L, ducatur BQ, tangatque ellipsim LMN recta HLG in punctum L. Quoniam igitur angulus ALG , est aequalis angulo BLH; erit & priori ad verticem PLH, aequalis posteriori ad verticem QLG; reflectetur igitur PL in Q: quod ostendendum erat.

[34]

Dioptrice.

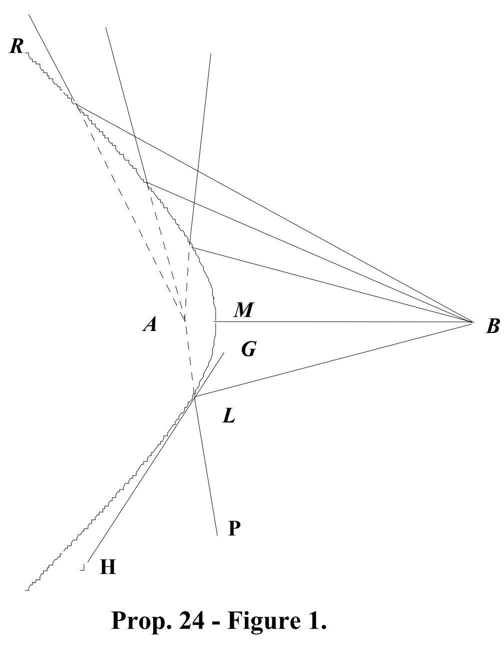
Sint radii ad punctum A convergentes, a puncto B diverergendi, lente cujus vertex incidentiae M, emersionis O. Radii ad punctum A convergentes intra lentem paralleli mittantur, ope conoidis vel sphaeroidis, cujus vertex M, & intra lentem paralleli, a puncto B divergentur, ope Conoidis vel Sphaeroidis cujus vertex O; quod erat faciendum.

Corollarium.

Hinc etiam sequitur, radios convergentes ad aliquod punctum circiter A, etiam divergi ab aliquo puncto circiter B, ope lentis vel speculi supradicti.

§4. Prop. 24.1. Prop. 24. Problem.

The rays converging to one given point are made to converge to another given point of the same medium with the help of a mirror or lens, the vertices of which are given too. It is necessary morover that the two points of convergence are collinear with the vertices of the lens or mirror.



Prop. 24 - Figure 1.

Reflection.

A convex hyperboloidal mirror LMR is described with foci A and B and vertex M.

[35]

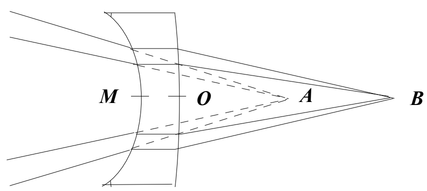
Let the rays that converge to some point A be made to converge at another point B by means of the mirror with vertex M. But only if the vertex M is nearer to the point of convergence A than B. If in fact the vertex M is nearer to the point of convergence B, then a concave hyperboloid mirror is described with foci A and B and vertex M.

[35]

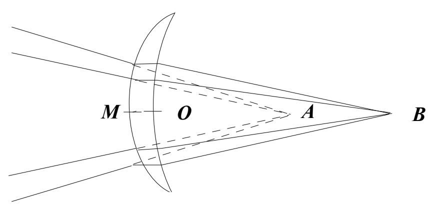
I say that all the rays converging to the focus A are reflected in the surface of the mirror LMR, and are concurrent at the focus B. For let the ray PL be incident on the focus A, which therefore is in the same plane as the axis of the mirror. Hence the hyperbola LMR is the surface of reflection for the ray PL (from the revolution of which the mirror has been generated) , by Prop. 10. The ray PL is produced to A, and from the point L the line LB is drawn. Let the line HLG touch the hyperbola at the point L, and hence the angle ALG *i.e.* HLP, is equal to the angle GLB. Therefore PL is reflected through the point B. Q.e.d.

This conclusion can be shown for a hyperbolic concave mirror by a similar argument.

[36]



Prop. 24 - Figure 2.



Prop. 24 - Figure 3.

Refraction.

Let the rays that converge to the point A be made to converge in another point B, by the lens with incident vertex M and emergent vertex O. By Prop. 17, the rays converging to A are sent parallel on entering the lens, by means of the conoidal [Prop. 24 - Fig. 2] or spheroidal [Prop. 24 - Fig. 3] surface with vertex M; within the lens the rays are parallel. Subsequently by Prop. 14, the rays converge to the point B, by means of the second conoidal or spheroidal surface with vertex O. Q.e.d.

