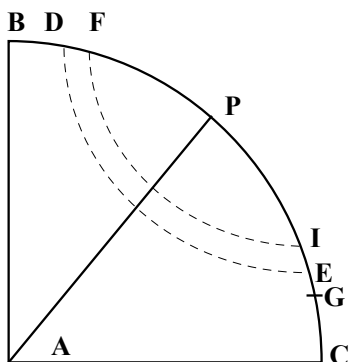[**96**]

Prop. 60. Problem

With the two given heights of a phenomenon around the pole, in a meridian with the height of the pole given; to find the sum or difference of the parallaxes of the given heights.

In this problem there are three different cases; the first case is where the meridian height is observed between the pole and the vertical ; in which ABC shall be the quadrant of the circle of the meridian, P the pole, D & E the true positions of the phenomena, above & below the pole, and these equidistant from the pole. Truly, the apparent positions shall be F, G. The arc PI is equal to the arc PF [PE in the text]. I say that the arc IG is the sum of the parallaxes sought. Since indeed, the arcs PD & PE are equal; and the arcs PI & PF are equal, & EI itself equal to DF; which DF is the parallax of the superior observation, and EG the parallax of the inferior. Therefore the arc IG, the sum of these, is the sum of the parallaxes sought. Which it was required to find.

The second case is when the zenith falls between the pole and the higher observation; in which BZC shall be the meridian circle, P the pole, Z the zenith, D & E the true locations of the phenomenon, above and below the pole; F & G the apparent locations. Let the arc PI be equal to the arc PF, which is smaller than the arc PG by IG itself. I say that the arc IG is the differences of the parallaxes sought. Indeed the arc PD is equal to the arc PE, from which, because D & E are the true locations of the phenomena.
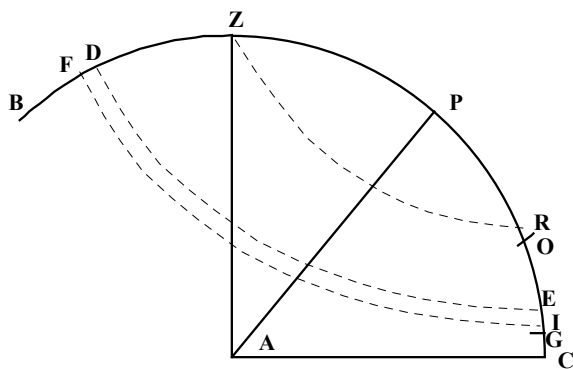
Therefore DF will be the upper parallax of the observation, and as EG shall be the lower parallax of the observation, IG will be the difference of the parallaxes observed; which it was required to find.

The third case, in which the phenomenon is observed in the zenith, is the easiest, since the difference of the apparent distances from the pole, surely RO, is equal, not only to the difference of the parallaxes, but also has no parallax in the vertical location.

Note: The diagrams show AP as the invariant direction of the polar axis; AB is the direction of the vertical initially, while AC is the vertical 12 hours later. The apparent height or angle measured relative to the polar axis AP is always less due to atmospheric refraction, while the dotted arcs show the motions of the true and apparent heights on the celestial sphere during the 12 hour period.

Prop. 60. Problema.

Datis duabus altitudinibus, Phaenomeni circumpolaris, in meridiano; cum altitudine poli : datarum altitudinum, parallaxium summam, vel differentiam invenire.

In hoc Problemate, tres sunt casus diversi; primus casus, est quando altitudo meridiana, supra polum, observatur inter polum & verticem ; in quo sit ABC, quadrans meridiani circuli, P polus, D & E vera loca Phaenomeni, supra & infra polum, adeoq; a polo aequidistantia : Loca vero apparentia sint F, G, infra loca vera; sitq; PI arcus, arcui PE, aequalis. Dico IG arcum esse summam parallaxium quaesitam. Quoniam enim, arcus PD, & PE, sunt aequales; & arcus PI, arcui PF; erit & EI ipsi DF aequalis; qui DF est parallaxis, superioris observationis; & EG, parallaxis inferioris; arcus ergo IG, eorum summa; est summa parallaxium quaesita. Quam invenire oportuit.

Secundus casus est, quando zenith cadit inter polum & altiorem observationem; in quo sit BZC, circulus meridianus; P polus, Z zenith, D, & E, vera loca Phaenomeni, supra & infra polum, F, & G, loca apparentia; sit PI arcus, aequalis arcui PF; qui minor est arce PG, per ipsum IG. Dico arcum IG, esse differentiam parallaxium quaesitam. Est enim arcus PD, aequalis arcui PE, ex eo quod D, & E sunt vera loca Phaenomeni; erit igitur, & DF

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parallaxis superioris observationis, aequalis EI; cumq; EG, sit parallaxis inferioris observationis, erit earum differentia IG, differentia parallaxium quaesita; quam invenire oportuit.

Tertius casus, in quo Phaenomenon observatur in zenith, est facillimus; quoniam differentia distantiarum apparentium a polo, nimirum RO, aequatur, non solum differentiae parallaxium, sed etiam in vertice nullam habet parallaxem.
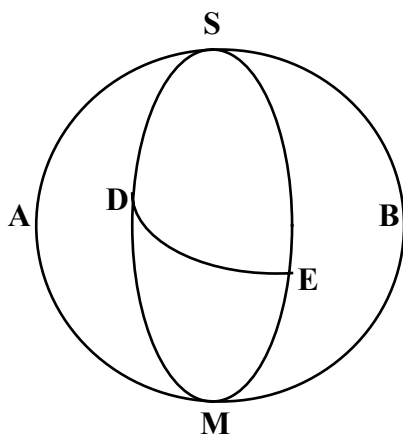
Prop. 61. Problem.

Given the longitude and latitude of two places on the terrestrial globe ; to find the angles with the common azimuth which the meridian of the places makes and the distance between them.

First, if those places shall be placed on opposite ends of a diameter, it is agreed that all the azimuth circles are common.

Secondly, if the places differ only with latitude, the meridian circle is the common azimuth.

Thirdly, if they differ with so much longitude , of if they differ with latitude and longitude; thus the proposition is investigated. ASBM shall be a hemisphere of the earth in which there are the locations D & E with different latitude and longitude. The meridian circles SDM, SEM and common azimuth DE are drawn. In the spherical triangle DSE, the sides DS, ES are given, with the angle DSE taken from these; from which the angles SDE & SED are found, with the length DE of the separation of the places, which it was necessary to find.

Prop. 61. Problema.

Datis longitudinibus, & latitudinibus, duorum locorum, in globo terreno; angulos, quos commune Azimuth, cum Meridianis locorum facit ; & locorum distantiam invenire.

Primo, si loca ista sint ex diametro opposita; constat, omnes circulos Azimuthales esse communes.

Secundo, si sola latitudine differant, meridianus circulus est azimuth commune.

Tertio, si longitudine tantum; vel longitudine, & latitudine differant ; sic investigatur propositum. Sit ASBM terrae Hemisphaerium, in quo, sunt loca D, & E,

longitudine, & latitudine, differentia; ducantur meridiani circuli, SDM, SEM, & azimuth commune DE ; in triangulo sphaerico, DSE, dantur latera DS, ES, cum angulo ab iis comprehenso DSE ; ex quibus, inveniantur anguli SDE, & SED, cum latere DE locorum distantia ; quae invenire oportuit.

Prop. 62. Problem.

To find the sum or difference for the observed parallaxes P from two altitudes of the same phenomenon brought together at the same time in different locations with a common azimuth.

Let the circle ABDE be the common azimuth of the locations, B and D shall be the vertical points [the local zeniths]. In the first case, there shall be some [astronomical] phenomenon between the verticals B and D, of which the true position shall be O; and for the observer beyond B, truly it would seem to be at P. Also, for the observer beyond D, it would appear to be at R. Therefore the observed arcs BP and DR, if they should be added together at the same time, make an arc that exceeds the arc BD, the distance between the [true] positions, by the sum of the parallaxes RP.

In the second case, the true location of any phenomenon shall be I, outside the intercepted arc BD, appearing at M for the observer beyond B, and appearing at N for the observer truly beyond D. Therefore with the arcs DB, BM subtracted at the same time from the arc DN, the difference MN of the parallaxes sought will remain.

Prop. 62. Problema.

Ex collatis, duabus altitudinibus, ejusdem Phaenomeni; ad idem tempus, in diversis locis, & eorum azimuth communi, observatis P parallaxium summam, vel differentiam, invenire.

Sit circulus ABDE, azimuth commune locorum; puncta verticalia sint B, D: sitq; primo inter vertices B, D Phaemomenon aliquod, cujus verus locus O; observatori vero infra B,

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videatur in P; & observatori infra D, videatur in R; arcus ergo BP, DR, observati; si simul addantur, efficiunt arcum, excedentem arcum BD, distantiam locorum, per RP, summam parallaxium.

