§7. Synopsis of Propositions 44 - 51:

This section is probably the most important in the book, and deals with image formation by lenses and mirrors of all kinds with conoidal surfaces.

Prop. 44: If rays from individual points of any visible object are rendered parallel by a lens or mirror, and these parallel rays are viewed with a normal relaxed eye, then a visible image of these points is always seen to appear at infinity. The visible image is seen with the same angle of vision which the object subtends at the incident vertex of the lens or mirror.
A detailed explanation of the ray diagrams for this proposition and the next are given below.

Prop. 45: Rays from the individual points of some real image are made to converge to other points by means of a lens or mirror. If the eye is placed between the lens or mirror and the points of convergence of the new image, then from the view-point of that angle of vision, for which the apices of the pencils of rays from the extreme points of the object appear at the centre of the eye, a confused image is always observed. However, the rays from the end points of the image subtend the same angle at the vertex of emergence as the rays from the object subtend at the vertex of incidence.

Prop. 46: Consider the same arrangement, but with the eye now placed beyond the points of concurrence of the points of the object at the apex of their pencil. An image will be seen for people with myopic eyes which is useful in determining their bounds of distinct vision. However, for long-sighted people the image is still confused. The image of the object also appears inverted, and at that angle corresponding to where the apices of the pencils of the extreme points appear from the centre of the eye.
This theorem gives a number of extra examples of image formation, and also includes some information regarding long and short sighed people. The myopic eye by its nature can form an image from rays with a greater degree of divergence than the normal eye, and attention is drawn to this fact. An analysis of the refraction ray diagrams is presented via ray diagrams.

Prop. 47: Rays from the individual points of some visible image are made to diverge from other points by a lens or mirror. The image of any of these visible points will appear in the apex of the radiant cone of the eye receiving these rays. In accordance with the previous theorem for myopic eyes the bounds of distinct vision can be determined, while a confused image will always appear for long-sighted eyes. And the visible image will appear from that angle of vision, from which the apices of the cones of rays of the extreme visible points, appear with the same angle from the vertex of emergence, with which the visible object appears from the vertex of incidence.
This theorem is complementary to the previous one in that the corresponding diagrams correspond to the opposite situation: concave and concave mirrors are interchanged, as are dense and rare mediums for lenses of the same geometry, so that the object and image are interchanged in each case. An analysis of the reflection ray diagrams is presented via ray diagrams.

Prop. 48: To measure the size or diameter of the pupil.

Prop. 49: Parallel rays are not weakened on traversing any distance, and equally illuminate the same object placed in the same way at any point along the path .

Prop. 50: If the object distance from the incident vertex to the object distance from centre of the eye [placed before the lens or mirror] is in the same ratio as the image distance from emergent vertex to the image distance from centre of the eye [placed after the lens or mirror] , then the image appears with the same angle of vision as the object appears to the naked eye, when viewed through a lens or mirror.

Prop. 51: With the same positions of object and image, I say that the image appears equally illuminated with the eye at O, as the object appears with the eye at L, provided the rays illuminate the whole pupil of the eye.

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Extended Note on Prop. 44:

A detailed discussion of the imaging processes in the various lenses presented in Prop. 44 is given using both Gregory's scheme and modern methods. The cases of the dense convex-planar and plano-concave lenses are set out initially, followed by image formation for less dense lenses in a denser medium, and for reflection by a parabolic mirror.
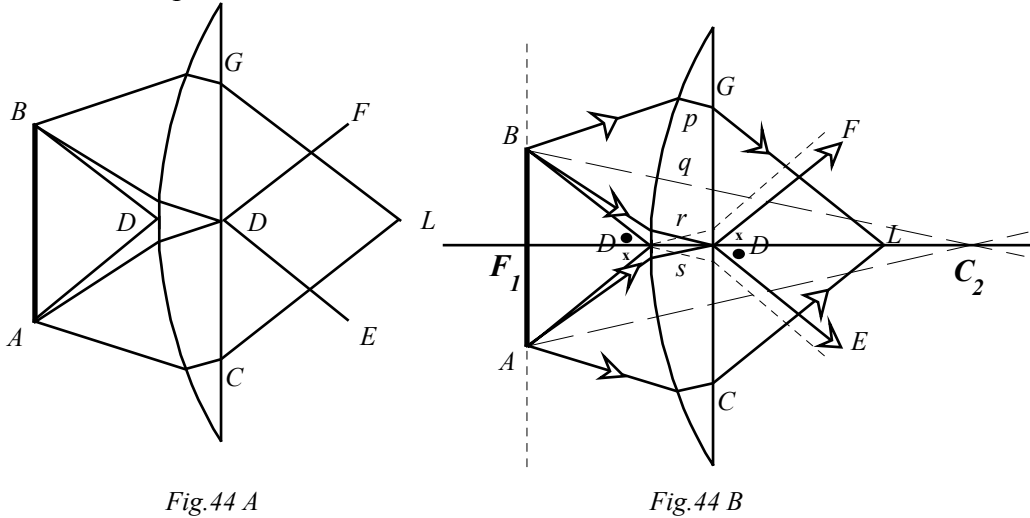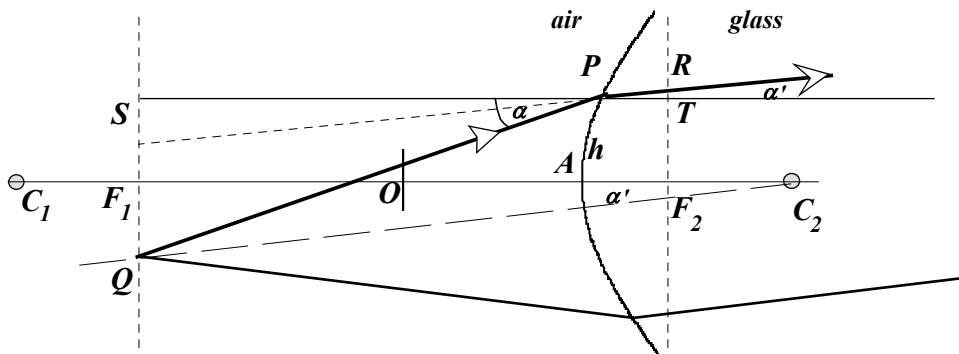
A: Dense convex-planar lens:

Fig.44 A

Fig.44 B

Fig.44 C

First we note that the scheme presented here for image formation by conoidal surfaces anticipates the discussions presented in modern optics text such as *Fundamentals of Optics,* Jenkins and White [McGraw-Hill, 1976] - where single convex and concave *spherical* surfaces are considered in turn (Chapter III). Thus, in the modern case, rays entering a spherical convex glass surface from a focal point on the axis are refracted and sent through the glass as rays parallel to the axis, while rays sent from all off-axis points suffer from spherical aberration. In the hyperbolic case, rays parallel to the optic axis are refracted without spherical aberration whatever their height, though of course there is still aberration for rays refracted parallel to an auxiliary axes.

Figure 44A shows the initial diagram of the plano-convex lens in Prop. 44, rotated by 90^0 for convenience. The curved surface is a hyperbola of revolution, and in this first example an object AB lies in the focal plane of the left-hand branch of the hyperbola (not

shown), and placed slightly asymmetric about the optic axis. Such objects are always shown much larger in diagrams for convenience and the angles are much larger than for true paraxial ray tracing. Gregory has already established in Proposition 27 the manner in which rays from A & B may be traced through such a lens.

Within the lens medium, rays parallel to some auxiliary axis only are to be considered, according to the principle that parallel paraxial rays within the lens are refracted either to or from the same point in the focal plane through which the auxiliary axis passes - the actual focus used depending on whether the refracting surface makes the rays converge or diverge. Thus, if B is such a point in Fig. 44B, where the vertical dotted line indicates the focal plane, then the ray refracted at the nearest vertex of the lens D is parallel to the auxiliary axis within the lens. This axis is shown dashed and labeled *s* within the lens. Conversely, a ray emerging at the same angle (these angles are marked with a dot) from the other vertex D and below the axis is also parallel to the auxiliary axis within the lens, labeled *r*, and this ray also originates from the point B, according to the parallel paraxial ray theorem. A similar argument applies for the ray from A to D. There are hence two parallelograms, one with a vertex at D (in) and the other at D (out). For Gregory, the angles of refraction at the left-hand vertex D for the rays coming from A & B could be found from a table of experimental results for the glass medium concerned. The angles for the emergent rays are equal to these incident rays. The total angle BDA subtended by AB at D is thus equal to that formed by the emergent rays at the right-hand vertex FDE.

It is a great simplification to consider the cusp of the hyperbolic curve as a nodal point of the lens. Hence, the almost horizontal sides of these parallelograms are parallel to the auxiliary optic axes, marked with dashed lines in Figure 44B, passing through the cusp C_2. In most diagrams, the closely spaced rays near the vertices of the lens are omitted, as presumably they would not be resolved in the printed diagram. The rays drawn entering a vertex are hence not the same as those drawn leaving the other vertex. See also Figure 44C - note that Gregory does not explicitly show any auxiliary axes, and considers only the refracted rays through D to define the direction. Thus, the line marked *r* in the medium is parallel to the auxiliary axis *q* in Fig. 44B. Any other paraxial ray from B such as *p* travelling through the lens from A is also parallel to the axes *q*. Conversely, each ray corresponding to the same auxiliary axis has come from the same point in the focal plane, such as A or B. An eye placed at L will see parallel rays from A or B, or from both. The image will appear to be at infinity.

Figure 44C shows the path of any ray through the hyperbolic surface using a paraxial ray in modern notation, as set out in §0.4 for M_6. The thin convex-plano lens is given by

the matrix product $M_{po}M_6 = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{-1}{an(n+1)} & \frac{1}{n} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{-1}{a(n+1)} & 1 \end{pmatrix}$, where M_{po} is the

plane surface matrix for the ray going out of the medium, and the focal length of the lens F_1A is equal to $a(n+1)$ as required.
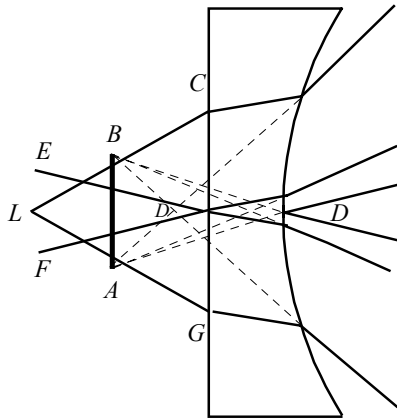
B: Dense plano-concave lens:
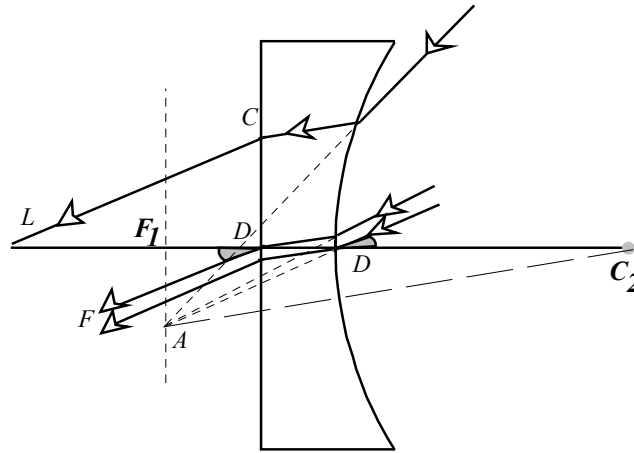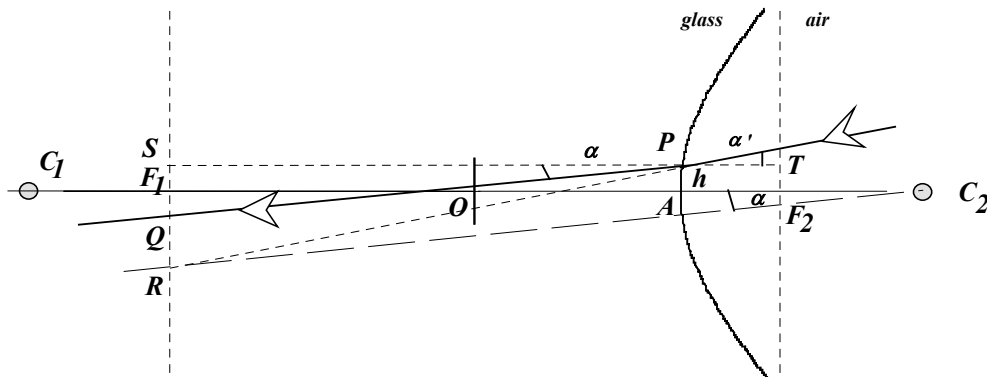

Figure 44D

Figure 44E

Figure 44F

    The plano-concave lens that follows is concerned with rendering parallel rays that converge in the absence of the lens to image points A and B in the focal plane of the lens, as in Fig. 44D. AB thus lies in the focal plane of the lens where the curved surface is again hyperbolic but now concave. The in-going rays shown are rendered parallel to the auxiliary axis through A or B. An eye placed at L will hence see an image of A or B at infinity. Figure 44E shows a simplified version for the rays converging to A, where the equality of the angles at the vertices can be determined from a single parallelogram.