Corollary.

Thus it follows, the rays converging to another point near A, also are to congregate in one point around B, with the aid of the lens or the mirror mentioned above.

§4. Prop. 24.2. Note.

The diverging lens in Prop. 24 - Fig. 2 can be represented by the matrices M_5 and M_7 for a thin lens, giving the transfer matrix :

$$M_5 M_7 = \begin{pmatrix} 1 & 0 \\ \frac{-1}{a_2(n+1)} & n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{a_1 n(n+1)} & \frac{1}{n} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{(n+1)}\left[\frac{1}{a_1} - \frac{1}{a_2}\right] & 1 \end{pmatrix}.$$

The focal length of the lens is thus $\frac{-1}{(n+1)}\left[\frac{1}{a_1} - \frac{1}{a_2}\right]$, where $a_2 > a_1$, while the focal lengths A & B associated with the hyperboloidal surfaces are at distances $a_1(n + 1)$ and $a_2(n + 1)$ from their associated vertices M & O. The effect of this transfer matrix is to change the slope of an incident ray going towards the focus A so that it appears to come from B. Similarly, the lens in Prop. 24 - Fig. 3 can be represented by the matrices M_6 and M_8, resulting in the transfer matrix:

$M_6 M_8 = \begin{pmatrix} 1 & 0 \\ \frac{-1}{a_2 n(n+1)} & \frac{1}{n} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{a_1(n+1)} & n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{n(n+1)}\left[\frac{1}{a_1} - \frac{1}{a_2}\right] & 1 \end{pmatrix}$ with the same change of direction of the rays. The focal length associated with this lens is $\frac{-1}{n(n+1)}\left[\frac{1}{a_1} - \frac{1}{a_2}\right]$.

§4. Prop. 24.3. Prop. 24. Problema.

Radios ad unicum punctum datum convergentes, in aliud punctum datum ejusdem diaphani congregare ; datis quoque lentis, vel speculi verticibus : Opertet tamen, ut punctum convergentiae, punctum concursus, & vertices lentis, vel speculi, sint in eadem recta linea.

Catoptrice.

Sint radii convergentes ad aliud punctum A, ad aliud punctum B congregandi, speculo cujus vertex M. Focis A, & B, & vertice M, describatur speculum Hyperbolicum convexum LMR; si modo vertex M, propior fuerit puncto convergentiae A: Si vero vertex M, propior fuerit puncto concursus B; focis A & B, & vertice M, describatur speculum hyperbolicum concavum.

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Dico omnes radios ad focum A convergentes, in superficie speculi LMR reflecti, & in focum B, congregari; sit enim radius PL ad focum A vergens, qui igitur est in eodem plano cum axe speculi, hyperbole igitur LMR (ex cujus revolutione genitum est speculum), est superficies reflectionis radii PL, qui radius producetur in A; & a puncto L ducatur recta LB, tangatque Hyperbolam, recta HLG, in puncto L. angulus igitur angulus ALG , hoc est HLP, est aequalis angulo GLB; reflectetur igitur PL in B: quod erat ostendendum . Eodem quoque modo, demonstratur haec conclusio, in speculo hyperbolico concavo.

Dioptrice.

Sint radii ad punctum A convergentes, in aliud punctum B congregandi,

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lente cujus vertex incidentiae M, emersionis vero O. Radii ad A convergentes intra lentem paralleli mittantur, ope conoidis vel sphaeroidis, cujus vertex M, & intra lentem paralleli, in punctum B congregentur , ope conoidis vel sphaeroidis, cujus vertex O; quod erat faciendum.

Corollarium.

Hinc sequitur, radios convergentes ad aliquod punctum prope A, etiam congregari in unum puncto circiter B, ope lentis vel speculi supradicti.

§4. Prop. 25.1. Prop. 25. Problem.

The rays diverging from one given point are to diverge from another given point. With the vertices of the required lens or mirror given too: nevertheless it is necessary that the points of divergence and the vertices of the lens or mirror are collinear.

This problem is the converse of the antecedent, and is solved by the same method, as is shown from *Prop. 9, Book 10*, Witelo.

Corollary.

From this problem it has also been clear that the rays diverging from another point near B, can also to be made to diverge from one point near A, with the aid of the lens or the mirror mentioned above.

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Some Obvious Properties:

1.