Secundo, extra arcum interceptum, BD, sit I, verus locus, alicujus Phaenomeni; observatori infra B, apparentis in M ; observatori vero infra D, apparentis inferius in N. Subductis ergo arcubus, DB, BM, simul, ex arcu DN, remanebit MN, differentia parallaxium quaesita.
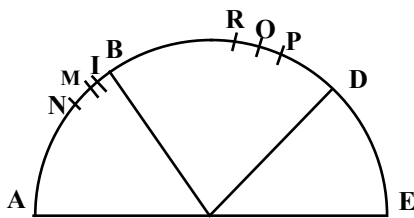

Prop. 63. Theorem.

The distance of the phenomena, from the centre of the earth, is to the radius of the earth as the sine of the angle of the apparent distance to the vertical to the sine of the parallaxes.

Let A be the centre of the earth, the position of the observer on the surface B, of which the zenith is D; and let the distance of the phenomenon from the centre of the earth be AO. The separation DBO therefore is [the angle] appearing from the vertical, and BOA the angle of the parallax. In the plane triangle BOA: as the distance AO of the

phenomenon from the centre of the earth, to BA, the radius of the earth ; thus the sine of the angle DBO, or the [angular] separation from the vertical, to the sine of the angle BOA, or the parallax, which was to be demonstrated.

Prop. 63. Theorema.

Distantia Phaenomeni, a centro terrae, est ad semidiametrum terra ; ut sinus, distantiae appatentis a vertice; ad sinum parallaxeos.
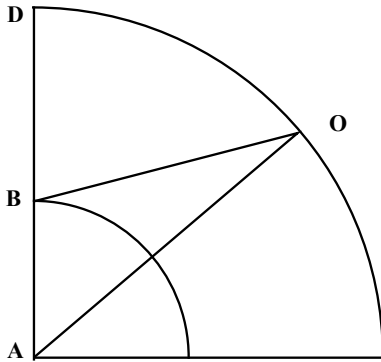
Sit centrum terrae A, locus observatoris in terrae superficie B; cujus zenith D ; sitque distantia phaenomeni O, a centro terrae AO, est igitur DBO distantia apparens a vertice, & BOA, parallaxis

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; & in triangulo rectilineo BOA; ut AO, distantia phaenomeni a centro terrae; ad BA, semidiametrum terrae ; ita sinus anguli DBO, seu distantiae apparentis a vertice ; ad sinum anguli BOA, seu parallaxeos. Quod demonstrandum erat.

Prop. 64. Theorem.

The sines of the parallaxes, to the sines of distances appearing from the vertical are directly proportionals.

Indeed, the sines of the angles of the distances appearing from the vertical always are to the sines of the parallaxes, as the distances of the phenomenon from the centre of the earth, to the radius of the earth. Thus, as the sine of the distance appearing from the vertical for one observer, to the sine of the same parallax, so the sine of the distance appearing from the vertical for another observer, to the sine of their parallax. By being exchanged: as the sine of the distance appearing from the vertical of the one observer, to the sine of the distance appearing from the vertical of another observer; thus the sine of the parallax of the one observer, to the sine of the parallax of the other observer, which was to be demonstrated.

Prop. 64. Theorema.

Sinus Parallaxium, sinubus diftantiarum apparentiam a vertice, sunt directe proportionales.
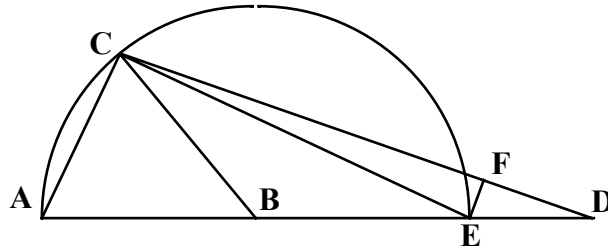
Sinus enim, distantiarum apparentium a vertice, semper sunt ad sinus parallaxium , ut distantia Phenomeni, a centro terrae, ad semidiametrum terrae; ergo ut sinus, distantiae apparentis a vertice, unius observationis; ad sinum parallaxeos ejusdem; ita sinus, distantiae apparentis a vertice, alterius observationis; ad sinum suae parallaxeos: & permutando ; ut sinus distantiae apparentis a vertice unius observationis ; ad sinum distantiae apparentis a vertice alterius observationis; ita sinus parallaxeos unius observationis ; ad sinum parallaxeos alterius observationis ; quod demonstrandum erat.

Prop. 65. Problem.

Given the sum or difference of two arcs, with the ratio of the sines, to find the arcs themselves.

Let the angle ABC be the given sum of the arcs, and the ratio of the sines AB to BD; with the smaller AB as radius, the semicircle ACE is described; and the straight lines CD, CE and CA are drawn. The line EF is drawn parallel to CA. And in the triangle CBD, as CB (i. e. AB) will be to BD, thus the sine of the angle CDB will be to the sine of the angle BCD ; therefore it is clear that the angles BCD and BDC themselves are to be

sought. Indeed, the sum of
these is the given angle CBA,
and the given ratio of the
sines AB to BD. It is clear
too, that the angle at the
circumference CEA is equal
to half the angle at the centre,
CBA. Therefore the angle
ECF, for which the greater of the angles sought exceeds half the sum ECB, is half the
difference of the same angles sought. Therefore, with the radius [meaning the adjacent
side] CE placed, on account of the right angles ACE and CEF, AC will be as the tangent
of half the sum of the angles sought, and EF as the tangent of half the difference of the
same angles. Since the lines AC and EF are parallel: as AD to ED, the sum of the terms
of the ratio of the sines to the difference of the same; thus, AC to EF, the tangent of half
the sum of the given arcs to the tangent of half the differences. Which added to half the
given sum reveals the greater angle ; or from the same subtracted to bring about the
smaller angle. Or, as ED to AD, the difference of the terms of the ratio of the sines to the
sum of the same; thus EF to AC, the tangent of half the difference of the given arcs to the
tangents of half the sum: from which the given arcs are found as before. Which was to be
accomplished.

Prop. 65. Problema.

Data summa, vel differentia, duorum arcuum; cum sinuum ratione; ipsos arcus invenire.

Sit data arcuum summa, angulus ABC, & sinuum ratio, AB, ad BD; minore AB semidiametro,
describatur semicirculus ACE;

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& ducantur lineae rectae CD, CE, CA; atq rectae CA parallela ducat EF. Eritq; in triangulo rectilineo,
CBD; ut CB, (id est AB) ad BD, ita sinus anguli CDB; ad sinum anguli BCD; manifestum est igitur istos
angulos, nempe BCD, BDC, esse quaesitos; eorum enim summa, est angulus CBA,datus, & sinuum ratio,
AB, ad BD data: manifestum quoq; est, angulum CEA in circumferentia, esse semissem anguli CBA, in
centro; & igitur angulus ECF, quo major quaesitorum, excedit semissem summae ECB, est semissis
differentiae eorundem angulorum quaesitorum. Posito igitur CE radio ; ob angulos ACE, CEF, rectos ; erit
AC tangens semissis summae angulorum quaesitorum, & EF, tangens semissis differentiae eorundem
angulorum ; & quia rectae AC, EF, sunt parallelae ; erit ut AD, summa terminorum rationis sinuum; ad ED,
differentiam eorundem; ita AC, tangens semissis summae arcuum datae ; ad EF, tangentem semissis
differentiae ; qua addita, ad semissem summae datae, producitur angulus major ; vel ab eadem subducta,
efficitur angulus minor. Vel, ut ED, differentia terminorum rationis sinuum ; ad AD summam eorundem;
ita EF, tangens semissis differentiae arcuum datae; ad AC, tangentem semissis summae : qua data
inveniuntur arcus ut prius. Quod erat faciendum.

Commentary on Prop. 65.

α < β

C

(β − α)/2

β

F

A   α + β B   (α + β)/2   α   D

E

y - x   *y + x*
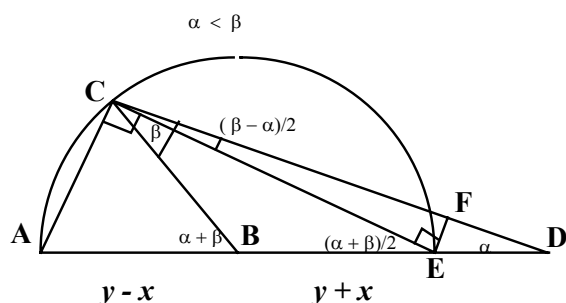
In the diagram, BD = $y + x$, and AB = $y - x$, the sum and difference of the arcs $x = R\alpha$ and $y = R\beta$, for some radius R. Now, in triangle CBD, CB/BD = sin α/sin β , and y/x= AD/ED = AC/EF = $\frac{\tan(\beta+\alpha)/2}{\tan(\beta-\alpha)/2}$.