    Figure 44F shows the path of any ray through the hyperbolic surface using a paraxial ray in modern notation, as set out in §0.4 for M_8. The thin convex-plano lens is given by

the matrix product $M_{pi}M_7 = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{an(n+1)} & n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{a(n+1)} & 1 \end{pmatrix}$, where M_{pi} is the

plane surface matrix for the ray going into the medium. The focal length of the lens is $-a(n + 1)$ as required.

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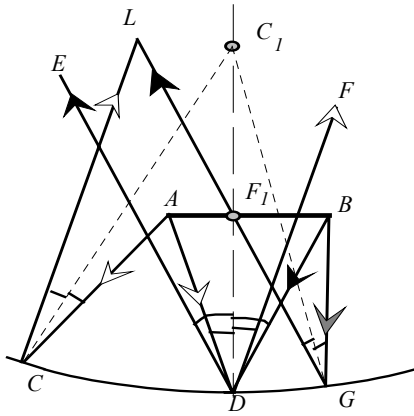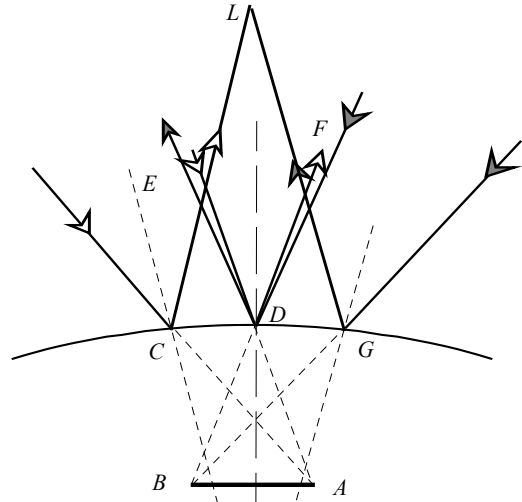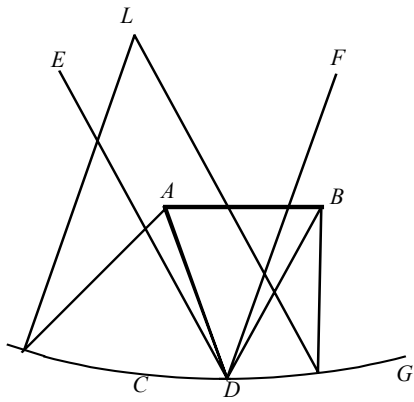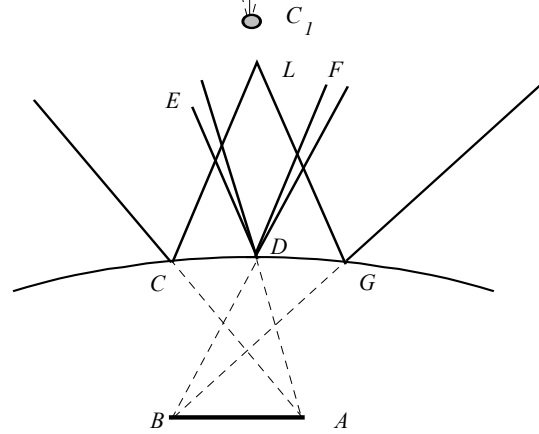C: Concave \ Convex Parabolic Mirrors:

Figure 44G

Figure 44H

Figure 44I

Figure 44J

These latter diagrams for reflection are left as exercises.

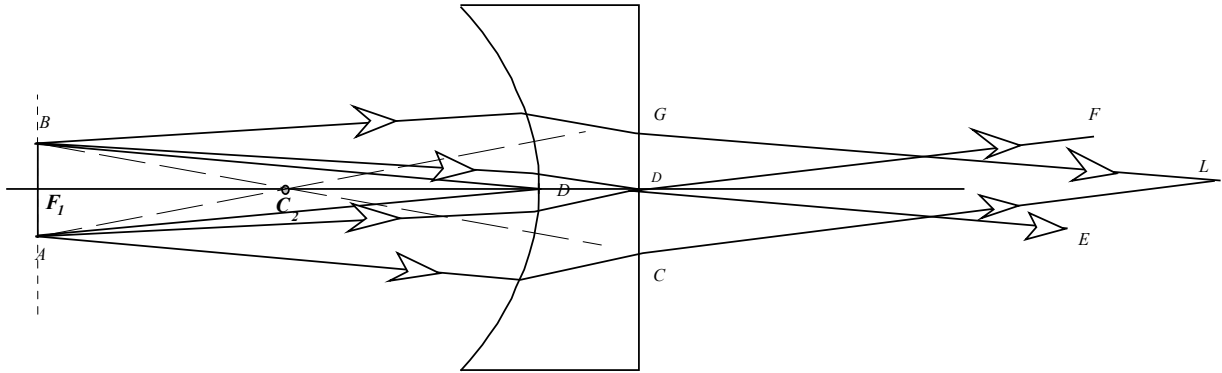D: Less dense concave-planar and convex-planar lenses in denser medium:
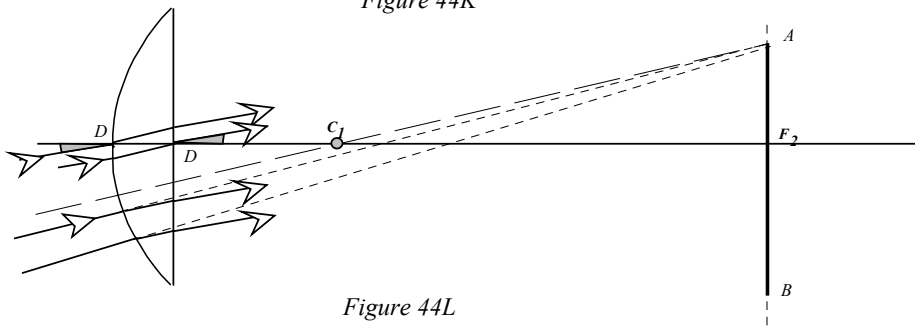

Figure 44K

Figure 44L

Figure 44K shows the path of rays through the ellipsoidal surface for paraxial rays, as set out in §0.4 for M_2. The thin convex-plano lens in the denser medium has the ABCD

matrix given by: $\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{n} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-n^2}{a(n+1)} & n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{-n}{a(n+1)} & 1 \end{pmatrix}.$

The focal length of the lens is $a(n + 1)/n$ as required.

Figure 44L shows the path of rays through the ellipsoidal surface for paraxial rays, as set out in §0.4 for M_3. The thin convex-plano lens in the denser medium has the ABCD

matrix given by: $\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{n} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{n^2}{a(n+1)} & n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{n}{a(n+1)} & 1 \end{pmatrix}.$ The focal length of the diverging

lens is $a(n + 1)/n$ as required.

End of Extended Note on Prop. 44.

[58]
Prop. 44. Theorem.

If rays from individual points of a visible object are rendered parallel by a lens or mirror, and these parallel rays are viewed by a normal relaxed eye, then a visible image of these points is always seen to appear at infinity. The image subtends the same angle with the eye that the object subtends at the incident vertex of the lens or mirror.
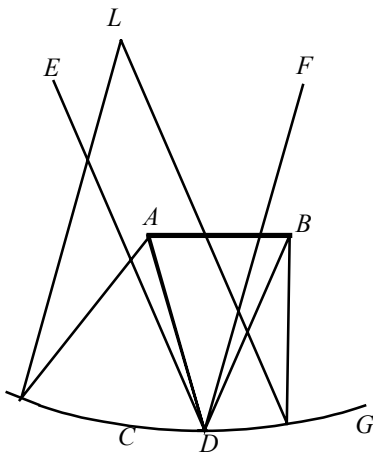
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Let AB be any visible object, or radiating matter, either before or behind the eye [i.e. the rays may be diverging from an actual object AB, or converging from some previous lens or mirror to form a prior image AB]. In any case, the object lies in the focal plane of the present lens or mirror CDG, and the end points are named A and B. The rays from individual points are rendered parallel by CDG, the axis of which is co-planar with the points A & B, but perpendicular to the plane of the visible object. The rays AD & BD are drawn from the end points A & B to the lens (or mirror) vertex of incidence D, and the reflected or refracted rays DF & DE are drawn from the vertex of emergence D.

I declare that an eye, placed at L in the same plane as the axis of the lens or mirror and intercepting the parallel rays produced by the lens or mirror from AB, shall always capture an image of that object through an angle of vision equal to the angle ADB. A ray GL is drawn to the eye from B parallel to DE. An equivalent ray, namely CL, is drawn from A, parallel to DF. These four lines, namely CL, DF, GL, DE, form a parallelogram, with equal opposite angles EDF &CGL; but the angle EDF equals the angle ADB, and therefore the angle CLG is equal to the angle ADB. But the eye at L captures the object AB viewed through the angle of vision CLG, as the extreme points A & B emit rays that give the angle CLG in the centre of the eye L, which therefore is equal to the angle ADB. This is the same angle by which the object can be seen from the vertex of incidence of the lens or mirror. The image appears at an infinite distance, and is distinct to a normal relaxed eye, since each point appears to radiates parallel rays. QED.

For Reflections.

Diverging Rays.   Converging Rays.
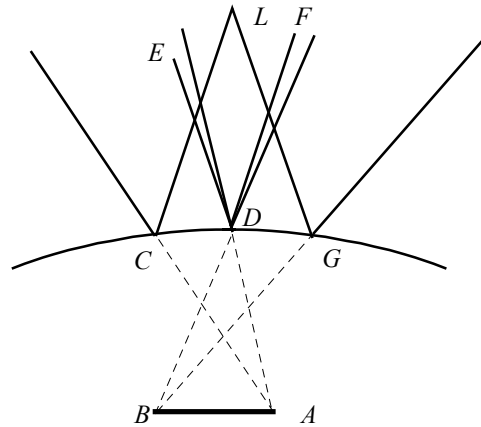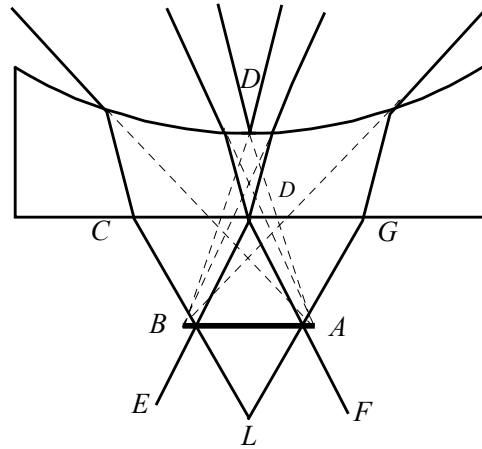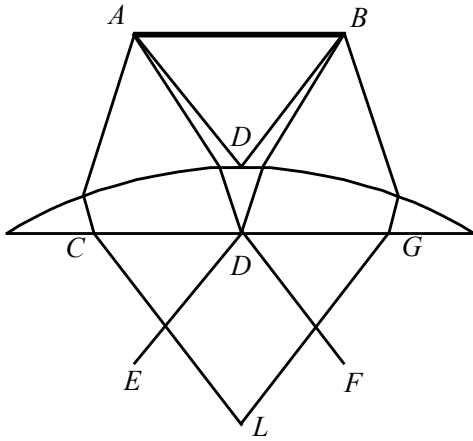
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For Refraction with a denser intermediate medium.
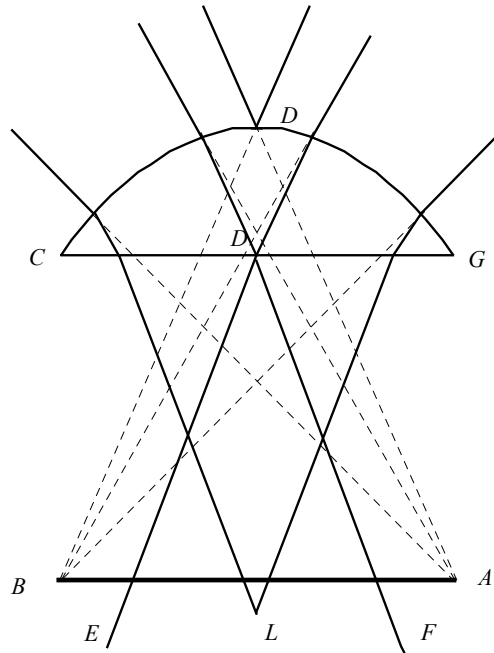
Diverging Rays.   Converging Rays.
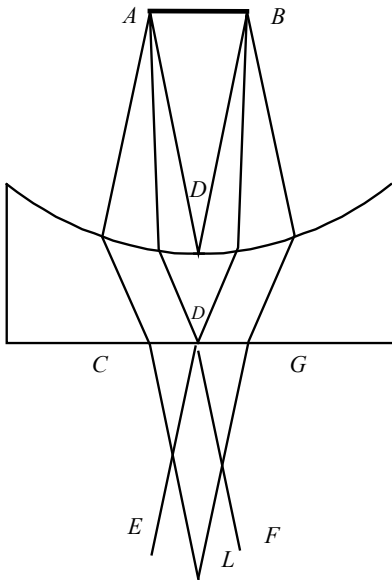
For Refraction with a less dense intermediate medium.