Through the determination of the properties which attend the resolution of these problems, it is clear in the case of refraction with small parts of conoids and spheroids thus joined together to make lenses, that the axis of the lens is the common axis of both parts, and that the vertices of the lens are the vertices of the parts.

2.

Secondly, it is clear that for the resolution of all these problems for refraction and corollaries, that the rays outside the lens are either parallel, or are converging to/ diverging from a point, while within the lens they are always parallel.

§4. Prop. 25.2. Prop. 25. Problema.

Radios ab unico puncto dato divergentes, ab alio puncto duco divergere ; datis quoq; lentis, vel speculi verticibus : opertet tamen, ut divergentiae puncta, & vertices lentis, vel speculi, sint in eadem recta linea.

Hoc Problema est conversum antecedentis, eodemque modo solvitur, ut patet ex *Prop. 9. lib. 10*, Vitellionis.

Corollarium.

Ex hoc problemate, etiam manifestum est ; radios divergentes ab uno puncto, circiter B, etiam divergi ab unico puncto circiter A, ope lentis vel speculi supradicti.

Manifestum 1.

Per limitationes,quae resolutioni horum problematum inserviunt,

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Manifestum est in Dioptricis, Conoideon, & Sphaeroideon portiunculas ; ita in lentibus connexas esse; ut axis lentis sit communis axis utriusq; portiunculae, & vertices lentis portiuncularum vertices.

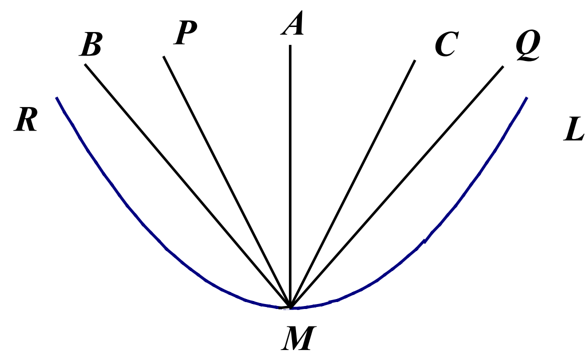
Manifestum 2.

Secundo, Manifestum est in omnibus horum problematum resolutionibus Diopticis, & Corollariis; radios extra lentem vel parallelos, vel ad unum punctum convergentes, vel ab unico puncto divergentes; intra lentem semper esse parallelos.

§4. Prop. 26.1. Prop. 26. Theorem.

If two rays are incident on the vertex of a mirror coplaner with the axis, by which they are reflected, then the angle of incidence is equal to the angle of reflection for the rays.

Let the two rays BM & CM be incident in the same plane of the mirror RML with vertex M, and the rays are reflected at M to Q and P. I say the angle BMC formed from the incident rays added together is equal to the angle PMQ formed from the reflected rays added together. Let the axis of the mirror be AM, and since BM & CM are coplaner with the axis, & for these rays there is one and the same surface of reflection, namely the figure RML, from which the mirror arises by rotation, from Prop. 10.



Prop. 26 - Figure 1.

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The angle BMA is equal to the angle AMQ; & likewise the angle CMA is equal to the angle AMP; therefore BMA & CMA likewise added together without doubt BMC, is equal to the angle AMP. Likewise the angle AMQ , that is, to the angle PMQ; which it was necessary to show.

§4. Prop. 26.2. Prop. 26. Theorema.

Si duo Radii, in eodem cum axe plano, incidant in speculi verticum, & ab ea reflectantur; erit angulus a radiis incidentibus , aequalis angulo a radiis repercussis.

Sint duo radii BM, CM, in eodem cum axe plano, incidentes, in speculi RML, verticem M, & reflectentur in Q, & P : Dico angulum BMC, a radiis incidentibus comprehensum, esse aequalem angulo PMQ a radiis reflexis comprehenso. Sit speculi axis AM, & quoniam BM, CM sunt in eodem cum axe plano, erit illis una, & eadem superficies reflectionis, figura nempe RML, ex cujus revolutione genitum est speculum ; eritq

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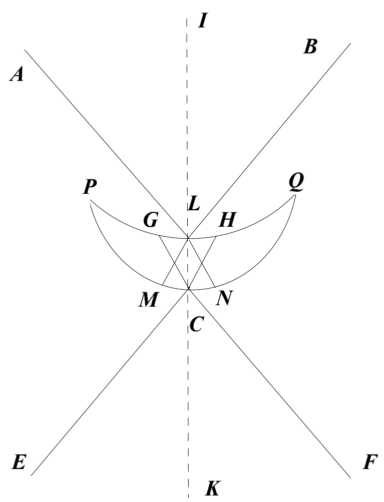
angulus BMA, aequalis angulo AMQ; & CMA, aequalis angulo AMP;igitur BMA & CMA simul, nimirum BMC, aequalis erit angulo AMP, & angulo AMQ simul, id est angulo PMQ; quod demonstrare oportuit.