Hence, $(x + y)/x$ = BD/y = 1 + tan(half-sum)/tan(half-diff.), from which y can be found, and x, etc.

Prop. 66. Problem.

For a given spherical triangle, and with the arc of a great circle given: from an angle of the spherical triangle, any arc of a great circle is drawn you please to the opposite side. To find the segments of the given side and [the length] of the arc given, from the given ratio of the sines from the sides of one segment, and the difference of the arcs.

L   P   C

B

D   A

O   E
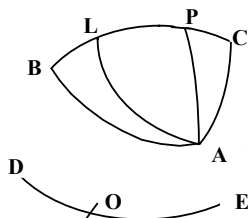
Let ABC be the spherical triangle, [of which] all the sides and angles are given; and DE shall be a given great arc of the circle. In the spherical triangle ABC, an arc AL is drawn to the side BC placed opposite; then EO, [an arc] equal to the arc AL, is taken from DE. The sine of the arc DO to the sine of the arc BL is the given ratio; and the arcs DO and BL are sought. A perpendicular arc AP is sent from the angle A to the side BC placed opposite, which will be given with the segments BP and PC, since all the sides and all the angles are given in the triangle ABC. Thus the sine of the arc AL may be put surely as 1 ♃ [i. e. an unknown amount], and the sine of the arc LP will be given in terms of the unknown number. Then the sine of the arc BL will be given in related numbers from the given difference of these, the sine of the arc LP and the sine of the arc BP. And in the same way ; from the sine of the given arc DE, and from the sine of the arc OE, surely 1 ♃, the sine of the arc DO will be given by the difference of these in related numbers. And since the sine of the arc DO to the sine of the arc BL will be the given ratio, and hence an equation will be given, from the resolution of which the value of the root will become known, without doubt the sine of the arc AL, or the arc EO ; from which being given, the rest are easily given, which were to be found.

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Prop. 66. Problema.

Dato triangulo sphaerico, & arcu circuli maximi ; ductoq; ab angulo, trianguli sphaerici in latus oppositum, arcu circuli maximi quolibet ; e data ratione sinuum, unius segmenti lateris, & differentia arcuum; lateris & arcus dati segmenta invenire.

Sit triangulum sphaericum ABC, data habens, omnia latera, & omnes angulos; sitq; datus arcus maximi circuli DE, & in triangulo sphaerico ABC, demittatur arcus AL, in latus oppositum BC ; deinde arcui AL aequalis EO, auferatur a DE ; & e data ratione, sinus arcus DO, ad sinum arcus BL ; quaeritur, & arcus DO

& arcus BL. Ab angulo A, in latus oppositum BC, demittatur perpendicularis arcus AP; qui dabitur, cum segmentis BP, PC; quoniam omnia latera, & omnes anguli dantur in triangulo sphaerico ABC. Ponatur itaque sinus arcus AL ;nempe 12; dabitur & sinus arcus LP, in numeris cossicis. Deinde datis, sinu arcus LP, & sinu arcus BP, dabitur & sinus arcus BL eorum differentiae, in numeris cossicis. Et eodem modo ; e datis, sinu arcus DE, & sinu arcus OE, nempe 12; dabitur & sinus arcus DO, eorum differentiae, in numeris cossicis. Cumq; sit data ratio, sinus arcus DO, ad sinum arcus BL; dabitur hinc aequatio, ex cujus resolutione, innotescet valor radicis, sinus nimirum arcus AL, vel arcus EO ; quo dato, facile dantur reliquo, quae invenienda erant.
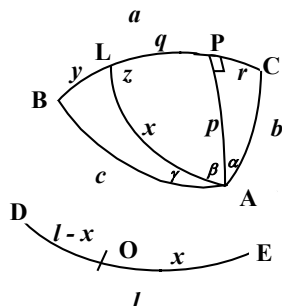
Commentary on Problem 66.

Fig. 66 - 1.

We are given the angles A, B, C ; the sides a, b, c ; together with the arc DE (= l), and $\sin(l - x) / \sin y = k$. The lengths of the arcs y (and a - y) and the length of the arc LA, or EO (= x), are to be found.

The basic formulae for the angles A, B, C and the sides a, b, c of a spherical triangle are the cosine rules for sides (I) and angles (II), and the sine rule (III):

$\cos a = \cos b \cos c + \sin b \sin c \cos A$ (I), and similarly for $\cos b$ and $\cos c$ in a cyclic manner ;

$\cos A = - \cos B \cos C + \sin B \sin C \cos a$ (II), and similarly for $\cos B$ and $\cos C$ in a cyclic manner ; and

$\sin a : \sin b : \sin c = \sin A : \sin B : \sin C$ (III).
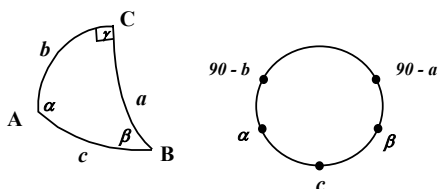

Fig. 66 - 2.

In addition, Napier's rules for right - angled spherical triangles are of assistance in solving this problem: For any arc on the circle, such as c, the cosine of the arc is equal to the product of the cotangents of the adjacent angles and also to the product of the sines of the complements of the opposite arcs. Thus, e.g. in Fig. 66 - 2,

$\cos c = \cot \alpha \cot \beta = \sin(90 - b) \sin(90 - a) = \cos a \cos b$.

Napier's rules applied to triangle APL give :

$\cos(90 - p) = \sin p = \sin b \sin C$; $\cos x = \cos z \cos p$; $\sin z = \sin x \sin \beta$;

while in triangle APB, $\cos(90 - q) = \sin q = \cot B \tan p$. Hence p and q are known.

Now, $\sin(\ell - x) / \sin y$ is given, or $\sin(\ell - x) = k \sin y$; also, from the arcs, $y = q - z$, and $\sin y = \sin(q - z) = \sin q \cos z - \cos q \sin z$.

Hence, $k \sin q \cos z - k \cos q \sin z = \sin \ell \cos x - \cos \ell \sin x$, giving on substitution from above: $k \sin q \cos x / \cos p - k \cos q \sin x \sin \beta = \sin \ell \cos x - \cos \ell \sin x$.

Hence, $k \sin q/\cos p = \sin \ell$, and $k \cos q \sin \beta = \cos \ell$; giving $\sin \beta = \cot \ell \cos p \cot q$; hence β is known, and as $\cos \beta = \cot x \tan p$ from triangle APL, it follows that $\cot x$ and hence x is also known.

Prop. 67. Problem

For two given spherical triangles, and from one angle of each an arc has been drawn to the opposite side equal for both : to find the segments of each side from the given ratio of the sine of the segment of the side of one triangle to the sine of the segment of the side of the other triangle.
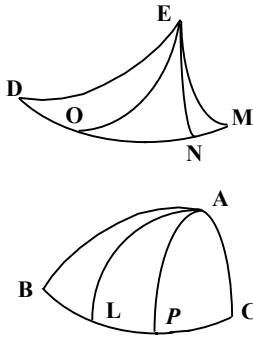
Let ABC, EDM be two spherical triangles with all the sides and angles having been given. From one angle of each, say A and E, equal arcs AL, EO are dropped to each to the opposite sides, certainly BC, DM. And ratio is given of the sine of the arc DO to the sine of the arc BL. Each arc BL and DO is sought. The sine of the arc AL, or the sine of the arc EO, (indeed both are equal by hypothesis) is put as 1⅟ : then from the same method , by which in the above problem, both the sine of the arc DO and the sine of the arc BL may be found in related numbers, whenever the ratio of these may be given from hypothesis. Hence an equation will be given, from the resolution of which will give the sine of the arc EO, or of the arc AL, the value of the root ; and from this being found, the remainder will be easily given, which were to be found.

Scholium.

By the same means, other problems are able to be solved, even if AL, EO are not equal, if the ratio of their sines is given.

[**103**]
Prop. 67. Problema.

Datis duobus triangulis sphaericis, & ab uno utriusque angulo, in latus oppositum, ducto arcu in utroque triangulo aequalis : e data ratione sinus, segmenti lateris unius trianguli ; ad sinum, segmenti lateris alterius trianguli ; utriusque lateris segmenta invenire.