[**60**]

Diverging Rays.   Converging Rays.

Further Comment on Prop. 44.

In the cases of the concave mirror and the convex hyperbolic lens, A and B are the end points of a radiating object, or of two discrete radiating points A and B, which lie in the focal plane, close to, but distinct from the focus of a lens or mirror on the optic axis.

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Hence, rays from A and B reflected or refracted by the vertex are sent out in a parallel beam at some angle to the axis. An eye capable of focusing a parallel beam will hence see the image at this angle. The entire range of angles, or the angle into which the rays are radiated, is the same as the angle subtended by the object from the vertex of incidence.

[**58**]

Prop. 44. Theorema.

Si cujuscunque visibilis, singulorum punctorum radii, ad parallelismum reducantur : oculo radios parallelos recipienti, semper videbitur visibilis imago, eodem angulo visorio, quo videtur ex vertice incidentiae lentis, vel speculi. Apparetq; imago infinite distans, & presbytis distincta.

Sit visibile quodlibet AB, sive materia radians, sive imago ante, sive post oculum ; dummodo sit planum ; cujus extrema puncta A, & B: Reducantur singulorum punctorum radii, ad parallelismum, lente vel speculo CDG, cujus axis sit in plano cum punctis A, B plano visibilis perpendicularis : & ab extremis punctis A, B, ad verticem incidentiae D, ducantur radii AD, BD ; & ad verticem emersionis D, radii refracti AD, BD ; qui reflectentur vel refrangentur in DF, DE. Dico oculum, radios visibilis AB parallelos, intercipientem, semper comprehendere illius imaginem, per angulum visorium, aequalem angulo ADB. Sit enim centrum oculi L, in eodem plano cum axe lentis, vel speculi, & punctis extremis AB ; ad quod ducatur radius puncti B, nempe GL, ipsi DE parallelus ; idem radius puncti A, nempe CL, ipsi DF parallelos ; igitur quatuor istae rectae, nimitum CL, DF, GL, DE, elliciunt parallelogrammum, cujus anguli oppositi, EDF, CGL, sunt aequales : Est autem angulus EDF, aequalis angulo ADB ; ergo & angulus CLG, est aequalis angulo ADB, sed oculus in L, apprehendit imaginem visibilis AB, per angulum visorium CLG ; quoniam puncta extrema AB, emittunt radios, facientes in centro oculi L, angulum CLG, qui igitur aequalis est angulo ADB, nimirum eodem angulo, quo videtur visibile, ex vertice incidentiae lentis, vel speculi. Quod demonstrandum erat. Apparet infinite distans, & presbytis distincta, quoniam unumquodque punctum radiat parallele.

Prop. 45. Theorem.

Rays from the individual points of some real image are made to converge to other points by means of a lens or mirror. If the eye is placed between the lens or mirror and the points of convergence of the new image, then from the view-point of that angle of vision, for which the apices of the pencils of rays from the extreme points of the object appear at the centre of the eye, a confused image is always observed. However, the rays from the end points of the image subtend the same angle at the vertex of emergence as the rays from the object subtend at the vertex of incidence.

Consider an object with end points A and B as in the previous theorem. The rays from the individual points of AB are either converging or diverging. They are made to converge by a lens or mirror CDG, the axis of which lies in the same plane as the points AB, in the plane of the perpendicular object. The apices of the pencils of rays are M and N. The rays CLN and GLM are drawn from the extreme points A and B. I am saying that with the eye at L, an image of the object is apparent with the angle of vision MLN or

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CLG. For since the ray CN extended to N comes from the extreme point A of the object; and by the same reason GM comes from the end B of the object. Therefore the angle CLG, taken from the rays of the extreme points of the object intersecting each other at the centre of the eye, is the angle of vision, within which the image of the object AB is apparent. The image does not appear sharp but is confused, since the rays from individual points converge to different points behind the eye. The position of the image is indeed apparent in these points, if by some means it has been found. Let ADB be the angle of vision of the object AB, from the vertex of incidence D, which is equal to the angle MDN of the apex of the pencils of the extreme visible points from the vertex D of the emergent ray ; as is apparent from *Prop. 26* and *Prop. 27* of this work, which it was required to show.
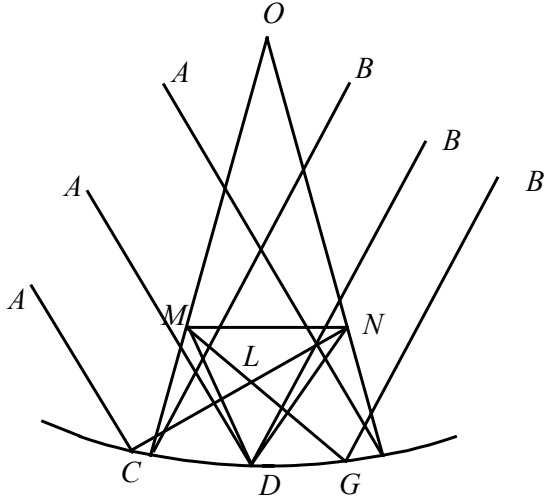
Scholium.

But if the line ML produced does not fall on the lens or mirror CDG, then the image of this point of this point will not be seen by the lens or mirror, of which the apex of the pencil is M.
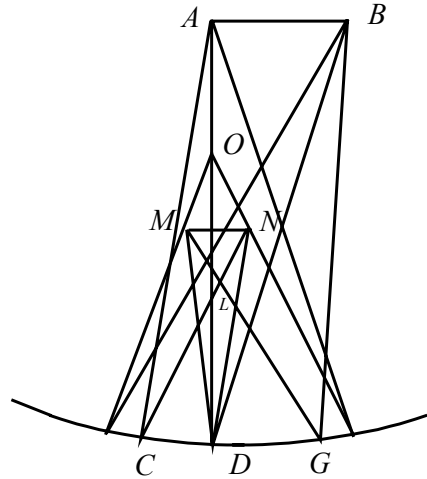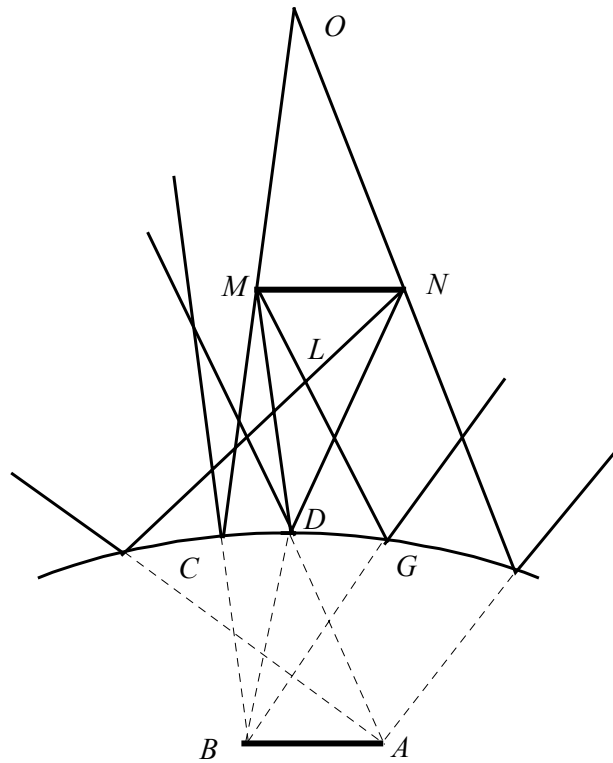
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Reflection.

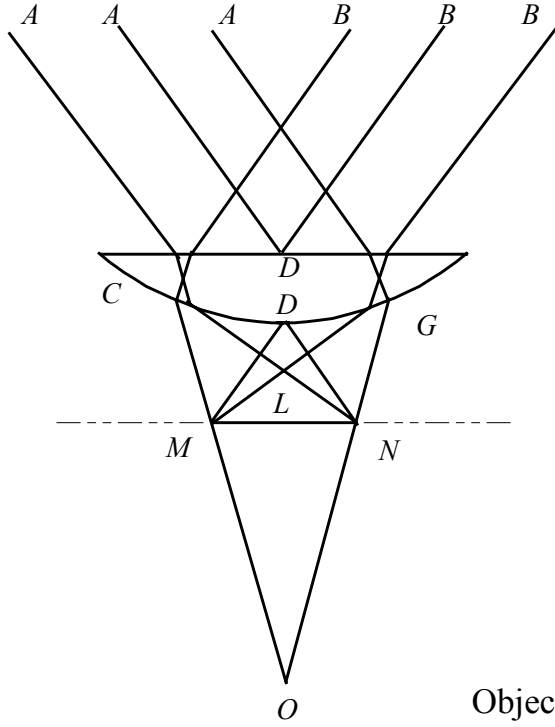Object at infinite distance :

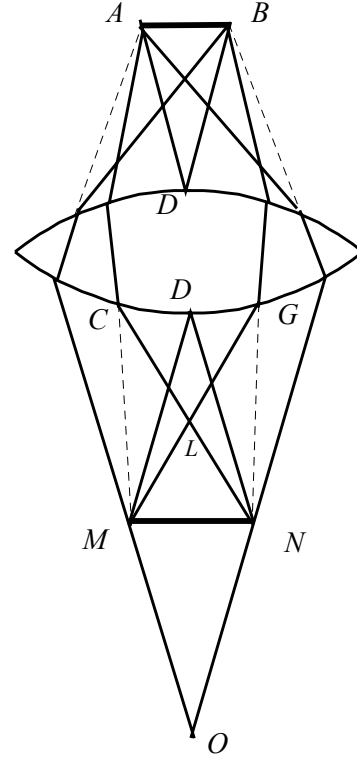Object before the eye :

Object behind the eye :

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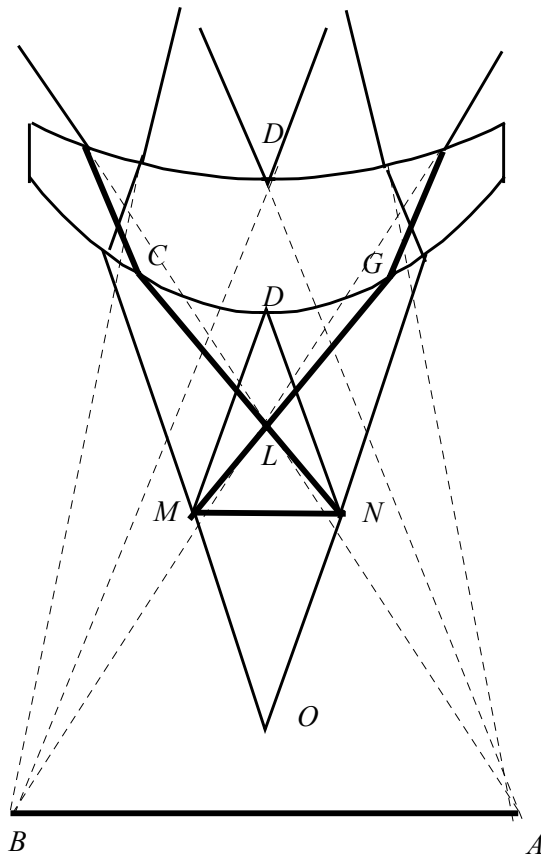For Refraction with a denser intermediate medium.
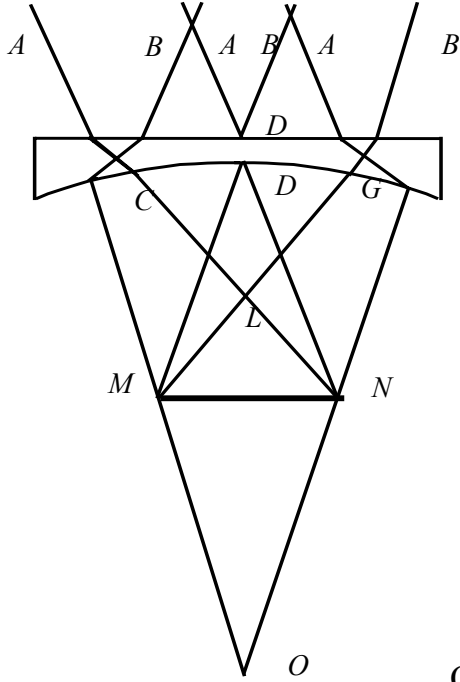
Object at infinite distance :

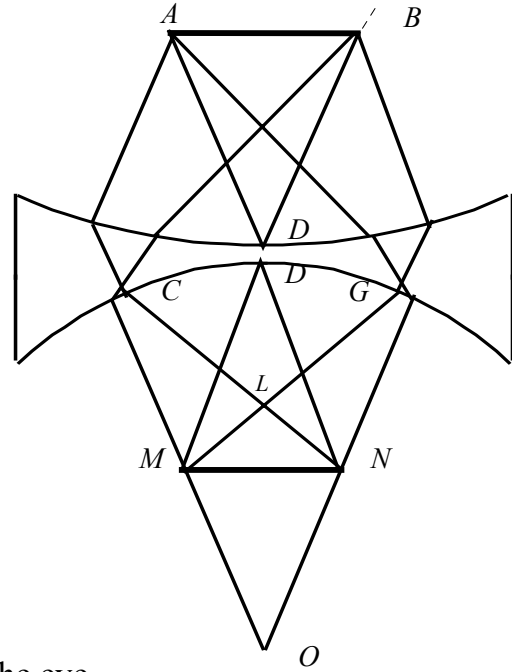Object before the eye :

Object behind the eye:

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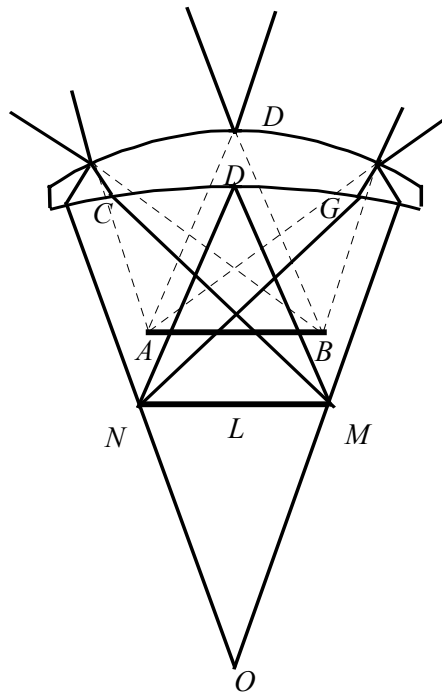For refraction with a less dense intermediate medium.

