A Translated Account of Viète's *Ad Angulares Sectiones.*

Ian Bruce

1. Introduction.

François Viète (1540 -1603) was the greatest French mathematician of the 16th century. A glimpse of Viète as a person can be found by dipping into Eves [Howard Eves. *An Introduction to the History of Mathematics,* (Holt, Rinehart and Winston), (1966), p. 224 onwards.], as well as finding (of course) a brief outline of his mathematical achievements. Certainly the work to be considered here, although firmly based on Euclidean geometry, was forward looking at the time of its composition (*c*. 1590 or later) and helped establish the modern way of thinking about some aspects of mathematics: indeed, one could be forgiven for believing parts of it come from a later age, if the sophisticated iterative notions introduced by Viète in geometrical terms are recast analytically, for his work predates some standard results of the 18th century. The subject matter is centred around the polynomial expansions of $\sin n\alpha$ and $\cos n\alpha$ in powers of $\sin\alpha$ and $\cos\alpha$, and associated trigonometrical identities: the aim being a prescription for the production of a table of sines. (Viète had already produced such tables from Pythagorian triplets). The work was presented posthumously for publication by Alexander Anderson in 1614, an admiring associate of Viète, who was a professor of mathematics at the University of Paris at this time, and incidently the great-uncle of James Gregory of telescope fame [Agnes Grainger Stewart, *The Academic Gregories*, (Oliphant, etc), Edinburgh & London, (1901). Chapter 1.]. The closely worded text in crisp Latin runs to some 17 pages, and is available in Viète's *Opera Mathematica* [Georg Olms Verlag, Hildesheim & N.Y., (1970), pp. 286 - 304.], originally produced by Fran van Schooten (Leyden, 1646).

Viète's work in general had an influence on the English mathematicians such as Henry Briggs (1559 -1631), the English table-maker, and Thomas Harriot (1560 -1621). Briggs provided his own proofs of the multiple angle formulae from chord lengths in the initial chapters of his *Trigonometria Britannica* [Henry Briggs & Henry Gellibrand. *Trigonometria Britannica, in Two Books*, (1633). A copy is held by the Rare Books Department at Cambridge University Library.], before Viète's *Ad Angulares …* was published; however, he made use of Theorem IV below in his developments in Ch. 11, where an alternative method of finding chord lengths is presented. In any case, Briggs accomplished the table making activities anticipated by Viète; that the same equations emerge for the lengths of chords of multiple angles by the two quite different methods of generation proposed by Viète and himself must have been reassuring to Briggs; though he makes no mention of this in the *Trigonometria,* which finally got published posthumously in 1633, some 30 years after he had composed his first table of sines (according to Gellibrand). Harriot, on the other hand, in the posthumous *Artis Analyticae Praxis* [Thomas Harriot, *Artis Analyticae Praxis,...* (1631). Available on microfilm from University Libraries originally from Microfilms Inc., Ann Arbour, Michigan.] assembled mainly by his friend and associate Walter Warner, extended Viete's tentative beginnings of algebra by recasting some of his work symbolically, thus making it accessible to a wider audience. The second part of the *Praxis* showed how to set out neatly Viete's arguments for finding the positive roots of polynomials up to the 5th powers. Briggs, to his credit, circumvented this time-consuming procedure, and went on to use the method to be rediscovered by Newton for finding the roots of equations.

At least one translation of the *Ad Angulares …*is available already, by T. Richard Witmer, being part of *The Analytic Art* , a compendium of some of Viète's works. Also, to some extent Newton cut his mathematical teeth on Viète, under the keen eye of Isaac Barrow, and Vol. 1 of D.T. Whiteside's monumental *Mathematical Papers of Isaac Newton* contains some relevant material from Newton's notes: if nothing else, these demonstrate the calibre of the young Newton, who had transcribed some of the theorems of this work of Viète and others to aid his own understanding. Witmer, though an admirable Latin scholar, has not done full justice to the underlying mathematics for the modern reader (in this writer's opinion), by not providing sufficient commentary or background notes: for example, Theorem III appears to be a geometrical statement and proof of de Moivre's Theorem, a remarkable achievement. Occasionally he has failed to discover typographical errors of a serious nature in the text: for there are occasions when the letters in geometrical diagrams are assigned to the wrong points! At other times meaningful ideas (such as the 'ambiguous' or multi-valued nature of the roots of the polynomials) have been ignored. This author's other 'sin' (again, in the present writer's point of view) has been to abandon Viete's archaic way of writing equations, which is surely inappropriate if one wants to reproduce a work which is a true reflection of the time when it was composed. On the other hand, Whiteside has been content appropriately enough to consider only the theorems examined by Newton, and then to place them in a modern form perhaps more abstract than the material warrants.

There cannot, of course, be any *new* mathematics presented here for *us*, the question answered to some extent instead being: *How did geometry progress from the notions of ratio and similarity associated with the geometry of Euclid, and expounded by people such as Gregory of St. Vincent at the time, to the actual measurement of lengths and angles associated with modern trigonometry?* Viète was a path finder in this transformation, who used inductive reasoning extensively: many of the diagrams show lines in some sort of repetition, the hall mark of an iterative process, carrying Euclidean geometry to new heights. Finally, one is left marvelling at Viète's dexterity in the face of the grossly inadequate mathematical resources of the time, in pushing the development so far, and to achieve what can be accomplished now in a mundane manner with a few trigonometrical identities! What the modern reader gets out of this work then, is the insight of just how this transformation came about. It is not an easy business to take a Latin text, and to transcribe and repackage it while retaining the original spirit of the work. Thus, one has to plough through a lot of wordy descriptions of the proofs of the theorems if the original presentation is to be kept intact — while the theorems could be reduced to a brief collection of mathematical formulae at the other extreme. To improve the readability, notes are inserted after or at relevant points in the discussion, in which ratios are considered as quotients of lengths in simple equations. and relevant lengths carried through in the development are highlighted. Finally, results are presented in modern notation involving sine and cosines of multiple angles, etc. The true die-hard will of course have mastered the Latin and perhaps the Greek also in his or her quest for a final understanding of the evolution of mathematical through out the last two and a half millennia.

François Viète
ON ANGULAR SECTIONS

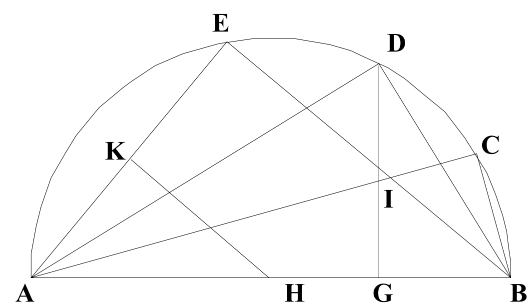
General Theorems
Demonstrated by Alexander Anderson, and here rendered into English by Ian Bruce.
Theorem I.

If there are three right-angled triangles, of which the first acute angle differs from the acute of the second, by the acute of the third, and with the first given largest, then the sides of the third triangle have the following similitude [1].

The hypotenuse: shall be similar to the hypotenuse of the first by the second.

The perpendicular, as the rectangle with the perpendicular of the first and the base of the second, less the rectangle from the perpendicular of the second, and the base of the first.

The base, from the rectangle, with the bases of the first and second, plus the rectangle from the perpendiculars of the same.



If the acute angle is known, by which that specified side called the perpendicular is subtended [EB, etc]. *The base is the remaining line* [AE, etc] *from the right-angle. And indeed the hypotenuse, the side subtending the right-angle.*

Let the triangles be AEB, ADB, ACB, of which the bases shall be AE, AD, AC, with the perpendiculars EB, DB, CB. EB shall cut the base AC in the point I: and the perpendicular DG sent to the line AB.

And as AD is to DB, thus AE is to EI, that is EB less IB. And the rectangle DB by AE, is equal to the rectangle AD by EB less AD by IB, and on being added together AD by IB: DB by AE plus AD by IB, is equal to AD by EB [2]. Also, as AD is to AB so CB is to IB, and the rectangles AB by CB is equal to AD by IB: hence, DB by AE plus AB by BC, are themselves equal to AD by EB [3]. And by taking the common amount (AE by DB), AB by BC is equal to AD by EB less AE by DB, from which AB itself is to arise from the width BC, and the ratio is AB squared to AD by EB less AE by DB, as AB to BC. [4] Q.E.D.

Again, as AG to AD, thus AE to AI, and AD by AE itself is equal to AG by AI; but AB by AC is equal to AG by AI plus AG by IC plus GB by AC [5]: and GB by AC, itself equal to GB by AI plus GB by IC. It is also AG by IC plus GB by IC, equal AB by IC, and therefore AG by AI plus AB by IC plus GB by AI, is equal to AB by AC: but GB by AI, is equal to DB by EB less DB by IB, (for it is GB to DB, as EI or EB less IB, to AI.). And DB by IB is equal to AB by IC (for it is IB to IC, as AB to DB:), hence AG.AI plus AB.IC plus DB.EB less AB.IC, that is AG.AI or AD.AE plus DB.EB, is equal to AB.AC, and this therefore is applied to AB itself, gives rise to the width AC, and the ratio is AB squared to AD by AE plus EB by DB , as AB to AC. Which in the second place had to be shown.

Scholium.

The proposition can be demonstrated in the same way, when there are different hypotenuse for the triangles, as in AKH, ADB, ACB, for on account of the similitude of the triangles, it is as AB squared to AE by AD plus EB by DB, thus AB by AH to

AD by AK plus DB by KH: if indeed it is as AB to AH, thus AE to AK, and EB to KH, and likewise as AB squared to AB by AH, so EB by AD less DB by AE to KH by AD less DB by AK.

Let the perpendicular of the first triangle be 1, the base 2.
Of the second the perpendicular 1, the base 3.
The perpendicular of the third triangle shall be 1, the base 7

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Notes on Theorem 1:

1. The difference between the first and second angles is equal to the third angle: in particular, the angles BAD and CAE are equal. The largest angle EAB is the first, the second DAB, and the third CAB: (CAB = BAE - BAD). The hypotenuse of the third shall be proportional to the hypotenuse of the first and the second, i.e. AB^2 ; The perpendicular of the third shall be proportional to AD.EB - AE.DB (see 4 below); the base shall be proportional to AD.AE + EB.DB (see 5 below).

To reduce confusion, the terms that continue in the argument are printed in bold type.

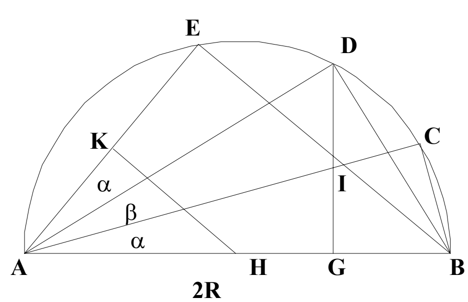
2. AD/DB = AE/EI = AE/(EB - IB), giving DB.AE + AD.IB = **AD.EB**;
3. Also, AD/AB = CB/IB: AB.BC = AD.IB, so DB.AE + AB.CB = **AD.EB**.
4. AB.BC = AD.EB - DB.AE ; **AB^2/(AD.EB - AE.DB) = AB/BC**.
5. AG/AD = AE/AI, or **AG.AI** = AD.AE; but (AB - GB).(AC - IC) = AD.AE, giving **AB.AC** = AD.AE + GB.(AC - IC) + AB.IC = **AG.AI** + GB.AC - GB.IC + AB.IC = **AG.AI** + GB.AC +(AB - GB).IC = **AG.AI + GB.AC + AG.IC**: and **GB.AC** = **GB.AI** + GB.IC. Also **AG.IC** = **AB.IC** - GB.IC, and therefore **AG.AI + AB.IC + GB.AI** = **AB.AC**: but **GB.AI** = **DB.EB** - DB.IB, (for it is GB/DB =EI/AI.). And DB.IB = **AB.IC** (for IB/IC = AB/DB:), hence **AG.AI + *AB.IC* + DB.EB - *AB.IC***: that is AG.AI (i.e. **AD.AE**) + **DB.EB = AB.AC**, and this therefore when applied to AB itself, gives rise to the width AC, and the ratio is **AB^2/(AD.AE + EB.DB)** = AB^2/AB.AC = **AB/AC.**

Scholium.

On account of the similitude of the triangles AKH, ADB, ACB: AB/AH = AE/AK = EB/KH; and from 5, AB/AC = AB.AB/ (AE. AD + EB.DB) = (AB.AH)/(AD.AK + DB.KH); And likewise:
AB^2 /(AB.AH) = (EB.AD - DB.AE)/(KH.AD - DB.AK), as, from 4 above,
AB^2/(EB.AD - DB.AE) = AB.AH/(KH.AD - DB.AK) = AB/BC

Let the perpendicular of the first triangle be 1, the base 2.
Of the second the perpendicular 1, the base 3.
The perpendicular of the third triangle shall be 1, the base 7.