Sint duo triangula sphaerica, ABC, EDM, data habentia, omnia latera, & onmes angulos ; & ab uno utriusq, angulo nempe A & E demittantur, in utriusque latus oppositum, nimirum BC, DM, arcus aequales AL, EO: sitque data ratio, sinus arcus DO, ad sinuum arcus BL. Quaeritur uterque arcus, BL & DO. Ponatur sinus arcus AL, seu sinus arcus EO, (sunt enim ex hypothesi aequales) 1⅟ : deinde, eodem modo, quo in superiore problemate, inveniatur & sinus arcus DO, & sinus arcus BL, in numeris cossicis : cumque detur eorum ratio, ex hypothesi ; Dabitur hinc aequatio ; quae resoluta, dabit sinum arcus EO, seu arcus AL, valorem radicis ; & eo reperto, facile dabuntur reliqua, quae invenienda erant.

Scholium.

Iisdem mediis, possunt resolvi utraq; haec problemata ; etiamsi AL, EO, non sint aequales, si detur ratio eorum sinuum.

Prop. 68. Problem.

From two given altitudes, the given azimuths of the phenomenon and the altitude of the pole, to show clearly the parallaxes of each altitude.

Let PZR be the arc of a meridian, P the pole, and Z the zenith. The true parallel of the phenomenon shall be AIL. R shall be the first one of the apparent positions of the phenomenon, in a meridian of which the altitude shall be known (and therefore its distance from the vertical, ZR, shall be known). Another of the apparent positions shall be at N, the distance of which from the vertical ZN and the angle NZP of the azimuth circle to the meridian are given. But the true positions shall be in the common intersections of the given azimuths with the true parallel, surely A and L. Since indeed in the spherical triangle NZP, the sides ZN and ZP are given, and the angle NZP, then the whole triangle NZP is given [*i. e.* from the two arcs and the included angle]. Therefore the spherical triangle NZP, and the arc of the great circle PR are given. The arc PL is drawn from the angle P to the side ZN. The ratio: sine of the segment LN to sine of the arc RA is given; for the difference of the arcs PR and PL, as PL is equal to PA (by hypothesis). (Indeed it is the same ratio which is between the sines of the given arcs ZR,
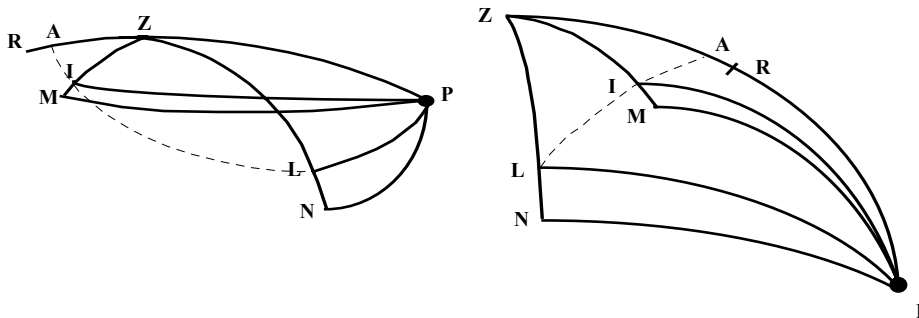
ZN). Therefore from Prop. 66, the arcs LN and AR shall be give, the parallaxes surely of the altitudes sought.

[**105**]

In the second case, both the apparent positions M and N shall be beyond the meridian ; but the true position (as was stated hitherto) will be in the common intersection of the given azimuth with the true parallel, needless to say, in I and L. And since the arcs ZM, ZN, ZP, and the angles MZP, NZP are known, the two triangles ZPM, ZPN, are given. The equal arcs PI, PL are drawn to the sides of which, surely ZM, ZN from the opposite angles; and the ratio of the sine of the arc IM, to the sine of the arc LN is given : (for it is the same, which the sine of the distance appearing from the vertex ZN) therefore both the arcs IM and LN are given, the parallaxes sought, from Prop. 67 ; which were to be found.

Prop. 68. Problema.

Ex Datis duobus altitudinibus ; cum Azimuthis Phaenomeni ; & altitudine Poli ; utriusque altitudinis parallaxes, enucleare.

Sit PZR, arcus meridiani, P polus, Z zenith ; verus Phaenomeni parallelus, AIL. Sitque primo unus apparentium locorum phaenomeni R, in meridiano, cujus altitudo, (& igitur ipsius a vertice distantia, ZR) sit cognita ; alter apparentium locorum in N, cujus distantia a vertice ZN, & angulus circuli azimuth, cum meridiano NZP, sint data ; loca autem vera, erunt in communi intersectione , azimuthorum datorum, cum parallelo vero, nempe A, & L. Quoniam igitur in triangulo sphaerico, NZP, dantur latera ZN, ZP, & angulus NZP; datur & totum triangulum NZP: Datur igitur triangulum sphaericum NZP, & arcus circuli maximi PR ; duciturque arcus PL, ab angulo P, in latus ZN; & datur ratio, sinus segmenti LN, ad sinum arcus RA, differentiae arcuum PR, & PL, seu illi aequalis (ex hypothesi) PA ; (est enim eadem ratio quae est inter sinus arcuum datorum ZR, ZN) dabitur igitur arcus LN, & arcus AR, parallaxes nempe altitudinum quaesitae. {66. Hujus. }

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Secundo, sint ambo apparentia loca, M, & N extra meridianum; loca autem vera (ut hactenus dictum) erunt in communi intersectione azimuthorum datorum cum parallelo vero ; nimirum in I, & L. Et quoniam cognita sunt ZM, ZN, ZP, MZP, NZP; dabuntur duo triangula sphaerica ZPM, ZPN, in quorum latera nempe ZM, ZN, ab angulis oppositis ducuntur arcus aequales PI, PL: daturq; ratio sinus arcus IM, ad sinum arcus LN: (eadem enim est, quae sinus distantiae apperentis a vertice ZN,) datur ideo & arcus IM, & arcus LN, parallaxes quaesitae { 67. Hujus.}; quae inveniendae erant.

Prop. 69. Theorem.

The ratio of the sine of the parallax of one phenomenon to the sine of the parallax of another phenomenon is composed from the reciprocal proportion of the distance from the centre of the earth, and from the direct proportion of the sine of the apparent distance from the vertical.

Let A be the centre of the earth, B the position of observation on the surface of the earth, of which the zenith is D. O and M shall be two phenomena. I say the ratio of the sine of the parallax BOA, to the sine of the parallax BMA, to be composed from the ratio AM to AO; and from the ratio, the sine of the angle DBO, to the sine of the angle DBM. BO is produced in N, and AN is equal to AM itself. It is therefore clear that the phenomenon M arising at N has the parallax BNA. And as AN, that is AM, shall be to AO: so the sine of the parallax BOA shall be to the sine of the parallax BNA. And, as the sine of the angle DBO to the sine of the angle DBM, thus the sine of the parallax BNA, to the sine of the parallax BMA [applying the sine rule to triangles BAM & BAN, noting that AM = AN]. If then there shall be several quantities, namely the sine of the angle BOS, the sine of the angle BNA, and the sine of the angle BMA ; the ratio of the first, surely the sine of the parallax BOA, to the last, surely the sine of the parallax BMA, is composed from the middle ratio;
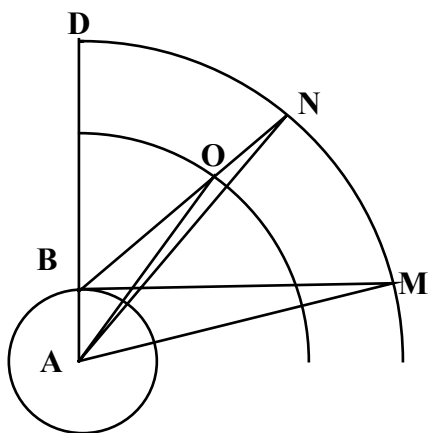
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surely the sine of the angle BOA, to the sine of the angle BMA, or AN to AO ; and the sine of the angle BNA, to the sine of the angle BMA, or the sine of the angle DBO, to the sine of the angle DBM ; which was to be shown[1].

Corollary.

From the demonstration of the theorem it is readily deduced that the sine, either of the horizontal parallax or of the apparent radius is in the inverse proportion to the distance of the

phenomenon from the centre of the earth ; and therefore the sines of the parallax are directly proportional to the sines of the apparent diameters[2].

Notes on Prop. 69.

[1] The relevant triangles are (1) ABO ; (2) BAN ; (3) BAM. From these we obtain in turn:
sin BOA/ sin DBO = AB/ AO ; sin DBO/ sin ANB = AN/ AB ;
sin BMA/ sin DBM = AB/ AM. From these it follows that
sin BOA/ sin BMA = (AN/ AO) . sin DBO/ sin DBM = (AN/ AO) . sin BNA/ sin AMB.