§4. Prop. 27.1. Prop. 27. Theorem.



Prop. 27 - Figure 1.

Let there be two rays co-planar with the axis outside the lens which concur and cross at the vertex of the lens. Indeed, there shall be two other rays, which are coplanar with the former rays, which are each parallel to a previous ray crossing within the lens, and meeting at the other vertex of the lens, and emerging from that vertex. The angle subtended by the two rays before entering will be equal to the angle subtended by the two final rays upon emerging.

Let the two rays AL & BL outside the lens incident at L, and in the same plane as the axis of this lens, be refracted as the rays LM and LN in crossing the lens PMNQL . Let there be two other rays crossing parallel to the previous rays within the lens: GC parallel to LN, HC parallel to LM, and with all of these four rays arising from the one kind of refracting surface. AL, BL are incident at the one vertex of the lens L, while HC & GC are refracted by the other vertex of the lens C to give the emerging rays CF & CE. I assert that the angle ALB is equal to the angle ECF. For the lens PMNQL is drawn, with the axis ILCK, through which (it is evident all of these rays are in the same plane as the axis, by supposition) the surface of common refraction is drawn PMCNQHLG; indeed this is the plane from which the lens has been generated by rotation, by Prop. 10. MLC & LCH will be the equal angles of incidence of the parallel rays ML & HC within the lens (plainly by supposing ML to be refracted in LB; for if BL is refracted in LM, then conversely

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ML is refracted in LB). Therefore the refracted angles are equal too, which angles taken from or added to the same equal angles of incidence MLC & LCH, gives the equal angles ILB & ECK. In the same way, the equality of the angles ALI, KCF will be approved. Therefore the whole angle ALB is equal to the whole angle ECF, which had to be shown.

§4. Prop. 27.2. Note.

At first sight, this theorem and the previous seem a little unusual. However, they form part of the plan for locating images formed be reflection or refraction by lenses and mirrors. It is usual to draw the object with some degree of asymmetry, and hence the angles of incidence are from different sides of the axis, and of different sizes. As Gregory has pointed put, he intends to consider parallel rays within the lens, and it remains at this stage for Gregory to connect up the rays that have been abruptly terminated at the in-going and out-going surfaces of the lens, using auxiliary axes.

§4. Prop. 27.2. Prop. 27. Theorema.

Si duo radii in eodem cum axe plano,extra lentem, concurrant in vertice lentis, eam transeuntes; duo vero alii, in eodem plano cum prioribus, intra lentem prioribus etiam in transitu paralleli, concurrans in altera lentis vertice, ex illa egredientes : angulus comprehensus a duobus primis ante ingressum, aequalis erit angulo comprehenso a duobus postrimis post egressum.

Sint duo radii extra lentis AL, BL, in eodem tamen plano cum illius axe, lentem PMNQL transeuntes, refracti in LN & LM; sintq; duo alii, intra lentem prioribus in transitu Paralleli, nempe GC ipsi LN, & HC ipsi LM ; omnibus hisce quatuor radiis in uno refractionis superficie existentibus; incidantq; AL, BL in unam lentis verticem L, egrediantur vero in HC, GC ex alia lentis vertice C, in CF, CE. Dico angulum ALB esse aequalem angulo ECF. Ducatur enim lens PMNQL, axis ILCK, per quem (quoniam omnes hi radii sunt in eodem plano cum axe ex suppositione) ducatur communis superficies refractionis, PMCNQHLG, planum nempe, ex cujus revolutione grenita est lens ; eruntq; MLC, LCH anguli incidentiae radiorum parallelorum ML, HC (supposendo nimium ML refringi in LB; nam si BL refringatur in LM, & e converso,

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ML refringetur in LB) aequales : & ideo refractiones quoq aequales, quae ab aequalibus incidentiae angulis MLC, LCH subtractae, vel iisdem additae, efficiunt angules ILB, ECK aequales. Eodem modo probabitur aequalitas angulorum ALI, KCF ; Totus igitur angulus ALB, est aequalis toti angulo ECF; quod demonstrare oportuit.