Object at an infinite distance.

Object before the eye.

Object behind the eye.

Notes on Prop. 45:

Finally we are presented with the fully-fledged method of ray tracing for lenses with hyperboloidal surfaces placed in a less dense medium, paraboloidal mirrors, and lenses with ellipsoidal surfaces placed in a more dense medium. In order to appreciate the method, we set out some of the ray diagrams in detail. The first figure in each pair of

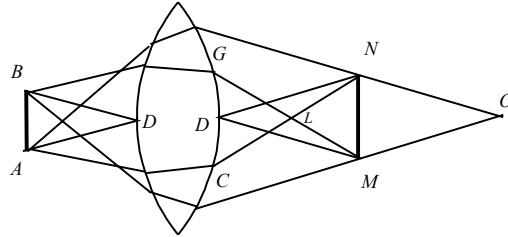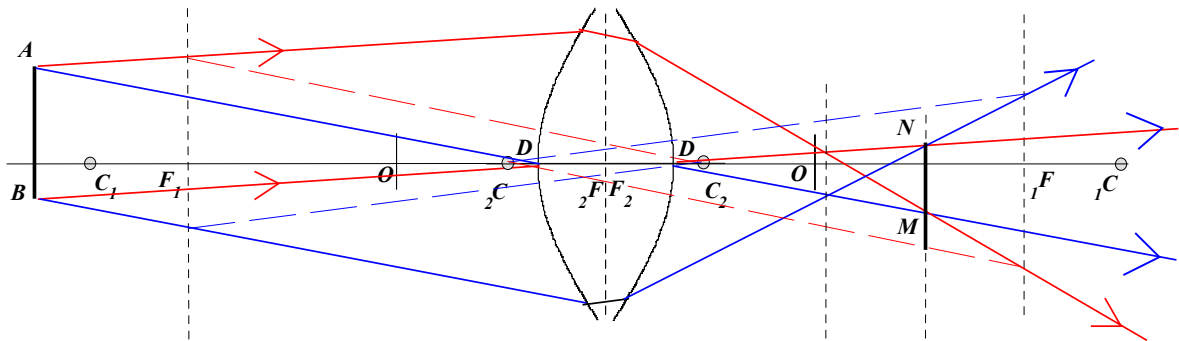**Figure 45 A.**

Figure 45 B.

diagrams is taken from Gregory's text, while the latter shows focal planes, auxiliary axes, cusp points, etc, as in Figures 44A & B.
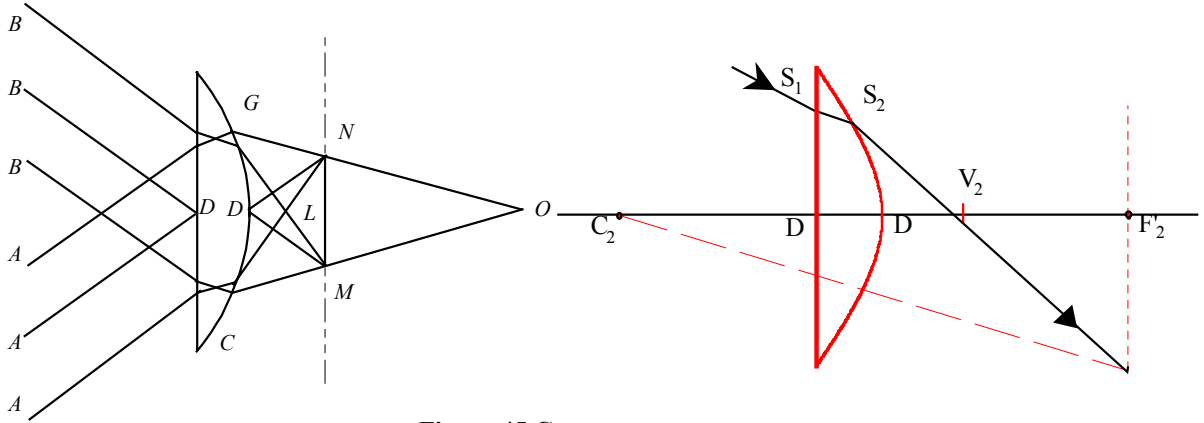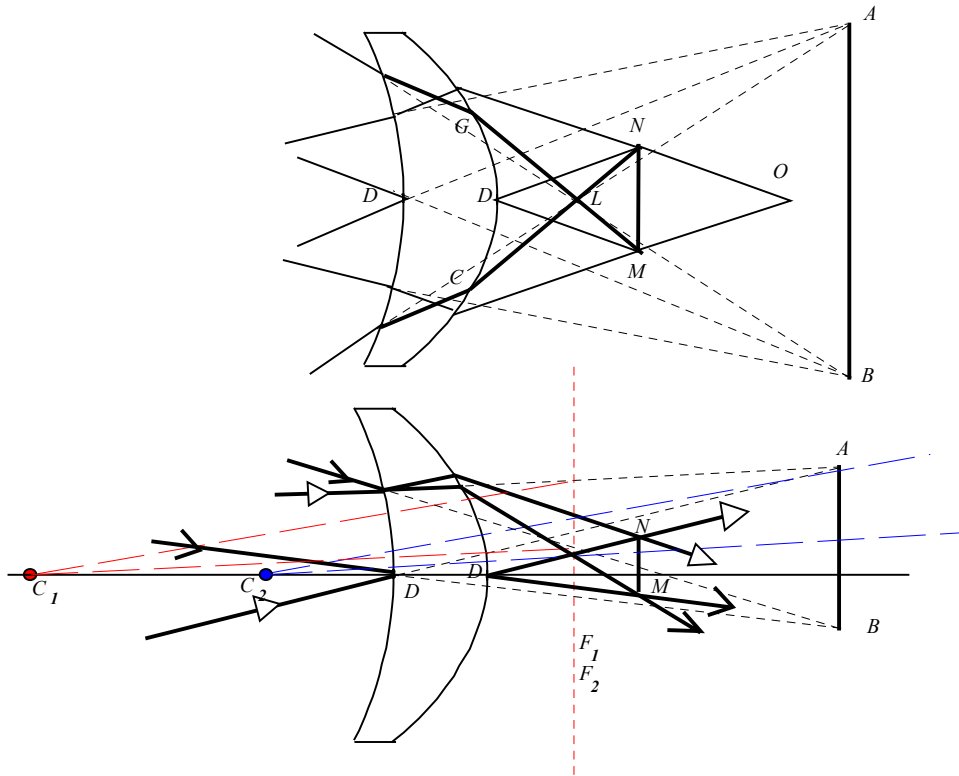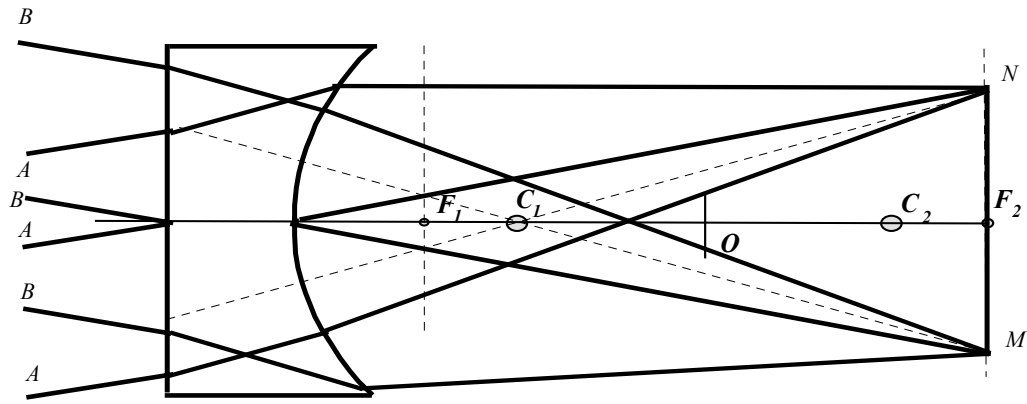
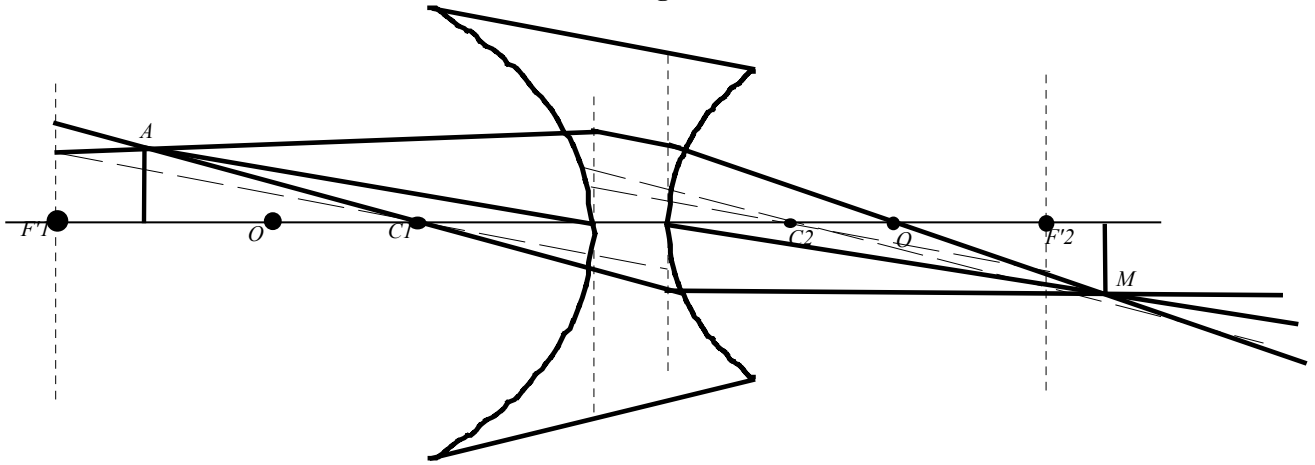Figure 45 C.
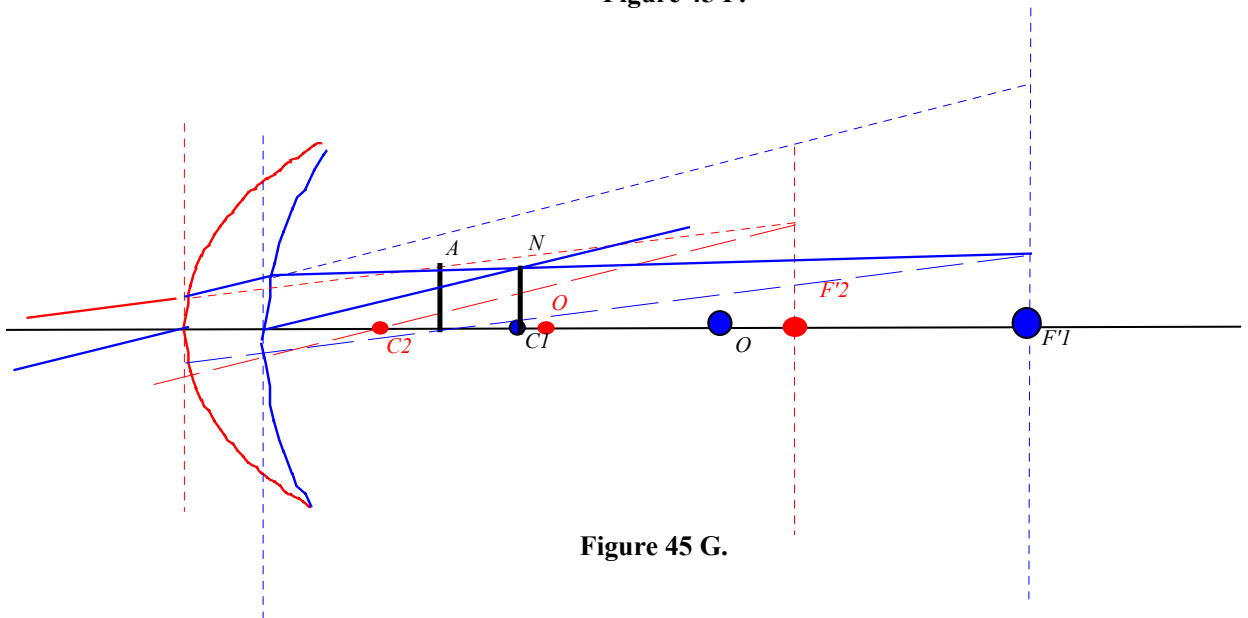
Figure 45 D.