[2] From triangle BAM, sin BMA/ AB = sin DBM/ AM = sin BAM/ BM. Now, AB is the radius of the earth R_E; the angle ABM or θ corresponds to the apparent radius, as we are looking down on the earth from a pole; BMA is the horizontal parallax α, and BM is the apparent radius R_A. Hence:
sin α / R_E = sin θ / AM = sin BAM/ BM, giving sin α = (R_E sin θ)/AM, from which the results follow.

Prop. 69. Theorema.

Ratio sinus parallaxeos unius Phaenomeni, ad sinum parallaxeos alterius Phaenomeni, est composita, ex reciproca proportione, distantiarum a centro terrae, & directa proportiione sinuum, distantiarum apparentium a vertice.

Sit centrum terrae A, locus observatoris in terrae superficie B, cujus zenith D : Sintque duo phaenomena, O & M. Dico rationem sinus parallaxeos BOA, ad sinum parallaxeos BMA; esse compositam, ex ratione AM, ad AO ; & ex ratione, sinus anguli DBO, ad sinum anguli DBM. Producatur BO, in N; sitque AN, aequalis ipsi AM: Manifestum est igitur phaenomenon M, in N existens habere parallaxem BNA : eritque , ut AN, hoc est AM, ad AO, ita sinus parallaxeos BOA, ad sinum parallaxeos BNA ; &, ut sinus anguli DBO, ad sinum anguli DBM, ita sinus parallaxeos BNA, ad sinum parallaxeos BMA. Si igitur fuerint quotcunq; quantitates, nempe sinus anguli BOA, sinus anguli BNA, & sinus anguli BMA ; ratio primae, nempe sinus parallaxeos BOA ; ad ultimam, nempe sinum parallaxeos BMA ; componitur ex rationibus mediarum ; nempe , sinus anguli

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BOA, ad sinum anguli BMA, vel AN ad AO ; & sinus anguli BNA, ad sinum anguli BMA vel sinus anguli DBO, ad sinum anguli DBM ; quod demonstrandum erat.

Corollarium.

Ex Theorematis demonstratione facile deducitur, sinus, sive parallexium horizontalium, sive semidiametrorum apparentium, esse in reciproca proportione distantiarum phaenomeni a centro terrae ; & igitur sinus parallaxium, sunt directe proportionales sinubus semidiametrorum apparentium.

Prop. 70. Theorem.

From two given longitudes and latitudes of the motion of a given phenomenon in a great circle, to find the point of intersection and the angle of that great circle with the ecliptic.

ACB shall be the ecliptic of which F is the pole and A the first [principal
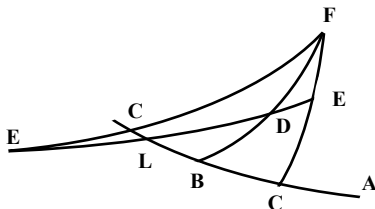
star] of Aries, from which two longitudes of the phenomenon AB, AC may be considered; and BD, CE shall be latitudes, and D, E shall be the positions of the phenomenon. Then the great circle is produced in which the phenomenon is considered to move, and L shall be

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the point of intersection with the ecliptic. In the spherical triangle DEF, the sides DF, FE are given, together with the angle of interception DFE, and therefore the angle DEF is given. Then in the spherical triangle ECL, with the right angle at C, the side EC is given, and the angle LEC, and hence the side CL will be found, the distance of the point of intersection from C, and the angle ELC, which it was required to find.

Prop. 70. Theorema.

Ex datis, duabus longtudinibus, & latitudinius Phaenomeni, in circulo maximo moventis ; punctum intersectionis ; & angulum, illius circuli maximi cum ecliptica, invenire.

Sit ACB ecliptica cuius polus F, A principium arietis ; a quo numerentur duae phaenomeni longitudines, AB, AC ; sintque ; latitudines BD, CE, & loca phaenomeni D, E. Deinde producatur circulus maximus, in quo moveri supponitur phaenomenon;

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sitque punctum intersectionis cum ecliptica L. In triangulo sphaerico DEF, dantur latera DF, FE, una cum angulo intercepto DFE; & igitur datur angulus DEF: Deinde in triangulo sphaerico ECL, rectangulo ad C, datur latus EC, & angulus LEC, & proinde dabitur latus GL, distantia puncti intersectionis a C; & angulus ELC ; quae invenire oportuit.

Prop. 71. Problem.

From three given longitudes and latitudes of a phenomenon, from the movements in a small circle ; to find the distance of the small circle from its pole, and from which to find the position of the pole, according to the longitude and latitude.

Let ACBL be the ecliptic, the pole of which is F; A shall be the first star of Aries, from which the three longitudes of the phenomenon AC, AB, AL are measured ; and the altitudes shall be CD, BN, LM ; & the positions of the phenomenon D, N, M. And because of course the complements of the latitudes DF, NF, MF are given, with the angles of interception DFN, MFN from the differences of the longitudes. Both the arcs DN, MN and the angles DNF, MNE, NMF are given ; therefore the angle MND will be given. Then two arcs EO, IO can be raised, perpendicular to the arcs DN, NM; and cutting these in two, at E and I: and the point O shall be the common intersection of these; which by necessity will be the pole of the minor circle in which the phenomenon is moving. The arc IE is drawn, and in the triangle INE, the sides IN, NE are given, and the common angle INE; and hence the angles

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NEI, NIE and the side IE are given. Then in the triangle OIE the angles OEI, OIE, surely from the complements of the angles EIN, IEN, and from the side IE, the side OI will be given. Then in the spherical triangle MIO, from the given right angle at I, with sides IO, IM ; the side OM will not be forgotten , the distance of the minor circle from its own pole. The angle OMI taken from the angle FMI leaves the given angle FMO. Therefore in
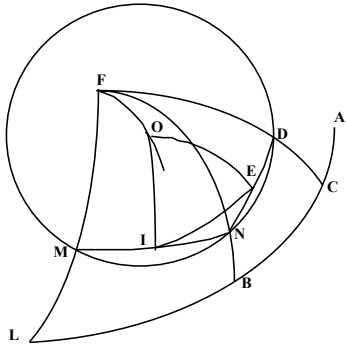
triangle FMO, from the given sides FM, MO, with the common angle FMO, the side FO will be given, the complement of the latitude of the pole of the minor circle, or its distance from the pole of the ecliptic; and the angle MFO will be given, the difference of this longitude from the longitude of the phenomenon M. Which were to be found.

Prop. 71. Problema.

Ex datis tribus longitudinibus, & latitudinius Phaenomeni, in circulo minore moventis ; distantiam circuli minoris a suo polo ; & poli locum, quo ad longitudinem, & latitudinem, invenire.

Sit ACBL, ecliptica; cujus polus F : A principium arietis, a quo innumerentur tres phaenomeni longitudines, AC, AB, AL ; sintq; atitudines, CD, BN, LM ; & loca phaenomeni D, N, M. & quoniam, dantur, DF, NF, MF, complementa nimirum latitudinum, cum angulis interceptis, DFN, MFN, differentiis longitudinum; dabuntur & arcus DN, MN ; & anguli DNF, MNE, NMF ; dabitur igitur angulus MND. Deinde*,* erigantur duo arcus EO, IO, perpendiculares arcubus DN, NM ; & eos bifariam secantes, in E, & I: sitque eorum communis intersectio, punctum O; quod necessario erit polus circuli minoris, in quo movetur Phaenomenon ; ducanturq; arcus IE ; & in triangulo sphaerico INE, dantur latera IN, NE, & angulus interceptus INE ; ac proinde, dabuntur anguli NEI, NIE, & latus IE. Tunc in triangulo OIE; e datis , angulis, angulis OEI, OIE, complementis nimium angulorum

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EIN, IEN, & latere IE, dabitur latus OI. Deinde in triangulo sphaerico MIO, rectangulo ad I, e datis , IO, IM, lateribus ; non ignorabitur latus OM, distantia circuli minoris, a suo polo; & angulus OMI, qui ablatus ab angulo FMI relinquit angulum FMO datum. In triangulo igitur FMO, e datis, lateribus FM, MO, cum angulo intercepto FMO ; dabitur latus FO, complemenum latitudinis poli circuli minoris, seu distantia ejus a polo eclipticae ; & angulus MFO, differentia illius longitudinis a longitudine Phaenomeni M. Quae invenienda erant.

Prop. 72. Problem.
To observe precisely the apparent diameter of the sun, moon, or any star you please.