For, in the first triangle, we set AK = 2 and KH = 1; while in the second triangle, we set AD = 3 and DB = 1. Then in the third triangle:
AB/AC = (AB.AH)/(3.2 + 1.1), or 1/AC = AH/7; while
AB/BC = AB.AH/(AD.KH - AK.DB) = AB.AH/(3.1 - 2.1), or 1/BC = AH/1; hence,



$BC/AC = {}^1/_7$.

The bases:

$AC = 2R\cos\alpha$; $AD = 2R\cos(\alpha + \beta)$;
$AE = 2R\cos(2\alpha + \beta)$.

The perpendiculars:

$BC = 2R\sin\alpha$; $BD = 2R\sin(\alpha + \beta)$;
$BE = 2R\sin(2\alpha + \beta)$.
$AB^2/(AD.EB - AE.DB) = AB/BC$ becomes: $(AD.EB - AE.DB)/AB = BC$,
or $\sin\alpha = \sin(2\alpha + \beta).\cos(\alpha + \beta)$
$- \cos(2\alpha + \beta).\sin(\alpha + \beta) = \sin((2\alpha + \beta) - (\alpha + \beta))$.
While $AB^2/(AD.AE + EB.DB) = AB/AC$ becomes:
$(AD.AE + EB.DB)/AB = AC$,
or $\cos\alpha = \cos(2\alpha + \beta).\cos(\alpha + \beta) + \sin(2\alpha + \beta).\sin(\alpha + \beta) = \cos((2\alpha + \beta) - (\alpha + \beta))$.

End of Notes.

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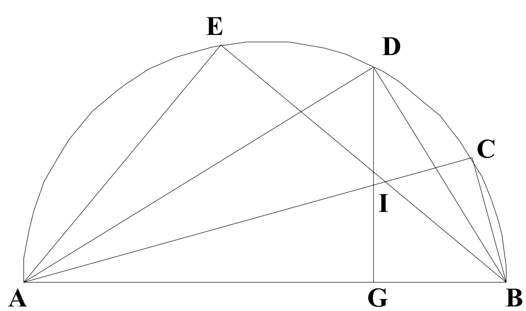
Theorem II.

If there are three right-angled triangles, of which the first acute angle is added to the acute angle of the second, is equal the acute angle of the third, the sides of the third shall receive this similitude.

The [third] hypotenuse is similar to the rectangle of the first by the second hypotenuse.

The [third] perpendicular, is similar to the rectangle by the first perpendicular and the base of the second plus the rectangle by the perpendicular of the second and the base of the first.

The [third] base is similar to the rectangle by the base of the first and the second, less the rectangle by the perpendiculars of the same.



The diagram of the above theorem is repeated, in which AG is to AD, as CB to IB, and the rectangle AD by CB, is equal to AG by IB.

But AB by EB, by itself is equal to AG by IB, AG by IE and GB by BE that is GB by BI, plus GB by IE: and AB by DB, as AI to IE, and the rectangle AB by IE, is equal to AG by IE, plus GB by IE: to which is added GB by IB, that is DB by IC (for it is GB to DB, as IC to IB), AG by IE, plus GB by IE, plus GB by IB, is equal AI by DB, plus IC by DB, that is DB by AC. Hence AG by IB, that is AD by CB, plus DB by AC, is equal to AB by EB; and with all is applied to AB itself, the square of AB is to AD by CB, plus DB by AC, as AB to EB. Q.E.D. [1].

Again, AB to AD, is as AI that is AC minus IC to AE, and the rectangle AB by AE, equal to the rectangle AD by AC, less the rectangle AD by IC: but the rectangle AD by IC, is equal to the rectangle CB by DB, (for it is AD to DB, as CB to IC.) therefore the rectangle AB by AE, is equal to the rectangle AD by AC, less the rectangle CB by DB, and with all is applied to AB itself, is AB squared to AD by AC, less CB by DB, as AB to AE. Q.E.D. And in the same way it is permitted the hypotenuses could have been different, as was considered previously [2].

Let the perpendicular of the first triangle be 1, the base 7.
Of the second the perpendicular 1, the base 3.
The perpendicular of the third triangle shall be 1, the base 2.

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Notes:

1. i.e. $AB^2/(AD.CB + DB.AC) = AB/EB$;
or $EB/AB = (AD/AB).(CB/AB) + (DB/AB).(AC/AB)$ (1).
Giving: $\sin(2\alpha + \beta) = \sin\alpha.\cos(\alpha + \beta) + \cos\alpha.\sin(\alpha + \beta)$
2. i.e. $AB^2/(AD.AC - DB.BC) = AB/AE$;
or $AE/AB = (AD/AB).(AC/AB) - (DB/AB).(BC/AB)$ (2).
Giving: $\cos(2\alpha + \beta) = \cos(\alpha + \beta).\cos\alpha - \sin(\alpha + \beta).\sin\alpha$.

End of Notes.

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Theorem III

If there should be two right-angled triangles, of which the acute angle of the first shall be a sub-multiple of the acute angle of the second.

The sides of the second receive this similitude.

The hypotenuse shall be similar to the agreed power of the first hypotenuse: so it is the power of the conditions; which follows the step of multiple proportions, clearly the square with the double ratio, the cube with the triple, bi-quadratic with the fourth, the square - cube with the fifth, and with this in an infinite progression.

But for the similitude of the sides around the right angle of the congruent hypotenuse, is effected from the base and the perpendicular of the first as with the square root, the powers with equal height, and with the single homogeneous product is distributed in two successive parts, and on one side then the other the first positive, then negative, and of these the first part shall be similar to the base of the second, the perpendicular of the other.

Thus with the duplicate ratio; the hypotenuse of the second shall be similar to the square of the hypotenuse of the first, or otherwise to the sum of the squares around the right angle; the base to the difference [of the same squares]; the perpendicular by double of the aforementioned sides for the rectangle.

In the triple ratio; the hypotenuse of the second shall be similar to the cube of the hypotenuse of the first; the base to the cube of the first base, less the solid [*] of three by the square of the perpendicular of the first triangle and the base of the same; the perpendicular similar to the solid of three by the perpendicular of the first and the square of the base of the same, less the cube of the perpendicular.

[* Vieta likens the terms of his equations to geometrical shapes of the same degree, either real or imaginary: thus cube for 3, plane-plane for 4, etc.]

In the quadruple ratio; the hypotenuse of the second shall be similar to the square-square of the hypotenuse of the first; the base to the square-square of the base of the first, less the plane-plane of six by the square of the perpendicular of the first triangle and the square of the base of the same plus the square-square of the perpendicular; the perpendicular similar to the plane-plane of four by the perpendicular of the first and the cube of the base of the same, less four by the plane-plane cube of the perpendicular of the first and the base of the same.

In the quintuple ratio; the hypotenuse of the second shall be similar to the square-cube of the hypotenuse of the first; the base similar to the square-cube of the base of the first, less the plane-solid of ten by the cube of the perpendicular of the first triangle and the square of the base of the same plus five by the plane-solid of the perpendicular of the first and the square-square of the base of the same; the perpendicular [similar to] the plane-solid of five by square-square of the first perpendicular and the base of the same, less ten by the plane-solid square of the perpendicular of the first and the cube of the base of the same, plus the square-cube of the same base.

Let the triangle be any right-angled triangle whatsoever, of which the hypotenuse Z, the perpendicular B, the base D. Therefore, from the demonstration of the second Theorem, it is for the triangle with double the angle:
(when the same duplicate differs from the half through a half itself.)

As Z squared to D squared – B squared thus Z to the base of double the angle: and from these as Z squared to D by two B, thus Z to the perpendicular of the double angle.
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Note: Here $\beta = 0$, AB = Z = 2R; D = AC = AD = $2R\cos\alpha$, and B = BC = BD = $2R\sin\alpha$;
giving $AB^2/(AD.AC - DB.BC) = AB/AE$; or $Z^2/(D^2 - B^2) = Z/AE$, giving
$AE = (D^2 - B^2)/Z = 2R\cos 2\alpha$; Again, $AB^2/(AD.CB + DB.AC) = AB/EB$, giving
$Z^2/(2B.D) = Z/EB$, leading to $EB = 2B.D/Z = 2R\sin 2\alpha$.
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And once more, as Z cubed to D cubed less D by three B squared, thus Z to the base of the triangle with the triple angle. And from the same thing, Z cubed to D squared by three B less B cubed, as Z to the perpendicular of the same triangle of the triple angle.
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This follows, probably in line with the thinking at the time, by setting $Z^2/(D^2 - B^2) = Z/AD$ and $Z^2/(2B.D) = Z/BD$ with BC = B and AC = D in (1) and (2) of the Notes on Theorem II:
$AE/AB = (AD/AB).(AC/AB) - (DB/AB).(BC/AB) = ((D^2 - B^2)/Z^2).(D/Z) - 2B^2.D/Z^3$
$= (D^3 - 3D.B^2)/Z^3 = (\cos^3\alpha - 3\cos\alpha.\sin^2\alpha) = \cos 3\alpha$; while
$EB/AB = (AD/AB).(CB/AB) + (DB/AB).(AC/AB) = ((D^2 - B^2)/Z^2).\ B/Z +$
$(2B.D/Z^2).(D/Z) = \cos 2\alpha.\sin\alpha + \sin 2\alpha.\cos\alpha = \sin 3\alpha$.
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And Z squared-squared to D squared-squared less D squared by six B squared plus B squared-squared, as Z to the base of the triangle with the quadruple angle. And Z squared-squared to D cubed by four B less B cubed by four D, as Z to the perpendicular of the same triangle of the quadruple angle.
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Thus, in Figure 2, $\beta = 2\alpha$; hence $AD/AB = (D^3 - 3D.B^2)/Z^3 = \cos 3\alpha$;
$DB/AB = ((3B.D^2 - B^3)/Z^3) = \sin 3\alpha$; while $(AC/AB) = D/Z$ and $BC/AB = B/Z$ as before. Hence, $AE/AB = (AD/AB).(AC/AB) - (DB/AB).(BC/AB)$
$= ((D^3 - 3D.B^2)/Z^3)(D/Z) - ((3B.D^2 - B^3)/Z^3)(B/Z) = (D^4 - 6D^2.B^2 + B^4)/Z^4 =$
$\cos 3\alpha.\cos\alpha - 3\sin 3\alpha.\sin\alpha = \cos 4\alpha$;
while $EB/AB = (D^3 - 3D.B^2)/Z^3.(B/Z) + ((3B.D^2 - B^3)/Z^3).(D/Z) = ((4B.D^3 -$
$4D.B^3)/Z^2). = \cos 3\alpha.\sin\alpha + \sin 3\alpha.\cos\alpha = \sin 4\alpha$.
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The same as Z squared-cubed to D squared-cubed less D cubed by ten B squared, plus five D by B squared-squared, thus Z to the base of the triangle with the quintuple angle. And Z squared-cubed to D squared-squared by five B less D squared by ten B cubed, plus B square-cubed thus Z to the perpendicular of the same triangle of the quintuple angle.
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Thus, in Figure 2, $\beta = 4\alpha$; hence $AD/AB = (D^4 - 6D^2.B^2 + B^4)/Z^4 = \cos 4\alpha$;
$DB/AB = ((4B.D^3 - 4D.B^3)/Z^2) = \sin 4\alpha$; while $(AC/AB) = D/Z$ and $BC/AB = B/Z$ as before. Hence, $AE/AB = (AD/AB).(AC/AB) - (DB/AB).(BC/AB)$
$= (D^4 - 6D^2.B^2 + B^4)/Z^4(D/Z) - ((4B.D^3 - 4D.B^3)/Z^2)(B/Z)$
$= (D^5 - 10D^3.B^2 + 5DB^4)/Z^4 = \cos 4\alpha.\cos\alpha - \sin 4\alpha.\sin\alpha = \cos 5\alpha$;

while EB/AB = $(D^4 - 6D^2.B^2 + B^4)/Z^4.(B/Z) +((4B.D^3 - 4D.B^3)/Z^2).(D/Z) = ((5B.D^4 - 10D^2.B^3 + B^5)/Z^2). = \cos4\alpha.\sin\alpha + \sin4\alpha.\cos\alpha = \sin5\alpha$.