Figure 45 E.

Figure 45 F.

Figure 45 G.

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Prop. 45. Theorema.

Si cujuscunque visibilis, singulorum punctorum radii, ad alia puncta convergantur ; oculo inter lentem, vel speculum, & puncta concursuum posito, semper apparebit imago visibilis confusa, & eo angulo visorio, quo apices pencillorum, extremorum visibilis punctorum, ex oculi centro. Apices autem pencillorum, extremorum visibilis punctorum, eodem angulo apparent ex vertice emersionis, quo visibile ex vertice incidentiae.

Sit visibile quodlibet AB, sive materia radians, sive imago ante, sive post oculum, dummodo sit planum , cuius extrema puncta A, B. Convergantur singulorum punctorum radii, ad alia puncta, lente vel speculo CDG, cujus axis sit in plano cum punctis AB, plano visibilis perpendicularis. Sintq; punctorum extremorum A, B, pencillorum apices M, N ducantur radii, CLN, GLM. Dico oculo in L, imaginem visibilis AB apparere, cum angulo visorio MLN, seu CLG. Quoniam enim radius CN, tendit ad N, provenit ab extremo visibilis puncto A; & ob eandem rationem GM provenit, ab extremo visibilis puncto B; angulus igitur CLG, comprehensus radiis, extremorum visibilis punctorum, se invicem in centro oculi secantibus, est angulus visorius, quo apparet imago visibilis AB. Apparet autem confusa, quoniam singulorum punctorum radii, ad alia puncta post oculum converguntur ; in quibus etiam punctis, est apparens imaginis locus, si modo ullus detur : estq; ADB, angulus visorius visibilis AB, ex vertice incidentiae D, aequalis MDN, angulo apicum pencillorum, extremorum visibilis punctorum ex vertice emersionis D; ut patet per hujus 27 & 26 Prop. Quae omnia demonstranda erant.

Scholium.

Si autem recta ML producta, in lentem vel speculum CDG non incidat, tunc non videbitur imago istius puncti, per lentem vel speculum CDG, cujus apex pencilli est M.

Prop. 46. Theorem.

Consider the same arrangement, but with the eye now placed beyond the points of concurrence of the points of the object at the apex of their pencil. A clear image will be seen for people with myopic eyes which is useful in determining their bounds of distinct vision. However, for long-sighted people the image is still confused. The image of the object also appears inverted, and at that angle corresponding to where the apices of the pencils of the extreme points appear from the centre of the eye.

In the above figures, let the centre of the eye be placed at O beyond the point of concurrence of the rays. Thus the rays coming from the points of an object AB to the eye O first converge to the apex of their pencil, and then diverge from it as if from a fountain. The eye is applied at O in order to observe the image depicted on the retina by these rays diverging from the apex of their pencil; and by applying the eye the position of the points is judged to be in the apex of its own pencil of rays - if truly the apex of the pencil, or rather of the cone of rays, may be different for the myopic eye for that interval, and for which the retina itself is accustomed to be clearly depicted by the rays. In this case the aformentioned myopic eye will see the image of the point distinctly.

But for long-sighted eyes, (since the image on the retina of these is only depicted clearly by parallel rays) there is no clear vision formed by these diverging rays. Which in all respects is made clear enough, by Prop. 29 of this work onwards. I say also that the image of the object appears inverted : i.e. the image of the point A is apparent in the part corresponding to B, and formed in the opposite direction, while the image of point B is apparent in the part of the object corresponding to A . For A appears at N, from the apex of its pencil, which is apparent in the part B, with the eye surely placed at O ; as can be readily deduced from Prop. 26 and 27 of this work; and in the same way, the image of point B is apparent in the parts A. But since the extremes points of the object always appear at M and N - the image formed will always appear with the angle of vision NOM: that is, with the angle by which the apices of the pencils of the extreme points of the image appear from the centre of the eye. QED.

Scholium.

But if the line OM produced is not incident on the lens or mirror, then the image of this pencil with apex is M will not be seen through the lens or mirror CDG. Which is also to be understood in the following theorem.

Prop. 46. Theorema.

Iisdem positis ; oculo post puncta concursuum, apparebit imago, cujuslibet visibilis puncti, in apice sui penicilli ; myopibus, in determinata sua distantia, distincta ; Presbytis autem semper confusa. Videbitur quoque visibilis imago, everso sitis, & eo angulo visorio, quo apices penicillorum, extremorum visibilis punctorum, ex oculi centro apparent.

In Figuris superioribus, sit oculi centrum O, post puncta concursuum ; radii igitur cujuslibet puncti visibilis AB, oculum O ferentes, primo congregantur in apice sui penicilli, & tunc divergunt ab apice praedicta, tanquam a fonte; Oculus igitur O, se applicat, ut pingatur ipsius retina, istis radiis, ex apice peniocilli divergentibus, ex qua applicatione existimat videns locum puncti, esse in apice sui penicilli: si vero apex penicilli, vel potius coni radiosi, eo intervallo ab oculo myopis distet, quo solet ipsius retina a radiante distincte pingi ; in hoc inquam casu myops praedicti puncti imaginem, distincte videbit : Presbytis autem, (quoniam eorum retina radiis solummodo parallelis distincte pingitur) hisce radiis divergentibus nulla sit distincta visio; Quae omnia, satis patent, per *Prop. 29.* hujus & ejus consectaria. Dico imaginem quoque visibilis AB, apparere everso situ ; hoc est imaginem puncti A, apparere ad partes B, & e contra imaginem puncti B, ad partes A. A enim apparet in N, apice sui penicilli, quae apparet ad partes B, oculo nimirum in O posito ; ut facile deducitur ex *26 & 27 Prop.* hujus ; eodemq; modo imago puncti B apparet ad partes A : Quoniam autem extrema visibilis puncta apparent semper in M, N; apparebit imago visibilis, angulo visorio NOM, hoc est angulo, quo apices penicillorum, extremorum visibilis punctorum, ex oculi centro apparent. Quae omnia demonstrare opportuit.

Scholium.

Si autem recta OM producta, in lentem vel speculum CDG non incidat, tunc non videbitur imago istius puncti, cujus apex pencilli est M, per lentem vel speculum CDG: Quod in sequente etiam intelligendum.

James Gregory's Optica Promota

Prop. 47. Theorem.

Rays from the individual points of some visible image are made to diverge from other points by a lens or mirror. The image of any of these visible points will appear in the apex of the radiant cone of the eye receiving these rays. in accordance, for myopic eyes, with determining the bounds of their distinct vision; but for long-sighted eyes it will always appear confused. And the visible image will appear from that angle of vision, from which the apices of the cones of rays of the extreme visible points, appear with the same angle from the vertex of emergence, with which the visible object appears from the vertex of incidence.

　　Let there be some image, [formed by a previous lens or mirror] which is formed either before or behind the eye or actual object before the eye, AB. This object shall lie in a plane with end points A and B. The rays that diverge from the individual points of this object are made to converge to other points, by the action of the lens or mirror CDG. The axis of CDG lies in a plane perpendicular to the points A and B in the plane of vision. The eye therefore, receiving these rays on its application to the apices of the radiant cones from the individual points of the visible image, is able itself to judge where the points A and B are to be seen, from the apices of the aforementioned visible points, according to *cor.*1, *Prop*. 29 of this work. Thus, since the rays from neighbouring individual points of the image are diverging, a visible image is apparent for short-sighted people, within the bounds of their distinct vision. Long-sighted people however will always see a confused image for this situation.

　　 Let M and N be the apices of the emergent cones of rays from the points of extreme vision. Also let the centre of the eye be placed at L, where the two rays CL and GL of the extreme visible points concur, as they diverge from the points M and N. The angle NLM or CLG will be the angle of vision, by which the image is seen by the eye with centre L. This indeed is the angle by which the apices of the radiant cones of the extreme visible points appear from the centre of the eye. Also the angle ADB, which is indeed the angle subtended by the object at the vertex of incidence, is equal to the angle NDM or RDS, without doubt the angle which the apices of the radiant cones of the extreme visible points subtend from the vertex of emergence. All of which is apparent from *Prop. 27 & 26* of this work. QED.

Corollary.

For these theorems, the image is always seen to appear for the eye in the same place, with the help of a lens or mirror; and therefore the tangents of the visual semi-angle are in the reciprocal ratio of the direct distances from the eye.

Note on Prop. 47:

　　As indicated in the synopsis, inverse cases are considered in this theorem. As we have already examined the refraction diagrams in detail in Prop. 46, the interchange of more and less dense mediums for the lenses has been left as an exercise for the reader. We do, however, examine the reflection case briefly.

James Gregory's Optica Promota

Reflection

Object at infinity.

Object before the eye.

Centre of curvature

Object behind the eye.

Centre of curvature
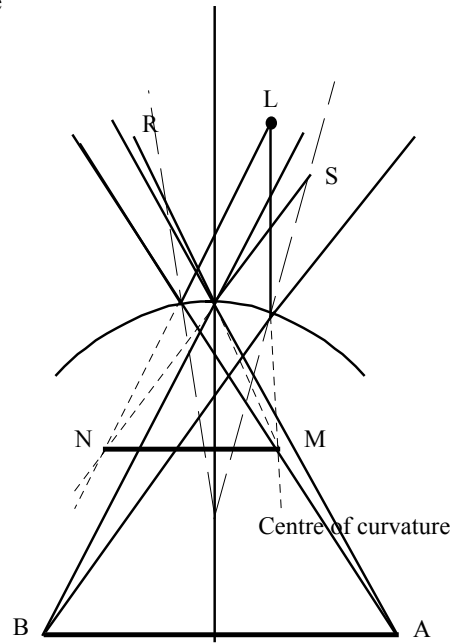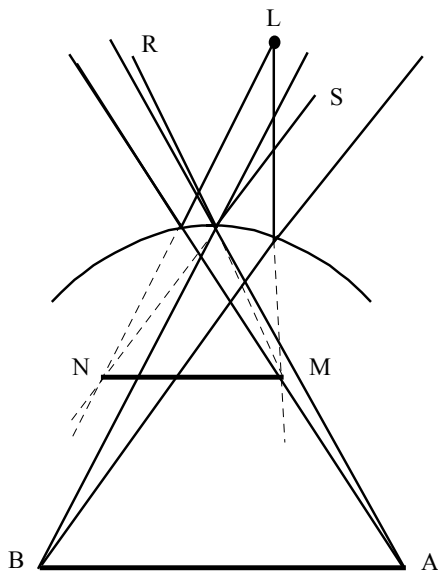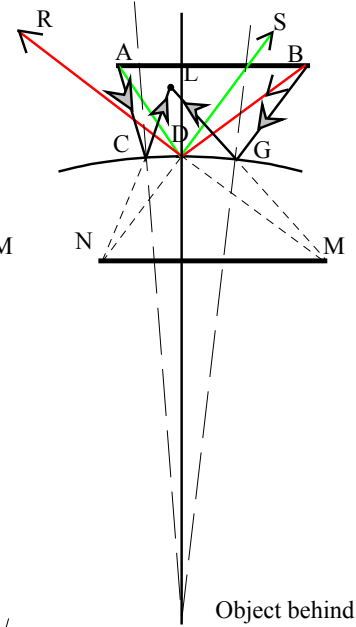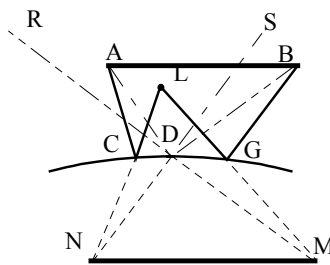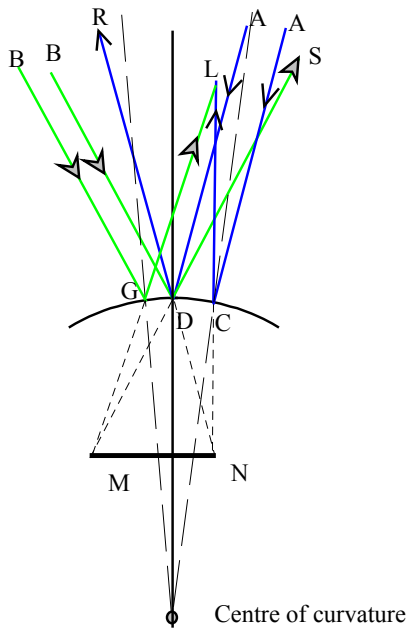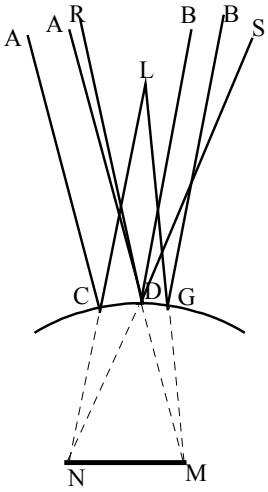
James Gregory's Optica Promota
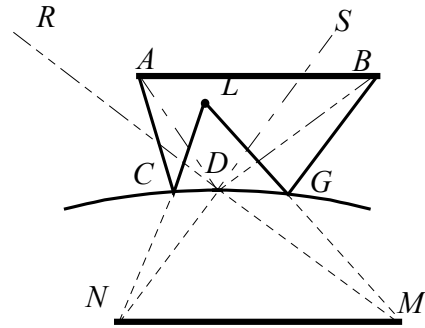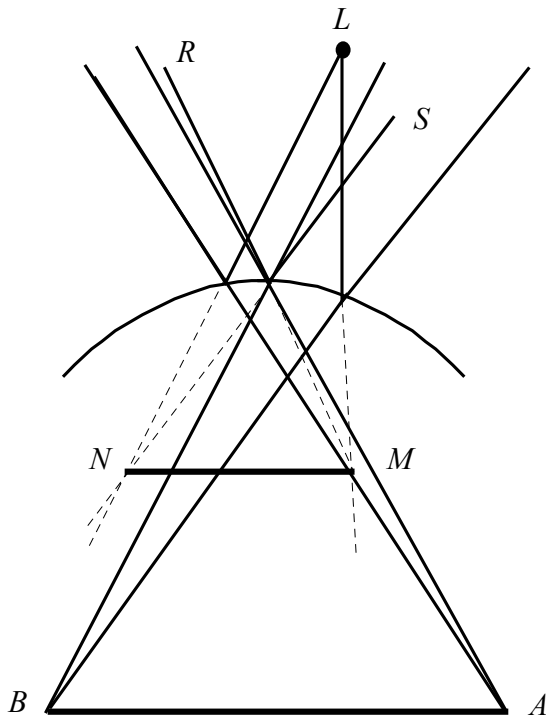
For Reflection.