The image of that [source] may be projected from one lens or mirror into another in order that it may result in a sensible enough size, and the rays of a star may be strongly shining, representing the image of that in a white plane, as we have taught in *Prop. 52* of this work, and from its notes. Then the diameter of the image, which is measured most precisely, from which was given projected with the help of some lenses or mirrors, and from the given distances themselves of the lenses, mirrors, and the image in turn, the angle of vision is found, of the visible phenomenon itself (except for stars) from its own vertex of incidence, this is the diameter of the star appearing by *Prop 53* of this work. But if the image shall not appear clear and bright enough; it may be made clear in some given ratio, by *Prop. 54* of this work; also in the observation of more obscure locations, a more distinct image is seen there; and with a more enlarged image (with everything else), there the observation shall be more exact. Also eclipses of all kinds are observed with great precision by the same method.
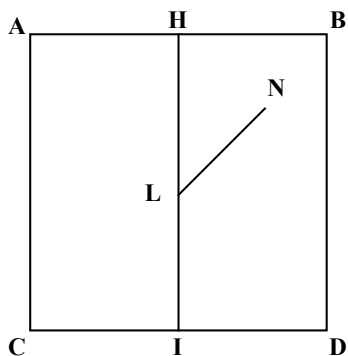
Prop. 72. Problema.

Diametrum apparentem, solis, lunae, vel stellae cujuslibet, exactissime observare.

Projiciatur illius imago, ab una lente, vel speculo, in aliam, ut tandem evadat in magnitudinem satis sensibilem ; & in plano albo, radios sideris fortiter revibrante, repraesenterur ejus imago, ut in *Prop. 52* hujus, & ejus scholiis, docuimus. Deinde mensuretur quam exactissime imaginis diameter ; qua data, ope quotcunque speculorum, vel lentium, projecta ; datisque, speculorum, vel lentium, & imaginum, a se invicem distantiis ; inveniatur angulus visorius, visibilis (nempe sideris) ex vertice suae incidentiae, hoc est diameter apparens sideris per *Prop 53* hujus. Si autem imago, non sit satis clara, & lucida ; illustretur, in quacunque ratione data, per *Prop 54* hujus ; quo etiam obscurior fuerit locus, in quo sit observatio, eo distinctius videtur imago ; & quo magis amplificatur imago (caeteris paribus) eo exactior sit observatio. Eodem etiam modo observantur quam exactissime omnium generum eclipses.

Prop. 73. Problem.

To observe the separation of two stars close to each other, and the angle of the great circle drawn through their centres from any desired vertical drawn through the centre of one star.

The axis of an icoscope is to be made in the plane of that vertical, of which the angle is sought for the great circle drawn through the centres of the stars, so that it will be able to be moved freely to and fro, and always keeping the axis precisely in the vertical plane. ABCD shall be the image plane of this icoscope, in which the right line HI is noted, the common intersection of the image plane and of the aforementioned [great] vertical circle, in which the axis of the iconoscope produced is incident in the point L. Then the axis is turned most carefully until the star touches the vertical circle, the centre of which should be noted in the vertical circle ; and the icoscope is moved up and down until the centre of the image of that star can be seen in the point L. And being close to the same, the centre of the other star is noted in the image plane in the point N. Therefore with the image LN given, projected with the help of some lenses or mirrors; and from the given image of the lenses or mirrors, the visible angle of vision is found from the separations between themselves, from the vertex of its incidence, which will be the separation of the stars, the images of which are L, N. Then the angle HLN is measured most precisely, which will be the angle of the vertical circle (in which the axis of the icoscope is present) with the great circle, in which the centres of the stars are present. For the line HI is the common intersection of the vertical with the plane of the image, and the line LN is the common intersection of the great circle drawn through the centres of the stars with the image plane.

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And with the common intersection of the vertical circle and the great circle through the centres of the stars; indeed the axis of the iconoscope is at right angles to the plane of the image; the angle HLN shall be equal to the angle of incidence of the plane of the vertical circle and of the great circle through the centres of the stars: this is the spherical angle understood from these that was required to be found.

Comment on Prop. 73.

Although the construction details of the icoscope are not provided, we consider it to consist of a telescope with a lens or lenses that projects the final image on to a screen: by means of which measurements can be made in lieu of cross-wires. We regard the image plane ABCD as being tangential to the celestial sphere at a point on a great circle, such as the point L. The azimuth and altitude angles of L and N are measured by the associated telescope, hence LN lies on a great circle.

Prop. 73. Problema.

Distantiam , duorum siderum, sibi invicem propinquorum , & angulum circuli maximi, per eorum centra ducti, cum verticali quolibet, per centrum unius siderum ducto, observare.

Fiatur icoscopii axis, in plano illius verticalis, cujus angulus cum circulo maximo per centra siderum ducto quaeritur ; ita ut sursum , & deorsum, libere moveri poterit ; semper tenens axem, quam exactissime, in plano verticalis. Sitque hujus icoscopii planum imaginis ABCD, in quo noterur recta HI, communis intersectio, plani imaginis, & circuli verticalis praedicti : in quam incidat, axis icoscopii productus, in puncto L. Deinde diligentissime attendatur, donec sidus cujus centrum in circulo verticali notari debeat, circulum verticalem attingat : & moveatur icoscopium sursum & deorsum, donec cernatur centrum imaginis illius sideris, in puncto L. Et eodem instante, notetur in plano imaginis, centrum alterius sideris, in puncto N, Data igitur imagine LN, ope quotcunque lentium, vel speculorum , projecta ; & datis, lentium, vel speculoram, & imaginum, a se invicem distantiis; inveniatur angulus visorius visibilis, ex vertice suae incidentiae; qui erit distantia siderum, quorum imagines sunt L, N, Deinde mensuretur quam exactissime angulus HLN, qui erit angulus circuli verticalis, (in quo existit axis icoscopii) cum circulo maximo, in quo existunt siderum centra: Est enim recta HI, communis intersectio verticalis cum plano imaginis, & recta LN communis intersectio circuli maximi per centra siderum ducti, cum plano imaginis :

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cumque communis intersectio circuli verticalis, & circuli maximi per centra siderum ; nempe icoscopii axis, sit ad planum imaginis rectus ; erit angulus HLN, aequalis angulo inclinationis planorum, circuli verticalis, & circuli maximi per centra siderum ; hoc est angulo sphaerico, ab illis comprehenso. Quem invenire oportuit.

Prop. 74. Problem.

To investigate exactly the parallax of any desired planet .

SIB shall be the common azimuth of two locations, of which the vertices are S, B; & the common azimuth SIB is to be observed from locations on both sides, a joint occurrence of the body of the planet, of which the true location is I, from a fixed star of
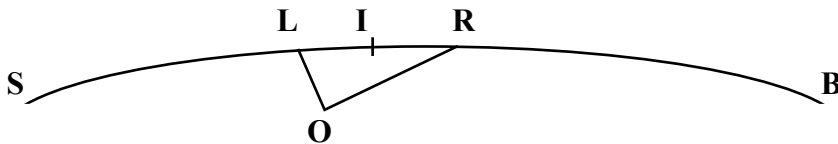
which the position is O. But the apparent position of the planet shall be the point L from the location with vertex B ; and the position of the planet appearing to be at the point R,

from the location with vertex S. Then by *Prop. 73* of this work, from the locations of which the vertices are S, B, the arcs LO, RO are found, and the angles RLO, LRO : the length LR can be found in four ways, the sum of the parallaxes of the planet of which the true position is I. But given the ratio of the sines of the same parallaxes; indeed it is the same as for the ratio: sine arc BL to sine arc SR, and therefore will give the parallaxes LI, IR separately, which had to be found.

Scholium.

The parallaxes of Saturn itself can be found with the help of this problem, only if the exact observations are used.

Prop. 74. Problema.

Parallaxem Planae cujuslibet, exactissime investigare.

Sit SIB azimuth commune duorum locorum, quorum vertices S, B ; & azimuth commune SIB, observetur, ex utroque loco, conjunctio corporalis planetae, cujus verus locus I, cum stella fixa, cujus locus O : Sit autem apparens locus planetae, ex loco, cujus vertex B, punctum L; & apparens locus planetae, ex loco, cujus vertex S, punctum R. Deinde per *Prop. 73* hujus, ex locis quorum vertices S, B, investigentur arcus LO, RO, & angulis RLO, LRO; quatuor modis potest inveniri latus LR, summa parallaxeon planetae cujus verus locus I. Datur autem

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ratio sinuum earundem parallaxium ; eadem enim est cum ratione, sinus arcus BL, ad sinum arcus SR ; dabuntur igitur & parallaxes LI, IR seorsum : quas investigare oportuit.

Scholium.

Ope hujus problematis, poterunt & ipsius Saturni parallaxes inveniri ; si modo obsevationes exactae adhibeantur.

Prop. 75. Lemma.