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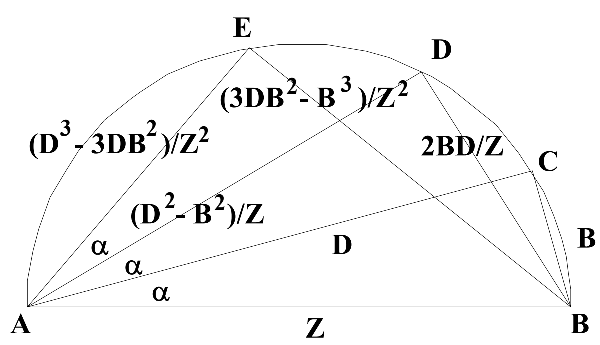
Thus, from the drawing of the hypotenuses, and the sides around the right angle [is multiplied by] the ratio of the similitude already is shown, there will appear from the multiple angles of the triangles the sides in the same manner endlessly, which has been proposed by this method, as is to be observed from the table set out more clearly below.

Right - Angled Triangles.

Powers of ratio	*simple angles*			*multiple angles*		
	Hypotenuse	*Sides around* *Base*	*right angle* *Perpendicular*	*Hypotenuse*	*Base*	*Perpendicular*
	Z.	*D.*	*B.*		*Double*	
	double	*Z sq.*	*D squared* *D by 2B* *B squared*	*Z sq.*	*D sq.* *- B sq.*	*D by 2 B.*
					Triple	
	triple	*Z cub.*	*D cubed* *D sq. by 3B* *D by 3B sq.* *B cubed*	*Z cub.*	*D cub.* *-D by 3B sq.*	*D sq. by 3B* *-B cub.*
					Quadruple	
	quadruple	*Z sq. sq.*	*D sq. sq.* *D cub. by 4B* *D sq. by 6B sq.* *D by 4B cub.* *B sq. sq.*	*Z sq. sq.*	*D sq. sq.* *-D sq. by 6B* *+B sq. sq.*	*D cub. by 4B* *-D by 4B cub.*
					Quintuple	
	quintuple	*Z sq. cub.*	*D sq. cub.* *D sq. sq. by 5B* *D c. by 10B sq.* *D sq. by 10B c.* *D by 5B sq. sq.* *B sq. c.*	*Z sq. cub.*	*D sq. cub.* *-D c. by 10B sq.* *+ D by 5B sq. sq.*	*D sq. sq. by 5B* *-D sq. by 10B c.* *+ B sq. c.*



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Note: the numbers in the right - hand column are a summary of the development. We indicate the double and triple angle cases in the diagram. Now, $(2BD/Z)^2 + ((D^2 - B^2)/Z)^2 = Z^2$; or $(D^2 + B^2)^2 = (Z^2)^2$;

Again, for the triple angle case: the relation is $Z^6 = (D^3 - 3DB^2)^2 + (3DB^2 - B^3)^2$; or $(Z^2)^3 = (D^2 + B^2)^3$, and so on inductively. These results are of course well-known trigonometric identities that may be derived from do Moivre's theorem: the double angle case is the usual formula for Pythagorian Triplets, though the Z, D, and B need to be squared quantities in the binomial, to give $(Z^2)^n = (D^2 + B^2)^n$, for any positive integer n. The right - hand column contains the binomial coefficients, which are selected below, in a slightly different presentation.

End of Note.

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And with this progressing indefinitely, the ratio is given of the sides with the ratio of the angle to the multiple angle, as has been prescribed. Q.E.D.

The right - angled triangle is proposed of which the base is 10, the perpendicular 1, and the simple acute angle of the same is understood.

As regards the triangle of double the angle , the base 99 is established , the perpendicular 20.

As regards the triangle of triple the angle , the base 970 is established , the perpendicular 299.

As regards the triangle of quadruple the angle , the base 9401 is established , the perpendicular 3960.

As regards the triangle of quintuple the angle , the base 90050 is established , the perpendicular 4900.

But with a factor that cannot be made from subtraction, the argument is the multiple angle to be obtuse, and for the same reason, no excess factors [products] *can be assigned to the sides, and the angle subtended is understood* [to be] *outside the multiple* [angles].

THE SAME OTHERWISE.

Geometrical phrases are adapted.

If there were any number of right - angled triangles, and the second of these the acute angle should be double of the first acute, the third three times, the fourth four times, the fifth five times, and for this to be continuing naturally in a progression, so the first proportion is established from the perpendicular of the first triangle, the second from the base of the same, and from this the series is continued.

For the second, the base is to the perpendicular as the third less the first to double the second. [This refers to the right- hand table above, though the '2' is already in the table, and so on for the other higher order terms].

For the third, as the fourth less the second by three to three times the third less the first. [i.e. $D^3 - 3DB^2 : 3D^2B - B^3$; the ratio has been written for the perpendicular to base].

For the fourth, as the fifth less six times the third, plus the first, to four of the fourth, less four of the second. [the latter is written with the signs reversed].

For the fifth, as the sixth, less the fourth by ten, plus the second by five, to the fifth by five, less the third by ten, plus the first [the perpendicular to base ratio].

For the sixth, as the seventh, less the fifth by fifteen, plus the third by fifteen, less the first, to the sixth by six, less the fourth by twenty, plus the second by six. [The signs are inverted for the first ratio, which should be altogether $(-D^6 + 15B^2D^4 - 15B^4D^2 + B^6) : (6BD^5 - 20B^3D^3 + 6B^5D)$ = base: perpendicular].

For the seventh, as the eighth less the sixth by twenty one, plus the fourth by thirty five, less the second by seven, to the seventh by seven, less the fifth by thirty five, plus the third by twenty one, less the first. [i.e. $D^7 - 21B^2D^5 + 35B^4D^3 - 7B^6D : 7BD^6 - 35B^3D^4 + 21B^5D^2 - B^7$ = base : perpendicular, which is correct].

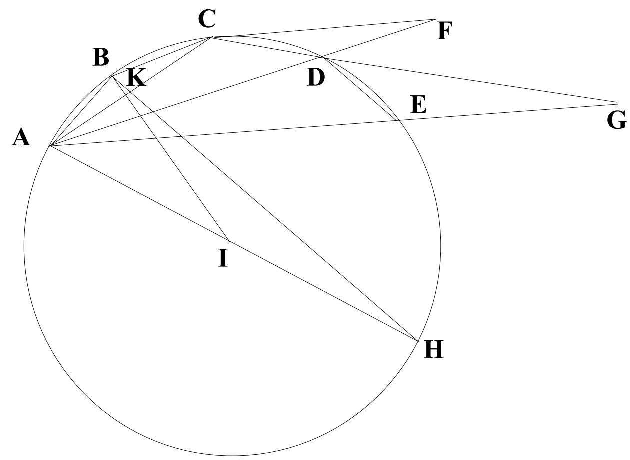
[<u>*Note*</u>: these ratios all come from the expansion $(\cos\alpha + i\sin\alpha)^n$, where $B = \cos\alpha$ and $D = i\sin\alpha$; thus, the base contains the real terms, and the perpendicular the imaginary terms.]

And thus indefinitely, with successive [terms] is distributed in two proportional parts, following this series, on both sides [of the ratio] the first plus* then minus, and of multiples is taken, as the order of the steps ingeniously generating the power, from which these is added on demand. [*This is not always a correct assumption, as we have seen.]

Which indeed are all clear, the above expository table is examined.

Theorem IV.

If from a point on the periphery of a circle, any number of equal segments is taken, and from the same point to each section lines is drawn, [the ratio] isas the smallest [length] to [that] nearest itself; thus the rest is sought from the smallest successively, to the sum of the two themselves nearest on either side.
[i.e. AB: AC :: AC : (AB + AD)]



Let AE be some amount as great as you please from the circumference of the circle, cut into some number of equal parts, from which the lines AB, BC, CD, DE are is subtended, and the lines AC, AD, and AE is drawn out : and the lines CF, DG is drawn, themselves equal to CA, DA.

Therefore it is as AB to AC, thus AC to AF, and AD to AG, from the similitude of the isosceles triangles ABC, ACF, ADG. But the line AF is itself equal to [the sum of] AD, AB [as follows]: for in the triangle with equal legs ACF, the angle CFA is equal to the angle CAF, that is the angle BAC, truly the angle CDA is the double of the angle BAC, (if indeed it stands on the double arc) the angle CDA is therefore the double of the angle CFD, nevertheless is equal to the two angles CFD, FCD. And therefore the angles CFD, FCD are equal, and the sides CD, DF are equal: but the side CD is itself equal to AB, and therefore FD itself is equal to AB; and for the equal line AF to be made from AD, AB. Similarly with the angles of the isosceles triangle ADG, the angles to the base DAG, DGA are equal, and thus the angle DGA equal to the angle CAD, and the external angle DEA of the triangle DGE, equal to the triple of the same angle CAD or DGE: if indeed it stands on the triple [part of] the circumference, therefore such as the angle DGE is the single part, so as the angle EDG is the double [part]. Therefore the triangle EDG is equiangular to the triangle ACD, and the side DE the equal of the side CD; therefore as AB to AC, thus AC to the sum of AB and AD; and therefore AD to the sum of AC and AE, and thus in succession if there were more segments. That which had to be shown. And hence

An Investigation.

In a circle, two arcs to be taken in the ratio of a given multiple, which [ratio] the squares of the lines is subtended by the arcs themselves shall have also.

Let the given circle be above, ABH, of which the diameter shall be AH, the semi-diameter BI, and the line BH is drawn, and let the arcs AB, AC be in the duplicate ratio; AB, AD in the triplicate; AB, AE in the quadruple ratio, etc. Therefore BI to BH will be, as AB to AC, on account of the similitude of the triangles BIH, ABC: hence $\frac{\text{BH by AB}}{\text{BI}}$ is equal to AC itself.

But as AB to $\frac{\text{BH by AB}}{\text{BI}}$, so $\frac{\text{BH by AB}}{\text{BI}}$ to $\frac{\text{BHsq. by AB sq.}}{\text{AB by BI sq.}}$ because itself is multiplied by AB, gives $\frac{\text{BHsq.by AB sq.}-\text{AB sq. by BI sq.}}{\text{AB by BI sq.}}$ that is $\frac{\text{BHsq.by AB.}-\text{BI sq. by AB.}}{\text{BI sq.}}$ equal to AD itself, from the preceding preposition. [Thus, AB/AC = AC/(AB + AD), according to the above theorem, giving

$AC^2 = AB^2 + AB.AD$, leading to $AD = (AC^2 - AB^2)/AB = [\frac{BH^2 AB^2}{BI^2} - AB^2]/AB$, as given; and consequently, $AD/AB = BH^2/BI^2 - 1$, leading to the next result.] Therefore as BI sq. to BH sq. - BI sq. thus AB to AD: or since it is as AH to AB, thus AB to BK, therefore the square BK, is

$\frac{\text{AB sq.sq.}}{\text{AH sq.}}$ & $\frac{\text{AB sq. by AH sq.} - \text{AB sq.sq.}}{\text{AH sq.}}$ is equal to AK^2 :

[i.e. $BK^2 = AB^4/AH^2$; $AK^2 = AB^2 - BK^2 = \frac{AB^2 AH^2 - AB^4}{AH^2}$]

& $\frac{\text{AB sq. by 4AHsq.} - \text{4AB sq.sq.}}{\text{AH sq.}}$ is equal to the square of AC [as $AC = 2AK$]: but this from the preceding Theorem is itself equal to AB sq. + AB by AD. Therefore with is taking away in common AB squared, $\frac{\text{AB sq. by 3AH sq.} - \text{4AB sq.sq.}}{\text{AH sq.}}$ shall be equal to AB by AD

[$AC^2 - AB^2 = \frac{4AB^2.AH^2 - 4AB^2 - AB^2.AH^2}{AH^2} = \frac{3AB^2.AH^2 - 4AB^2}{AH^2} = AB.AD$]: and with this itself is applied to AB, $\frac{\text{AB sq. by 3AH sq.} - \text{4AB sq.sq.}}{\text{AB by AH sq.}}$ is equal to AD, that is $\frac{\text{AB by 3AH sq.} - \text{4AB cubed}}{\text{AH sq.}}$ Therefore as AH sq. to 3AH sq. - 4AB sq., so it will be AB to AD. Again, as AH to BH, thus AB to AK: and $\frac{\text{BH by AB}}{\text{AH.}}$ itself shall equal AK, therefore $\frac{\text{BH by AB}}{\text{AI.}}$ is itself AC [as AH/2 = AI]. But as AB to [AC =] $\frac{\text{BH by AB}}{\text{AI.}}$, so this to $\frac{\text{BH sq. by AB sq.}}{\text{AB by AI sq.}}$, that is $\frac{\text{BH sq. by AB}}{\text{AI sq.}}$: but this less AB itself is equal to $\frac{\text{BH sq. by AB} - \text{AI sq. byAB}}{\text{AI sq.}}$, that is from is shown before itself AD

[For $AC^2/AB = BH^2.AB^2/(AB.AI^2) = AB + AD$; hence, $AD = \frac{BH^2.AB - AI^2 AB}{AI^2}$].