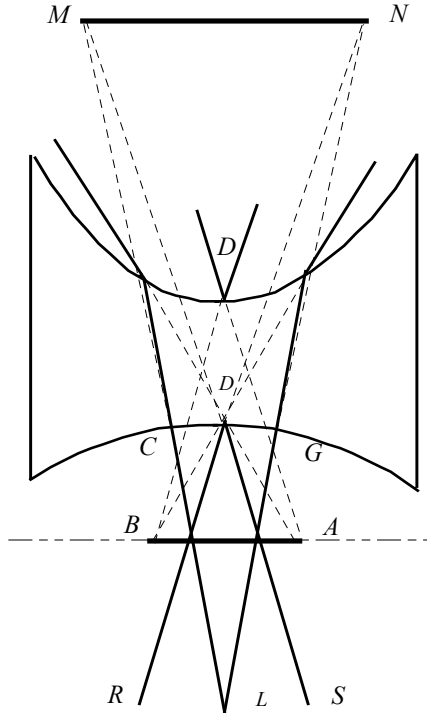Object at infinite distance.
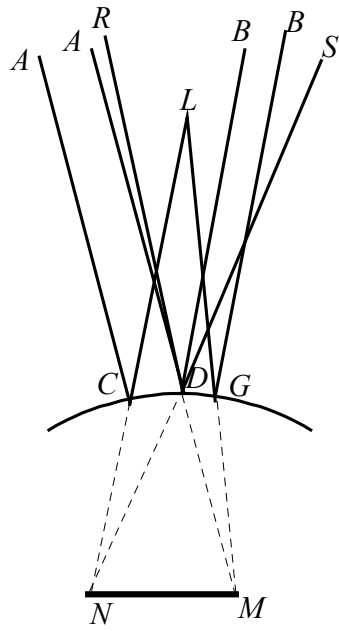
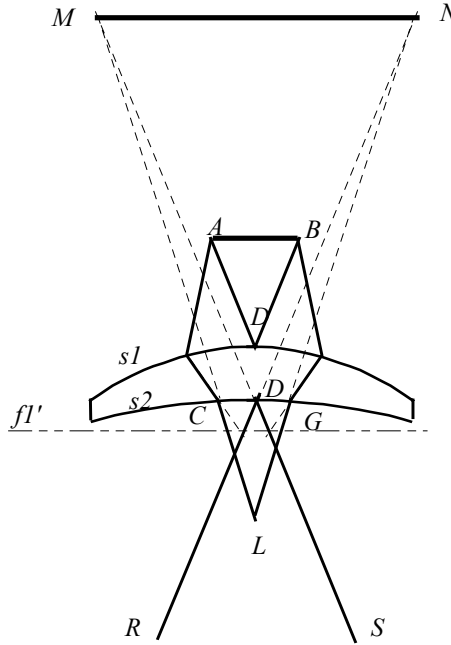Object before the eye.

Object behind the eye.

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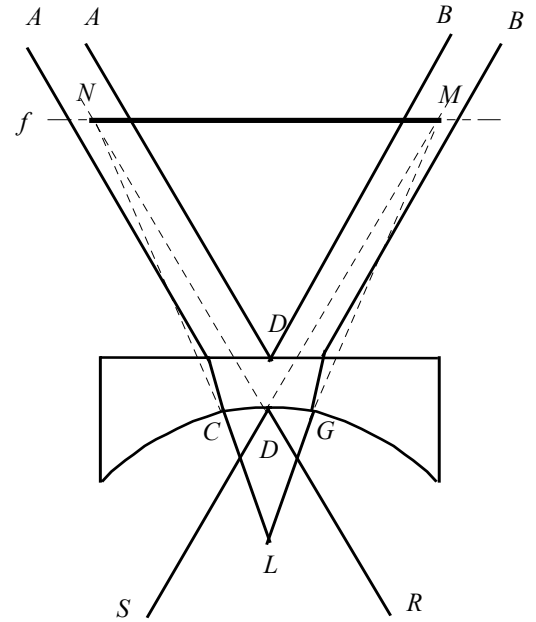For Refraction with a more dense intermediate medium.

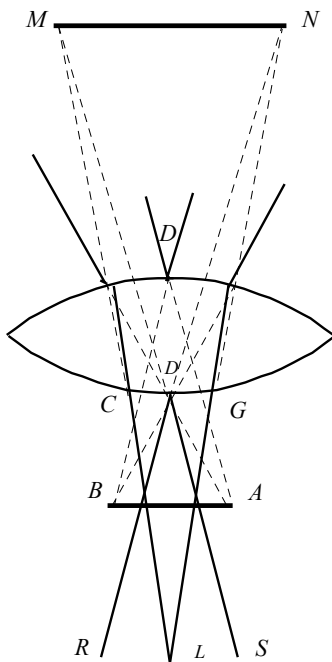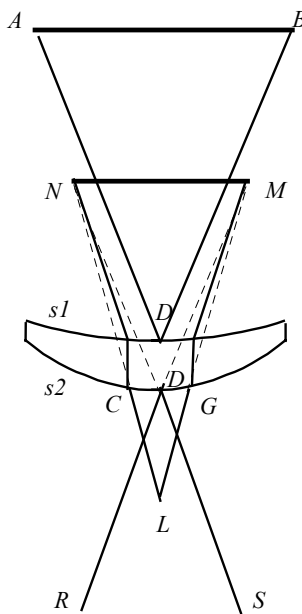Object past the eye. Object before the eye. Object at infinite distance.

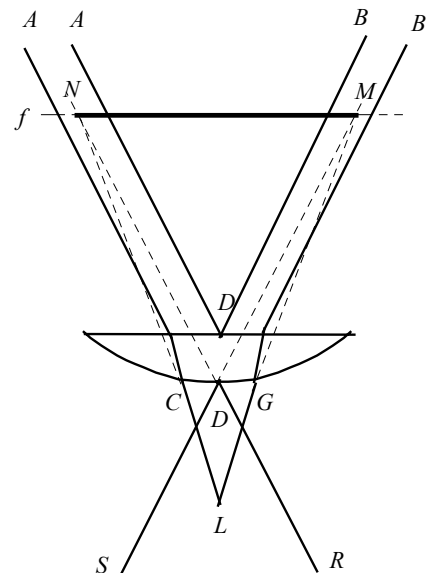For Refraction with a less dense intermediate medium.

Object before the eye. Object past the eye. Object at infinite distance.

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Prop. 47. Theorema.

Si cujuscunq; visibilis, singulorum punctorum radii, ab aliis punctis divergantur ; oculo hos radios recipienti, apparebit imago cujuslibet visibilis punctis in apice sui coni radiosi, myopibus, in determinatis suis distantiis distincta, presbytis autem semper confusa. Et imago visibilis, apparebit eo angulo visorio, quo apices conorum radiosorum, extremorum visibilis punctorum, ex centro oculi apparent : apices autem conorum radiosorum, extremorum visibilis punctorum, eodem apparent angulo ex vertice emersionis, quo visibile ex vertice incidentia.

Sit visibile quodlibet AB; sive materia radians, sive imago ante, sive post oculum, dummodo sit planum; cujus extrema puncta A, B. Divergantur singulorum punctorum radii, ab aliis punctis, lente vel speculo CDG, cujus axis sit in plano, cum punctis A, B, plano visibilis perpendicularis : Oculus igitur, hos radios recipiens, ex sua applicatione ad apices conorum radiosorum, singulorum visibilis punctorum, existimat se videre, puncta visibilis A, B, in praedictis conorum radiosorum apicibus {cor.1. 29 Hujus}. Quoniam itaque singulorum punctorum radii, a punctis propinquis diverguntur, apparebit imago visibilis, myobibus, in determinatis suis distantiis distincta ; Presbytis vero semper confusa. Sint M, N apices conorum radiosorum, extremorum visibilis punctorum; sitq; centrum oculi L, in quod concurrunt duo extremorum visibilis punctorum radii CL, GL, a punctis N, M divergentes, eritq; NLM, seu CLG, angulus visorius, quo videtur imago visibilis ab oculo, cujus centrum L; angulus nimirum, quo apices conorum radiosorum, extremorum visibilis punctorum, ex centro oculi apparent. Estq; angulus ADB, angulus nimirum quo visibile apparet ex vertice incidentiae, aequalis angulo NDM, seu RDS, angulo nempe, quo apices conorum radiosorum, extremorum visibilis punctorum, ex vertice emersionis apparent; ut patet per *Prop. 27 & 26* hujus; quae omnia demonstranda erant.

Corollarium.

Ex his Theorematibus, patet imaginem visibilis ope lentium vel speculum, oculo apparere, semper in eadem loco ; & igitur tangentes semiangulorum visualium sunt in reciproca ratione distantiarum directarum ab oculo.

Prop. 48. Problem.

To measure the size or diameter of the pupil.

Let the plate ABCD be made of copper or some other metal, in which there shall be the thinnest of cracks BC [i.e. a narrow slit] at right angles to the horizontal. The cylinder GMN is at right angles to the horizontal too, in order that a line drawn parallel to the horizontal from the axis of that cylinder to the middle of the crack is perpendicular to the plate. Let GN be the radius of the base of the cylinder, and GT the distance between the cylinder axis and the centre of the crack; the width of the slit BC is measured and noted. The line CM is drawn parallel to the horizontal from the point C at the edge of the slit

tangent to the cylinder GMN at some point M. In the same way, from point B at the other edge of the crack, BN is drawn in the same plane as CM, i. e. parallel to the horizontal, tangent to the cylinder at the point N. [See the diagram below]. The lines NB and MC are produced from B and C and these intersect at R. In triangle GTB with a right angle at T, from the given GT and TB, it follows that both BG the angle BGM can be found. Again in triangle GBN, with the right angle at N, the angle BGN is found from the given sides BG and GN; then from BGN if the angle BGT is taken away then the angle RGN is left. The length GR can be found in the isosceles triangle BRC, given the angle BRC and the perpendicular RT. The diameter of visibility is OP in the same plane as the lines NRB and MRC, as nothing can be seen through the crack beyond the extremities of the cylinder GMN. Therefore the lines NRB and MRC on being produced exactly determine the diameter OP. For consider the isosceles triangle ORP, (it is necessary indeed that the axis of the eye and the plane ABCD are perpendicular) with the given angle ORP, and with the perpendicular from the vertex R to the base OP - which is the distance of the eye from the point R - both the base OP and the visual diameter can be found, which it was necessary to observe.