If there are two series of magnitudes in arithmetical proportion, and if the first [*term*] *of the first series is added to the first term of the second series : and the second term of the first, to the second term of the second; and thus henceforth ; or if subtracted from the same ; the sum or difference will be also a series of arithmetical proportionals.*

Let there be the same difference between the magnitudes A and BC which there is between the magnitudes D and EF, surely H ; and let there be the same difference between the magnitudes I and KL which there is between the magnitudes M and NO, surely R. In the first case [involving subtraction], I say that A - I is deficient from BC - KL or that it is exceeded by the same difference that D - M is deficient or is exceeded by EF - NO. For PC = H is taken from BC, & TL = R is taken from KL ; and BP = A , KT = I ; therefore there will be no difference between A - I & BP - KT : Therefore the total difference between A - I, & BC - KL is the same as that between PC & TL, or H & R. And in the same way the difference is shown between D - M, & EF - NO to be equal to the difference between H & R.

In the second case [involving addition], I say that the difference between A + I & BC + KL, likewise the difference between D + M & EF + NO is equal to H + R: for there is no difference between A + I & BP + KT ; therefore the total difference between A + I & BC + KL

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is PC + TL = H + R ; which it was necessary to show.

Comment on Prop. 75.

We may regard this result as showing the state of algebra at the time:

$$\underline{\quad A \quad} \qquad \underline{B \quad \overset{P}{|} \quad C} \qquad \underline{\quad D \quad} \qquad \underline{E \quad \overset{S}{|} \quad F}$$
$$a \qquad\qquad a+h \qquad\qquad d \qquad\qquad d+h$$

$$\underline{\quad I \quad} \qquad \underline{K \quad \overset{T}{|} \quad L} \qquad \underline{\quad M \quad} \qquad \underline{N \quad \overset{V}{|} \quad O}$$
$$i \qquad\qquad i+r \qquad\qquad m \qquad\qquad m+r$$

Prop. 75. Lemma.

Si fuerint duae series magnitudinium arithmeticae proportionalium, & prima, primae seriei, addatur primae, secundae: & secunda primae, secundae secundae serei; & sic deinceps: vel si ab isdem substrahantur; facta, vel relicta, erunt etiam series arithmeticae proportionalium.

$$\underline{\quad A \quad} \qquad \underline{B \quad \overset{P}{|} \quad C} \qquad \underline{\quad D \quad} \qquad \underline{E \quad \overset{S}{|} \quad F}$$

$$\underline{\quad I \quad} \qquad \underline{K \quad \overset{T}{|} \quad L} \qquad \underline{\quad M \quad} \qquad \underline{N \quad \overset{V}{|} \quad O}$$

Sit eadem differentia , inter magnitudines A, & BC; quae inter magnitudines D, & EF ; nempe H ; & sit eadem differentia inter magnitudines I, & KL ; quae inter magnitudines M, & NO; nimirum R. Dico primo A - I deficere a BC - KL, vel eandem excedere, eadem differentia, qua D - M excedere, vel deficit ab EF - NO. Auferatur a BC, PC = H ; & a KL, TL = R ; eritque BP = A ;& KT = I ; ideo nulla erit differentia inter A - I, & BP - KT : Tota igitur differentia inter A - I, & BC - KL; eadem est, quae inter PC, & TL, seu H, & R. Eodemq; modo ostenditur differentiam inter D - M, & EF - NO esse aequalem differentiae inter H & R.

Dico secundo, differentiam, inter A + I, & BC + KL; item & differentiam inter D + M, & EF + NO esse aequalem H + R: nulla enim est differentia inter A + I & BP + KT ; tota igitur differentia
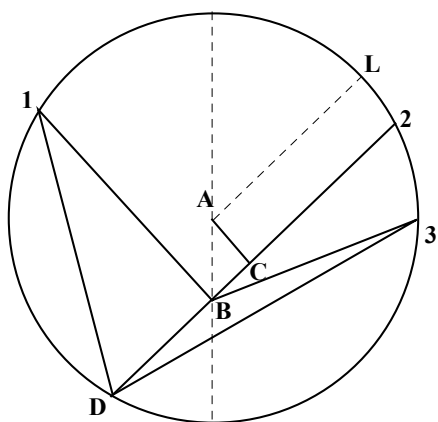
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inter A + I & BC + KL est PC + TL = H + R ; quae demonstrari oportuit.

Prop. 76. Problem.

From the mean motion of a given planet, together with three observations from the centre of its apparent motion; the eccentricity and the position of the aphelion is found, with the supposition that the planet is moving in an eccentric circle.

With centre A, the circle 1 2 3 D is described , and in the same circumference, the following intervals 1 2, 2 3 shall be the arcs of the given observations of the central motion; thus the first [point] 1, the second 2, and the third 3, represent the positions of the planet in the circle. Then from the centre of the apparent motion B, the lines B1, B2, B3 are drawn ; thus the motions to be observed appear as the angles 1B2 & 2B3. From the given arcs 12, 23, surely for the central motions; and from the angles 1B2, 2B3 from the apparent motions, the eccentricity AB can be found, and the position of the aphelion AB. 2B is produced as far as the circumference D; & the line AL is drawn through the centre A parallel to 2D ; & a perpendicular AC is sent from the centre A to the line 2D, which divides the line 2D in two equal parts at C; and AC will be the sine of the arc L2: then the lines 1D and 3D are joined: & in the triangles 1BD, 3BD, the angles to D are

given ; *viz.* half of the mean motions, or of the arcs 12 and 23; & the angles at B, surely the supplements of the observed angles 1B2, 2B3 to the two lines : therefore the angles B1D, B3D are given, and hence the ratio BD to ID; and BD to D3 : therefore the ratio1D to 3D is given : & from the given ratio of the chords 1D, 3D, & the sum of the arcs 1D3 ; (surely of the remaining motion to complete the whole circle) the arcs 1D, 3D may be found, & the lines 1D, 3D, BD, in parts of which the radius AL is 1000000 : & the difference of the arc 1D + 12, or 23 + 3D, & the semicircle, will be equal to twice L2 itself; which will hence give the arc L2 & the sine of this arc AC, with the sine of the complement CD; from which the line DB is taken, BC is left. Finally, from the right angled triangle ABC, with the given sides AC, BC; the eccentricity AB can be found, & the angle ABC, for the location of the aphelion. Which it was necessary to find.

Prop. 76. Problema.

Ex dato planeta motu medio, una cum tribus observationibus ex centro sui motus apparentis ; ejus excentricitatem, & aphelii locatus enucleare: supposito cum un circulo excentrico moveri.

Centro A, describatur circulus 1 2 3 D, & in ejusdem peripheria, secundum observationum intervalla , sint arcus motus medii dati 1 2, 2 3; ita ut 1, locum primum ; 2 , secundum ; 3 , tertium planetae locum, in circulo repraesenter. Deinde, a

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centro motus apparentis B, ducantur rectae B 1, B 2, B 3 ; ira ut anguli 1 B 2, 2 B 3, sint motus apparentes observati ; & ex datis arcubus 12, 23, nimirum motibus mediis; & angulis 1B2, 2B3, motibus apparentibus ; quaeritur excentricitas   AB, & apheli positio AB: Producatur 2B, usque ad peripheriam in D; & ducatur per centrum A, rectae a 2D parallela, AL ; & demittatur ex centro A, rectae 2D parallela, AL; & demittatur ex centro A, in rectam 2D perpendicularis AC, quae dividet rectam 2D bifariam in C; eritque sinus arcus L 2: deinde, jungantur lineae 1 D, 3 D: & in triangulis 1BD, 3BD, dantur anguli ad D ; *viz.* dimidia motuum mediorum, seu arcuum 12, 23; & anguli ad B, nempe residua angulorum observatorum 1B2, 2B3, ad duos rectos : dabuntur igitur, & anguli B1D, B3D ac proinde ratio BD, ad ID; & BD, ad D3 : ergo datur ratio 1D ad 3D: & data ratione chordatum 1D, 3D, & arcuum summa, 1D3 ; (residuum nempe motuum mediorum ad integrum circulum) inveniantur arcus 1D, 3D; & rectae 1D, 3 D, BD, in partibus, quarum radius AL est 1000000 :& differentia arcus 1 D + 1 2, vel 23 + 3D, & semicirculi, aequalis erit duplo ipsius L2, dabitur proinde arcus L2, & ejus sinus AC, cum sinu complementi CD; a quo, recta DB ablata, relinquitur BC: ex datis denique, in triangulo rectangulo ABC, lateribus AC, BC; inveniantur AB excentricitas, & angulus ABC, aphelii positio, ab observationea , Quae invenire oportuit.

Prop. 77. Theorem.