It is too as AB to $\frac{\text{BH by AB}}{\text{AI.}}$, $\frac{\text{BH sq. by AB} - \text{AI sq. by AB}}{\text{AI sq.}}$ to $\frac{\text{HB cub. by AB sq.} - \text{BH by AI sq.by AB sq.}}{\text{AB by AI cub.}}$

[i.e. AB:AC :: AD:AG] that is

[AG =] $\frac{\text{HB cub.} - \text{BH by AI sq.}}{\text{AI cub.}}$ by AB; because itself a multiple of AC, or $\frac{\text{BH by AB}}{\text{AI.}}$ it is

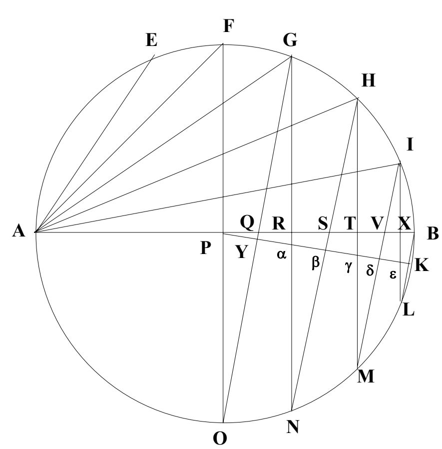
$\frac{\text{HB cub. by AB by AI} - \text{BH by AI cub. by 2AB}}{\text{AI sq.sq}}$ or $\frac{\text{HB cub. by AB} - \text{BH by AI sq. by 2AB}}{\text{AI cub.}}$ itself is equal to AE.

[For $AB/AC = AC/(AD + AB) = AD/(AE + AC)$; hence, $AB.(AE + AC) = AC.AD$,
or $AE = (AD/AB - 1)AC = ((BH^2 - AI^2)/AI^2 - 1).BH.AB/AI = (BH^3 - 2BH.AI^2)AB/AI^3$].

As therefore AI cub. to BH cub. - BH by 2AI sq., so AB to AE. And by the same method is taken for the others, for the ratio of the multiple given. Which had to be established. And this is related to that general Analytical theory of the areas of lunules [i.e. little moons], which are addressed in Book 8 of the Book by Viete, *Variorum de Rebus Mathematicis,* [Concerning various Mathematical Things], and in Chapter 9 [of the same].

THEOREM V.

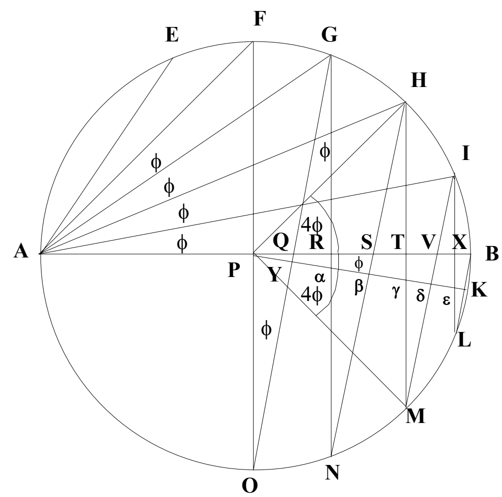
If from the end [B] of the diameter [AB] of a circle is taken any number of equal arcs, and from the other end [A] right lines is taken to the ends of the equal arcs, it is as the semi-diameter (AB/2) to the line already mentioned is drawn with the end nearest the diameter [i.e. AI], thus to whatever intermediate



[i.e. the vertical or slanting chords], to the sum of the two, in the same semi- periphery nearest to itself on either side. But if the arcs taken are equal, to be greater than a semi-circle, thus with the smallest is drawn, to the difference of the two nearest to itself on either side.

Let there be the circle of which the diameter is AB, centre P, and of which the arc from the point B, is cut in any number of equal parts BI, IH, HG, GF, FE, with which too BL, LM, MN, NO are equal, and from the other end of the diameter A shall be drawn the lines to the ends of the equal sections AI, AH, AG, AF, AE, etc, and from the lines is drawn the points BL, IL, IM, HM, HN, GN, GO is connected, etc., which from the first point A drawn are one by one equal, with the same amount indeed to the equal segments subtended, and cut the semi-diameter PB in the points P, Q, R, S, T,V, X: as the smallest BL cuts the line PK from the centre at right angles, and cutting the rest parallel to BL itself in the points Y, β, δ and to the right angle, and as itself GN, HM, IL in the points α, γ, ε.

And since the lines IL, HM, GN, FO connect the more removed points from the end of the diameter B equally on either side, these will be perpendicular to the diameter, and as AB to AI, thus QO to OP, and GQ to GR, hence thus the whole of GO to the components from OP, GR: thus HN to the component from RN, HT: and so of the rest of the intermediates to the composition from the half of the two nearest itself on either side [i.e. the triangles AIB, POQ, GRQ, RNS, ….., IKV, KLB are all similar, and AB/AI = QO/OP = GQ/GR; hence, GO is equal to the sum of the vertical half chords on either side, GQ + QO = (OP + GR) AB/AI = $^1/_2$(OF + NG) AB/AI = (OF + NG) (AB/2)/AI; in the same way, HN = (HT + NR) AB/AI, etc.]: similarly, as AB to AI, so Gα to GY, and αN to Nβ, therefore as AB to BI, so the whole of GN to the composition from the halves of GY, Nβ, of the nearest to itself on either side: and so HM to the composition from the halves nearest itself on either side Hβ, Mδ, and so for the rest [Again we have similar triangles: AIB, GYα, Nαβ, Hβγ, Mδγ,…; AB/AI = Gα/GY = Nα/Nβ = Hγ/Hβ = Mγ/Mδ =…; HM = Hγ + Mγ = (Hβ + Mδ) AB/AI, etc]. But as any you wish of the intermediaries to the half of the two nearest on either side, so the double of the intermediaries to the composition from the same; hence as the semi-diameter to the diameter [i.e. the chord AI] of the nearest, so the double of the intermediate to the composition from the two nearest itself on either side , and as the semi-diameter to the nearest diameter, so the simple intermediary to the composition from the two nearest itself on either side. [i.e. GO/(OF + NG) = (AB/2)/AI, etc.]. Q. E. D.



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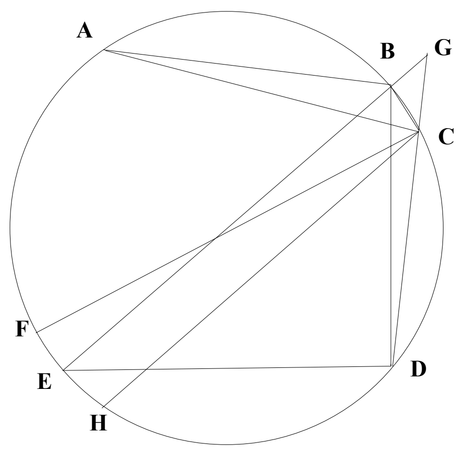
Note: Viete has established geometrically two trigonometric identities for multiple angles, the second familiar to us as 'components' have been taken. Consider AG = AB cos 3φ and AF = AB cos 4φ; then GN =2AG sin 3φ = 2Zsin 6φ;
FO = 2AF sin 4φ = 2Z sin 8φ. Consequently:
GO = (GR + OP)/cos φ = (GN + OF)/2cos φ
= (AB/cos φ)(cos 3φsin 3φ + sin 4φcos 4φ) =
(Z/cos φ).(sin 6φ + sin 8φ) = 2Z sin 7φ.

The result: GO/(OF + NG) = (AB/2)/AI becomes, in this case,
$\frac{\sin 7\varphi}{\sin 6\varphi+\sin 8\varphi} = \frac{1}{\cos\varphi}$, or simply
sin 6φ + sin 8φ = 2sin 7φ cos φ.

Again, HM = Hγ + Mγ = (Hβ + Mδ) AB/AI.
Now, HM = 2Z sin(4φ); Hβ = Z sin(5φ); Mδ = Z sin(3φ), and AB/AI = 1/(cosφ).
Hence, 2Z sin(4φ) = Z (sin(5φ) + Z sin(3φ))/cosφ;
or 2sin4φ cos(φ)/(cosφ) = 2 sin4φ, as required.

End of note.



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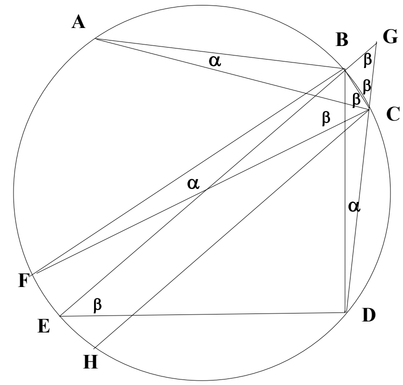
Let the periphery to the following circle of which the diameter [is] FC, be cut into equal parts FA, AB, BD, DH, which are greater than the semi-periphery, and let the smallest of the others is inscribed on either side of the semi-circle be BC or CD: I say, as the semi-diameter to the largest [chord] subtended [as above], so BC to the difference of AC, CD themselves; or CD to the difference of BC, CH themselves.



For [the chord] AC is is subtended, and BG, BC shall be made equal, [note that the angles ABC and DBG are made equal also] (without doubt by producing DC in G), and extending GB in E, and ED is drawn: therefore it will be the angle BCG that is BGC, and the angle BED that is equal BCA. (Indeed they have been taken from the equal arcs BD, BA). The angles BAC, BDC are equal too, and the sides BG, BC is equal from the construction. Hence AC and DG are equal too [For the triangles DBG and ABC are congruent]: so it is as the semi-diameter to the largest subtended [chord, here BD], so BC to CG the difference of AC, CD themselves. For the angle BCG is equal to the angle BED, which makes the diameter too of the largest subtended [chord]. By the same way is shown too to be DC to the difference of HC, CB, as the semi-diameter to the maximum of [these] is drawn.

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Note: For $\alpha + 2\beta = 90^0$ from triangle BFC, hence EB is a diameter. In this case, we have: BC/(AC - CD) = BC/CG = sinβ/sin2β = 1/(2cosβ) = (EB/2)/BD, or AC - CD = BC.(EB/2)/BD as required.

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Theorem VI.

If from the ends of a diameter is taken any number of equal arcs, and from the other end is drawn to the ends of the equal arcs is taken, [the lines] is drawn become the bases of triangles, of which the diameter is the common hypotenuse, and indeed the base nearer to the diameter is thought of as the base of the [simple] uncompounded angle, the succeeding of the double, and from this to be continuing in regular succession: so is constituted a series of lines in continued proportion, of which the first shall be equal to the semi-diameter, the second, the base of the simple angle, that will progress in regular succession for the rest of the bases.

The third by continued proportion, less double the first, is equal to the base of the second angle.

The fourth, less three times the second, for the base of the triple angle.

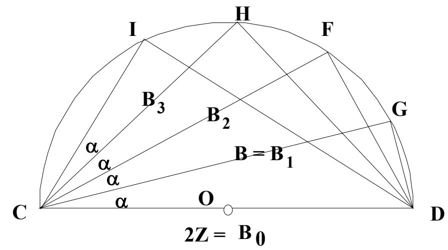
The fifth, less four times the third, plus double the first, for the base of the quadrupled angle.

The sixth, less five times the fourth, plus five times the second, for the base of the quintuple angle.

The seventh, less six times the fifth, plus nine times the third, less double the first, for the base of the seventh angle.

The eighth, less seven times the sixth, plus fourteen times the fourth, less seven times the second, for the base of the seventh angle.

The ninth, less eight times the seventh, plus twenty times the fifth, less sixteen times the third, plus double the first, for the base of the eighth angle.



The tenth, less nine times the eighth, plus twenty seven times the sixth, less thirty times the fourth, plus nine times the second, for the base of the ninth angle.

And so indefinitely, as the odd place of proportionality shall be succeeded by a new positive, for the affirmative by the negative, to the negative by the affirmative: and these proportionalities shall always be alternating, and indeed the multiple in the first section is incremented by one, in the second by the triangular numbers, in the third by the pyramidal numbers, the fourth by the triangulo-triangular numbers, the fifth by the triangulo-pyramidal numbers; indeed not from unity, as with the power is generated, but from two the increment itself is taken.