<div align="center">*Scholium.*</div>

But the pupil is not always the same size; as in strong light it is made smaller, while in weak light it is larger.
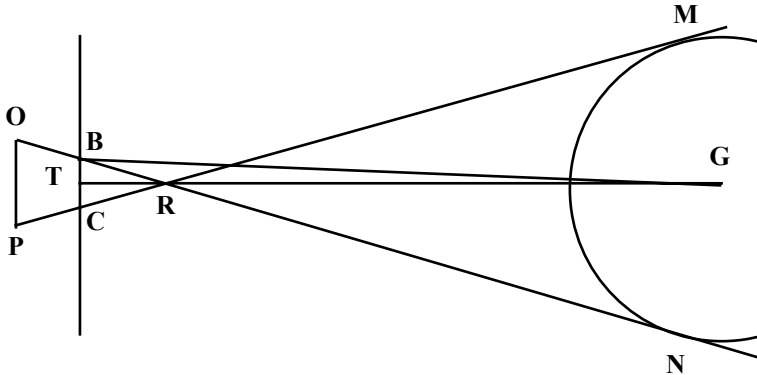
Note on Prop. 47:
    Gregory realises that the size of the entrance pupil of the eye itself is needed to maximise the light admitted via lenses and mirrors, by matching up the angles subtended by cones of rays, as shown in Prop. 50 and Prop. 51. The procedure adopted here is open to criticism, but we do not intend to explore this further!

<div align="center">

[**70**]
Prop. 48. Problema.

Quantitatem foraminis uveae, feu diametrum visualem, obfervari.

</div>

Sit ex aere, vel ex quovis alio metallo, lamina ABCD; in qua sit rimula tenuissima BC. Horizonti recta : sitq; cylindrus GMN, horizonti quoq; rectus ; ita ut linea, horizonti parallela, ab illius axe, ad rimulae medium ducta, sit laminae perpendicularis : sintq; basis

semidiameter GN; & distantia inter axem, & rimulae medium GT ; & latitudo rimulae BC, notae mensurae. Et ab extremo, rimulae puncto C, ducatur linea horizonti parallela CM, cylindrum GMN tangens ad alteras partes, in puncto M. Eodem modo, a B, altero extremo rimulae puncto, ducatur BN, in eodem plano, cum CM, horizonti parallelo, tangens cylindrum, in puncto N : & producantur lineae NB, MC, ad partes B, C; sitq;
[**71**]
earum intersectio in R; & in triangulo GTB, rectangulo ad T, e datis, GT, TB, datur & BG, & angulus BGM: Et rursus in triangulo GBN, rectangulo ad N, e datis, BG, GN, dabitur angulus BGN, a quo, angulus BGT ablatus, relinquit angulum RGN; datur latus GR :dantur igitur in triangulo isosceli BRC, angulus BRC, & perpendicularis RT. Sit diameter visualis OP, in eodem plano cum rectis NRB, MRC, nihil videns per rimulam CB, ultra extremitates cylinderi GMN: Lineae igitur NRB, MRC productae, diametrum visualem OP exacte comprehendunt ; & in triangulo isosceli ORP, (oportet enim ut axis oculi, laminae ABCD sit perpendicularis) datis angulo ORP, & perpendiculari a vertice R in basem OP, seu distantia oculi a puncto R; dabitur & basis OP, diameter visualis quaesita; quam observare oportuit.

Scholium.

Foramen autem uveae, non est semper ejusdem quantitatis ; sed in forti luce diminuitur, in debili autem ampliatur.

Prop. 49. Theorem.

Parallel rays are not weakened on traversing any distance, and equally illuminate the same object placed in the same way at any point along the path .

Let the parallel rays be AMNL, of which the outer rays are AL and MN, and which illuminate the same object BC placed in the same way at different distances from the source of radiation. I say that BC is always illuminated equally, since indeed the outer rays AL and MN which are illuminating the edges of the surface of the object BC are equidistant among themselves. Therefore the same object placed in the same way always intercepts each and every ray between the outer edges of the object BC, and since these rays give the same illumination, then indeed the strength of the rays cannot be diminished by distance alone.
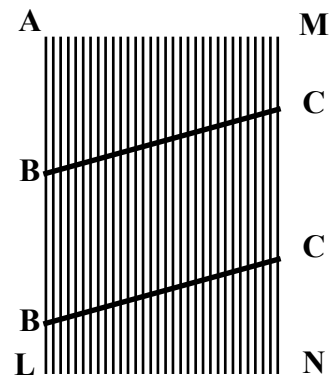
Corollary 1.

Thus it follows that the pupil of the eye is always equally illuminated by the rays emanating from individual points, whether they come from the vertex of incidence or emission of the lens. (Indeed for all these things concerning a lens, the rays from these vertices are parallel, as we have shown following Prop. 25 of this work). Only a small part of the illumination removed, which is drawn out by the opacity of the lens; and this

exception is always to be invoked in Dioptrics [Refraction]. [Gregory's optics relies on parallel beams either entering or leaving lenses, or are parallel within the lens.]

Corollary 2.

For from this theorem and the first corollary, (if the rays of individual points from any object are made parallel), and the eye can receive these rays, then not only is an image of the object formed by the eye for that angle of vision from the vertex of incidence (as we have shown from *Prop.* 44 of this work), but also it is received with the same illumination.

Prop. 49. Theorema.

Radii paralleli, non debilitati; in omni distantia, aequaliter illustrant idem objectum, eodem modo positum.

Sint radii paralleli AMNL; quorum extremi funt AL, MN, illustrantes, idem objectum B, C, eodem modo pofitum, fed in diverfis distantiis, a radiorum fonte. Dico BC, aequaliter femper illustrati. Quoniam enim extremi radii, e quibus sunt AL, MN, extremitates superficiei, objecti BC illustrantes, inter se aequidistant, idem igitur objectum, eodem modo pofitum, semper intercipiunt omnes igitur radii, inter extremos objectum BC, intercipientes, in objectum BC semper incidunt: iidem igitur radii eandem dant illustrandem ; sola enim distantia, radiorum vim non debilitat.

Corollarium 1.

Hinc sequitur, foramen uveae oculatis semper aequaliter illustrati, a singulis radiantis punctis ; sive fuerit in vertice incidentiae, sive emersionis ; (in omni enim lente, ejusdem puncti radii sunt paralleli, ut ad manifestum secundum *Prop.* 25 hujus diximus) dempta solummodo particula ista illustrationis, quae exhauritur opacitate lentis; quae exceptio in Dioptricis semper est adhibenda.

Corollarium 2.

Ex hoc Theoremate, & praemisso Corollario, (si cujuscunque visibilis, singulorum punctorum radii, ad parallelismum reducantur) oculum hos radios recipientem, non solum comprehendere visibile, eo angulo visorio, quo comprehenditur ex vertice incidentiae, (ut demonstravimus ad 44 *Prop.* hujus) sed etiam eadem illustratione.

James Gregory's Optica Promota

Prop. 50. Theorem.

If the object distance from incident vertex to the object distance from centre of the eye is in the same ratio as the image distance from emergent vertex to the image distance from centre of the eye, then the image appears with the same angle of vision as the object appears to the naked eye, when viewed through a lens or mirror.

    Let AVB be any real object or image [formed by a previous lens or mirror] as you please, with this object or image either before or after the eye. However, AVB should be in a plane normal to the axis of the lens or mirror. With the help of some mirror or lens, the rays from the object plane AVB appear in the image plane MIN. Through the planes AVB, MIN, and the axis of the lens of mirror, a plane is drawn making a common section with the object line AVB and the image line MIN. The object AVB appears with the angle of vision ALB from some axial point L. The ratio is thus as VD, the distance of the object from the vertex of incidence, to VL, the distance of the object to the centre of the eye placed at L, thus as VD, the distance of the image from the vertex of emergence, to IO, the distance of the image from the centre of the eye at O. The image IMN appears from the axial point O at the visual angle MON. I say that the angle MON is equal to the angle ALB.

Indeed, as   VD : VL: : ID : IO; and on
rearranging, as   VD : ID : : VL : IO;
but as   VD : ID : : VA : IN;
Therefore,   VL : IO : : VA : IN.

Therefore the triangles AVL and NIO are similar, having equal right angles AVL and NIO, and sides in proportion around these equal angles. Therefore the angles ALV and ION are equal, and the angles VLB and MOI are equal. Hence the whole angles ALB and MON are equal. QED.

Prop. 50. Theorema.

Si fuerit ut distantia visibilis, a vertice incidentia, ad distantia visibilis, ab oculi centro ; ita distantia imaginis, a vertice emersionis, ad distantiam imaginis, ab oculi centro : eodem angulo visorio, apparebit imago, ope lentis, vel speculi ; quo apparet visibile, nudo oculo visum.

Sit visibile quodlibet AVB; sive materia radians, sive imago ante, sive post oculum, modo sit planum ; cui axis lentis, vel speculi sit normalis : ope speculi cujuslibet, vel lentis, appareat imago, plani radiantis AVB, in plano MIN ; & per plana AVB, MIN, & axem lentis, vel speculi VID, ducatur planum, faciens cum visibili communem sectionem, rectam AVB; cum imagine vero, rectam MIN. Et a quovis axeos puncto L, appareat visibile AVB, angulo visorie ALB; sitque ut VD, distantia visibilis a vertice incidentiae; ad VL, distantiam visibilis ab oculi centro: Ita D, distantia imaginis a vertice emersionis; ad IO, distantiam imaginis ab oculi centro: Et a puncto axeos O, appareat imago MIN, angulo visorio MON. Dico angulum MON ; esse aequalem angulo ALB:

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Quoniam, ut   VD : VL: : ID : IO; erit &
    permutando, ut VD : ID : : VL : IO;
    ut autem   VD : ID : : VA : IN;
Igitur, ut   VL : IO : : VA : IN; triangula
igitur AVL, NIO, habentia angulos AVL, NIO, aequales, utpote rectos, & latera circa
aequales angulos proportionalia, sunt similia ; angulus igitur ALV, aequalis est angulo
ION. Eodem modo, demonstrabitur aequalitas, angulorum VLB, MOI: totus igitur
angulus ALB, est aequalis toti angulo MON. Quod erat demonstrandum.

Prop. 51. Theorem.
*With object and image positions unchanged , I say that the image appears equally
illuminated with the eye at O, as the object appears with the eye at L, provided the whole
pupil of the eye is illuminated by the rays.*

    The ratio of the diameters is indeed as VL to VD: *i. e.* the ratio of the diameter of the
pupil at L, to be illuminated by the paraxial rays from V, to the diameter of the same
cone of rays reaching the lens or mirror, which has the vertex at D. And the ratio IO to
ID is also as VL to VD.   Thus, the ratio is the diameter of the pupil at O, illuminated by
the rays from V, to the base diameter of the same cone of rays at the lens or mirror.
Therefore, if the diameters of the pupils are supposed equal at the two positions, then the
diameters of the bases of the cones of rays from V are also equal. Therefore the base
areas of these cones of rays are equal: for since they have a common axis VD, they too
are equal, in short, each cone contains the power of one and the same cone of rays. The
powers of the cones in illuminating or burning (for rays at equal distances passing into
equal pupils) are as the square ratio of the chords of their radiant semi-angles. Since the
radiant cones have equal radiant angles, the chords of these half-angles are equal, and so
the squares of chords are equal also [for the case of equal distances]. Thus, the
illuminations are equal.   Indeed, this is so for the illumination into the pupil L from the
rays coming from V, and the illumination due to the rays coming from the point V into
the pupil at O. Therefore, in like manner, the eye at L sees the illumination from the
object point V, with which the eye at O sees the illuminated image of this. In the same
way too, this equality of the illumination can be shown for all the points of the visible
object AVB. QED.

Scholium.
    But if VL to VD, or IO to ID has a larger proportion than the diameter of the pupil to
the diameter of the lens or mirror to be illuminated by the rays, then in this case it can be
said that the whole pupil is illuminated by the rays from the individual points. If the ratio
is equal, then the result is exact. If however the ratio is less then only a part of the pupil is
illuminated, and since it is sufficient to be warned, the demonstration is indeed clear [in
all cases].