If a planet may be moved along the line of an ellipse equally around one of the focuses of the ellipse, and truly appearing around the other focus with an equal motion to the focus; from the centre of the position of an eccentric circle, of which the radius is equal to the transverse axis of the ellipse. By supposing that the planet moves with the same intermediate motion in the eccentric circle, the motion appearing in the eccentric circle will be the arithmetic mean between the common motion about the centre [which is also the first focus] *and the apparent motion in the ellipse* [about the second focus] *: with the*

centre of the motion always appearing from the same position, truly from either focus of the ellipse.
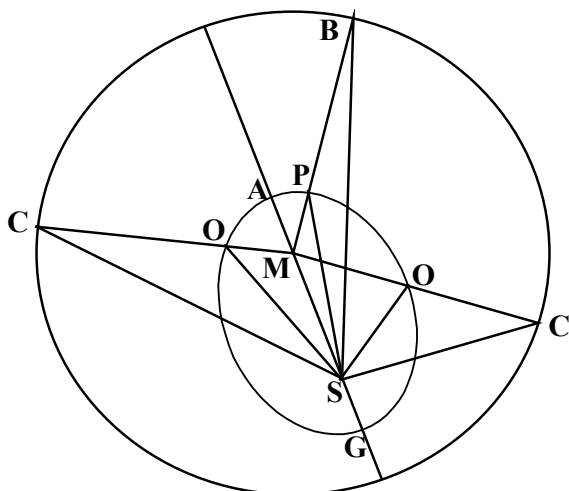
Let the ellipse be APOG, in the circumference of which the planet moves from P to O ; equally around the focus M, and around the focus S. And let

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BCC [the text has ABG] be the circle of which the centre is M, & radius MB, equal to the transverse axis of the ellipse AG; and the planet moves from B to C, with the same intermediate motion, with which it moves from P to O: thus, as MPB and MOC shall be made straight lines. The lines of the true motion shall be drawn so in the circle as in the ellipse: SP, SO, SB, SC. I say that the angles PSO, BSC, BMC, are in arithmetic proportion ; that is, there is the same difference between PSO and BSC, which there is between BSC and BMC. Since indeed AG is equal to MC ; and MO + OS will be equal to AG and MC; and MO taken from both sides; OC = OS; & OCS = OSC; & AMC - OCS = ASC; & ASC - OCS = ASO : therefore the three angles AMC, ASC, ASO, are

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in continued arithmetical proportion ; the common difference of which is the angle OCS. And by the same method it can be demonstrated that the three angles AMB, ASB, ASP, are in continued arithmetical proportion, & have as a common difference the angle PBS. Therefore if these two series in order in arithmetical proportion are themselves added or subtracted in turn; the angles BMC, BSC, PSO emerge in arithmetical proportion ; quod demonstrandum erat.

Corollary.

Hence the method of the Geometrician is clear ; finding the eccentricity, or the separation of the foci, & the position of the aphelion of the ellipse, following the hypothesis CL : V: & in all the writings of the most learned *Dr Seth Ward*. If indeed three observations are given of the ellipse, with the intermediate motions ; the arithmetical means are taken, among the mean motions, and the apparent motions of two observations ; which puts the arithmetical means in the place of the motion appearing in the circle, and with these apparent motions, and with the same intermediate motions, which in the ellipse, the eccentricity of the circle can be found , which will be the same as the separation of the foci of the ellipse ; from the radial position of the eccentricity with the axis of the ellipse, and it will be the position of the aphelion from the line of the mean motion in the eccentricity, with the same position of the aphelion, or of the transverse axis, from the same line of the mean arc in the ellipse. I had decided to lift the hand from the table, but because concerning the inequalities of the planets, *Dr. Seth Ward*

will make public some most acute propositions, it has pleased to make this little tract known with these, and to add certain concerning the same material from our own store.

Prop. 77. Theorema.

Si Planeta, in linea elliptica aequaliter moveatur circa unum ellipseos focorum ; apparenter vero circa alterum ; foco motus aequalis, centro circuli excentrici posito, cujus radius aequalis est transverso axi ellipseos ; & supposito, planetam eodem motu medio moveri in circulo excentrico; erit motus apparens in circulo excentrico, media arithmetica, inter motum medium communem, & motum apparentem, in ellipsi : eodem semper posito, centro motus apparentis, nempe altero ellipseos foco.

  Sit ellipsis APOG, in cujus peripheria, moveatur planeta, a P, ad O ; aequaliter circa focum M ; & apparenter circa focum S. Sitque

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circulus ABG eujus centrum M, & radius MB, aequalis axi transverso ellipseos AG; & moveatur planeta, a B, ad C, eodem motu medio, quo a P, ad O: ita, ut MPB, MOC, fiant lineae rectae : ducanturq; lineae veri motus, tam in circulo, quam in ellipsi SP, SO, SB, SC. Dico angulos PSO, BSC, BMC, esse arithmetice proportionales ; hoc est eandem esse differentiam inter PSO, & BSC, quae est inter BSC, & BMC, Quoniam enim AG, est aequalis MC ; erit & MO + OS = AG = MC; & ablata MO utrinque; erit OC = OS; & OCS = OSC; & AMC - OCS = ASC; & ASC - OCS = ASO : tres igitur anguli AMC, ASC, ASO, sunt

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in continue proportiones arithmeticae; quorum communis differentia est angulus OCS. Eodemque modo demonstrabitur, tres angulos AMB, ASB, ASP, esse continue proportionales arithmeticae, & habere communem differentiam , angulum PBS, Igitur si hae duae series, arithmeticae proportionalium, ordine sibi invicem addantur, vel a se invicem subtrahantur ; emergent anguli BMC, BSC, PSO arithmeticae proportionales ; quod demonstrandum erat.

Corollarium.

    Hinc patet modus Geometricus; inveniendi excentricitatem, seu focorum distantiam, & positionem aphelii elliptici, secundum Hypothesin CL : V: & in omni literatura doctissimi *D. Sethi Wardi*. Si enim dentur tres observationes in ellipsi,cum motibus mediis ; Sumatur media arithmetica, inter motum medium, & motum apparentem duarum observationum ; quae media arithmetica ponatur loco motus apparentis in circulo, & cum hisce motibus apparentibus, & motibus medibus iisdem, qui in ellipsi ; inveniatur circuli excentricitas, quae eadem erit cum distantia focorum ellipseos ; a posito excentrici radio, axe ellipseos ; eritque positio aphelii a linea medii motus in excentrico; eadem cum positione aphelii, seu axeos transversi, ab eadem linea medii arcus in ellipsi. Decreveram manum de tabula tollere : sed quoniam de planetarum inaequalitatibus, acutissimas aliquot propositiones evulgabit *D. Sethus Ward* ; placuit hunc tractatulum illis nobilitare, & quasdam de eadem materia e nostra penu adjungere.

Prop. 78. Problem.

From the given mean motion of a planet, with the position of the aphelion, and the ratio of the axis to the separation of the foci of the ellipse given ; to find the apparent motion of the planet for the eye in agreement with the sun.

Let AP be the transverse axis of the ellipse, A the aphelion, P the perihelion, F the focus of the mean motion, S the sun, or the focus of the apparent motion, Q the locus of the planet, of which the mean motion from the aphelion is AFQ. The true motion ASQ from the aphelion is sought. FQ is produced to R, and QR is equal to QS; & SR is drawn. Therefore, since in the triangle SFR the two sides FS, FR are given, together with the contained angle; we have the angles at S and R ; but FSR - FRS = ASQ, for the angle that it was necessary to find. Because if the distance SQ between the sun and the planet is

A

F

[Q]

S

[R]

P
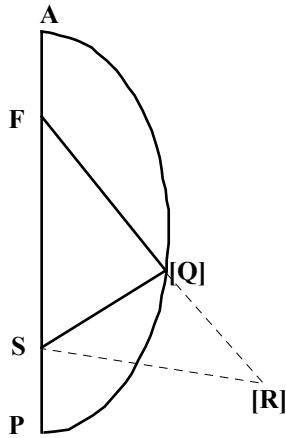
sought ; as in triangle FSQ, all the angles are given, with the side FS ; which will also give the side SQ.

Prop. 78. Problema.

E datis, Planetae motu medio, Aphelii loco, & ratione axeos transveri, ad distantiam focorum ellipseos; planetae motum apparentem invenire pro oculo in sole constituto.

Sit AP axis transversus ellipseos, A aphelium, P perihelium, F focus medii motus, S Sol, sive focus motus apparentis, Q locus planetae, cujus medius motus ab aphelio AFQ. Quaeritur ipsius verus motus ab aphelio ASQ ? Producatur FQ ad R, & sit

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QR = RS; & ducatur SR. Quoniam igitur in triangulo SFR dantur duo latera FS, FR , una cum angulo comprehenso ; habemus angulos ad S & R ; at FSR - FRS = ASR, angulo quem invenire oportuit. Quod si quaeratur SQ distantia inter solem, & planeram ; quoniam in triangulo FSQ,dantur omnes anguli, cum latere FS ;dabitur etiam latus SR.