Let the semicircle of which the periphery is is cut in some number of equal parts, of which indeed the semi-diameter shall be Z, and from the end of the diameter is drawn lines to the point of any section you please, of which lines the first shall be B. And so it shall be from the preceding Theorem, as Z to B, so B to the composition from the diameter and from that itself which follows nearest to B [i.e. B_1]: but this is $\frac{\text{B squared}}{\text{Z}}$ which with the diameter or the semi-diameter twice is taken away, leaves $\frac{\text{B sq. -2Z sq}}{\text{Z}}$ equal to the third [B_1] [chord]; and successively, as Z to B, so $\frac{\text{B sq. -2Z sq}}{\text{Z}}$ to that composed from the second [B] and the fourth [B_3], from which is taken away from the following B, from which isleft

$\frac{\text{B cubed - Z sq. by3B}}{\text{Z sq.}}$ to be equal to the fourth [chord]. And thus if that [found] shall be up to the one before and the next beyond, with Z itself is repeated, or for the step is carried out for the next succeeding power is applicable, and the nearest previous should have been multiplied, that which is left will come [out] to be proportional to that which has been said to be aspired to by this means, indefinitely.

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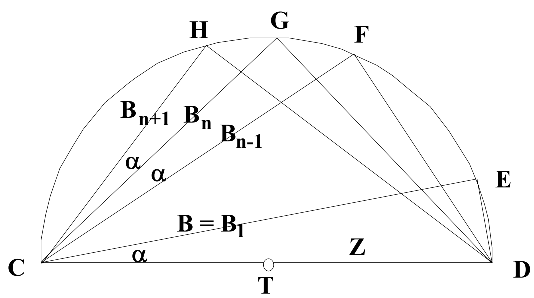
Note: This is a statement of $Z(B_{n-1} + B_{n+1}) = B\,B_n$, an expansion in terms of cosines of the multiple angle, as shown in the following Note: There are a number of ways of establishing this result: thus, for any three consecutive triangles with bases B_{n-1}, B_n, and B_{n+1}, with equal arcs subtending equal angles α at C, for the triangles CDF, CDG, and CDH, we have: $B_{n-1} = 2Z\cos(n-1)\alpha$; $B_n = 2Z\cos n\alpha$; and $B_{n+1} = 2Z\cos(n+1)\alpha$. Now,

$B_{n-1} + B_{n+1} = 2Z\cos(n-1)\alpha + 2Z\cos(n+1)\alpha = 4Z\cos n\alpha.\cos\alpha = 2B_n \cos\alpha$.

Also, $B/2Z = 1/\cos\alpha$: Hence, $B_n/(B_{n-1} + B_{n+1}) = Z/B$, and by extricating B_{n+1}:



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$B_{n+1} = \frac{BB_n - ZB_{n-1}}{Z}$, the results follow in an inductive manner, where $B = B_1$, and $B_0 = 2Z$. For n = 0 gives $B_1 = B$; n = 1 gives $B_2 = (B^2 - 2Z^2)/Z$; n = 2 gives $B_3 = (B^3 - 3BZ^2)/Z^2$, etc. If Z =1, then $B_{n+1} = BB_n - B_{n-1}$], giving $B_2 = B^2 - 2$; $B_3 = B^3 - 3B$, etc.

End of note.

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Thus $\frac{\text{B sq.sq. - Z sq. by 4B sq. thus +2Z sq. sq.}}{\text{Z cu.}}$ is equal to the fifth.

$\frac{\text{B sq. cu. - Z sq. by 5B cu. +Z sq. sq.by5B}}{\text{Z sq.sq.}}$ is equal to the sixth.

$\frac{\text{B sq.sq.sq. - Z sq. by 6B sq.sq. +Z sq. sq.by 9B sq. - 2Z sq.sq.sq.}}{\text{Z sq.cu.}}$ seventh.

$\frac{\text{B sq. sq.cu. - Z sq. by 7B sq.cu. +Z sq. sq.by 14 B cu. - Z sq.sq.sq.b y7B}}{\text{Z cu.cu.}}$ eighth.

$\frac{\text{B sq. cu.cu. - Z sq. by 8B cu.cu. +Z sq. sq.by 20 B sq.sq - Z cu.cu.by 16 B sq. + 2Z sq.cu.cu}}{\text{Z sq.sq.cu.}}$ ninth.

$\frac{\text{B cu. cu.cu. - Z sq. by 9B sq. sq.cu. +Z sq. sq.by 27 B sq.cu. - Z sq.sq.sq.. by 30 B cu. + Z sq.sq.sq.s q. by 9 B}}{\text{Z sq.cu.cu.}}$ tenth.

And so henceforth. Q.E.D.

In note [form], let the semi-diameter be 1, the first base 1 N. It will be

1 Q	– 2						*Double*
1 C	– 3 N						*Treble*
1QQ	– 4 Q	+ 2					*Quadruple*
1QC	– 5 C	+ 5 N				*For the*	*Quintuple*
1CC	– 6 QQ	+ 9 Q	– 2			*base of*	*Sextuple*
1QQC	– 7 QC	+ 14 C	– 7 N			*the*	*Septuple*
1QCC	– 8 CC	+ 20 QQ	– 16 Q	+ 2		*angle*	*Octuple*
1CCC	– 9 QQC	+ 27 QC	– 30 C	+ 9 N			*Nontuple*

And thus by continuing from the square root of two, with that nearest itself to be added on [without signs, diagonally up from the left: thus, 14 = 9 + 5, etc.]*, and the arrangement with the nearest number to these is put together successively, the numbers with the desired multiplicity is created indefinitely.* [As a special case, by setting the equation equal to zero, $B_{n+1} = 2Z\cos(n+1)\alpha = 0$, or $(n+1)\alpha = \pi/2$ or 90^0; *hence,* $\alpha = 90/(n+1)^0$. Thus, n = 1 gives $B_2 = 0$ and $\alpha = 45^0$ with $B = \sqrt{2}$; n = 2 gives $B_3 = 0$ and $\alpha = 30^0$, etc.]

THE DESIRED NUMBER MULTIPLICITY

1st Neg.								
2	*2nd Pos.*							
3								
4	2	*3rdt Neg.*						
5	5							
6	9	2	*4th Pos.*					
7	14	7						
8	20	16	2	*5th Neg.*				
9	27	30	9					
10	35	50	25	2	*6th Pos.*			
11	44	77	55	11				
12	54	112	105	36	2	*7th Neg.*		
13	65	156	182	91	13			
14	77	210	294	196	49	2	*8th Pos.*	
15	90	275	450	318	140	15		
16	104	552	660	672	336	64	2	*9th Neg.*
17	119	442	935	1122	714	204	17	
18	135	546	1287	1782	1386	540	81	2
19	152	665	1729	2717	2508	1254	187	19
20	170	800	2275	4604	4290	2640	825	100
21	189	952	2940	5733	7007	5148	1079	385

THEOREM VII.

If from a point on the circumference of a circle any number of equal parts is taken, and from the same is drawn lines to the ends of the equal arcs is taken, so is constituted a series of lines from continued proportionality, of which the first shall be equal to the smallest is drawn, the second from the smallest following, this is the progress of the rest of the succeeding lines is drawn.

The third in continued proportion, less the first, for the third [line].

The fourth less double the second, for the fourth [line].

The fifth less three times the third, for the fifth [line].

The sixth, less four times the fourth, plus three times the second, for the sixth [line].

The seventh, less the five times the fifth, plus six times the third, less the first, for the seventh [line].

The eighth, less six times the sixth, plus ten times the fourth, less four times the second, for the eighth [line].

The ninth, less seven times the seventh, plus fifteen times the fifth, less ten times the third, plus the first, for the ninth [line].

The tenth, less eight times the eighth, plus twenty one times the sixth, less twenty times the fourth, plus five times the second, for the tenth [line].

And so indefinitely, as the desired new odd place shall arise by means of proportionality, for positive negative, for negative positive; and proportions for these shall always be alternating, and indeed by the multiplicity the first desired [number] is increased by unity, in the second by the triangular numbers [2, 3, 4, 5, …], in the third by the pyramidal numbers [3, 6, 10, ..], in the fourth by the triangulo-triangular numbers [4, 15,..], in the fifth by the triangulo- pyramidal numbers; from unity, with the power is generated leading the increase.

Let the periphery of the circle be cut in any number of equal parts from any taken point, from which to the ends of the equal arcs straight lines is drawn, of which the smallest shall be Z, from this truly the second shall be B. Therefore, from Theorem four, as the first to the second, so the second to the composition of the first and the third: and thus it will be, the third is equal to $\frac{\text{B sq. -Z sq}}{\text{Z}}$. And by the same way, with the method we have used in the preceding, the fourth is found $\frac{\text{B cubed - Z sq. by 2 B}}{\text{Z sq.}}$

$\frac{\text{B sq.sq. - Z sq. by 3B sq. thus +Z sq. sq.}}{\text{Z cu.}}$ for the fifth.

$\frac{\text{B sq. cu. - Z sq. by 4B cu. +Z sq. sq.by3 B}}{\text{Z sq.sq.}}$ for the sixth.

$\frac{\text{B cu.cu.. - Z sq. by 5B sq.sq. +Z sq. sq.by 6B sq. - Z cu.cu.}}{\text{Z sq.cu.}}$ for the seventh.

$\frac{\text{B sq. sq.cu. - Z sq. by 6B sq.cu. + Z sq. sq.by 10 B cu. - Z cu.cu.by4B}}{\text{Z cu.cu.}}$ for the eighth.

$\frac{\text{B sq. cu.cu. - Z sq. by 7B cu.cu. + Z sq. sq.by 15 B sq.sq - Z cu.cu.by 10 B sq. + Z sq.cu.cu}}{\text{Z sq.sq.cu.}}$ for the ninth.

$\frac{\text{B cu. cu.cu. - Z sq. by 8 B sq. sq.cu. + Z sq. sq.by 21 B sq.cu. - Z sq.sq.sq.. by 20 B cu. + Z sq.sq.sq.s q. by 5 B}}{\text{Z sq.cu.cu.}}$ for the tenth.

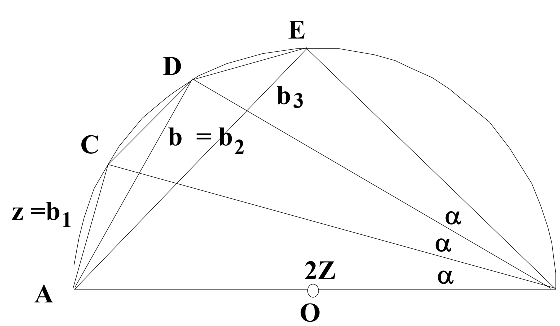
And in the same way, for the ratio and for the rest of the proportionals indefinitely, from that which has been proposed for the manner of the desired [number], for the equal lines is drawn on the circle is produced. Q.E.D.

In brief the smallest shall be 1, the following N. It will be:

1 Q	– 1					*Treble*
1 C	– 2 N					*Quadruple*
1QQ	– 3 Q	+ 1				*Quintuple*
1QC	– 4 C	+ 3 N			*Is equal to the*	*Sextuple*
1CC	– 5 QQ	+ 6 Q	– 1			*Septuple*
1QQC	– 6 QC	+ 10 C	– 4 N			*Octuple*
1QCC	– 7 CC	+ 15 QQ	– 10 Q	+ 1		*Nontuple*
1CCC	– 8 QQC	+ 21 QC	– 20 C	+ 5 N		*Dectuple*

And thus the square root of one with that nearest itself to be added [without signs, diagonally up from the left: thus, 16 = 9 + 7, etc.*, and the arrangement with the nearest number to these is put together successively, the numbers with the desired multiplicity is created indefinitely: which if it pleases to place in front of the eyes in the table, that indeed will have been easily made, just as has been shown by the preceding proposition.*

Note: For the first three consecutive triangles with opposite sides $z = b_1$, $b = b_2$, and b_3, with equal arcs subtending equal angles α at B, for the triangles ACB, BCD, and BDE, we have: $b_1 = 2Z\sin\alpha$; $b_2 = 2Z\sin 2\alpha$; and $b_3 = 2Z\sin 3\alpha$.



Now, according to Theorem four, $b_2/(b_1 + b_3) = b_1/b_2$, and by extricating b_3: $b_3 = \frac{b_2^2 - b_1^2}{b_1}$

$= \frac{4Z^2 \sin^2 2\alpha}{\sin\alpha} - 2Z\sin\alpha = 2Z(3\sin\alpha - 4\sin^3\alpha) = 2Z\sin 3\alpha$, as required. Hence, the general result follows in an inductive manner:

$b_n/(b_{n-1} + b_{n+1}) = b_1/b_2 = z/b$,

for which $\sin 2\alpha.\sin n\alpha = \sin\alpha.[\sin(n+1)\alpha + \sin(n-1)\alpha] = 2\sin\alpha.\cos\alpha.\sin n\alpha$, and from which we have: $b_{n+1} = (b_2/b_1)b_n - b_{n-1}$ as the recurrence relation. By setting $b_1 = 1$, and $b_2 = b$, we then have $b_3 = b^2 - 1$; $b_4 = bb_3 - b_2 = b^3 - 2b$; $b_5 = bb_4 - b_3 = b^4 - 3b^2 + 1$, etc, as in the table. Hence we have an expansion in terms of sines of the multiple angle.