Comment:
The point V is the common vertex of two cones, with a common axis VD. The pupil is
the base of the small cone located at L, while the lens or mirror is the base of the large
cone: thus, each receives the same amount of radiation, while the intensities vary as the
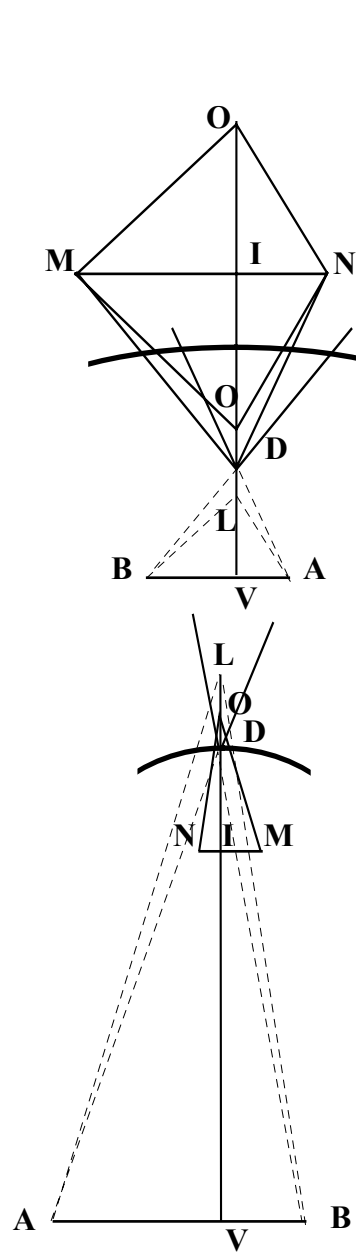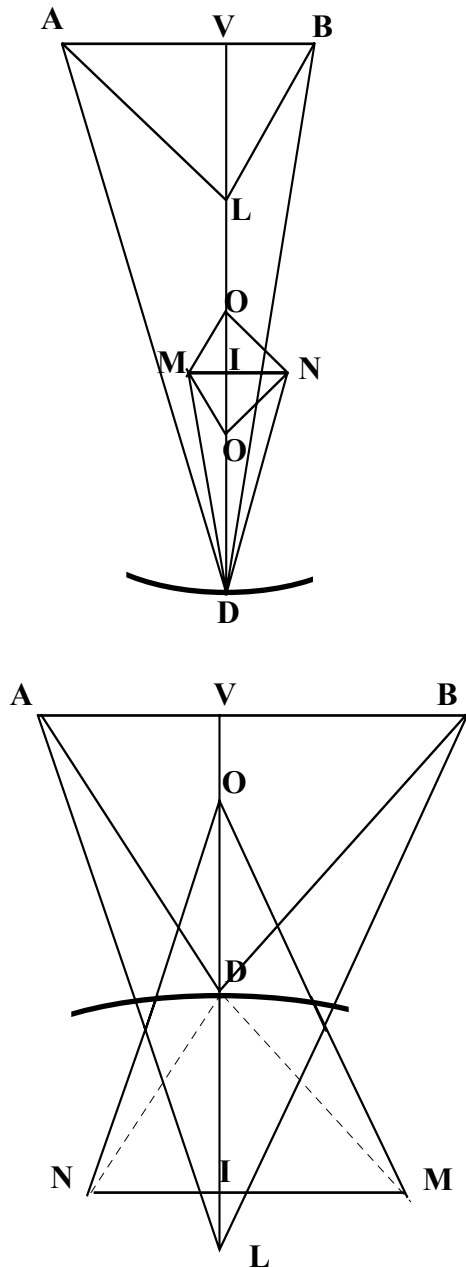
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squares of the diameters. The same reasoning applies for the image point I, the new position of the eye at O, and the lens or mirror. A summation is carried out over all the points of the object, from A to B, and all the paraxial rays in the angle ALB, the angle of vision, is assumed to enter the eye. If the ratio of the lengths is greater than the ratio of the diameters then the whole pupil and lens or mirror is bathed in illumination; and only in part if the ratio is less; the theorem is thus true only for the case of equality. The theorem is concerned with the possible loss of light intensity due to a miss-match of the size of the lens or mirror used when an image is processed for viewing.
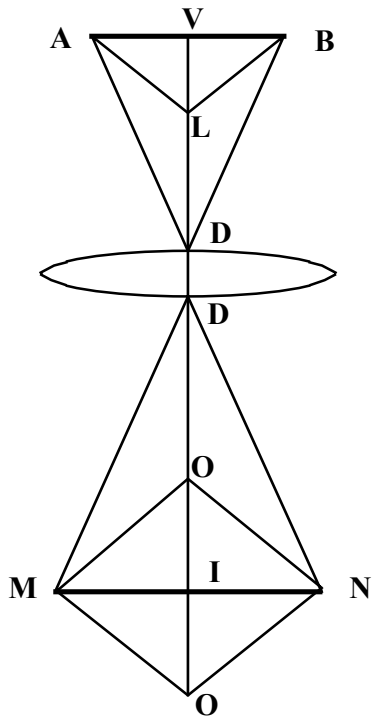
[74]

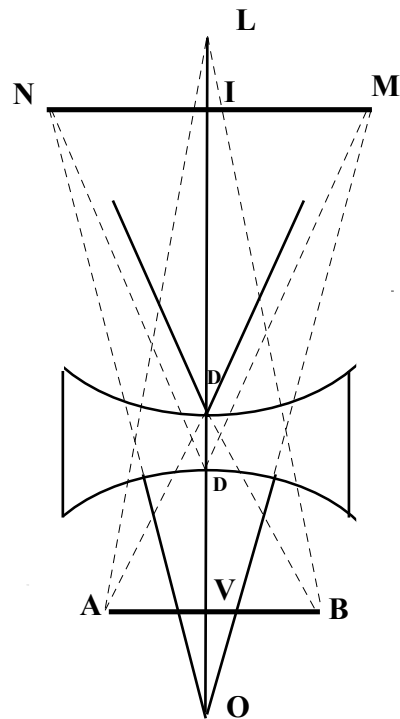or Reflection: with the object before the eye.   The object behind the eye.

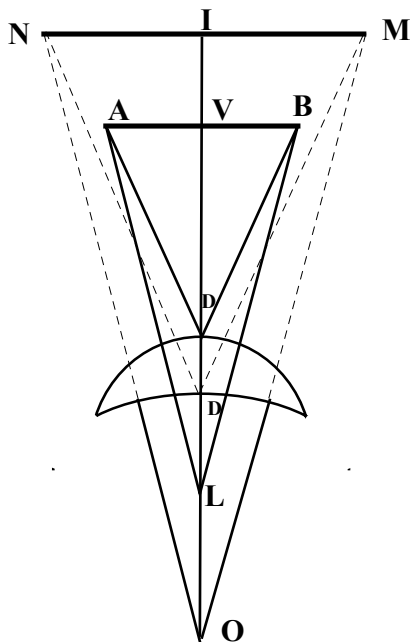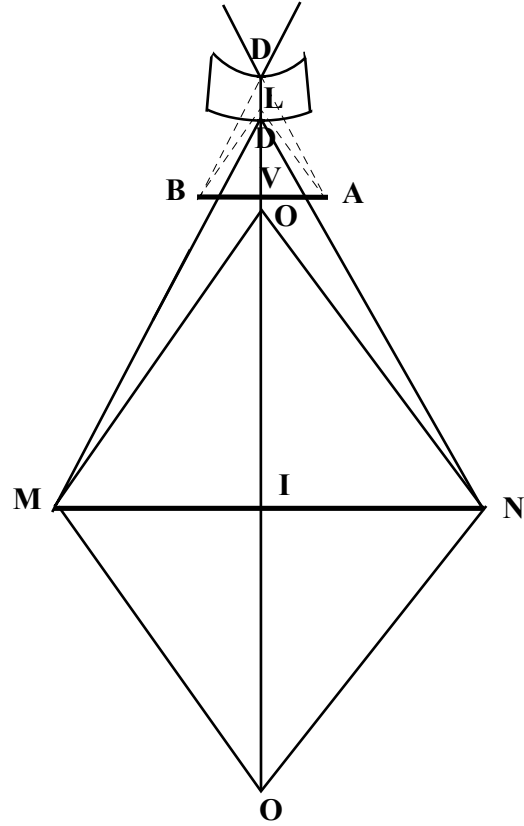# *James Gregory's Optica Promota*

[75]

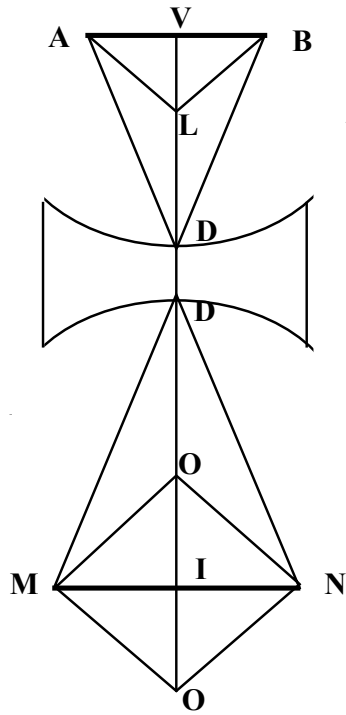For Refraction with the intermediate medium denser.

Object before the eye.

Object behind the eye.
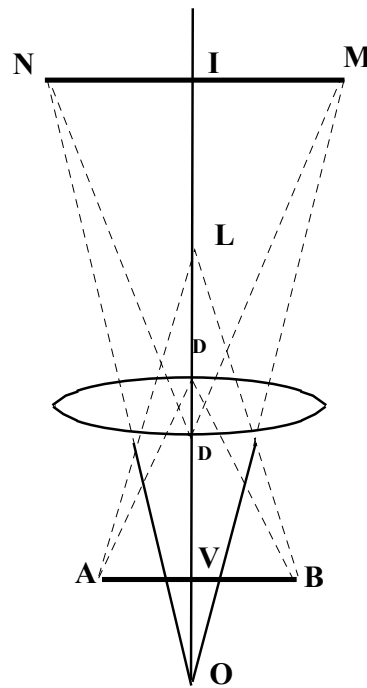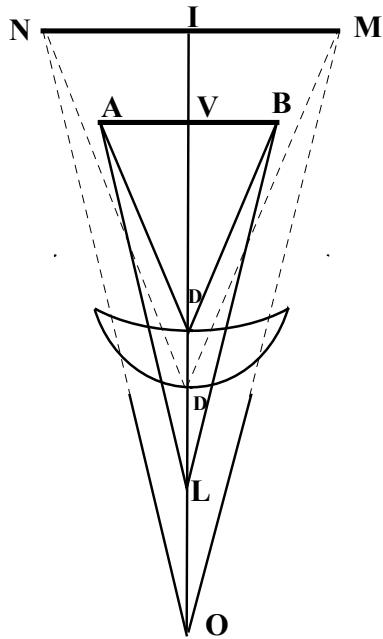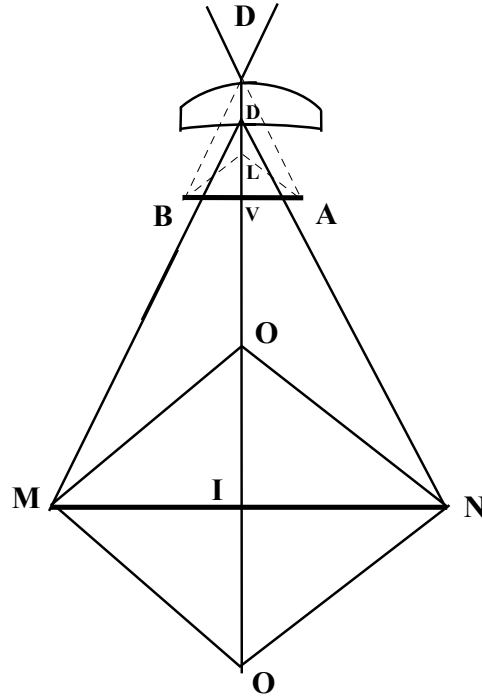
For Refraction with the intermediate medium less dense.

Object before the eye.

Object after the eye.

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[**76**]
Prop. 51. Theorema.

Iisdem positis; dico imaginem, eodem modo apparere illustratam, oculo in O; quo apparet visibile, oculo in L : dummodo totum foramen uvea, radiis illustretur.

Est enim, ut VL, ad VD; ita diameter foraminis uveae, radiis puncti visibilis V, illustrati, in L ; ad diametrum , baseos coni radiorum eorundem, in lente vel speculo; atq; ut IO ad ID ; hoc est VL, ad VD; ita diameter foraminis uveae, radiis puncti visibilis V, illustrati in O; ad diametrum baseos coni, radiorum eorundem, in praedicto quoque speculo vel lente : cum igitur diametri foraminum uveae supponantur aequales; aequales quoq; erunt, in lente vel speculo, diametri basium, conorum, radiorum puncti V; quibus, illustratur foramen uveae in L, & foramen uveae in O. Bases igitur horum conorum aequales erunt, cumq, eundem communem habeant axem VD, prorsus aequales erunt; vel potius unus, & idem conus radiosus; eruntq; vires conorum radiosorum, in illustrando, vel comburendo (radiis in spatia aequalia nempe in foramina uvearum aequalia reductis) in duplicata ratione, chordarum, suorum semiangulorum radiosorum; cumque coni radiosi, aequales habeant angulos radiosos; erunt & chordae semissium horum angulorum aequales;

[**77**]
adeoq; & quadrata chordaram aequalia; unde, illustrationes aequales erunt : Nempe, illustratio radiorum puncti V, in foramine uveae L; & illustratio radiorum puncti V, in foramine uveae O: oculus igitur in L, eodem modo videt punctum visibile V, illustratum ; quo oculus in O videt illius imaginem illustratam. Eodem quoque modo haec aequalitas illustrationis in omnibus punctis visibilis AVB manifesta fiet; quod erat demonstrandum.

Scholium.

Si autem VL, ad VD; vel IO, ad ID; majorem habeat proportionem ; quam diameter foraminis uveae, ad diametrum lentis, vel speculi, radiis illustrati; in hoc inquam casu, totum foramen uveae, singulorum punctorum radiis illustrabitur; si aequalem, praecise ; Si vero minorem, totum non illustrabitur; quod admonuisse sufficiat, demonstratio enim est manifesta.