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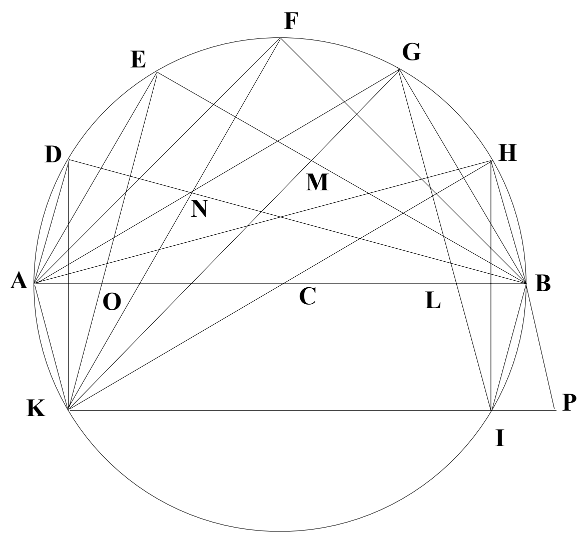
THEOREM VIII.

If from the end of a diameter is taken any number of equal parts, and from the end of the same diameter is drawn lines to the individual points of the sections: as the semi-diameter to the subtended of the part is equal to one [i.e. the ratio of the radius to the shortest chord], is thus to any of those you please remaining from the other end of the diameter is drawn, except the diameter, but near to the diameter because is incident on the section itself to the difference of the two to the remaining end is drawn to the sections themselves nearest on either side [e.g. HC/HB = GB/GM =FB/FN = BE/EO, etc.] : but thus because the diameter itself cuts in equal section, or because it is not cut, to that nearest in the section nearest is cut to the sum of the two from the other end of the diameter, to the nearest on either side of the sections is drawn.



About the diameter AB, with centre C, the circle is described the periphery of which is cut in any number of equal parts AD, DE, EF, RG, GH, HB, BI, and the lines is drawn from the points B, A to the individual sections, and so AK is itself taken equal to AD too, and the lines HI, GI, GK, FK, EK, DK, AK, is drawn, and the semi-diameter shall be CH, and the line GI shall cut the diameter in the point L.

Thus indeed the line HI dividing the angle BIL in two parts, perpendicular to the base, the triangle BIL is isosceles, and similar to the triangle HCB: so the triangle BLI is similar to the triangle GAL [for AG is parallel to KH], and the triangle GAL is isosceles, and the sides GA, AL equal. Thus as CH to HB, so HB or BI to BL, the difference of the sides AG, AB [for LB = AB - AL (or AG)]. Similarly indeed the angles GBE, EKG are equal to the angle HCB, (since for these the double arc on the circumference, here the simple [i.e. single angle] stands at the centre.) and the angles KEB, BGK is equal to the angle CBH, by equal standing on the circumference. The lines GK, EB cut themselves in M, the triangles EKM, GBM is similar to the triangle HCB, and as HC to HB, so GB to GM the difference of themselves GK that is HA (for it is subtended by equal arcs) and EK that is AF [i.e. GK (or HA) - KM (or EK)] (which subtend equal arcs too.)

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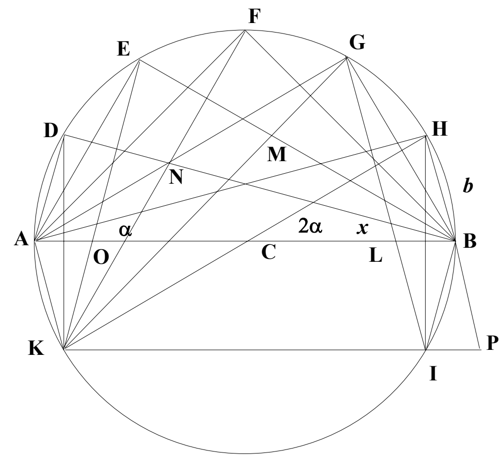
Note: It is probably a good idea to pause here in Anderson's exposition, and to consider the iterative scheme that is produced, which consists of two alternate kinds of terms, successive odd perpendiculars and even bases:



1. From the similar triangles BIL and BCH we have the first perpendicular with ΔBAH, (p_1; α): $b = 2x \sin\alpha$ and BL = $2b\sin\alpha = 4x\sin^2\alpha = b^2/x$;
2. While from the sequence of triangles similar to BCH, ΔAGL, AL = AG = $2x(1 - 2\sin^2\alpha) = (2x^2 - b^2)/x$, is the second base ($b_2$; 2α) for ΔBAG.
3. Also, GL = AG. $b/x = (2x^2 - b^2)b/x^2$, and GI = AF = $(2x^2 - b^2)b/x^2 + b = (3bx^2 - b^3)/x^2$ = FB is the third perpendicular (p_3; 3α); and $p_3 = p_1 + b_2.b/x$.
4. Now, from similar triangles, FN/FB = b/x; hence FN = $b^2(3x^2 - b^2)/x^3$. Also, AE = KD = KF - FN = AG - FN = $(2x^2 - b^2)/x - b^2(3x^2 - b^2)/x^3 = (2x^4 - 4x^2b^2 + b^4)/x^3$; or $b_4 = b_2 - p_3.b/x$, the base of the 4th triangle (b_4; 4α).

Hence, in an inductive manner, we have algorithms for the even bases and odd perpendiculars:
$b_{2n} = b_{2n-2} - b/x.\ p_{2n-1}$; and $p_{2n+1} = p_{2n-1} + b/x.\ b_{2n}$; In addition, these can be written as trigonometric identities: for, $\cos 2n\alpha = \cos 2(n-1)\alpha - 2\sin\alpha.\sin(2n-1)\alpha$, is the standard identity: $\cos 2n\alpha - \cos 2(n-1)\alpha = -2\sin\alpha.\sin(2n-1)\alpha$; while
$\sin(2n+1)\alpha = \sin(2n-1)\alpha + 2\sin\alpha.\cos(2n\alpha)$, is the standard identity
$\sin(2n+1)\alpha - \sin(2n-1)\alpha = 2\sin\alpha.\cos(2n\alpha)$.

End of Note.

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And by the same manner, the lines FK [EK in text], DB cut each other in the point N, the triangles FBN, DKN will have equal legs and similar to the triangle HCB, and as HC to HB, so FB to FN the difference of the subtended FK, DK, that is as the above GA, EA [as the first is subtended by 4 equal segments, the other by 2]. Similarly the lines EK, AB cut each other in the point O, and HC to HB is as EB to EO the difference of the lines EK, AK that is FA, DA. And in the same way to be shown as HC to HB, thus GA to the difference FB, HB; and FA to the difference GB, EB; and EA to the difference FB, DB. And if the triangles HCK, KIP, HBP is drawn, the triangles HCB, HKP is similar: for the angles HCB, HKP, for that in the centre, here on the circumference, are equal, and the angle CHB in common with the other, and therefore that left is equal to that remaining. Therefore as CH to HB is thus HK to HP, but HP itself is double HB, for the angle IBP is equal to the angle HKI, and the angle BPI to the angle BHC, therefore the triangle BIP is isosceles, and the legs BP, BI, that is BP, BH are equal.

[HC/HB = EB/(FA - DA): this amounts to cos3α - cos5α = 2sin4α. sinα; similarly HC/HB = GA/(GB - EB) gives sin3α - sinα = 2sinα. cos2α, and so on with the rest.

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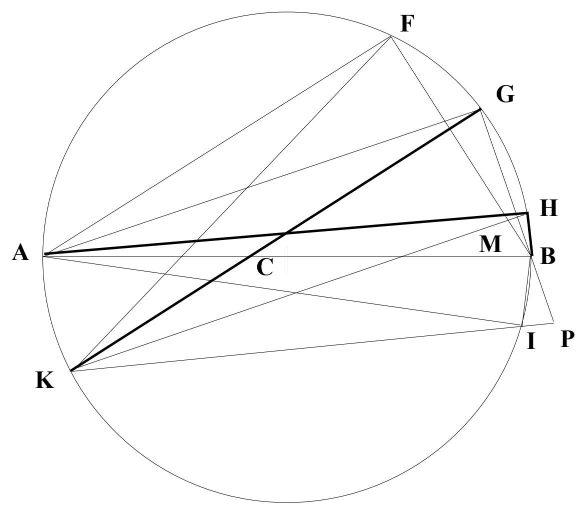
Now the diameter is not cut by equal sections, but between sections the circumference shall be cut in the point B: with the lines GK, KP, GBP drawn as above.



Since the arcs GAK, AGH are equal, (with the placing of the equal segments AK, GK) and the subtended [chords] AH, GK is equal: and it is as previously, the angle ABG to be equal to that which the chord makes to any of these equal arcs with the diameter, and GKP equal to the angle at the centre [is the double arc at the periphery and the single arc at the centre for equality]. Therefore the triangle GKP is isosceles, it shall be similar to that from the two semi-diameters, and the line equal to one of the subtended segments: therefore as the semi-diameter to the said subtended [chord; these are not shown in the present diagram], so KG that is AH to GP. But BP is itself equal to BI, for in the inscribed quadrilateral KGBI, the exterior angles PBI, BIP is equal to the interior [angles] GKI, KGB: therefore the triangle BIP is isosceles, similar to the triangle GKP, and with the equal sides BI, BP. But as the semi-diameter to the equal subtended part, and so here too BH to that equal segment, for the angle HBF is equal to that which shall be from the diameter and by is subtended by any equal segment, and the angle HBF, equal to the angle in the centre of the section is equal to the one section is insisted upon. Then the triangle BMH is isosceles, and the triangle FKM similar to this, and thus as the radius to subtended [chord] of any of the equal segments, thus BH to HM the difference of the lines HK, KF; but HK is itself equal to AI, (for the segments AK, HI are equal) by which with the addition of KI is in common, will become KIH, AKI equal, and with KF itself equal to AG, for the segments AK, FG, is put equal, by the addition of the common AF, KAF, AFG, shall become equal. In the same way, so AG or KF the difference of the lines BF, BM or BH. Q.E.D.

THEOREM IX.

If there shall be right angled triangles of equal hypotenuse, of which the first acute angle shall be a sub-multiple ratio to the acute angles of the succeeding triangles in order, for the acute clearly half of the second, to the third a third, to the fourth a quarter, and with that order continuing: so is constructed a series of lines in continued proportion, of which the first shall be equal to the semi-diameter, the second to the perpendicular of the first angle, between succeeding continued proportionals, and the succession of the base of triangles and perpendiculars, this is the equality.

Twice the first, less the third continued proportion, is equal to the base of the second triangle.

Three times the second, less the fourth, for the perpendicular of the third triangle.

Twice the first, less four times the third, plus the fifth, for the base of the fourth triangle.

Five times the second, less five times the fourth, plus the sixth, for the perpendicular of the fifth triangle.

Twice the first, less nine times the third, plus six times the fifth, less the seventh, for the base of the sixth triangle.

Seven of the second, less fourteen times the fourth, plus seven times the sixth, less the eighth, for the perpendicular of the seventh triangle.

Twice the first, less sixteen times the third, plus twenty times the sixth, less eight times the seventh, plus the ninth, for the base of the eighth triangle.

Nine times the second, less thirty the fifth, plus twenty seven times the sixth, less nine times the eighth, plus the tenth, for the perpendicular of the ninth.

And so indefinitely, by inverting those which have been shown in Theorem six, in order.

Let the semicircle be such as the above, of which the periphery shall be cut into any number of equal parts, and from the ends of the diameter is drawn the sides of right angled triangles; and let the semi-diameter be X, the perpendicular of the sub-multiple triangle truly shall be B: and let it become as X to B, so B to B sq./X, which is taken away from the diameter or twice X , the base of the second triangle is $\frac{\text{X sq.2 - B sq.}}{\text{X}}$ from the preceding Theorem. Thus, X to B shall become, as $\frac{\text{X sq.2 - B sq.}}{\text{X}}$ to $\frac{\text{X sq.by B 2 - B cub.}}{\text{X sq.}}$ this itself is added to B (since indeed with the bases is decreased, the perpendiculars are is increased) it shall become $\frac{\text{X sq.by B 3 - B cub.}}{\text{X sq.}}$ by is equal to the perpendicular of the third triangle. And by the same method:

$\frac{\text{X sq.sq.2 - B sq.by Xsq.4+Bsq.sq.}}{\text{X cub.}}$ the base of the fourth triangle.

$\frac{\text{X sq.sq.by B 5 - B cub.by Xsq. 5+Bsq.cub.}}{\text{X sq.sq.}}$ the perpendicular of the fifth triangle.

$\frac{\text{X sq.sq.sq. 2 - Xsq.sq.by B sq. 9+Xsq.by Bsq.sq.6 - Bcub.cub.}}{\text{X sq.cub.}}$ the base of the sixth triangle.

$\frac{\text{X sq.sq.sq. by B 7 - Xsq.sq.by B cub. 14+Xsq.by Bsq.cub. 7 - Bsq.sq.cub.}}{\text{X sq.sq.sq.}}$ the perpendicular of the seventh triangle.

$\frac{\text{X sq.sq.sq. sq. 2 - Xsq.sq.sq. by B sq. 16+Xsq.sq. by Bsq.sq. 20 - X sq. by B cub.cub. 8 + Bsq.sq.sq.}}{\text{X sq.sq.cub.}}$ the base of the eighth triangle.

$\frac{\text{X sq.sq.sq. sq. by B 9 - Xsq.sq.sq. by B cub. 30+Xsq.sq. by Bsq.cub. 27 - X sq. by B sq.sq,.cub . 9 + B cub.cub.cub.}}{\text{X sq.cub.cub.}}$ the perpendicular of the ninth triangle.

And by this indefinite progression, if it pleases the table of Theorem six is adopted. In note form the first of the continued proportions shall be 1 And with the same common hypotenuse of the right-angled triangles.

Truly the second continued proportion 1N. And the same perpendicular is understood to pertain to the sub-multiple angle triangle.

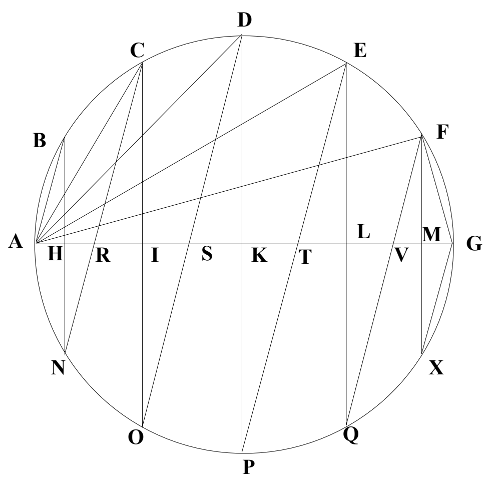
			2	–1 Q		*Base*		*Double*
			3 N	–1 C		*Perp.*		*Treble*
		2	– 4 Q	+ 1QQ		*Base*		*Quadruple*
		5 N	– 5 C	+ 1QC	*Will Be*	*Perp.*	*of*	*Quintuple*
	2	– 9 Q	+ 6 QQ	– 1CC	*equal to*	*Base*	*the*	*Sextuple*
	7 N	– 14 C	+ 7 QC	– 1QQC		*Perp.*	*Angle*	*Septuple*
2	–16 Q	+ 20 QQ	– 8 CC	+ 1QCC		*Base*	*of the*	*Octuple*
9 N	–30 C	+ 27 QC	– 9 QQC	+ 1CCC		*Perp.*		*Nontuple*

And so henceforth, by inverting the order of Theorem six, according as it has been determined there, except that in two places[in each column] *there should be alternating changes of sign .*

THEOREM X.

If the semi-circumference of a circle should be cut in some number of equal parts, and from the end of the diameter is drawn [chords] to any number of points of section, it is as the smallest [chord] is drawn to the diameter, thus is the sum from the diameter and the minimum, and besides that [chord AF] of which the square is added to the minimum [chord FG] squared gives the square of the diameter, to double the sum from all [the chords from the end of the diameter] is drawn.

Let the diameter in the points A, B, C, D, E, F, G, H be cut in any number of equal parts , and from the end of the diameter A, lines to the sections is drawn AB, AC, AD, AE, AF, AG, and dividing too the remaining semi-circle in just as many equal segments as before AN, NO, OP, PQ, QX, XG, then the points equally distant from the ends of the diameter is connected by lines, because the diameter is is cut at right angles, and let these be BHN, CIO, DKP, ELQ, FMX, of which the alternate ends, is connected by the transversals CRN, DSO, ETP, FVQ, GX.



Therefore BN is itself equal to CA, CN itself to AD, and CO itself to AE, and DO itself to AF, by the same way, and the lines EP, EQ, FQ, FX, from the same is taken is shown equal singly. And GX itself is equal to BA, and so the lines AB, BN, CN, CO, DO, DP, EP, EQ, FQ, FX, GX, are equal to double of AB, AC, AD, AE, AF themselves, and besides for the diameter DP or AG itself, is added with the other diameter AG, all of those is said with the diameter AG, to the double of all of AB, AC, AD, AE, AF, AG. So it is as AH to HB, it is AB or GF to FA, so HR to HN, and RI to IC, and IS to IO, and SK to KD, and KT to KP, and TL to LE, and VL to LQ, and VM to MF, and GM to MX: so therefore as AB to AF, so AG to all the perpendiculars together in the diameter AG, and interchanging, as AB to AG, so AF to all the perpendiculars together. Again, AH is to AB, that is FG or AB to AG, so HR to RN, and RI to RC, and IS to SO, and SK to SD, and KT to TP, and TL to TE, and LV to VQ, and VM to VF, and MG to GX, therefore as AB to AG, thus all AH, RI, etc.; that is AG to all the transversals together: but it was as AB to AG, so AF to all the perpendiculars, therefore it is as AB to AG so is the sum of AF, AG to all the transversals and perpendiculars, and with is taken together as AB to AG so is composed from the three AF, AG, AB to the composition from all the perpendiculars, with all the transversals, and the line AG [recall the diameter is added twice above], that is (as has been shown) to the double of the sum of AB, AC, AD, AE, AF. Q.E.D.

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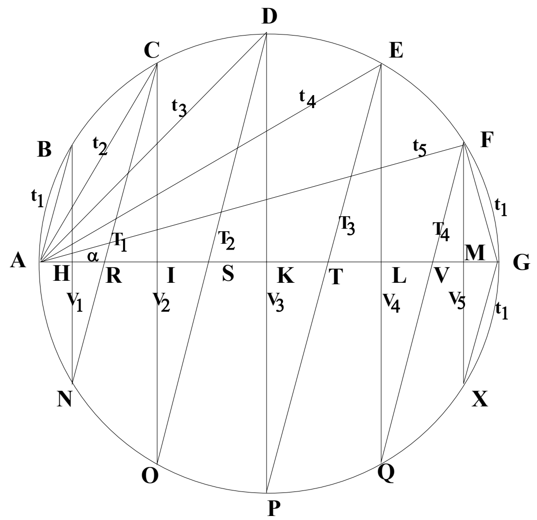
Note: The theorem states symbolically, that:
AB/AG = (AF + AG + AB)/[$2(t_1 + t_2 + t_3 + t_4 + t_5 + t_6)$], and is equivalent to an addition formula for $\sin\alpha + \sin 2\alpha + .. + \sin(n-1)\alpha$, or its cosine equivalent.
The diagram considers n = 5.
Equal arcs subtend equal angles on the circumference (see diagram on next page), and chords of equal length. Let the vertical chords from the left be V_1, V_2, V_3, V_4, and V_5, while the long transverse chords are T_1, T_2, T_3, T_4, and the chords in the semi-circle

from A are t_1, t_2, t_3, t_4, and t_5, as shown. We now follow the above argument: $V_1 = t_2$, $T_1 = t_3$, $V_2 = t_4$, $T_2 = t_5$. Also, the chords $T_3 = t_5$, $V_4 = t_4$, $T_4 = t_3$,



and $V_5 = t_2$. The sum of the chords $S = t_1 + V_1 + T_1 + V_2 + T_2 + 2V_3 + T_3 + V_4 + T_4 + V_5 + t_1 = t_1 + t_2 + t_3 + t_4 + t_5 + 2V_3 + t_5 + t_4 + t_3 + t_2 + t_1 = 2(t_1 + t_2 + t_3 + t_4 + t_5 + t_6)$, where the diameter, called t_6, is counted twice.

The angle subtended by the smallest chord gives $\tan\alpha = AH/HB = AH/(V_1/2)$, and also $\tan\alpha = AB/AF = t_1/t_5 = HR/HN = HR/(V_1/2) = RI/IC = RI/(V_2/2) = IS/IO = IS/(V_2/2) = SK/KD = SK/(V_3/2) = KT/KP = KT/(V_3/2) = TL/LE = TL/(V_4/2) = VL/LQ = VL/(V_4/2) = VM/MF = VM/(V_5/2) = GM/MX = GM/(V_5/2)$.

Hence, by adding the proportionalities,
$AG = AH + HR + \ldots + MG = (AB/AF).(V_1 + .. + V_5)$,
or **AF/(sum of perpendiculars) = AB/AG**.

Also, as $AH/AB = \sin\alpha = FG/AG = AB/AG = HR/RN = RI/RC = IS/SO = SK/SD = KT/TP = TL/TE = LV/VQ = VM/VF = MG/GX$; thus $AH = AB \sin\alpha$, $HR = RN \sin\alpha$, $RI = RC \sin\alpha$, etc. Hence, AG is the sum of $AH + HR + .. + MG = \sin\alpha.(t_1 + T_1 + T_2 + T_3 + T_4 + t_1) = (AB/AG)$.(sum of transversals). Hence,
AG/(sum of transversals) = AB/AG.
Now, $(t_1 + T_1 + T_2 + T_3 + T_4 + t_1 + V_1 + V_2 + V_3 + V_4 + V_5) + AG = (AG/AB).(AF + AG + AB) = 2(t_1 + t_2 + t_3 + t_4 + t_5 + t_6)$, which is Viete's Theorem X.

In terms of multiples of α:
The r.h.s. becomes:
$2AG.(1 + \cos\alpha + \cos2\alpha + \cos3\alpha + \cos4\alpha + \cos5\alpha) = AG.(1 + \sin\alpha + \cos\alpha)/\sin\alpha = (AG/AB).(AF + AG + AB)$, the l.h.s. [Recall that α and 5α are complementary angles]. The summation of a series of sines or cosines has thus been effected.

End of note.

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Hence from no one having knowledge of the Mystery before, so with Arithmetic as with Geometry, by Analysis the section of angles has been explained.

Problem I.

By is given the number for the ratio of the angles, to give the ratio of the sides.
This has been shown abundantly by Theorem 3.

Problem II.

To make as number to number, so as angle to angle.
In the ratio of the smaller to the larger or is unequal it can be sufficient from Theorems 3, 6, and 9: but in the ratio of the greater inequality is deduced in this way from Theorems 5 and 8.

Consequences.

Since the same line is inscribed in a circle, [which is] not a diameter, with two arcs is subtended, of which one is the smaller to the semi-circumference of the circle, the other larger, the equalities [equations] between the subtended [chord] of the minor or of the major [arc] and the subtended [chord] for the minor segment, it will apply for the similar larger segment too, and to the subtended [chord] and so for the rest of the arcs which is equal multiples, [in] the larger or smaller [arcs], that are compounded by the circle.

And not even that stands in the way with what Theorem 8 has considered, when the segment of the larger arc is distributed in equal parts, either if the diameter is incident on the [ends of] the sections, or otherwise, not even the sign [the text has '*adsectionum*' , which Witmer considers a misprint for '*affectionum*' , which I have called '*sign*'] of the pertaining amount to the perpendiculars is changed, or of the numbers from the order of the prescribed series [for the perpendiculars]: in the second figure of that Theorem, since it is allowed from the subtended [chord] AH to infer the sum of the chords GB, BI, after however from the difference of the said sum and of the chord GB inferred before, it is from the chord BI, finally is inferred the difference of the chords from the point A is drawn, to which in the section the point I itself and the other nearest, are incident: which therefore have been changed thus, by this operation henceforth having been restored.

But in the progression of the bases, when the equal segments exceed the semi-perimeter, (as has been shown by Theorem five) with the order of the homogenous [terms] is inverted under the step [i.e. the order of the iteration], and from the progression sometimes shall be the amount Theorem nine has uncovered, if which from twice the most analogous chord and the other of the smallest in the other semi-periphery [arc], to the difference of itself and of the others on either side is restored: because indeed from the prescription to the fifth Theorem the progression is made, is consistent enough, and hence the truth of the corollaries has been made known.

And for these sections of the given angles is deduced from Theorems six and nine Problems, the work made accessible providing many a use, and which is able to be extended indefinitely by the given reasoning. Let an example be proposed.

Exercises I.

To cut a given angle in three equal parts.

With the radius X placed, or half the diameter of the circle, B the chord of the angle to be subdivided, E the chord of the segment.

X squared by 3 E, less E cubed, is equal to X squared by B,
[Thus: $3EX^2 - E^3 = BX^2$; this follows from Theorem VII, where in modern terms we have: $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$, and on setting $E = 2X\sin\alpha$ and $B = 2X\sin 3\alpha$, the result follows], and [the value of] E shall become two-fold:
1. The chord of the third [part] of the arc;
2. The chord for the third [part] of the arc for the rest of the whole circle.

If indeed as has been shown above, the equality between the chord to the smaller or larger segment and the chord for the segment of the smaller, shall pertain too for the chord for the similar segment of the larger.

Exercises II.

To cut a given angle in five equal parts.

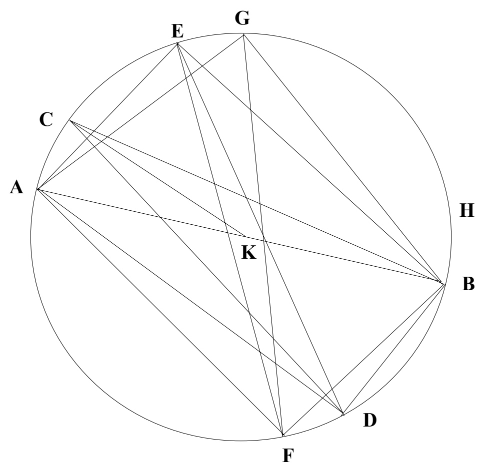
With the same as supposed before.
X squared - squared by E5 — X squared by 5 E cubed + E squared-cubed, is equal to X squared-squared by B. [I.e. $5EX^4 — 5E^3X^2 + E^5 = BX^4$; now, on substituting $E = 2X\sin\alpha$ and $B = 2X\sin 5\alpha$, we obtain $5\sin\alpha — 20\sin^3\alpha + 16\sin 5\alpha = \sin 5\alpha$, as required.]

And E shall become threefold [in meaning]:
1. The chord of the fifth part of the arc.
2. The chord of the fifth part of the remaining arc of the whole circle.

3. The chord of the arc is composed from the fifth of the arc, and from the double of the fifth [of the arc] of the whole circle.

Because finally, with the lack of expertise with Analysis probably [made] less with the practise from an example, thus it is worthwhile to been shown [this one].

Let the circle of which the diameter is AB be cut into unequal segments, the larger of which [is] BAG, and the smaller BHG, and let BH be the fifth part of the minor segment, to which is added the segment HGC, equal to the double of the fifth part of the four right angles [that is, the angle HGC is $4\pi/5$ or 144^0; and assume the angle subtended at the centre by the arc BH is α, and BG 5α: hence the angle at the centre by chord BC is $(\alpha + 4\pi/5)$]



And so for the segments; five times BGC is equal to twice the arc is subtended by four right angles going around the point K [the centre of the circle], and the arc BG itself besides; and five times the segment BC is taken, that sum is measured: is repeated five times, and the segments shall be BGC, CAD, DBE, EAF, FBG. [Thus, the radius KB rotates anti-clockwise by $(\alpha + 4\pi/5)$ to KC, and similarly KC $\rightarrow$ KD, repeated in total 5 times, eventually returning to the initial conditions; the chord BC $\rightarrow$ CD, with the angle BCD is $(\pi/5 - \alpha)$, then CD $\rightarrow$ DE $\rightarrow$ EF $\rightarrow$ FG. The arc AC subtends $(\pi/5 - \alpha)$ at K.] And thus the segment BD is left if BGC be taken twice from the whole circle, the double of CA itself to be left from the semicircle, by the segment is taken from BGC. [The arc BD remains after two rotations, with angle at K = $2(\pi/5 - \alpha)$; is double the arc AC] And therefore EC itself [shall be] equal to BD (for the lines BC, CD with the lines CD, DE themselves shall be equal) shall be the double too of CA itself, and also from this EA the triple of CA itself. [Alternately, the angle rotated about K by KB $\rightarrow$ KE is now $3(\alpha + 4\pi/5)$: hence, $3\alpha + 2\pi/5 +$ angle EKC = angle CKB = $4\pi/5 + \alpha$, giving EKC = $2(\pi/5 - \alpha)$, and AKE $3(\pi/5 - \alpha)$].

And by the same reasoning, since the lines CB, CD themselves are equal, ED, EF themselves too are equal, and the segments BD, AE are equal, [the text has BD, DF] and the segment BF is the quadruple of the segment CA itself; similarly, GF and EF are equal [the text has GE and EC, which again is nonsense: one wonders why Witmer did not comment on this, or make the correction, in his translation.], hence GA is five times the segment CA. Therefore, ACB is the right-angled triangle of the simple angle [ABC], BAD of the double, BAE the triple, BAF the quadruple, and BGA the quintuple.

[Alternately, the angle rotated about K by KB $\rightarrow$ KF is now $4(\alpha + 4\pi/5)$: hence, $4\alpha + 6\pi/5 + BKF = 2\pi$, giving BKF $= 4(\pi/5 - \alpha)$. Similarly, KB $\rightarrow$ KG rotates by $5(\alpha + 4\pi/5)$, hence the angle BKG = 5α].

With the semi-diameter therefore for the first place in continued proportion, and CB itself for the second; and this series is continued: from Theorem six GB is the equal to the sixth, less five times the fourth, plus five times the second.

In note [form], *let* CK *be* 1, CB 1 N. 1 QC - 5C + 5N, *is equal to GB itself.*

Exercises III.

To divide a given angle in seven equal parts.

With the above assumed.

X cubed-cubed by 7E — X squared-squared by 14E cubed + X squared by 7 E squared-cubed —E squared-squared-cubed, is equal to X cube-cubed by B.

E shall be fourfold [in possible meaning].

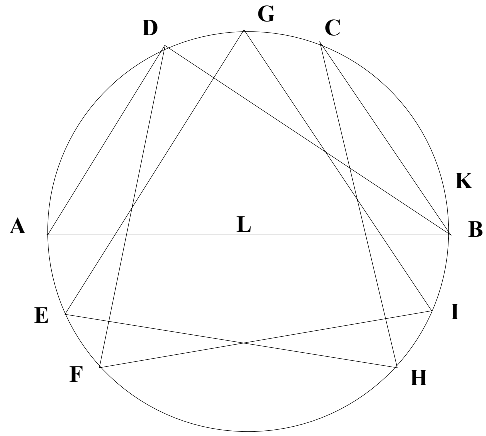
1. The seventh part of a chord of an arc.

The seventh part of the remaining arc to the whole circle.

3. The chord of the arc made from the seventh part and twice the seventh part of the whole circle.
4. The chord of the arc made from the seventh part and the quadruple of the seventh part of the whole circle.

Which is made clear too, and thus. Let the circle of which the diameter shall be AB, in which the line CB is is subtended, and let the seventh part of the arc CB be BK, to which is is added the arc KD, equal to twice the seventh part of the whole circumference, and BD, DA is drawn. I say seven times the angle DLB [the text has DBA] to be equal to four right angles, and beyond this of the arc BLC [the text has ADC], and hence from Theorem 6 the equation between the lines BC, BK may be explained too to be from the terms BD, BC.

BD, DF, FI, IG, GE, EH, HC, is equal, and indeed the arc DB is taken seven times, four right angles is measured, that is the whole arc of the circle AB [taken] twice (as the angles are is found on the circumference), and besides the arc BC [in addition]: therefore DB by is inscribed in continuation seven times, finally returns to the point C. And indeed the lines BD, DF is put equal, the arc FB is the double of DA itself, (for FB is the complement of twice DB itself to the whole circle.) For the arc FB truly is equal to the arc DBI; when indeed BD, DF themselves are equal to the lines DF, FI: and thus the arc ADI is the triple of AD itself. In the same manner, because the lines IG, GE themselves too are equal to DF, FI: the arc IE is the double of AD itself, and the whole ADBE five times AD itself: and the lines EH, HC the same equal too, accordingly the total ADHEC is seven times AD. Therefore from Theorem six,



With the semi-diameter AL is placed first in continued proportion, DB itself the second, and this series is continued, the line CB is equal to seven times the second, less fourteen times the fourth, plus seven times the sixth, less the eighth.

In note [form] *it will be: AL* 1. *DB* 1N. 7 N — 14 C + 7 QC — 1 QQC, *is equal to CB itself. And similarly with the four* [terms] *determined the same equality is explained.*

But truly from the preface of the fifth and eighth Theorems, it is consistent with the ambiguous Problems of the propositions indefinitely. Who should wish an example, consult the reply of Viete to the problem of Adrian Roman.

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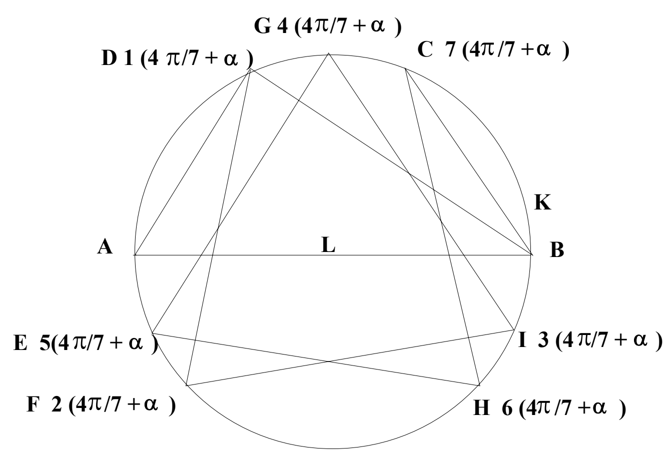
Note: Let BK subtend the angle α at the centre L, with 7α is the angle for BC. The arc KD subtends $4\pi/7$. Hence, the arc DLB subtends the angle $(4\pi/7 + \alpha)$ at the centre, while 7 times this arc gives $4\pi + 7\alpha$. On rotation by $(4\pi/7 + \alpha)$ repeatedly, the equal chords DF, FI,…., EH, HC are generated. Thus, the arc DA = $\pi - \alpha - 4\pi/7 = 3\pi/7 - \alpha$, while for the second rotation, the arc FB = $2\pi - (2\alpha + 8\pi/7) = 2(3\pi/7 - \alpha)$ = $2 \times$ arc DA. The third rotation results in the chord FI, where the reflex angle BLI is $3(4\pi/7 + \alpha)$, with the acute angle BLI = $2\pi/7 - 3\alpha$; the arc DBI is the sum of the arcs DB and BI, which is $(4\pi/7 + \alpha) + 2\pi/7 - 3\alpha = 2(3\pi/7 - \alpha)$ = arc FB. The obtuse angle subtended by the chord AI is (π + arc BLI) = $9\pi/7 - 3\alpha = 3(3\pi/7 - \alpha) = 3 \times$ arc DA; and the rest in like manner, or following the scheme suggested by Viete, where the arcs of equal chords are simply added as one goes around the circle twice in seven steps.



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Problem III.

Lines from the arcs of the circle is subtended in arithmetical progression, is inscribed from a largest or smallest is given, and with the second from the first, to be reckoned from symmetric numbers.

And this work to be deduced from Theorems 6, 7 & 9, because is clearly set out and shown there.

Problem IV.

To find the sum of lines is subtended from the arcs of a circle in arithmetical progression, from a largest or smallest is given.

This is shown by Theorem 10.

Corollary.

Therefore a Mathematician can safely and easily construct a table from Analysis, and the construction is examined and guided by these principles of analysis, and the method is given for any powers to be resolved, either with or without signs.

But to the construction, first is sought a single small perpendicular, so that it is able to made accurate, by this method.

1. From the root of the extreme and mean ratio from a hypothetical [line], to be given of 18 parts [i.e. degrees].
2. From this by the work of quinquisection, should be found the perpendicular of 3.36'.
3. From the work of trisection, is given the perpendicular of 20 parts.
4. And thus from is trisected, the perpendicular of the parts 6.40'.
5. By the work of bisection, the perpendicular of the parts 3.20'.
6. From the difference of the perpendiculars of the parts 3.36' and 3.20', is given the perpendicular of the small part 16', from the first Theorem.
7. By bisection are produced the perpendiculars of the small parts 8'. 4'. 2'. 1'.

And by stepping back to angles in multiples of the ratio, the remainder is completed with symmetric numbers from the rule of Theorem six.

This work indeed constitutes the elements of angular sections, drawn from the well of purest Analysis. The main propositions about these and many other fine investigations have been deduced by the greatest mathematician for many an age, Francis Vieta. Proposed and thought out long ago, but communicated to others without demonstration; now at last the same propositions are here presented complete and perfect, derived from the principles of Geometry. Mathematicians can accept these, confirmed by my own study, and with equal good men consult.

The End.